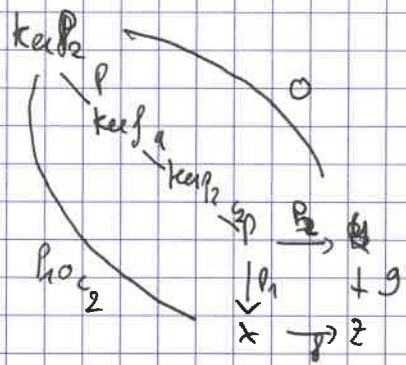
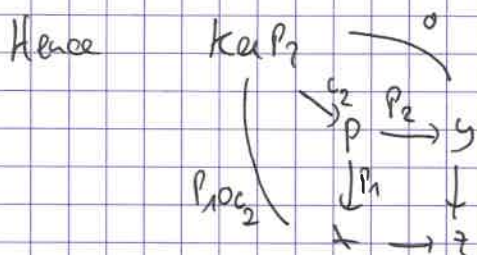


Now consider the Pullback Diagram:



$$P_1 \circ c_2 \circ q \circ p = c_1 \circ p = P_1 \circ c_2$$



are the same pullback so we have

$$c_2 = c_2 \circ q \circ p \text{ so } q \circ p = \text{Id since}$$

c_2 is a monomorphism.

(5) Let $P \xrightarrow{P_2} Y$ be a pullback. Then we have a

$$\begin{array}{ccc}
 P & \xrightarrow{P_2} & Y \\
 \downarrow P_1 & \hookrightarrow & \downarrow g \\
 X & \xrightarrow{f} & Z
 \end{array}$$

exact sequence $P \xrightarrow{\begin{pmatrix} P_1 \\ P_2 \end{pmatrix}} X \oplus Y \xrightarrow{(-f, g)} Z$

Moreover saying that P is a pullback is equivalent to $\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \text{ker}(-f, g)$

indeed

$$\begin{array}{ccc}
 P & \xrightarrow{\begin{pmatrix} P_1 \\ P_2 \end{pmatrix}} & X \oplus Y \xrightarrow{(-f, g)} Z \\
 \uparrow \text{ker} & & \uparrow \text{ker} \\
 \exists! \alpha & \xrightarrow{\begin{pmatrix} a \\ b \end{pmatrix}} & \begin{pmatrix} a \\ b \end{pmatrix} \in \begin{pmatrix} X \\ Y \end{pmatrix}
 \end{array}$$

$$(-f, g) \begin{pmatrix} a \\ b \end{pmatrix} = 0 \Leftrightarrow af = gb$$

$$\exists \alpha \text{ s.t. } \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} \alpha = \begin{pmatrix} a \\ b \end{pmatrix}$$

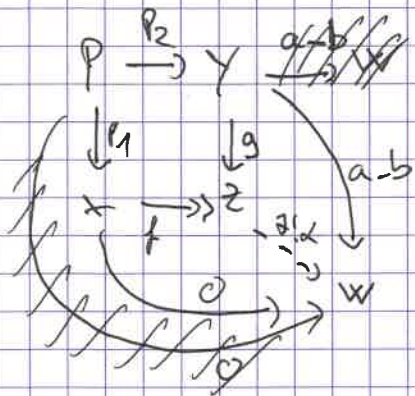
is equivalent to $\begin{cases} \alpha P_1 = a \\ \alpha P_2 = b \end{cases}$

If f is an epimorphism then $(-f, g)$ is an epimorphism so we have

a short exact sequence $0 \rightarrow P \xrightarrow{\begin{pmatrix} P_1 \\ P_2 \end{pmatrix}} X \oplus Y \xrightarrow{(-f, g)} Z \rightarrow 0$ saying $\text{ker} \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$.

by an argument dual of above $\begin{array}{ccc} P & \xrightarrow{p_1} & Y \\ p_1 \downarrow & & \downarrow p_2 \\ X & \xrightarrow{f} & Z \end{array}$ is also a pushout square.

