Def 7.19 $(X,A) \in Top^2$ is called a good pair if A is closed in X and $\exists$ V open with $A \subseteq V \subseteq X$ s.t. A is a "deformation retract" of V

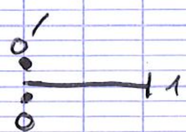
i.e. $\exists$ $R : V \to A$ s.t. $\begin{cases} R \circ i \simeq Id_V & \text{as morphisms } (V,A) \to (V,A) \\ i \circ R = Id_A \end{cases}$

where $i : A \hookrightarrow V$ is the inclusion.

Ex (1) $(\mathbb{D}^n, \mathbb{S}^{n-1})$ is a good pair with $\mathbb{S}^{n-1} \subset \mathbb{D}^n \setminus \{0\} \subset \mathbb{D}^n$

(2) $\bar{I}$ the unit interval with 2 origins $\bar{I} = (I \times \{0\} \sqcup I \times \{1\})/_\sim$ where $(x,1) \sim (x,0)$ iff $x \neq 0$, open are given by the quotient topology

$\mathcal{U}$ is open iff $\pi^{-1}(\mathcal{U})$ is open in $I \times \{0\} \cup I \times \{1\}$
neighbour of $0'$ are $[0', k[= \{0'\} \cup]0,k[$



Then $A = \{0,0'\} \subseteq \bar{I}$ is not a good pair since neighbour of A are of the form $[0,k[\cup \{0'\}$ cannot retract to $\{0,0'\}$.

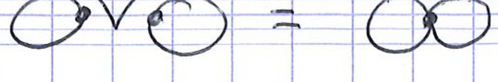
Thm 7.20 (1) If (X,A) is a good pair, then the quotient map $(X,A) \to (X/_A, A/_A)$ induces an isomorphism in homology

(2) If (X, x_0), (Y, y_0) are two good pairs, then

$$\tilde{H}_\bullet(X \vee Y) \simeq \tilde{H}_\bullet(X) \oplus \tilde{H}_\bullet(Y)$$

Here $X \vee Y = \text{pushout}\left(X \xleftarrow{x_0} \{*\} \xrightarrow{y_0} Y \right)$

$$= X \sqcup Y /_{(\{x_0 \sim y_0\})}$$

eg 

Thm 7.21 X, Y two topological spaces, then

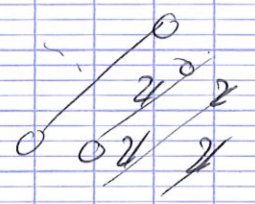
(1) $C_*^{sing}(X \times Y)$ and $C_*^{sing}(X) \otimes_{\mathbb{Z}} C_*^{sing}(Y)$ are homotopy equivalent

(2) $0 \to \bigoplus_{p+q=n} H_p(X) \otimes H_q(Y) \to H_n(X \times Y) \to \bigoplus_{p+q=n-1} \mathrm{Tor}_1^{\mathbb{Z}}(H_p(X), H_q(Y)) \to 0$

Ex (1) $H_n(S^1 \times S^1) = \begin{cases} \mathbb{Z} & n = 0,2 \\ \mathbb{Z} \oplus \mathbb{Z} & n = 1 \\ 0 & \text{otw} \end{cases}$

indeed $H_i(S^1) = \begin{cases} \mathbb{Z} & \text{if } i = 0,1 \\ 0 & \text{otw} \end{cases}$ $\qquad \mathrm{Tor}_1(\mathbb{Z}, -) = \mathrm{Tor}(-, \mathbb{Z}) = 0$

so get



(2) $(S^1, \cdot)$ is a good pair so $\tilde{H}_n(S^1 \vee S^1 \vee S^2)$
$(S^2, \cdot)$ also

$$\simeq \tilde{H}_n(S^1)^2 \oplus H_2(S^2) = \begin{cases} \mathbb{Z}^2 & \text{if } n = 1 \\ \mathbb{Z} & \text{if } n = 2 \end{cases}$$

So non reduced homology : $H_n(S^1 \vee S^1 \vee S^2) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}^2 & n = 1 \\ \mathbb{Z} & n = 2 \end{cases}$

but the two spaces are not homotopy equivalent since

$$\pi_1(S^1 \vee S^1 \vee S^2) \simeq \pi_1(S^1) * \pi_1(S^1) * \pi_1(S^2) \simeq \mathbb{Z} * \mathbb{Z}$$

but $\pi_1(S^1 \times S^1) \simeq \mathbb{Z} \times \mathbb{Z}$ is commutative.

