Chap 3 Tensor product

All rings have units, if A is a ring then an left A-module is an abelian group $(M,+)$ with $\bullet : A \times M \to M$, $(a, m) \mapsto a \bullet m$ s.t.

(1) $a\cdot(m+n) = am + a\cdot n$
(2) $(a+b)(m) = am+bm$
(3) $(ab)\cdot m = a\cdot(bm)$
(4) $1_A \cdot m = m$

$\forall a,b \in A$, $m,n \in M$

symmetrically: get notion of Right A-module.

An A-B-bimodule is an abelian gp $(M,+)$ with a left action of A and a right action of B s.t. $\forall a \in A, b \in B, m \in M$

we have $a\cdot(m\cdot b) = (a\cdot m)\cdot b$

Goal M_A, ${}_AN$ a right and a left A-module $\rightsquigarrow M \underset{A}{\otimes} N$ an abelian gp which is universal in some sense with respect to bilinear maps

Def 3.1 M_A, ${}_AN$ two Amodules. A bilinear map $f : M \times N \to G$ where G is an abelian group (ab gps) is a map s.t.

$$f(m_1+m_2, n) = f(m_1,n) + f(m_2,n) \qquad \forall m, m_1, m_2 \in M$$
$$f(m, n_1+n_2) = f(m,n_1) + f(m,n_2) \qquad \forall n, n_1, n_2 \in N$$
$$f(ma, n) = f(m, an) \qquad \forall a \in A.$$

Thm 3.2 Il existe un groupe abélien, $M \underset{A}{\otimes} N$ together with $t : L \times M \to M \underset{A}{\otimes} N$ universal in the following:

$\forall$ $g : L \times M \to X$ bilinear, $\exists$ $\bar{g} : M \underset{A}{\otimes} N \to X$ gp homo s.t. $g = \bar{g} \circ t$.

$$\begin{tikzcd} L \times M \arrow[r, "g"] \arrow[d, "t"'] & X \\ L \otimes M \arrow[ur, dashed, "\exists \bar{g}"'] & \end{tikzcd}$$

Proof we have to construct an $M \otimes_A N$ and maps t bilinear.

$L = \mathbb{Z}[M \times N]$ free abelian gp over $M \times N$ (have basis (m,n); $m \in M$ and $n \in N$.

$I = \langle (m_1+m_2, n) - (m_1,n) - (m_2,n)\ ;\ (m, n_1+n_2) - (m,n_1) - (m,n_2)$
$(ma, n) - (m, an) \rangle$ ← subgp generated by

Set $M \otimes_A N := L/I$ $\quad t : M \times N \hookrightarrow \mathbb{Z}[M \times N] \xrightarrow{\pi} L/I$

$(m,n) \longmapsto [(m,n)]$

(•) $M \otimes_A N$ is an abelian gp
(•) t is bilinear

Now if we have
$$\begin{tikzcd} L \times M \arrow[r, "b"] \arrow[d, "t"] & A \\ L \otimes_A M & \end{tikzcd}$$
b bilinear

then
$$\begin{tikzcd} L \times M \arrow[r, "b"] \arrow[d, "i"] & A \\ \mathbb{Z}[L \times M] \arrow[ur, dashed, "\exists!\,\tilde b"] \arrow[d, "\pi"] & \\ L \otimes_A M \arrow[uur, dashed, "\exists!\,\bar b"'] & \end{tikzcd}$$
$\tilde b$: univ prop of free abelian gps ← additive

$\bar b$ ← univ prop of quotient

now $\tilde b\big((m_1+m_2, n) - (m_1,n) - (m_2,n)\big) = \tilde b(m_1+m_2, n)$
$- \tilde b(m_1,n) - \tilde b(m_2,n) = b(m_1,n) + b(m_2,n) - b(m_1,n)$
$- b(m_2,n) = 0$

De même on voit que $I \subseteq \ker(\tilde b)$ so $\tilde b$ goes through the quotient. $\square$

Unicity is easy

$$\begin{tikzcd} L \times M \arrow[r, "b"] \arrow[d, "i"] & A \\ \mathbb{Z}[L \times M] \arrow[d, "\pi"] & \\ L \otimes M \arrow[uur, "\gamma"'] & \end{tikzcd}$$

then ~~...~~
$\gamma \circ \pi$ is a gp morphism o.t.
$\gamma \circ \pi \circ i = b \Rightarrow \gamma \circ \pi = \tilde b$
and $\gamma = \bar b$