It remains to see that p_2 is an epi



$$(a-b) \circ p_2 = 0$$

we have $(a-b) \circ p_2 = 0 = 0 \circ p_1$

so by universal property of pushout

$$\exists! \alpha: Z \rightarrow W \text{ s.t. } \begin{cases} \alpha \circ f = 0 \\ \alpha \circ p_2 = 0 \end{cases}$$

but f epi $\Rightarrow \alpha \circ 0 = 0 \Rightarrow \alpha = 0$

□

Thm 5.21 Let \mathcal{A} be an abelian category

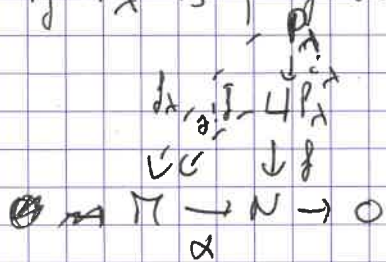
(1) Let $(P_\lambda)_{\lambda \in \Lambda}$ be a family of objects of \mathcal{A} . Then $\coprod_{\lambda \in \Lambda} P_\lambda$

is projective iff P_λ is projective $\forall \lambda \in \Lambda$

(2) P is projective iff $\forall X \xrightarrow{d} P \exists s: P \rightarrow X; fs = 1_P$

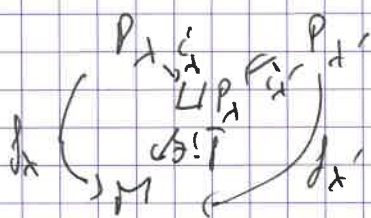
+ dual for injective

Proof (1) if P_λ is projective $\forall \lambda \in \Lambda$



by projectivity $f \circ i_\lambda$ lifts along $\alpha \exists f_\lambda$ Now by unic prop

of coproduct



and $\forall \lambda \exists f_\lambda \circ i_\lambda = \alpha \circ f_\lambda$
 $f_\lambda \circ i_\lambda = f \circ \beta_\lambda$
 $= f \circ \beta_\lambda$

So $\alpha \circ f$ and f are two solutions for

$$\begin{pmatrix} P \\ \downarrow f \\ N \end{pmatrix} \xrightarrow{\alpha} \begin{pmatrix} P \\ \downarrow f \\ N \end{pmatrix} \circ \alpha^{-1} \quad (18)$$

So equal.

Conversely assume that $\bigcup_{\lambda \in I} P_\lambda$ is projective and consider

$$\begin{array}{ccc} & P_\lambda & \\ & \downarrow f & \\ M & \xrightarrow{\alpha} & N \rightarrow 0 \\ & \downarrow \alpha & \\ & \bigcup_{\lambda \in I} P_\lambda & \\ & \downarrow f & \\ M & \xrightarrow{\alpha} & N \rightarrow 0 \end{array}$$

Then

$$\begin{array}{ccc} P_\lambda & \xrightarrow{\alpha} & P_{\lambda'} \quad \lambda \neq \lambda' \\ & \downarrow f & \\ & \bigcup_{\lambda \in I} P_\lambda & \\ & \downarrow f & \\ & N & \end{array} \rightarrow 0$$

So

$$\begin{array}{ccc} \bigcup_{\lambda \in I} P_\lambda & \xrightarrow{\alpha} & N \rightarrow 0 \\ \downarrow f & & \\ M & \xrightarrow{\alpha} & N \rightarrow 0 \end{array}$$

Then

$$\begin{array}{ccc} \bigcup_{\lambda \in I} P_\lambda & \xrightarrow{\alpha} & N \rightarrow 0 \\ \downarrow f & & \\ M & \xrightarrow{\alpha} & N \rightarrow 0 \end{array}$$

$$\alpha \circ \bigcup_{\lambda \in I} P_\lambda \circ \alpha^{-1} = f \circ \bigcup_{\lambda \in I} P_\lambda = f$$

(2) If P is projective and $\begin{array}{c} \exists \alpha \\ \downarrow \\ \alpha \end{array} \parallel \begin{array}{c} P \\ \downarrow f \\ N \end{array}$

Conversely if every epi towards P splits

$$\begin{array}{ccc} & P & \\ & \downarrow f & \\ M & \xrightarrow{\alpha} & N \rightarrow 0 \end{array} \quad \begin{array}{ccccccc} 0 & \rightarrow & \ker(P_1) & \rightarrow & P_1 M & \xrightarrow{P_2} & P \rightarrow 0 \\ & & \downarrow \alpha & & \downarrow P_1 & \swarrow \alpha & \downarrow f \\ 0 & \rightarrow & \ker \alpha & \rightarrow & M & \xrightarrow{\alpha} & N \rightarrow 0 \end{array}$$

Take pull back

by previous lemma the first row is exact and the injection splits \square

Thm 5.22 Let A be an associative ring with 1. Then $\text{Mod } A$ and $A\text{-Mod}$ have enough projectives and injectives

Proof (i) - A is projective since $\text{Hom}_A(A, -) \simeq \text{Id}_{\text{Mod } A}$ exact