8- Singular cohomology

Recall that for G an abelian gp $H^*(X, G) = H^\bullet(\mathrm{Hom}_{\mathbb{Z}}(C_n^{sing}(X), G))$

Thm 7.22: X is a topological space, G abelian grp. We a short exact sequence

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}(X), G) \to H^n(X,G) \to \mathrm{Hom}_{\mathbb{Z}}(H_n(X),G) \to 0$$

Moreover, the sequence splits $\forall n \geq 0$.

Coro 7.23 (1) $H^0(X,G) \simeq \mathrm{Hom}_{\mathbb{Z}}(H_0(X),G) \simeq \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{\text{*conn comp } X}, G)$
$\simeq G^{\text{cc of } X}$

(2) If $H_n(X) = \mathbb{Z}^R \oplus T_n$
$H_{n-1}(X) \simeq \mathbb{Z}^P \oplus T_{n-1}$ torsion

Then $H^n(X, \mathbb{Z}) \simeq T_{n-1} \oplus \mathbb{Z}^R$

Proof (2) $0 \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}^P \oplus T_{n-1}, \mathbb{Z}) \to H^n(X,G) \to \mathrm{Hom}(\mathbb{Z}^R \oplus T_n, \mathbb{Z}) \to 0$

$\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}^P \oplus T_{n-1}, \mathbb{Z}) \simeq \mathrm{Ext}^1_{\mathbb{Z}}(T_{n-1}, \mathbb{Z})$, $\mathrm{Hom}(\mathbb{Z}^R \oplus T_n, \mathbb{Z}) \simeq \mathbb{Z}^R$

$\hookrightarrow$ $\mathrm{Ext}^1_{\mathbb{Z}}(T_{n-1}, \mathbb{Z}) \simeq T_{n-1}$: do it for $T_{n-1} = \mathbb{Z}/m\mathbb{Z}$, take direct sum $\square$

Chap 8 Morita theory (Appendix)

Questions (1) Can we recover A from $\mathrm{Mod}\,A$?
(2) When do we have $\mathrm{Mod}\,A \simeq \mathrm{Mod}\,B$?

Lem 8.1 Let $F: \mathcal{E} \to \mathcal{D}$ be an equivalence of categories with quasi inverse G. Then (F,G) and (G,F) are two adjoints pairs

Def 8.2 An A-module P satisfying

(1) P is fg proj
(2) $\forall M \in \mathrm{Mod}\,A$, $\exists I$, $\pi : P^{\oplus I} \twoheadrightarrow M$

is called a progenerator of Mod A.

Thm 8.3 (Morita 1958) Mod A and Mod B are equivalent iff $A \cong \mathrm{End}_B(P)^{op}$ for a progenerator P of Mod B

Proof $\Rightarrow$ is more or less easy: F is an equivalence from Mod A $\to$ Mod B
Then (1) $F(A)$ is a progenerator
(2) $\mathrm{End}_B(F(A)) \cong \mathrm{End}_A(A_A) \cong A^{op}$

For (1) need to see that $F(A)$ is projective: clear $\mathrm{Hom}_B(F(A), -) \cong \mathrm{Hom}_A(A, G(-)) \cong G(-)$

G is exact as it is left and right adjoint to F.

Lem 8.4 P be a projective B-module Then P is fg iff $\mathrm{Hom}_B(P,-)$ commutes with coproducts

Here this is clear since $\mathrm{Hom}(F(A), -) \cong \mathrm{Hom}(A, G(-)) \cong G(-)$ commutes with coproducts.

$\Leftarrow$ We assume that $A \cong \mathrm{End}_B(P)^{op}$, we know then ${}_AP_B$ has a structure of A,B bimodule for $f \cdot p = f(p)$.

So $F = \mathrm{Hom}_B(P,-) : \mathrm{Mod}\,B \to \mathrm{Mod}\,A$ is a functor between these two categories. Si P_B is projective it is an exact functor and since P is fg it commutes with arbitrary coproducts.

Now $G = - \otimes_A P : \mathrm{Mod}\,A \to \mathrm{Mod}\,B$ is a left adjoint to F

As a left adjoint G is right exact and commutes with arbitrary coproducts

Recall that unit η : $Id_{ModA} \longrightarrow \cancel{EG}\, FG$ is ~~a~~ the natural transformation given by

$$\eta_X : X_A \longrightarrow Hom_B(P, X \otimes_A P)$$
$$x \longmapsto (p \longmapsto x \otimes p)$$

and counit $\varepsilon : GF \Longrightarrow Id_{ModB}$ is the natural transformation given by

$$\varepsilon_Y : Hom_B(P,Y) \otimes^A P \longrightarrow Y$$
$$\phi \otimes^A p \longmapsto \phi(p)$$

