Now consider the Pullback diagram:

$$\begin{tikzcd}
\ker p_2 \arrow[r, "p"] \arrow[rrrr, bend left, "0"] \arrow[rrrd, bend right, "p_1 \circ c_2"'] & \ker f_1 \arrow[r, "q"] & \ker p_2 \arrow[r, "c_2"] & P \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d, "g"] \\
 & & & X \arrow[r, "f"] & Z
\end{tikzcd}$$

$$\underbrace{p_1 \circ c_2 \circ q}_{c_1 (*)} \circ p = c_1 \circ p = p_1 \circ c_2$$

Hence

$$\begin{tikzcd}
\ker p_2 \arrow[r, "c_2"] \arrow[rr, bend left, "0"] \arrow[dr, bend right, "p_1 \circ c_2"'] & P \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d] \\
 & X \arrow[r] & Z
\end{tikzcd}$$

are the same Pullback so we have $\boxed{c_2 = c_2 \circ q \circ p}$ so $q \circ p = Id$ since c_2 is a monomorphism.

(5) Let

$$\begin{tikzcd}
P \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "g"] \\
X \arrow[r, "f"'] \arrow[ur, phantom, "\circlearrowleft"] & Z
\end{tikzcd}$$

be a pullback. Then we have a exact sequence

$$P \xrightarrow{\binom{p_1}{p_2}} X \oplus Y \xrightarrow{(-f,g)} Z$$

Moreover saying that P is a pullback is equivalent to $\binom{p_1}{p_2} = \ker(-f,g)$ indeed

$$\begin{tikzcd}
P \arrow[r, "\binom{p_1}{p_2}"] & X \oplus Y \arrow[r, "(-f{,}g)"] & Z \\
 & T \arrow[ul, "\exists! \alpha"] \arrow[u, "\binom{a}{b}"] \arrow[ur, "0"'] &
\end{tikzcd}$$

$(-f,g)\binom{a}{b} = 0 \iff af = gb$

$\exists \alpha$ s.t. $\binom{p_1}{p_2}\alpha = \binom{a}{b}$ is equivalent to $\begin{cases} \alpha p_1 = a & \text{and} \\ \alpha p_2 = b \end{cases}$

If f is an epimorphism then $(-f,g)$ is an epimorphism so we have a short exact sequence $0 \to P \xrightarrow{\binom{p_1}{p_2}} X \oplus Y \xrightarrow{(-f,g)} Z \to 0$ saying $\mathrm{Coker}\binom{p_1}{p_2} = (-f,g)$.

by an argument dual of above
$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, two heads, "g"] \\ X \arrow[r, two heads, "f"'] & Z \end{tikzcd}$$
is also a pullback square.

It remains to see that p_2 is an epi

$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[drr, bend right=60, "0"'] & Y \arrow[d, "g"] \arrow[dr, bend left, "a-b"] & \\ X \arrow[r, two heads, "f"'] \arrow[rr, bend right, "0"'] & Z \arrow[r, dashed, "\exists!\alpha"] & W \end{tikzcd}$$

$(a-b)\circ p_2 = 0$

We have $(a-b)p_2 = 0 = 0\circ p_1$
so by universal property of pushout
$\exists!\ \alpha : Z \to W$ s.t. $\begin{cases} \alpha g = a-b \\ \alpha f = 0 \end{cases}$

but f epi $\Rightarrow \alpha = 0 \Rightarrow a-b=0$ □

<u>Thm 5.21</u> Let $\mathcal{A}$ be an abelian category

(1) Let $(P_\lambda)_{\lambda\in\Lambda}$ be a family of objects of $\mathcal{A}$. Then $\coprod_{\lambda\in\Lambda} P_\lambda$ is projective iff P_λ is projective $\forall \lambda\in\Lambda$

(2) P is projective iff $\forall\ X \overset{f}{\twoheadrightarrow} P$ $\exists\ s : P \to X$; $fs = 1_P$

+ dual for injective

<u>Proof</u> (1) if P_λ is projective $\forall\lambda\in\Lambda$

$$\begin{tikzcd} & P_\lambda \arrow[d, "i_\lambda"] \arrow[ddl, bend right, "f_\lambda"'] \\ & \sqcup P_\lambda \arrow[dl, dashed, "\exists!\bar{f}"'] \arrow[d, "f"] \\ M \arrow[r, two heads, "\alpha"'] & N \arrow[r] & 0 \end{tikzcd}$$

by projectivity $f\circ i_\lambda$ lifts along α $\exists\ f_\lambda$ now by univ prop of coproduct

$$\begin{tikzcd} P_\lambda \arrow[r, "i_\lambda"] \arrow[dr, bend right, "f_\lambda"'] & \sqcup P_\lambda \arrow[d, dashed, "\exists!\bar{f}"] & P_{\lambda'} \arrow[l, "i_{\lambda'}"'] \arrow[dl, bend left, "f_{\lambda'}"] \\ & M & \end{tikzcd}$$

and $\forall\lambda$ $\alpha\circ\bar{f}\circ i_\lambda = \alpha\circ f_\lambda$
$= f\circ i_\lambda$