- Free module $\bigcup_{\lambda \in I} A$ is projective

- Every module is quotient of a free module

$$\text{MEM } f_m: A \rightarrow M$$

$\coprod_{M \in \mathcal{M}} A \xrightarrow{\text{inj}} M$ is injective

(2) \triangleleft A criterion for injective is not dual $(\text{Mod } A)^{\text{op}} \neq \text{Mod } B$.

claim 1 (1) \mathbb{Q}/\mathbb{Z} is an injective abelian group (TD)

(2) $D = \text{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z}) : \text{Mod } A \rightarrow A\text{Mod}$ is exact
 $M_A \mapsto \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$

since \mathbb{Q}/\mathbb{Z} is injective

(3) $M \rightarrow DD(M)$ is injective
 $m \mapsto \text{ev}_m$

in deed if $\text{ev}_m = 0$ consider the subgp of M generated by m

$M' = \langle m \rangle \quad m \neq 0$

and set $\phi_m : M' \rightarrow \mathbb{Q}/\mathbb{Z}$
 $m \mapsto \begin{cases} \frac{1}{n} & \text{if } \alpha(m) = n < \infty \\ \frac{1}{2} & \text{else} \end{cases}$

then $M' \hookrightarrow M$ by injectivity of \mathbb{Q}/\mathbb{Z}
 $\phi_m \downarrow \quad \downarrow \phi$
 $\mathbb{Q}/\mathbb{Z} \hookrightarrow D(M)$

Moreover $\phi(m) = \text{ev}_m(\phi) = \phi(m) \neq 0$ this is a contradiction.

(4) $M \in \text{Mod } A$ then $D(M) \in A\text{Mod}$ so $\exists P \twoheadrightarrow D(M)$ with P projective. Since D is exact we have

$$M \hookrightarrow DD(M) \hookrightarrow D(P)$$

finally $D(P)$ is projective: choose $P = \coprod A$ then $D(P) = \prod D(A)$ and $D(A)$ is injective since

$$\text{Hom}_A(-, D(A)) \simeq \text{Hom}_A(-, \text{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z}))$$

$$\simeq \text{Hom}_{\mathbb{Z}}(- \otimes_A A, \mathbb{Q}/\mathbb{Z}) \simeq \text{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z}) \quad \square$$

Rem we see that $-\otimes_A A$ is exact since it is the identity functor

Def 5.21 An M -module M is flat if $M \otimes_A -$ is exact

Prop Projective \Rightarrow flat

proof $M = \bigoplus_{i \in I} M_i$ is flat iff M_i flat $\forall i$

since $M \otimes_A -$ is right exact flat $\Leftrightarrow M \otimes_A -$ preserves injective maps

$$\begin{array}{ccc}
 x \xrightarrow{\phi} y & M \otimes x \xrightarrow{\phi \otimes 1} M \otimes y & \text{so injectivity of } \phi \otimes 1 \\
 \text{IS } \subset \text{ IS} & & \Leftrightarrow \text{injectivity of } \phi: x \rightarrow y \\
 \bigoplus_{i \in I} M_i \otimes x \xrightarrow{\bigoplus \phi_i \otimes 1} \bigoplus_{i \in I} M_i \otimes y & & \forall i
 \end{array}$$

P projective $\Leftrightarrow \exists n \in \mathbb{N}; \exists Q \text{ s.t. } A^n = P \oplus Q$ then $A^n \text{ flat} \Rightarrow A^n \text{ flat} \Rightarrow P \text{ flat. } \square$

Chapter 6 Resolutions and derived functors

Def 6.1 (1) Let \mathcal{A} be an abelian category, and $M \in \mathcal{A}$

A projective resolution of M is

- a complex $(P_n)_{n \geq 0}$ of projective objects
- A morphism $\Sigma: P_0 \rightarrow M$ s.t. $\text{Im}(d_n) \subseteq \text{ker}(\Sigma)$
- The "augmented" complex

$$\dots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \xrightarrow{\Sigma} M \rightarrow 0$$

is exact

(2) A injective (co)resolution of M is a complex of codomains

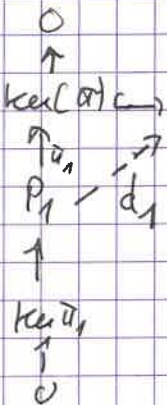
$(I^n)_{n \geq 0}$ of injective objects with a morphism $\Sigma: M \rightarrow I^0$ with $\text{Im}(\Sigma) \subseteq \text{ker } d^0$ s.t. the augmented complex

$$0 \rightarrow M \rightarrow I^0 \rightarrow I^1 \rightarrow \dots$$

is exact.

Prop 6.2 If \mathcal{A} has enough projectives (resp injectives) then every object has a projective (injective) resolution.

