

Chap 3

Tensor product

All rings have units, if A is a ring then an A -module is an abelian group $(M, +)$ with $\bullet : A \times M \rightarrow M$ s.t. $(a, m) \mapsto a \bullet m$

(1) $a \bullet (m+n) = a \bullet m + a \bullet n$

(2) $(a+b) \bullet m = a \bullet m + b \bullet m$

(3) $(ab) \bullet m = a \bullet (b \bullet m)$

(4) $1_A \bullet m = m$

$\forall a, b \in A$
 $m, n \in M$

symmetrically: get notion of right A -module.

An A - B -bimodule is an abelian gp $(M, +)$ with a left action of A and a right action of B s.t. $\forall a \in A, b \in B, m \in M$

we have $a \bullet (m \bullet b) = (a \bullet m) \bullet b$

Goal $M_A, {}_A N$ a right and a left A -module $\mapsto M \otimes_A N$ an abelian gp which is universal in some sense with respect to bilinear maps

Def 3.1 $M_A, {}_A N$ two A -modules. A bilinear map $f: M \times N \rightarrow G$ where G is an abelian group is a map s.t.

$f(m_1 + m_2, n) = f(m_1, n) + f(m_2, n)$

$f(m, n_1 + n_2) = f(m, n_1) + f(m, n_2)$

$f(ma, n) = f(m, an)$

$\forall m_1, m_2, n \in M$
 $\forall n_1, n_2 \in N$
 $\forall a \in A$

Thm 3.2 If exists an abelian group X together with $t: L \times M \rightarrow X$ universal in the following:

$\forall g: L \times M \rightarrow X$ bilinear, $\exists \tilde{g}: L \otimes M \rightarrow X$ gp homo s.t. $g = \tilde{g} \circ t$

Proof

we have to construct an $M \otimes N$ and maps t bilinear.

$L = \mathbb{Z}[M \times N]$ free abelian gp over $M \times N$ (have basis (m, n) ; $m \in M$ and $n \in N$).

$$I = \left\langle \begin{array}{l} (m_1+m_2, n) - (m_1, n) - (m_2, n) ; (m, n_1+n_2) - (m, n_1) - (m, n_2) \\ (ma, n) - (m, an) \end{array} \right\rangle \text{ subgp generated by}$$

$$\text{Set } M \otimes N := L/I \quad t: M \times N \hookrightarrow \mathbb{Z}[M \times N] \xrightarrow{\pi} L/I$$

$$(m, n) \longmapsto [(m, n)]$$

- (*) $M \otimes N$ is an abelian gp
- (*) t is bilinear

Now if we have $L \times M \xrightarrow{b} A$ b bilinear

$$\downarrow t$$

$$L \otimes M$$

then

$$\begin{array}{ccc} L \times M & \xrightarrow{b} & A \\ \downarrow i & \nearrow \exists! \tilde{b} & \uparrow \\ \mathbb{Z}[L \times M] & \xrightarrow{\tilde{b}} & \text{univ prop of free abelian gpe} \\ \downarrow \pi & \nearrow \exists! \tilde{b} & \uparrow \\ L \otimes M & \xrightarrow{\tilde{b}} & \text{univ prop of quotient} \end{array}$$

now $\tilde{b}((m_1+m_2, n)) - \tilde{b}(m_1, n) - \tilde{b}(m_2, n) = \tilde{b}(m_1+m_2, n) - \tilde{b}(m_1, n) - \tilde{b}(m_2, n) = b(m_1, n) + b(m_2, n) - b(m_1, n) - b(m_2, n) = 0$

de même on voit que $I \subseteq \ker(\tilde{b})$ so \tilde{b} goes through the quotient. \square

Unicity is easy

$$\begin{array}{ccc} L \times M & \xrightarrow{b} & A \\ \downarrow i & \nearrow \exists! \tilde{b} & \\ \mathbb{Z}[L \times M] & \xrightarrow{\tilde{b}} & \\ \downarrow \pi & \nearrow \exists! \tilde{b} & \\ L \otimes M & \xrightarrow{\tilde{b}} & \end{array}$$

then \tilde{b} is a gp morphism o.t
 $\tilde{b} \circ \pi \circ i = b \Rightarrow \tilde{b} \circ \pi = b$
 and $\tilde{b} = b$

Rem ① $M \otimes_A N$ is unique up to unique isomorphism that makes the square

$$\begin{array}{ccc} M \times N & \xrightarrow{\iota} & (M \otimes_A N) \\ \downarrow \cong & \searrow \cong & \\ M \otimes_A N & \xrightarrow{\text{fid}} & \end{array}$$

② we usually denote by $M \otimes_A N$ or just $M \otimes N$ the class of $[(m, n)]$ in $M \otimes_A N$. They generate $M \otimes_A L$ △ not uniquely
 $\hookrightarrow x \in M \otimes_A N \Leftrightarrow x = \sum_{i=1}^n m_i \otimes n_i$ $m_1 \otimes n_1 - n_2 \otimes n_2$
 $= (m_1 + m_2) \otimes n_1 - m_2 \otimes (n_1 - n_2)$