So $\alpha \circ f$ and f are two solutions for
$$\begin{tikzcd} P_\lambda \arrow[r, "i_\lambda"] \arrow[dr, bend right, "f\circ i_\lambda"'] & \coprod P_\lambda \arrow[d, "f"', shift right] \arrow[d, "\alpha\circ f", shift left] & P_{\lambda'} \arrow[l, "i_{\lambda'}"'] \arrow[dl, bend left, "f\circ i_{\lambda'}"] \\ & N & \end{tikzcd}$$

so equal.

Conversely assume that $\coprod_{\lambda\in\Lambda} P_\lambda$ is projective and consider

$$\begin{tikzcd} & P_\lambda \arrow[d, "f"] & \\ M \arrow[r, "\alpha"'] & N \arrow[r] & 0 \end{tikzcd}$$

then

$$\begin{tikzcd} P_\lambda \arrow[r] \arrow[dr, bend right, "f"'] & \coprod P_\lambda \arrow[d, dashed, "\exists! F"] & P_{\lambda'} \ \lambda \neq \lambda' \arrow[l] \arrow[dl, bend left, "0"] \\ & N & \end{tikzcd}$$

So

$$\begin{tikzcd} & P_\lambda \arrow[d, "i_\lambda"] & \\ & \coprod P_\lambda \arrow[dl, dashed, "\exists \tilde{F}"'] \arrow[d, "F"] & \\ M \arrow[r, "\alpha"'] & N \arrow[r] & 0 \end{tikzcd}$$

then

$$\begin{tikzcd} & P_\lambda \arrow[dl, "\tilde{F}\circ i_\lambda"'] \arrow[d, "f"] & \\ M \arrow[r, "\alpha"'] & N \arrow[r] & 0 \end{tikzcd}$$

and $\alpha \circ \tilde{F} \circ i_\lambda = F \circ i_\lambda = f$.

(2) If P is projective and
$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists s"'] \arrow[d, equal] \\ X \arrow[r, two heads, "f"'] & P \end{tikzcd}$$

Conversely if every epi toward P splits

$$\begin{tikzcd} & P \arrow[d, "f"] & \\ M \arrow[r, two heads, "\alpha"'] & N \arrow[r] & 0 \end{tikzcd}$$

Take pull back

$$\begin{tikzcd} 0 \arrow[r] & \ker(p_1) \arrow[r] \arrow[d, "\simeq"] & P\times_N M \arrow[r, "p_2"] \arrow[d, "p_1"] & P \arrow[r] \arrow[l, bend right, dashed, "\exists s"'] \arrow[d, "f"] \arrow[dl, dashed] & 0 \\ 0 \arrow[r] & \ker\alpha \arrow[r] & M \arrow[r, "\alpha"'] & N \arrow[r] & 0 \end{tikzcd}$$

by previous lemma the first row is exact and the surjection splits □

Thm 5.22 Let A be an associative ring with 1. Then $\mathrm{Mod}\,A$ and $A\,\mathrm{Mod}$ have enough projectives and injectives

Proof (1) - A is projective since $\mathrm{Hom}_A(A,-) \simeq \mathrm{Id}_{\mathrm{Mod}A}$ exact

- Free module $\coprod_{\lambda\in\Lambda} A$ is projective

- Every module is quotient of a free module

$m\in M$ $f_m : A \longrightarrow M$, $1 \longmapsto m$

$$\coprod_{m\in M} A \xrightarrow{(f_m)_{m\in M}} M \quad \text{is surjective}$$

(2) ⚠ Analogue for injective is not dual $(\mathrm{Mod}\,A)^{op} \not\cong \mathrm{Mod}\,B$.