Proof $M \in \mathcal{A} \exists \pi: P_0 \rightarrow M \rightarrow 0$ so get seq $0 \rightarrow \ker(\pi) \rightarrow P_0 \rightarrow \pi \rightarrow 0$
 Now $\exists P_1 \xrightarrow{\pi_1} \ker(\pi)$



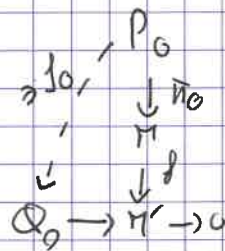
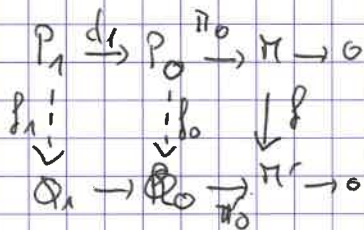
by induction we construct the resolution. □

Thm 6.3 ["factoriality" of resolutions] A category with enough projectives

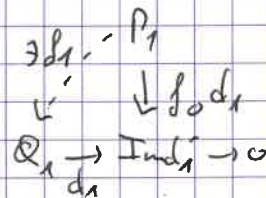
$f: M \rightarrow M' \in \mathcal{A}$ (P_\bullet) be a projective resolution of M
 (Q_\bullet) " " " " of M'

Then there exists $(f_n): P_n \rightarrow Q_n$ a morphism of chain complexes s.t. $H_0(f) = f$ and any two such chain maps are homotopic.

Proof

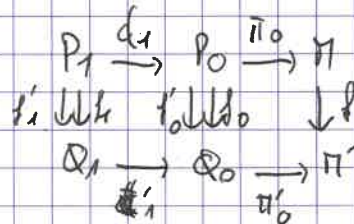


Now $\pi'_0 \circ f_0 \circ d_1 = f \circ \pi_0 \circ d_1 = 0$
 so $\text{Im}(f_0 \circ d_1) \subset \ker(\pi'_0) = \text{Im}(d'_1)$ hence



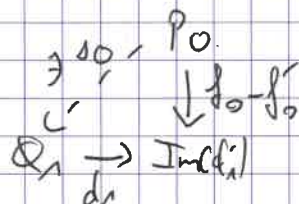
Then induction:

Now if we have two lifts



we have $\pi'_0 \circ f_0 = \pi'_0 \circ f'_0$ so $\pi'_0 (f_0 - f'_0) = 0$ so $\text{Im}(f_0 - f'_0) \subset \ker(\pi'_0) = \text{Im}(d'_1)$

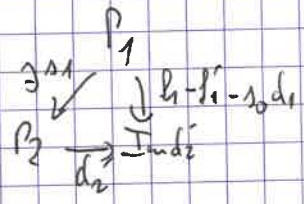
$\subset \text{Im}(d'_1)$



s.t. $f_0 - f'_0 = d_1$ so

Now $d_1(f_1 - f'_1 - s_0 d_1) = d_1 f_1 - d_1 f'_1 - d_1 s_0 d_1$
 $= (f_0 - f'_0) d_1 - (f_0 - f'_0) d_1 = 0$

so $\text{Im}(f_1 - f'_1 - s_0 d_1) \subseteq \text{ker } d_1 = \text{Im } d_2$



□

Rem (1) Only need top complex to be projective and bottom to be exact
 (2) Can do the dual for injective coresolution.

Thm 6.4 (a) Let \mathcal{E} be an abelian category with enough projectives.
 There exists a functor $P_R: \mathcal{E} \rightarrow K_+(\text{Proj}(\mathcal{E}))$ sending X to one of its projective resolutions. Moreover

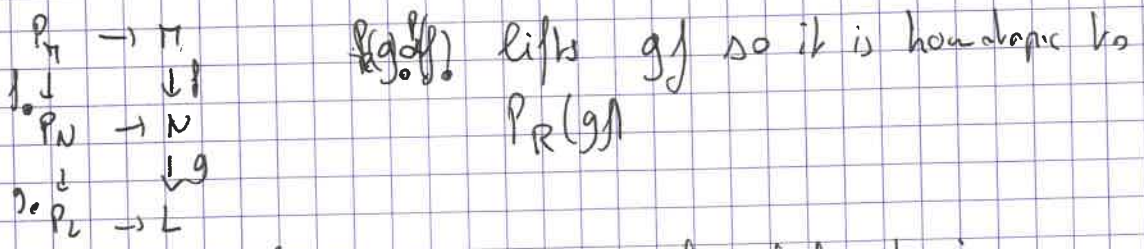
- (a) $H_0 \circ P_R \cong \text{Id}_{\mathcal{E}}$
- (b) $H_i \circ P_R = 0 \forall i \neq 0$