We have $\varepsilon_P : Hom(P,P) \otimes^A P \longrightarrow P$ is the natural isomorphism
$\phi \otimes^A p \longmapsto \phi(p)$ $(A = End(P))$

So ε_P is an isomorphism. Since $Y \in Mod B$, $\exists\, I_1, I_2$ s.t.

$$\begin{tikzcd} P^{\oplus I_2} \arrow[r] \arrow[dr] & P^{\oplus I_1} \arrow[r, "\pi"] & Y \arrow[r] & 0 \\ & \ker(\pi) \arrow[u] & & \end{tikzcd}$$

F being ex + commuting with direct sums
G being right — + — —
we get

$$\begin{tikzcd} GF(P)^{\oplus I_2} \arrow[r] \arrow[d, "\varepsilon_P^{\oplus I_2}"] \arrow[dr, phantom, "\circlearrowleft"] & GF(P)^{\oplus I_1} \arrow[r] \arrow[d, "\varepsilon_P^{\oplus I_1}"] \arrow[dr, phantom, "\circlearrowleft"] & GF(Y) \arrow[r] \arrow[d, "\varepsilon_Y"] & 0 \\ P^{\oplus I_2} \arrow[r] & P^{\oplus I_1} \arrow[r] & Y \arrow[r] & 0 \end{tikzcd}$$

two iso

So ε_Y is an iso by snake lemma. Hence $GF \simeq Id_{ModB}$

Similarly η_A : $A \xrightarrow{\simeq} Hom_B(P,P) \xrightarrow{\simeq} Hom_B(P, A \otimes P)$ is the map η_A

Hence it is an isomorphism. Moreover A is a progenerator of $\mathrm{Mod} A$, hence if X is an A-module, then $\exists\ A^{(E_2)} \to A^{(E_1)} \to X \to 0$ exact and with similar arguments we get Φ_X isomorphism and this finishes the proof. $\square$

Rem (1) One can prove that $\mathrm{Hom}_B(P,-) \simeq P_1 - \otimes_B \mathrm{Hom}_B(P_B,B_B)$
hence the two equivalences are given by tensor product of bimodules

(2) $\mathrm{Mod} A \simeq \mathrm{Mod} B$ iff $\mathrm{mod} A \simeq \mathrm{mod} B$ ($\leftarrow$ fg)
indeed $\Rightarrow$ just need to check that $\mathrm{Hom}_B(P,M)$ is fg when M is

(clear M fg $B^n \twoheadrightarrow M$ and $P^m \twoheadrightarrow B$ hence $P^{n'} \twoheadrightarrow M$ applying $\mathrm{Hom}_B(P,-)$ gives the result)

$\Leftarrow$ $G : \mathrm{mod} A \simeq \mathrm{mod} B$ then $G(B)$ is a progenerator.

Answer to Q1 : In general we cannot recover A from $\mathrm{Mod} A$: can recover all $\mathrm{End}_A(P)$ for progenerator

$\hookrightarrow$ in a particular case : if A is fd over k ($\Rightarrow$ artinian) notion of projective indecomposable which are in bijection with the simple modules. Then $P = P_1 \oplus \dots \oplus P_n$ is the smallest possible progenerator. Hence $\mathrm{End}(P)$ can be recovered from $\mathrm{Mod} A$. This is called the basic algebra of A.

$\mathrm{Mod} A \simeq \mathrm{Mod} B$ iff A and B have isomorphic basic algebras

Examples (1) TD 2 $M_n(A)$ is Morita equivalent to A

(2) Semisimple fd k-algebra, $k = \bar{k}$ is a product of matrix algebras
$\hookrightarrow$ Morita equivalent to $k \times k \times \dots \times k$

(3) [Gabriel] Any fd algebra over an algebraically closed field

is isomorphic to a quotient of the path algebra of its Ext^1-quiver

Q is a quiver (= digraph) $k[Q]$ path algebra: free k-module with basis paths in Q, product induced by concatenation

<u>Ex</u> (1) $k[x] = k \, \bullet\!\circlearrowleft$

(2) $k[x]/(x^2) \simeq k(\bullet\!\circlearrowleft^{\alpha})/(\alpha^2)$

(3) $\begin{pmatrix} k & \cdots & k \\ & \ddots & \vdots \\ 0 & & k \end{pmatrix} \cong k[1 \leftarrow 2 \leftarrow \cdots \leftarrow n] \cong \mathrm{Fun}([n-1], k)$

...

<u>Question</u>: What invariants of algebras are preserved by Morita equivalences?

– many: Center, #simples, proj.dim, Hochschild Homology, semisimplicity