<u>claim 1</u> (1) $\mathbb{Q}/\mathbb{Z}$ is an injective abelian group (TD)

(2) $D = \mathrm{Hom}_{\mathbb{Z}}(-,\mathbb{Q}/\mathbb{Z}) : \mathrm{Mod}A \longrightarrow A\mathrm{Mod}$ is exact

$$M_A \longmapsto \mathrm{Hom}_{\mathbb{Z}}(M_A,\mathbb{Q}/\mathbb{Z})$$

since $\mathbb{Q}/\mathbb{Z}$ is injective

(3) $M \longrightarrow DD(M)$ <u>is injective</u>

$$m \longmapsto ev_m$$

indeed if $ev_m = 0$ consider the subgp of M generated by m

$M' = \langle m \rangle$ $\boxed{m \neq 0}$

and set

$$\phi_m : M' \longrightarrow \mathbb{Q}/\mathbb{Z}$$

$$m \longmapsto \begin{cases} [\tfrac{1}{n}] & \text{if } o(m) = n < +\infty \\ [\tfrac{1}{2}] & \text{otw} \end{cases}$$

then

$$\begin{array}{ccc} M' & \hookrightarrow & M \\ {\scriptstyle \phi_m}\downarrow & \swarrow {\scriptstyle \exists \phi} & \\ \mathbb{Q}/\mathbb{Z} & & \end{array}$$

by injectivity of $\mathbb{Q}/\mathbb{Z}$

Moreover $\phi(m) = ev_m(\phi) = \phi(m) \neq 0$ this is a contradiction.

(4) $M \in \mathrm{Mod}A$ then $D(M) \in A\mathrm{Mod}$ so $\exists\, P \twoheadrightarrow D(M)$ with P projective. Since D is exact we have

$$M \hookrightarrow DD(M) \hookrightarrow D(P)$$

finally $D(P)$ is projective: choose $P = \coprod A$ then $D(P) = \prod D(A)$ and $D(A)$ is injective since

$$\mathrm{Hom}_A(-, D(A)) \simeq \mathrm{Hom}_A(-, \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z}))$$

$$\simeq \mathrm{Hom}_{\mathbb{Z}}(- \otimes_A A, \mathbb{Q}/\mathbb{Z}) \simeq \mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z}) .$$

∎

Rem we see that $-\otimes_A A$ is exact since it is the identity functor

Def 5.21 An M-module M is flat if $M \otimes_A -$ is exact

Prop Projective $\Rightarrow$ flat

proof $M = \bigoplus_{i\in I} M_i$ is flat iff M_i flat $\forall i$

since $M \otimes_A -$ is right exact flat $\Leftrightarrow M \otimes_A -$ preserves injective maps

$$X \xhookrightarrow{\phi} Y \qquad \begin{array}{ccc} M\otimes X & \xrightarrow{\phi\otimes 1} & M\otimes Y \\ \wr\| & \circlearrowleft & \wr\| \\ \bigoplus_{i\in I} M_i\otimes X & \xrightarrow[\bigoplus_{i\in I}\phi_i\otimes 1]{} & \bigoplus_{i\in I} M_i\otimes Y \end{array}$$

so injectivity of $\phi\otimes 1$ $\Leftrightarrow$ injectivity of $\phi_i \otimes 1$ $\forall i$

P projective $\Leftrightarrow \exists n \in \mathbb{N}, \exists Q$ $A^n = P\oplus Q$ then A^n flat $\Rightarrow A^n$ flat $\Rightarrow P$ flat. $\square$

Chapter 6 Resolutions and derived functors

Def 6.1 (1) Let $\mathcal{A}$ be an abelian category, and $M \in Ob(\mathcal{A})$.

A projective resolution of M is

- a complex $(P_n)_{n\geq 0}$ of projective objects
- A morphism $\varepsilon : P_0 \to M$ s.t. $Im(d_1) \subseteq Ker(\varepsilon)$
- The "augmented" complex

$$\cdots \to P_2 \to P_1 \to P_0 \xrightarrow{\varepsilon} M \to 0$$

is exact

(2) An injective (co)resolution of M is a complex of cochains $(I^n)^*_{n\geq 0}$ of injective objects with a morphism $\varepsilon : M \hookrightarrow I^0$ with $Im(\varepsilon) \subset \ker d^0$ s.t. the augmented complex

$$0 \to M \to I^0 \to I^1 \to \cdots$$

is exact.

Prop 6.2 If $\mathcal{A}$ has enough projectives (resp injectives) then every object has a projective (injective) resolution.

Proof $M \in \mathcal{A}$ $\exists \pi: P_0 \to M \to 0$ so get ses $0 \to \ker(\pi) \hookrightarrow P_0 \to M \to 0$

Now $\exists P_1 \overset{\pi_1}{\twoheadrightarrow} \ker(\pi)$

$$\begin{tikzcd}
 & 0 & & \\
0 \arrow[r] & \ker(\pi) \arrow[u] \arrow[r, hook] & P_0 \arrow[r] & M \arrow[r] & 0 \\
 & P_1 \arrow[u, "\pi_1"] \arrow[ur, dashed, "d_1"'] & & \\
 & \ker \pi_1 \arrow[u] & & \\
 & 0 \arrow[u] & &
\end{tikzcd}$$

by induction we construct the resolution. $\square$

Thm 6.3 ["functoriality" of resolutions] $\mathcal{A}$ category with enough projectives
$f: M \to M' \in \mathcal{A}$ $(P_\bullet)$ be a projective resolution of M
$(Q_\bullet)$ " " " of M'

then there exists $(f_n): P_\bullet \to Q_\bullet$ a morphism of chain complexes s.t. $H_0(f) = f$ and any two such chain maps are homotopic.