(a) Dual for injective coresolution

Proof If P_0, Q_0 are two projective resolutions of M then

$$\begin{array}{ccc}
 P_0 \rightarrow M & \text{ker } g \sim \text{Id}_M \text{ and } \text{coker } f \sim \text{Id}_M \\
 \downarrow & \downarrow \text{Id}_M \\
 Q_0 \rightarrow M & \\
 \downarrow g & \downarrow \text{Id}_M \\
 P_1 \rightarrow M &
 \end{array}$$

so P_0, Q_0 are isomorphic in $K_+(\text{Proj}(\mathcal{E}))$. Moreover once we choose a projective resolution for each M , this is functorial



Since a projective resolution is exact everywhere but at $i=0$
 we get $H_i(P_M) = 0 \forall i \neq 0$
 and $H_0(P_M) \cong M$

finally
$$\begin{array}{ccc} P_0 \xrightarrow{\varepsilon} M \rightarrow 0 \\ \downarrow \downarrow \\ P_1 \xrightarrow{\varepsilon'} N \rightarrow 0 \end{array}$$
 we get $f = H_0(f)$

So $H_0 \circ P_R \approx \text{Id}_E$ □

Rem in the proof we see that $\dots \rightarrow P_1 \rightarrow P_0 \rightarrow 0$ is a qis

$$\begin{array}{ccc} \downarrow & \downarrow \varepsilon \\ 0 & \rightarrow M & \rightarrow 0 \end{array}$$

so any M is qis to one of its projective resolutions but it is not homotopy equivalent to its projective resolution

we define category $D(E)$ where qis = is \circ called the derived category of E . Done by localization!

If $F: A \rightarrow B$ is an additive functor between abelian categories then F extends as

(b) $F: \mathcal{C}_1(A) \rightarrow \mathcal{C}_1(B)$

$$\begin{array}{ccc} \dots \rightarrow X_n \xrightarrow{d_n} X_{n-1} \rightarrow \dots & \xrightarrow{F(d_n)} & \dots \rightarrow F(X_n) \xrightarrow{F(d_n)} F(X_{n-1}) \rightarrow \dots \\ \downarrow \downarrow & & \downarrow \downarrow \\ Y_n \xrightarrow{f_n} Y_{n-1} & & F(Y_n) \xrightarrow{F(d_n)} F(Y_{n-1}) \end{array}$$

works since $F(d_n) \circ F(d_{n+1}) = F(d_n \circ d_{n+1}) = F(0) = 0$

• Also a functor $F: K(A) \rightarrow K(B)$

pro

$$\begin{array}{ccc} & \rightarrow X_n \rightarrow X_{n-1} \\ & \swarrow \downarrow d_n \searrow \\ Y_{n+1} & \rightarrow Y_n \rightarrow Y_{n-1} \end{array}$$

then $F(f_n) = F(s_{n+1})F(d_n^*) + F(d_{n+1}^*)F(s_n) \approx 0$

$$f_n = s_{n+1}d_n^* + d_{n+1}^*s_n$$

Def 6.5 Let $F: A \rightarrow B$ be an additive functor from A to B two abelian such that A has enough projectives

The nth derived functor $L_n F$ is the composition

$$A \xrightarrow{P_x} K_+(Proj A) \xrightarrow{F} K_+(B) \xrightarrow{H_n} B$$

Conceptually: If $x \in A$ then $L_n F(x) = H_n(F(P_x))$ where P_x is a projective resolution of x

And $L_n(f) = H_n(F(f_0))$ where f_0 is the lift of f along proj resolutions.
 $f: X \rightarrow Y$

Rem This is more a construction of $L_n F$ than a definition. For a definition one can use derived categories or "universal δ -functors" (Grothendieck 57) (Verdier 1963)

Def 6.6 If \mathcal{A} has enough injectives, the nth right derived functor $R^n F$ of F is the composite

$$A \xrightarrow{I_x} K^+(Inj A) \xrightarrow{F} K^+(B) \xrightarrow{H^n} B$$

$\hookrightarrow R^n F(x) = H^n(F(I_x))$ where I_x is an injective coresolution of x .

Thm 6.7: We assume that \mathcal{A} has enough projectives

(1) $L_n F$ is additive $\forall n \geq 0$

(2) F right exact then $L_0 F \cong F$

(3) If $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is a ses in \mathcal{A} then we have a long exact sequence

$$\dots \rightarrow L_2 F(C) \rightarrow L_1 F(C) \xrightarrow{\delta} L_0 F(A) \xrightarrow{\delta} L_0 F(B) \rightarrow L_0 F(C) \rightarrow 0$$

Moreover there is a naturality in the ses.

