$$\mathrm{Tor}_k^{\mathbb{Z}[G]}(\mathbb{Z},\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k=0 \\ \mathbb{Z}/n\mathbb{Z} & \text{if } k \text{ odd} \\ 0 & k \text{ even } k\neq 0 \end{cases}$$

we have $H_n(\overset{G}{\overset{\|}{\mathbb{Z}/n}} \times \overset{H}{\overset{\|}{\mathbb{Z}/m}}) = \bigoplus_{\substack{p+q=n \\ p,q\geq 0}} H^p(G)\otimes H^q(H) \oplus \bigoplus_{p+q=n-1} \mathrm{Tor}_1(H_p(G), H_p(H))$

H_0 need $p,q=0$

H_1 $p+q=1$ + $p+q=0$.

H_2 need $2,0 + 1,1 + 0,2$ $1,0 + 0,1$ etc

look at the complex of $H^p(G)\otimes H^q(H)$ $\mathbb{Z}/n\otimes\mathbb{Z}/m$ 5

	0	0	0	0	0	0	4
	$\mathbb{Z}/n\otimes\mathbb{Z}/m$	0	$\mathbb{Z}/n\otimes\mathbb{Z}/m$	0	$\mathbb{Z}/n\otimes\mathbb{Z}/m$	$\mathbb{Z}\otimes\mathbb{Z}/m$	3
	0	0	0	0	0	0	2
	$\mathbb{Z}/n\otimes\mathbb{Z}/m$	0	$\mathbb{Z}/n\otimes\mathbb{Z}/m$	0	$\mathbb{Z}/n\otimes\mathbb{Z}/m$	$\mathbb{Z}\otimes\mathbb{Z}/m$	1
0	$\mathbb{Z}/n\otimes\mathbb{Z}$	0	$\mathbb{Z}/n\otimes\mathbb{Z}$	0	$\mathbb{Z}/n\otimes\mathbb{Z}$	$\mathbb{Z}\otimes\mathbb{Z}$	0
6	5	4	3	2	1	0	

etc

$(0,0)$ $H_0 = \mathbb{Z}$

$H_1 = \mathbb{Z}/n \oplus \mathbb{Z}/m \oplus \overset{0}{\overset{\|}{\mathrm{Tor}_1(\mathbb{Z},\mathbb{Z})}}$

$H_2 = \mathbb{Z}/n\otimes\mathbb{Z}/m \oplus \mathrm{Tor}_1(\mathbb{Z}/n,\mathbb{Z}) \oplus \mathrm{Tor}_1(\mathbb{Z},\mathbb{Z}/m) = \mathbb{Z}/\gcd(m,n)$

$H_3 = \mathbb{Z}/n \oplus \mathbb{Z}/m \oplus \mathrm{Tor}_1(\mathbb{Z}/n,\mathbb{Z}/m) = \mathbb{Z}/n \oplus \mathbb{Z}/m \oplus \mathbb{Z}/\gcd(m,n)$

$H_4 = (\mathbb{Z}/n\otimes\mathbb{Z}/m)^{\oplus 2} \oplus \underbrace{\mathrm{Tor}_1(\mathbb{Z}/n,\mathbb{Z})}_{0} \oplus \underbrace{\mathrm{Tor}_1(\mathbb{Z},\mathbb{Z}/m)}_{0}$

$H_5 = \mathbb{Z}/n \oplus \mathbb{Z}/m \oplus \mathbb{Z}/\gcd(m,n)^{\oplus 2}$

n odd $\bigoplus_{p+q=n} H_p(G) \otimes H_q(H) = \underset{p=0}{\mathbb{Z}/n} \oplus \underset{q=0}{\mathbb{Z}/m}$ $\qquad$ $p+q$ odd $\Rightarrow$ p even or q even

$$\bigoplus_{\substack{p+q=\underbrace{n-1}_{\text{even}}}} \underset{\substack{\| \\ 0 \text{ ou } \mathbb{Z} \text{ p even}}}{\mathrm{Tor}_1(H_p(G), H_q(H))} \overset{\text{same}}{=} \bigoplus_{\substack{p+q=n-1 \\ p,q \text{ odd}}} \mathrm{Tor}_1(\mathbb{Z}/n, \mathbb{Z}/m) = \mathbb{Z}/\gcd(m,n)^{\oplus k}$$

where $k = \#\{p,q \ ; \ p+q=n-1 \ \ p,q \text{ odd}\} = \frac{n-1}{2}$

$\Rightarrow H_{2n+1} = \mathbb{Z}/n \oplus \mathbb{Z}/m \oplus \mathbb{Z}/\gcd(m,n)^{\frac{n-1}{2}}$

n even $\bigoplus_{p+q=n} H_p \otimes H_q = \bigoplus_{\substack{p+q=n \\ p,q \text{ odd}}} H_p \otimes H_q \oplus \cancel{\mathbb{Z}/m} \oplus \cancel{\mathbb{Z}/m}$

$$= \mathbb{Z}/\gcd(m,n)^{\frac{n}{2}} \ \cancel{\oplus \mathbb{Z}/m \oplus \mathbb{Z}/n}$$

$$\bigoplus_{\substack{p+q=\underbrace{n-1}_{\text{odd}}}} \underset{\substack{\| \\ 0}}{\mathrm{Tor}_1(H_p, H_q)} \quad p+q \text{ odd} \Rightarrow p \text{ odd } \underset{\substack{\Downarrow \\ H_q = 0 \text{ ou } \mathbb{Z}}}{q \text{ even}} \text{ ou l'inverse}$$

Hence $H_n = \begin{cases} \mathbb{Z} & n=0 \\ \mathbb{Z}/\gcd(m,n)^{\oplus \frac{n}{2}} & \text{if } n \text{ even} \neq 0 \\ \mathbb{Z}/n \oplus \mathbb{Z}/m \oplus \mathbb{Z}/\gcd(m,n)^{\oplus \frac{n-1}{2}} & \end{cases}$.

$\square$