Proof

$$\begin{tikzcd}
P_1 \arrow[r, "d_1"] \arrow[d, dashed, "f_1"'] & P_0 \arrow[r, "\pi_0"] \arrow[d, dashed, "f_0"] & M \arrow[r] \arrow[d, "f"] & 0 \\
Q_1 \arrow[r] & Q_0 \arrow[r, "\pi_0'"'] & M' \arrow[r] & 0
\end{tikzcd}
\qquad
\begin{tikzcd}
 & P_0 \arrow[d, "\pi_0"] \arrow[ddl, dashed, "\exists f_0"'] \\
 & M \arrow[d, "f"] \\
Q_0 \arrow[r] & M' \arrow[r] & 0
\end{tikzcd}$$

Now $\pi_0' f_0 d_1 = f \pi_0 d_1 = 0$

so $\mathrm{Im}(f_0 d_1) \subset \ker(\pi_0') = \mathrm{Im}(d_1')$ hence

$$\begin{tikzcd}
 & P_1 \arrow[d, "f_0 d_1"] \arrow[dl, dashed, "\exists f_1"'] \\
Q_1 \arrow[r, "d_1"'] & \mathrm{Im}\, d_1' \arrow[r] & 0
\end{tikzcd}$$

Then induction:

Now if we have two lifts

$$\begin{tikzcd}
P_1 \arrow[r, "d_1"] \arrow[d, shift right, "f_1'"'] \arrow[d, shift left, "f_1"] & P_0 \arrow[r, "\pi_0"] \arrow[d, shift right, "f_0'"'] \arrow[d, shift left, "f_0"] & M \arrow[d, "f"] \\
Q_1 \arrow[r, "d_1'"'] & Q_0 \arrow[r, "\pi_0'"'] & M'
\end{tikzcd}$$

we have $\pi_0' f_0 = \pi_0' f_0'$ so $\pi_0'(f_0 - f_0') = 0$ so $\mathrm{Im}(f_0 - f_0')$

$\subseteq \ker(\pi_0') = \mathrm{Im}(d_1')$

$$\begin{tikzcd}
 & P_0 \arrow[d, "f_0 - f_0'"] \arrow[dl, dashed, "\exists s_0"'] \\
Q_1 \arrow[r, "d_1"'] & \mathrm{Im}(d_1')
\end{tikzcd}$$

s.t. $f_0 - f_0' = d_1 s_0$

Now $d_1'(f_1 - f_1' - s_0 d_1) = d_1' f_1 - d_1' f_1' - d_1' s_0 d_1$
$= (f_0 - f_0') d_1 - (f_0 - f_0') d_1 = 0$

so $\mathrm{Im}(f_1 - f_1' - s_0 d_1) \subseteq \ker d_1' = \mathrm{Im}\, d_2'$

$$\begin{tikzcd} & P_1 \arrow[dl, dashed, "\exists s_1"'] \arrow[d, "f_1 - f_1' - s_0 d_1"] \\ P_2 \arrow[r, "d_2'"'] & \mathrm{Im}\, d_2' \end{tikzcd}$$

$\square$

<u>Rem</u> (1) Only need top complex to be projective and bottom to be exact

(2) Can do the dual for injective coresolution.

<u>Thm 6.4</u> (a) Let $\mathcal{E}$ be an abelian category with enough projectives. There exists a functor $P_R : \mathcal{E} \to K_+(\mathrm{Proj}(\mathcal{E}))$ sending X to one of its projective resolution. Moreover

(a) $H_0 \circ P_R \simeq \mathrm{Id}_{\mathcal{E}}$

(b) $H_i \circ P_R = 0 \ \forall i \neq 0$

(b) Dual for injective coresolution

<u>Proof</u> If $P_\bullet, Q_\bullet$ are two projective resolutions of M then

$$\begin{tikzcd} P_\bullet \arrow[r] \arrow[d, "f"] \arrow[dd, bend right=60, "\mathrm{Id}"'] & M \arrow[d, "\mathrm{Id}_M"] \\ Q_\bullet \arrow[r] \arrow[d, "g"] & M \arrow[d, "\mathrm{Id}_M"] \\ P_\bullet \arrow[r] & M \end{tikzcd}$$

then $gf \sim \mathrm{Id}_P$ and conversely $fg \sim \mathrm{Id}_Q$

so $P_\bullet, Q_\bullet$ are isomorphic in $K_+(\mathrm{Proj}(\mathcal{E}))$. Moreover once we choose a projective resolution for each M, this is functorial

$$\begin{tikzcd} P_M \arrow[r] \arrow[d, "f_\bullet"'] & M \arrow[d, "f"] \\ P_N \arrow[r] \arrow[d, "g_\bullet"'] & N \arrow[d, "g"] \\ P_L \arrow[r] & L \end{tikzcd}$$