(4) $\forall P$ projective $L_n F(P) = 0 \forall n > 0$

Proof (1) every functor F, H_n and P_x ~~are~~ additive, so the composition is

(2) $M \in \mathcal{A}$ $P_0 \xrightarrow{\pi} M$ projective resolution

then $0 \rightarrow \ker(\pi) \rightarrow P_0 \rightarrow M \rightarrow 0$ exact
"in \mathcal{A} "

so $F(\text{Ind}_1) \rightarrow F(P_0) \rightarrow F(M) \rightarrow 0$ exact

so $\text{Coker}(F(\text{Ind}_1)) = F(P_0) / F(\text{Ind}_1) \simeq F(M)$
 \parallel
 $H_0(F(P_0)) = L_0(F)(M)$

(3) (i) claim there is a short exact sequence $0 \rightarrow P_0 \rightarrow Q_0 \rightarrow R_0 \rightarrow 0$ of complexes of projective objects ~~and that~~ and a morphism of ses

$$\begin{array}{ccccccc} 0 & \rightarrow & P_0 & \rightarrow & Q_0 & \rightarrow & R_0 \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & A & \rightarrow & B & \rightarrow & C \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

such that ~~with~~ each vertical complex is exact and each row is exact (The Horseshoe Lemma).

idea each ses of projectives splits so $Q_0 = R_0 \oplus Q_0!$

to construct the resolution \mathcal{Q} :

Pick one P_0 for A one R_0 for C and

$$\begin{array}{ccccccc} 0 & \rightarrow & P_0 & \rightarrow & P_0 \oplus R_0 & \rightarrow & R_0 \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & A & \xrightarrow{\alpha} & B & \rightarrow & C \rightarrow 0 \end{array}$$

check that this is a morphism of ses

The snake lemma gives: $B \rightarrow C$ lifts

$$\begin{array}{ccccccc} 0 & \rightarrow & k_0 f & \rightarrow & k_0 h & \rightarrow & k_0 g \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & P_0 & \rightarrow & P_0 \oplus R_0 & \rightarrow & R_0 \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & A & \rightarrow & B & \rightarrow & C \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & \rightarrow & \text{coker } \alpha & \rightarrow & 0 \end{array}$$

\Rightarrow is surjective + $0 \rightarrow k_0 f \rightarrow k_0 h \rightarrow k_0 g \rightarrow 0$ exact + inclusion.

(ii) F is additive so F is exact on split ses so

$0 \rightarrow F(P_0) \rightarrow F(Q_0) \rightarrow F(R_0) \rightarrow 0$ is exact and we have long

exact sequence in homology

$$\begin{array}{ccccccc} \dots & \rightarrow & H_1(F(R_0)) & \xrightarrow{\cong} & H_0(F(P_0)) & \rightarrow & H_0(F(Q_0)) \rightarrow H_0(F(R_0)) \rightarrow 0 \\ & & & & \parallel & & \parallel \\ \dots & \rightarrow & L_1(F(C)) & \xrightarrow{\cong} & L_0(F(A)) & \rightarrow & L_0(F(B)) \rightarrow L_0(F(C)) \rightarrow 0 \end{array}$$

(4) clear since $\dots \overset{0}{0} \rightarrow P \rightarrow 0$ is a projective resolution of P . \square

Thm 6.8 If \mathcal{A} has enough injectives

(1) $R^n F$ is additive $\forall n \geq 0$

(2) F is left exact $\text{ker } R^0 F = F$

(3) If $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ res then we have a long exact sequence

$$0 \rightarrow R^0 F(A) \rightarrow R^0 F(B) \rightarrow R^0 F(C) \xrightarrow{\delta} R^1 F(A) \rightarrow \dots$$

Case of contravariant functors

$F: \mathcal{E} \rightarrow \mathcal{D}$ contravariant iff

$F: \mathcal{E}^{op} \rightarrow \mathcal{D}$ covariant

Hence F is left exact iff $\forall 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ we have $\begin{matrix} \text{exact} & \xrightarrow{\delta} & F(C) \rightarrow F(A) \rightarrow R^1 F(A) \\ & & \text{exact} & & \downarrow \delta \\ & & & & 0 \end{matrix}$

Moreover Proj in $\mathcal{E} \leftrightarrow$ Inj in \mathcal{E}^{op}

Def 6.9 (1) $F: \mathcal{A} \rightarrow \mathcal{B}$ contravariant additive functor. Then

$$R^i F(A) = H^i(F(P)) \text{ where } P \text{ is a projective resolution of } A$$

$$(2) L_i F(B) = H_i(F(I)) \text{ where } I \text{ is an injective coresolution of } B.$$