Of course we want a (bi) functor $-\otimes_A - : \text{Mod } A \times A \text{Mod} \rightarrow \text{Ab}$
 $(L, M) \mapsto L \otimes_A M$

we have: for $f: L \rightarrow L'$, $g: M \rightarrow M'$ the following diagram

$$\begin{array}{ccc} L \times M & \xrightarrow{h} & L \otimes_A M \\ (f, g) \downarrow & & \downarrow f \otimes g \\ L' \times M' & \xrightarrow{h'} & L' \otimes_A M' \end{array}$$

then $h' \circ (f, g)$ is bilinear

indeed $h' \circ (f, g)(\alpha a, n) = h'(f(\alpha a), g(n))$
 $= h'(f(e)\alpha, g(n))$
 $= h'(f(e), \alpha g(n))$
 $= h'(f(e), g(\alpha n))$
 $= h'(f, g)(l, \alpha n)$

rest is similar
 so by universal property of $L \otimes_A M$ $\exists!$ $f \otimes g$ additive that makes the square commutative

Lem 3.3 $-\otimes_A -$ is a bifunctor from $\text{Mod } A \times A \text{Mod} \rightarrow \text{Ab}$

Cor 3.4 ① If M is an B - A -bimodule, then $M \otimes_A N$ is a B -Module.

② Si N est un A - C -bimodule alors $M \otimes N$ est un C -module à droite

③ Si ${}^B M_A$, ${}^A N_C$ sont des bimodules alors $M \otimes_A N$ est un B - C -bimodule

démo ① $b \in B$ on pose $\ell_b: M \rightarrow M$ alors ℓ_b est A -linéaire
 $m \mapsto bm$

et $\ell_b \otimes 1: M \otimes N \rightarrow M \otimes N$ donc une structure de B -module (vient de la propriété de distributivité)

B -module (vient de la propriété de distributivité).

② and ③ similaires. ($\ell_b \otimes 1$ and $1 \otimes \ell_c$ commutent)

Exemples ① ${}_A A_A \otimes_A N \simeq {}_A N$ as left modules

$$a \otimes n \mapsto a \cdot n$$

$$1 \otimes n \mapsto n$$

② If R is commutative (eg a field) then an R -module is an R - R -bimodule

$$R \times M \times R \rightarrow M$$

$$(x, m, y) \mapsto xmy = myx$$

\hookrightarrow In this case $M \otimes_R N$ is always an R -module

(special case $R =$ field the one you know)

Δ for a field V, W dim finite dim then $\dim(V \otimes W) = \dim(V) \times \dim(W)$ (solution to the universal property with R -bil maps)

but

③ if $\gcd(m, n) = 1$ then $\mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} = 0$

en effet pour $c \in \mathbb{Z}$ on a $c = a \otimes b = (ca) \otimes b = a \otimes cb$

donc en écrivant $1 = mu + nv$ on a

$$1 \otimes 1 = 1 \cdot (1 \otimes 1) = mu \otimes 1 + 1 \otimes nu = 0 + 0.$$

Comme 101 engender $\mathcal{U}/\mathcal{M} \otimes \mathcal{U}/\mathcal{N}$ on a le resultat.

Thm 3.5 [~~Hom~~ Isomorphisme chez Cartan / Schaprio Lemma
Nakayama, Frobenius...]

Let A, B two rings, then ${}_A M \otimes_B - : B\text{Mod} \rightarrow A\text{Mod}$

is left adjoint to $\text{Hom}_A({}_A M_B, -) : A\text{Mod} \rightarrow B\text{Mod}$

Rem first we need a B -module structure on $\text{Hom}_A({}_A M_B, N)$
 $\forall N \in A\text{Mod}$

$\phi \in \text{Hom}_A({}_A M_B, N)$ $b \cdot \phi$ is the map $b \cdot \phi(m) = \phi(mb)$

Proof we only need to give unit and counit of the adjunction.

$$\begin{array}{l} \Sigma : M \otimes_B \text{Hom}({}_A M_B, N) \rightarrow N \\ \quad m \otimes \phi \mapsto \phi(m) \end{array}$$

$$\begin{array}{l} \eta : N \rightarrow \text{Hom}_A({}_A M_B, M \otimes_B N) \\ \quad n \mapsto (m \mapsto m \otimes n) \end{array}$$

Exercise check details + try to build the natural isomorphism
see 101.