$(g_\bullet f_\bullet)$ lifts gf so it is homotopic to $P_R(gf)$

Since a projective resolution is exact everywhere but at $i=0$ we get $H_i(P_M) = 0 \ \forall i \neq 0$ and $H_0(P_M) \simeq M$

finally
$$\begin{array}{ccccc} P_0 & \xrightarrow{\varepsilon} & M & \to & 0 \\ {\scriptstyle f_0}\downarrow & & \downarrow f & & \\ Q_0 & \xrightarrow[\varepsilon']{} & N & \to & 0 \end{array}$$
we get $f = H_0(f_\bullet)$

So $H_0 \circ P_R \simeq Id_{\mathcal{E}}$ □

Rem in the proof we see that
$$\begin{array}{ccccc} \cdots \to P_1 & \to & P_0 & \to & 0 \\ \downarrow & & \downarrow \varepsilon & & \\ 0 & \to & M & \to & 0 \end{array}$$
is a qis

so any M is qis to one of its projective resolution but M is not homotopy equivalent to its projective resolution.

⇝ define category $\mathcal{D}(\mathcal{E})$ where qis = iso called the derived category of $\mathcal{E}$. Done by localization!

If $F: \mathcal{A} \to \mathcal{B}$ is an additive functor between abelian categories then F extends as

(•) $F: Ch_\bullet(\mathcal{A}) \to Ch_\bullet(\mathcal{B})$

$$\begin{array}{ccccc} \cdots \to X_n & \xrightarrow{d_n} & X_{n-1} \to \cdots & \longmapsto & \to F(X_n) \xrightarrow{F(d_n)} F(X_{n-1}) \to \\ {\scriptstyle f_n}\downarrow & & \downarrow {\scriptstyle f_{n-1}} & & \downarrow \quad \circlearrowleft \quad \downarrow \\ Y_n & \xrightarrow[d_n^Y]{} & Y_{n-1} & & F(Y_n) \xrightarrow[F(d_n)]{} F(Y_{n-1}) \end{array}$$

works since $F(d_n) \circ F(d) = F(d^2) = F(0) = 0$

• Also a functor $F: K(\mathcal{A}) \to K(\mathcal{B})$

$f \sim 0$
$$\begin{array}{ccccc} & \to X_n & \to & X_{n-1} & \\ & \swarrow s_n & \downarrow f_n & \swarrow s_{n-1} & \downarrow \\ Y_{n+1} & \to Y_n & \to & Y_{n-1} & \end{array}$$

$f_n = s_{n-1} d_n^X + d_{n+1}^Y s_n$

then $F(f_n) = F(s_{n-1}) F(d_n^X) + F(d_{n+1}^Y) F(s_n)$ ~ 0

Def 6.5 Let $F: \mathcal{A} \to \mathcal{B}$ be an additive functor from $\mathcal{A}$ to $\mathcal{B}$ two abelian such that $\mathcal{A}$ has enough projectives

The nth derived functor $L_n F$ is the composition

$$\mathcal{A} \xrightarrow{P_R} K_+(\mathrm{Proj}\,\mathcal{A}) \xrightarrow{F} K_+(\mathcal{B}) \xrightarrow{H_n} \mathcal{B}$$

Concretely: If $X \in \mathcal{A}$ then $L_n F(X) = H_n(F(P_X))$ where P_X is a projective resolution of X

And $L_n(f) = H_n(F(f_\bullet))$ where $f_\bullet$ is the lift of f along proj resolutions.
$f : X \to Y$

Rem This is more a construction of $L_n F$ than a definition. For a definition one can use derived categories (Verdier 1963) or "universal δ-functors" (Grothendieck 57)

Def 6.6 If $\mathcal{A}$ has enough injectives, the nth right derived functor $R^n F$ of F is the composite

$$\mathcal{A} \xrightarrow{I_R} K^+(\mathrm{Inj}\,\mathcal{A}) \xrightarrow{F} K^+(\mathcal{B}) \xrightarrow{H^n} \mathcal{B}$$

$\hookrightarrow R^n F(X) = H^n(F(I_X))$ where I_X is an injective coresolution of X.