2) Example of Ext functors

Def 6.10 (1) If \mathcal{E} has enough projective objects we denote

$$\forall A, B \in \text{ob}(\mathcal{E})$$

$$\text{Ext}^i(A, B) = [R^i(\text{Hom}(-, B))](A)$$

(2) If \mathcal{E} has enough injective objects we denote

$$\forall A, B \in \text{ob}(\mathcal{E})$$

$$\text{Ext}^i(A, B) = [R^i \text{Hom}(A, -)](B)$$

Thm 6.11 (Balancing ext) If \mathcal{E} has enough projectives and enough injectives we have $R^i \text{Hom}(-, B)(A) \simeq R^i \text{Hom}(A, -)(B)$

Proof not so easy see weibel or look at section 4. \square

Examples: $R = \mathbb{Z}$

M be a finite abelian gp. Then to compute $\text{Ext}^i(\pi, \mathbb{Z})$

Can choose: find an injective resolution of \mathbb{Z} or find a projective resolution of π

We have $0 \rightarrow \mathbb{Z} \xrightarrow{i} \mathbb{Q} \xrightarrow{\pi} \mathbb{Q}/\mathbb{Z} \rightarrow 0$ exact and $\mathbb{Q}, \mathbb{Q}/\mathbb{Z}$ are injective
so we get $0 \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0 \dots$ is a projective resolution

we apply $\text{Hom}(\pi, -)$ and we have

$$0 \rightarrow \text{Hom}(\pi, \mathbb{Q}) \xrightarrow{\pi_0} \text{Hom}(\pi, \mathbb{Q}/\mathbb{Z}) \rightarrow 0 \rightarrow 0 \dots (*)$$

since $\text{Hom}(\pi, -)$ left exact $\ker(\pi_0) = \text{Hom}(\pi, \mathbb{Z}) = 0$ in π is of finite.

$$\hookrightarrow \text{Ext}^0(\pi, \mathbb{Z}) = \text{Hom}(\pi, \mathbb{Z}) = 0$$

$$\text{Ext}^i(\pi, \mathbb{Z}) = 0 \quad \forall i > 0$$

what is $\text{Ext}^1(\pi, \mathbb{Z})$

look at the long exact sequence \rightarrow OR just look at $*$ $\text{Hom}(\pi, \mathbb{Q}) = 0$
so $\text{Ext}^1 = \text{Hom}(\pi, \mathbb{Q}/\mathbb{Z})$

$$0 \rightarrow \text{Hom}(\pi, \mathbb{Z}) \rightarrow \text{Hom}(\pi, \mathbb{Q}) \rightarrow \text{Hom}(\pi, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\cong} \text{Ext}^1(\pi, \mathbb{Z}) \rightarrow \text{Ext}^1(\pi, \mathbb{Q})$$

$\begin{array}{ccccccc} \parallel & & \downarrow & & & & \downarrow \\ 0 & & 0 & & & & 0 \end{array}$

$$\text{so } \text{Ext}^1(\pi, \mathbb{Z}) \cong \text{Hom}(\pi, \mathbb{Q}/\mathbb{Z})$$

since \mathbb{Q} injective

Thm 6.12 For an A -module X the following are equivalent

(1) X projective

(2) $\text{Hom}_A(X, -)$ exact

(3) $\forall i > 0 \quad \forall Y \in \text{Mod } A \quad \text{Ext}^i(X, Y) = 0$

(4) $\forall B \in \text{Mod } A \quad \text{Ext}^1(X, B) = 0$

Proof (1) \Leftrightarrow (2) by definition

(2) \Leftrightarrow (3) long exact sequence

(3) \Rightarrow (4) clear

(4) \Rightarrow (3) Ext^i can be computed by using Ext^1 if $i \geq 2$ "argument de décalage"

Indeed $0 \rightarrow B \rightarrow I \rightarrow I^1 B \rightarrow 0$ exact
long exact sequence:

$$\text{Hom}(X, I^1 B) \rightarrow \text{Ext}^1(X, B) \rightarrow \text{Ext}^1(X, I) \rightarrow \text{Ext}^1(X, I^1 B) \xrightarrow{\cong} \text{Ext}^2(X, B) \rightarrow \text{Ext}^2(X, I)$$

So $\text{Ext}^2(X, B) \cong \text{Ext}^1(X, I^1 B)$ and more generally
 $\text{Ext}^n(X, B) \cong \text{Ext}^{n-1}(X, I^1 B) \quad \forall n \geq 2.$ □

(3) Example of Tor functors

Def 6.13 A K -alg over commutative ring k . Then $\text{Tor}_i^A(M, N) := L_i(- \otimes_A N)(M)$

Thm 6.14 $L_i(- \otimes_A N)[M] \cong L_i(M \otimes -)[N]$

Example B abelian gp $\text{Tor}_i^{\mathbb{Z}/p\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, B)$?