Example: In general $\pi \otimes_B -$ does not have a left adjoint

$A=B=k$ field. Then $V \otimes_k -$ has a left adjoint iff $\dim V < \infty$

in this case $\text{Hom}_k(V, -)$ is a left and a right adjoint

idea \Rightarrow if it has a left adjoint $V \otimes_k -$ commutes with arbitrary product

one can show that this is true \forall fin dim

Conversely V finite dim then $V^* \otimes - \simeq \text{Hom}_K(V, -)$

$$\begin{aligned} \mathcal{Q}_W: V^* \otimes W &\rightarrow \text{Hom}_K(V, W) \\ w \otimes \phi &\mapsto \left(\begin{array}{l} V \rightarrow W \\ v \mapsto \phi(v) \cdot w \end{array} \right) \text{ natural transf.} \end{aligned}$$

Since $\dim V < \infty$ choose (e_i) basis of V and (e_i^*) dual basis

$$\begin{aligned} \text{then } \text{Hom}_K(V, W) &\rightarrow V^* \otimes W && \text{is the inverse bijection.} \\ f &\mapsto \sum_{i=1}^n e_i^* \otimes f(e_i) \end{aligned}$$

by Thm 3.5: $- \otimes V \simeq - \otimes V^{**} \simeq \text{Hom}(V^*, -)$ hence

$V^* \otimes -$ is a left adjoint but $\text{Hom}(V, -) \simeq V^* \otimes -$. So $\text{Hom}(V, -)$ is both left and right adjoint to $V \otimes -$.

Chap IV Additive categories

I. Preadditive and additive categories

Def 4.1 A zero object in a category \mathcal{E} is an object that is both final and initial

$\text{Ex } \{0\} \text{ in } R\text{Mod.}$

Def 4.2 (1) Let k be a commutative ring. A k -category is a category \mathcal{E} s.t.

- (a) $\forall x, y \in \mathcal{E} \quad \mathcal{E}(x, y) \in \text{Mod } k$
- (b) $- \circ -$ is k -bilinear

(2) $k = \mathbb{Z}$ we say that \mathcal{E} is preadditive

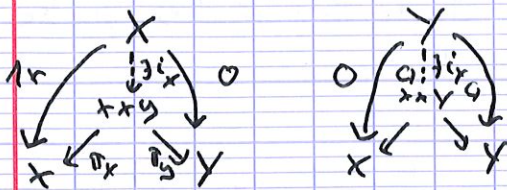
Rem One speaks about categories enriched over $\text{Mod } k$

Lem 4.3 Let A be a k -category. For $x, y \in \text{ob}(A)$ we have

- (1) The product $x \times y$ exists iff the coproduct exists
- (2) If so, they are isomorphic

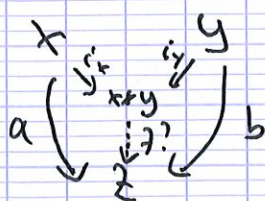
Proof. The key is the existence of 0 morphisms

Assume that $(x \times y, \pi_x, \pi_y)$ is a product. Then look at

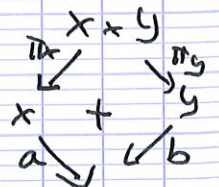


claim $(x \times y, i_x, i_y)$ is a coproduct.

indeed let



Step 1 Construct a map:



$$f = a \pi_x + b \pi_y$$

check the triangles commute

Step 2 unicity: if $g: X \times Y \rightarrow Z$ is another map that makes the square comm. triangles commute:

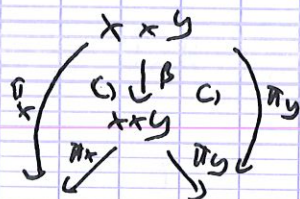
Also:

$$\begin{array}{ccc}
 X \times Y & & \\
 \pi_X \swarrow & & \searrow \pi_Y \\
 X & + & Y \\
 i_X \swarrow & & \searrow i_Y \\
 X \times Y & & \\
 \downarrow g & & \\
 Z & &
 \end{array}$$

gives $g \circ i_X \pi_X + g \circ i_Y \pi_Y = \alpha \pi_X + \beta \pi_Y = f$

so we are left to

Step 3 check that $\beta = i_X \pi_X + i_Y \pi_Y = \text{Id}_{X \times Y}$:
just look at



but $\text{Id}_{X \times Y}$ also makes this commute so by unicity we get $\beta = \text{Id}_{X \times Y}$

Def 3.5 $x, y \in \mathcal{E}$. A biproduct of x and y is the data of an object $x \oplus y \in \mathcal{E}$ with $x \xrightleftharpoons[\pi_X]{i_X} x \oplus y \xrightleftharpoons[\iota_Y]{\pi_Y} y$ so that

(1) $i_Y \pi_Y + i_X \pi_X = \text{Id}_{x \oplus y}$ (2) $\pi_i i_j = \delta_{ij} \text{Id}$

Def 3.6 Let k be a commutative ring. A k -additive category (or k -linear) is a k -category with finite products and finite coproducts.

Rem

- ① Only need to check that it has finite products and zero objects (= empty product)
- ② One can recover the structure of k -category from $x \oplus y$.
- ③ $k = \mathbb{Z}$ speak about additive categories.