Thm 6.7: We assume that $\mathcal{A}$ has enough projectives

(1) $L_n F$ is additive $\forall n \geq 0$

(2) F right exact then $L_0 F \simeq F$

(3) If $0 \to A \to B \to C \to 0$ is a ses in $\mathcal{A}$ then we have a long exact sequence

$$\cdots \to L_2F(C) \to L_1F(C) \xrightarrow{\delta} L_0F(A) \xrightarrow{} L_0F(B) \to L_0F(C) \to 0$$

Moreover there is a naturality in the ses.

(4) $\forall P$ projective $L_n F(P) = 0$ $\forall n > 0$

Proof (1) every functor F, H_n and P_R ~~are~~ is additive, so the composition is.

(2) $M \in \mathcal{A}$ $P_\bullet \xrightarrow{\pi} M$ projective resolution

then $0 \to \underset{\mathrm{Im}(d_1)}{\overset{\|}{\ker(\pi)}} \to P_0 \to M \to 0$ exact so $F(\mathrm{Ind}_1) \to F(P_0) \to F(M) \to 0$ exact

so $\mathrm{coker}(F(\mathrm{Ind}_1)) = F(P_0)/F(\mathrm{Ind}_1) \simeq F(M)$

$$\begin{gathered}\mathrm{coker}(F(\mathrm{Ind}_1)) \\ \parallel \\ H_0(F(P_\bullet)) = L_0(F)(M)\end{gathered}$$

(3) (•) ~~claim~~ there is a short exact sequence $0 \to P_\bullet \to Q_\bullet \to R_\bullet \to 0$ of complexes of projective objects and a morphism of ses

$$\begin{tikzcd} 0 \arrow[r] & P_\bullet \arrow[r] \arrow[d, "f"] & Q_\bullet \arrow[r] \arrow[d, "g"] & R_\bullet \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

such that each vertical complex is exact and each row is exact (The Horseshoe Lemma).

idea each ses of projectives splits so $Q_\bullet = R_\bullet \oplus Q_\bullet$!

to construct the resolution Q: Pick one P_0 for A one R_0 for C and

$$\begin{tikzcd} 0 \arrow[r] & P_0 \arrow[r] \arrow[d, "f"] & P_0 \oplus R_0 \arrow[r] \arrow[d, "\alpha f \oplus \tilde{g}"] & R_0 \arrow[r] \arrow[d, "g"] & 0 \\ 0 \arrow[r] & A \arrow[r, "\alpha"] & B \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

$\tilde{g} \quad R_0$

check that this is a morphism of ses. The snake lemma gives: $B \to C$ lifts

$$\begin{tikzcd} 0 \arrow[r] & \ker f \arrow[r] & \ker h \arrow[r] & \ker g & \\ 0 \arrow[r] & P_0 \arrow[r] \arrow[d, "f"] & P_0 \oplus R_0 \arrow[r] \arrow[d, "h"] & R_0 \arrow[r] \arrow[d, "g"] & 0 \\ 0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\ & 0 \arrow[r] & \mathrm{coker}\, h \arrow[r] & 0 & \end{tikzcd}$$

$\Rightarrow$ h surjective + $0 \to \ker f \to \ker h \to \ker g \to 0$ exact + induction.

(•) F is additive so F is exact on split ses so

$0 \to F(P_\bullet) \to F(Q_\bullet) \to F(R_\bullet) \to 0$ is exact and we have long exact sequence in homology

$$\begin{array}{ccccccccc} \cdots \to H_1(F(R_\bullet)) & \xrightarrow{\delta} & H_0(FP_\bullet) & \to & H_0F(Q) & \to & H_0F(R) & \to & 0 \\ & & \parallel & & \parallel & & \parallel & & \\ \cdots \to L_1F(C) & \xrightarrow{\delta} & L_0F(A) & \to & L_0F(B) & \to & L_0F(C) & \to & 0 \end{array}$$

(4) clear since $\cdots 0 \to \overset{0}{P} \to 0$ is a projective resolution of P. $\square$

Thm 6.8 If $\mathcal{A}$ has enough injectives

(1) $R^n F$ is additive $\forall n \geq 0$

(2) F is left exact then $R^0 F \simeq F$

(3) If $0 \to A \to B \to C \to 0$ ses then we have a long exact sequence

$0 \to R^0 F(A) \to R^0 F(B) \to R^0 F(C) \xrightarrow{\delta} R^1 F(A) \to \cdots$

Case of contravariant functors $F : \mathcal{E} \to \mathcal{D}$ contravariant iff
$F : \mathcal{E}^{op} \to \mathcal{D}$ covariant

Hence F is left exact iff $\forall\ \underset{\text{exact}}{0 \to A \to B \to C \to 0}$ we have $\underset{\text{exact}}{0 \to F(C) \to F(A) \to F(B)}$

Moreover Proj in $\mathcal{E}$ $\leftrightarrow$ Inj in $\mathcal{E}^{op}$

Def 6.9 (1) $F : \mathcal{A} \to \mathcal{B}$ contravariant additive functor. Then
$R^i F(A) = H^i(F(P))$ where P is a projective resolution of A

(2) $L_i F(B) = H_i F(I)$ where I is an injective coresolution of B.