$0 \rightarrow \mathbb{Z} \xrightarrow{xp} \mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z} \rightarrow 0$ exact sequence hence $0 \rightarrow \mathbb{Z} \xrightarrow{xp} \mathbb{Z} \rightarrow 0$
is a projective resolution of $\mathbb{Z}/p\mathbb{Z}$ so

$\text{Tor}_i^{\mathbb{Z}/p\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, B)$ is the i th homology of

$$0 \rightarrow \mathbb{Z} \otimes B \xrightarrow{xp} \mathbb{Z} \otimes B \rightarrow 0$$

$$0 \rightarrow B \xrightarrow{xp} B \rightarrow 0$$

$$\text{So } \text{Tor}_i^{\mathbb{Z}/p\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, B) = \begin{cases} B/pB & \text{if } i=0 \\ \{b \in B; pb=0\} & \text{if } i=1 \\ 0 & \text{otherwise} \end{cases}$$

Prop 6.15 Let B be an A -module. $\forall F \in \mathcal{F}$

- (1) B is flat
- (2) $- \otimes_A B$ is exact
- (3) $\text{Tor}_i^A(-, B) = 0 \quad \forall i \geq 1$
- (4) $\text{Tor}_i^A(-, B) = 0$

(4) Extensions à la Gorenstein

In this section \mathcal{A} is an abelian category. $A, B \in \text{Ob}(\mathcal{A})$

An extension E of B by A is an exact sequence

$$0 \rightarrow A \rightarrow E \rightarrow B \rightarrow 0$$

Two extensions of B by A are equivalent if:

$$\begin{array}{ccccccc} 0 & \rightarrow & A & \rightarrow & E & \rightarrow & B \rightarrow 0 \\ & & \downarrow 1_A & & \downarrow \varphi & & \downarrow 1_B \\ 0 & \rightarrow & A & \rightarrow & E' & \rightarrow & B \rightarrow 0 \end{array}$$

By short five lemma φ is an isomorphism. (\Rightarrow equivalence relation)

Def 6.17 $\text{Ext}(A, B) = \{ \text{extensions of } B \text{ by } A \} / \sim$ is called

the (Gorenstein)-Ext group

\hookrightarrow Make it a bifunctor: $\mathcal{A}^{\text{op}} \times \mathcal{A} \rightarrow \text{Set}$

$$f: B_1 \rightarrow B_2 \quad \text{Ext}(A, B_1) \xrightarrow{f_*} \text{Ext}(A, B_2)$$

$$\begin{array}{ccccccc} 0 \rightarrow B_1 \xrightarrow{\alpha} E \rightarrow A \rightarrow 0 & \xrightarrow{\text{pullback of } f \text{ and } \alpha} & & & & & \\ \downarrow f & & \downarrow & & \parallel & & \\ 0 \rightarrow B_2 \rightarrow E' \rightarrow A \rightarrow 0 & & & & & & \end{array}$$

$$g: A_1 \rightarrow A_2 \quad \text{Ext}(A_2, B) \xrightarrow{g^*} \text{Ext}(A_1, B)$$

$$\begin{array}{ccccccc} 0 \rightarrow B \rightarrow E \rightarrow A_2 \rightarrow 0 & \xrightarrow{\text{pullback}} & & & & & \\ \parallel & & \uparrow & & & & \\ 0 \rightarrow B \rightarrow E' \rightarrow A_1 \rightarrow 0 & & & & & & \end{array}$$

Many small things have to be checked eg $f_1 \circ f_2 = f_2 \circ f_1$ etc
good course

Lemma 6.18

Let $E \xrightarrow{\alpha} A_2 \rightarrow 0$ be two morphisms with α epi.

$$\begin{array}{ccc} E & \rightarrow & A_2 \\ \uparrow & & \uparrow \beta \\ E & \rightarrow & A_1 \end{array}$$

Then $E \rightarrow A_1$ is a pullback iff

$$\begin{array}{ccccccc} 0 \rightarrow \ker \alpha \rightarrow E \rightarrow A_2 \rightarrow 0 \\ \parallel & \parallel & \uparrow & \parallel & \uparrow \\ 0 \rightarrow \ker \alpha \rightarrow E' \rightarrow A_1 \rightarrow 0 \end{array}$$