2) Example of Ext functor

Def 6.10 (1) If $\mathcal{E}$ has enough projective objects we denote
$\forall A, B \in \mathrm{ob}(\mathcal{E})$

$$\mathrm{Ext}^i(A, B) = \left[R^i(\mathrm{Hom}(-, B)\right](A)$$

(2) If $\mathcal{E}$ has enough injective objects we denote
$\forall A, B \in \mathrm{ob}(\mathcal{E})$

$$\mathrm{Ext}^i(A, B) = \left[R^i \mathrm{Hom}(A, -)\right](B)$$

Thm 6.11 (Balancing ext) If $\mathcal{E}$ has enough projectives and enough injectives we have $R^i \mathrm{Hom}(-, B)(A) \simeq R^i \mathrm{Hom}(A, -)(B)$

Proof not so easy see Weibel, or look at section 4. $\square$

Examples: $R = \mathbb{Z}$

M be a torsion abelian gp. Then to compute $\mathrm{Ext}^*(M, \mathbb{Z})$

Can choose: find an injective resolution of $\mathbb{Z}$ or find a projective resolution of M

We have $0 \to \mathbb{Z} \xrightarrow{i} \mathbb{Q} \xrightarrow{\pi} \mathbb{Q}/\mathbb{Z} \to 0$ exact and $\mathbb{Q}, \mathbb{Q}/\mathbb{Z}$ are injective

so we get $0 \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0 \ldots$ is an injective resolution

we apply $\mathrm{Hom}(M, -)$ and we have

$$0 \to \mathrm{Hom}(M, \mathbb{Q}) \xrightarrow{\pi \circ -} \mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z}) \to 0 \to 0 \ldots \quad (*)$$

since $\mathrm{Hom}(M, -)$ left exact $\ker(\pi \circ -) = \mathrm{Hom}(M, \mathbb{Z}) = 0$ as M is of torsion.

$\hookrightarrow \mathrm{Ext}^0(M, \mathbb{Z}) = \mathrm{Hom}(M, \mathbb{Z}) = 0$

$\mathrm{Ext}^i(M, \mathbb{Z}) = 0 \ \forall i > 0$

what is $\mathrm{Ext}^1(M, \mathbb{Z})$

look at the long exact sequence $\nearrow$ OR just look at $*$ $\mathrm{Hom}(M, \mathbb{Q}) = 0$ so $\mathrm{Ext}^1 = \mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z})$

$$0 \to \underset{\substack{\| \\ 0}}{\mathrm{Hom}(M, \mathbb{Z})} \to \underset{\substack{\| \\ 0}}{\mathrm{Hom}(M, \mathbb{Q})} \to \mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\cong} \mathrm{Ext}^1(M, \mathbb{Z}) \to \underset{\substack{\| \\ 0}}{\mathrm{Ext}^1(M, \mathbb{Q})}$$

since $\mathbb{Q}$ injective

so $\mathrm{Ext}^1(M, \mathbb{Z}) \cong \mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z})$

Thm 6.12 For an A-module X the following are equivalent

(1) X projective

(2) $\mathrm{Hom}_A(X, -)$ exact

(3) $\forall i > 0 \ \forall Y \in \mathrm{Mod}A \ \mathrm{Ext}^i(X, B) = 0$

(4) $\forall B \in \mathrm{Mod}A \quad \mathrm{Ext}^1(X, B) = 0$

Proof (1) $\Leftrightarrow$ (2) by definition

(2) $\Leftrightarrow$ (4) long exact sequence

(3) $\Rightarrow$ (4) clear

(4) $\Rightarrow$ (3) Ext^i can be computed by using Ext^1 if $i \geq 2$ "argument de décalage"

indeed $0 \to B \to I \to \Omega^{-1}B \to 0$ exact

long exact sequence:

$$\mathrm{Hom}(X,\Omega^{-1}B) \to \mathrm{Ext}^1(X,B) \to \underset{\substack{\| \\ 0}}{\mathrm{Ext}^1(X,I)} \to \mathrm{Ext}^1(X,\Omega^{-1}B) \xrightarrow{\sim} \mathrm{Ext}^2(X,B) \to \underset{\substack{\| \\ 0}}{\mathrm{Ext}^2(X,I)}$$

So $\mathrm{Ext}^2(X,B) \simeq \mathrm{Ext}^1(X,\Omega^{-1}B)$ and more generally

$\mathrm{Ext}^n(X,B) \simeq \mathrm{Ext}^{n-1}(X,\Omega^{-1}B) \quad \forall n \geq 2.$ $\square$

(3) Example of Tor functor

Def 6.13 A k-alg over commutative ring k. Then $\mathrm{Tor}_i^A(M,N) := L_i(-\otimes_A N)(M)$

Thm 6.14 $L_i(-\otimes N)[M] \simeq L_i(M\otimes -)[N]$

Example B abelian gp $\mathrm{Tor}(\mathbb{Z}/p\mathbb{Z},B)$?

$0 \to \mathbb{Z} \xrightarrow{\times p} \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$ exact sequence hence $0 \to \mathbb{Z} \xrightarrow{\times p} \mathbb{Z} \to 0$

is a projective resolution of $\mathbb{Z}/p\mathbb{Z}$ so

$\mathrm{Tor}_i^{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z},B)$ is the ith homology of

$$\begin{array}{ccccccc} 0 & \to & \mathbb{Z}\otimes B & \xrightarrow{\times p} & \mathbb{Z}\otimes B & \to & 0 \\ & & \wr\| & & \wr\| & & \\ 0 & \to & B & \xrightarrow{\times p} & B & \to & 0 \end{array}$$

So $\mathrm{Tor}_i^{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z},B) = \begin{cases} B/pB & \text{if } i=0 \\ \{b\in B \,;\, pb=0\} & \text{if } i=1 \\ 0 & \text{otw.} \end{cases}$

Prop 6.15 Let B be an A-module. TFAE

(1) B is flat

(2) $-\otimes_A B$ is exact

(3) $\mathrm{Tor}_i^A(-,B)=0 \quad \forall i\geq 1$

(4) $\mathrm{Tor}_1^A(-,B)=0$

(4) Extensions à la Yoneda

In this section $\mathcal{A}$ is an abelian category. $A, B \in ob(\mathcal{A})$

An extension E of B by A is an exact sequence

$$0 \to A \to E \to B \to 0$$

two extensions of B by A are equivalent if:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "1_A"'] \arrow[dr, phantom, "\circlearrowleft"] & E \arrow[r] \arrow[d, "\exists\varphi"] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[r] \arrow[d, "1_B"] & 0 \\
0 \arrow[r] & A \arrow[r] & E' \arrow[r] & B \arrow[r] & 0
\end{tikzcd}$$

by short five lemma φ is an isomorphism. ($\Rightarrow$ equivalence relation)

<u>Def</u> <u>C.17</u> $\mathcal{E}xt(A,B) = \{\text{extensions of } B \text{ by } A\}/_{\sim}$ is called the (Yoneda)-Ext group

↳ Make it a $\boxed{\text{bi-functor}}$: $\mathcal{A}^{op} \times \mathcal{A} \to Set$

$f : B_1 \to B_2$ $\qquad \mathcal{E}xt(A, B_1) \xrightarrow{f\bullet-} \mathcal{E}xt(A, B_2)$

$$\begin{tikzcd}
0 \arrow[r] & B_1 \arrow[r, "\alpha"] \arrow[d, "f"] & E \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B_2 \arrow[r] & E' \arrow[r] & A \arrow[r] & 0
\end{tikzcd} \longmapsto \text{Pushout of } f \text{ and } \alpha$$

$g : A_1 \to A_2$ $\qquad \mathcal{E}xt(A_2, B) \xrightarrow{-\bullet g} \mathcal{E}xt(A_1, B)$

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r] & E \arrow[r] & A_2 \arrow[r] & 0 \\
0 \arrow[r] & B \arrow[r] \arrow[u, equal] & E' \arrow[r] & A_1 \arrow[r] \arrow[u] & 0
\end{tikzcd} \longmapsto \text{Pullback}$$

Many small things have to be checked eg $f_1 \cdot - \circ f_2 \cdot - = f_1 f_2 \cdot -$ etc

good to use:

<u>Lem</u> <u>6.18</u> Let $\begin{tikzcd} E \arrow[r, "\alpha"] & A_2 \arrow[r] & 0 \\ & A_1 \arrow[u, "\beta"] & \end{tikzcd}$ be two morphisms with α epi.

Then $\begin{tikzcd} E \arrow[r] & A_2 \\ E' \arrow[u] \arrow[r] & A_1 \arrow[u] \end{tikzcd}$ is a pullback iff $\begin{tikzcd} 0 \arrow[r] & \ker\alpha \arrow[r] & E \arrow[r] & A_2 \arrow[r] & 0 \\ 0 \arrow[r] & \ker\alpha \arrow[r] \arrow[u, equal] & E' \arrow[r] \arrow[u] & A_1 \arrow[r] \arrow[u] & 0 \end{tikzcd}$