Rem ① $M\otimes_A N$ is unique up to unique isomorphism that makes the square

$$\begin{tikzcd} M\times N \arrow[r, "t'"] \arrow[d, "t"] & (M\otimes_A N)' \\ M\otimes_A N \arrow[ur, dashed, "\exists!\,\alpha"'] \end{tikzcd}$$

② we usually denote by $m\otimes_A n$ or just $m\otimes n$ the class of $[(m,n)]$ in $M\otimes_A N$. They generate $M\otimes_A L$ ⚠ not uniquely

$\hookrightarrow$ $x\in M\otimes_A N \Leftrightarrow x=\sum_{i=1}^{n} m_i\otimes n_i$ $m_1\otimes n_1 - m_2\otimes n_2$
$= (m_1+m_2)\otimes n_1 - m_2\otimes(n_1-n_2)$

Of course we want a (bi) functor $-\otimes_A - : \mathrm{Mod}A\times A\mathrm{Mod}\to Ab$
$(L,M)\longmapsto L\otimes_A M$

ⓐ we have: for $f: L\to L'$, $g: M\to M'$ the following diagram

$$\begin{tikzcd} L\times M \arrow[r, "t"] \arrow[d, "(f,g)"] & L\otimes_A M \arrow[d, dashed, "f\otimes g"] \\ L'\times M' \arrow[r, "t'"] & L'\otimes M' \end{tikzcd}$$

then $t'\circ(f,g)$ is bilinear

indeed $t'\circ(f,g)(la,n) = t'(f(la),g(n))$
$= t'(f(l)a, g(n))$
$= t'(f(l), ag(n))$
$= t'(f(l), g(an))$
$= t'\circ(f,g)(l,an)$

rest is similar

so by universal property of $L\otimes_A M$ $\exists!$ $f\otimes g$ additive that makes the square commutative

Lem 3.3 $-\otimes_A -$ is a bifunctor from $\mathrm{Mod}A\times A\mathrm{Mod}\to Ab$

Coro 3.4 ① If M is an B-A-bimodule, then $M\otimes_A N$ is a B-Module.

② Si N est un A-C-bimodule alors $M \otimes N$ est un C-module à droite

③ Si ${}_B M_A$, ${}_A N_C$ sont des bimodules alors $M \underset{A}{\otimes} N$ est un B-C-bimodule

<u>démo</u> ① $b \in B$ on pose $\ell_b : M \to M$, $m \mapsto b \cdot m$ alors ℓ_b est A-linéaire

et $\ell_b \otimes 1 : M \underset{A}{\otimes} N \to M \underset{A}{\otimes} N$, $m \hat{\otimes} n \mapsto bm \otimes n$, donne une structure de B-module (vient de la fonctorialité).

② and ③ similar. ($\varphi_B \otimes 1$ and $1 \otimes \varphi_C$ commute)

<u>Examples</u> ① ${}_A A_A \underset{A}{\otimes} N \simeq {}_A N$ as left modules
$$a \otimes n \mapsto a \cdot n$$
$$1 \otimes n \leftarrow n$$

② If R is commutative (eg a field) then an R-module is an R-R-bimodule $R \times M_R \times R \to M$, $(x, m, y) \mapsto mxy = myx$

↳ In this case $M \underset{R}{\otimes} N$ is always an R-module

(special case R = field the one you know)
(solution to the universal property with R-bil maps)

⚠ for a field V, W ~~dim~~ finite dim then $\dim(V \otimes W) = \dim(V) \times \dim(W)$

<u>but</u>

③ if $\gcd(m,n) = 1$ then $\mathbb{Z}/m\mathbb{Z} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}/n\mathbb{Z} = 0$

en effet pour $c \in \mathbb{Z}$ on a $c \cdot a \otimes b = (ca) \otimes b = a \otimes cb$
donc en écrivant $1 = mu + nv$ on a
$$1 \otimes 1 = 1 \cdot (1 \otimes 1) = mu \otimes 1 + 1 \otimes nv = 0 + 0.$$

Comme $1 \otimes 1$ engendre $\mathbb{Z}/_{m\mathbb{Z}} \otimes \mathbb{Z}/_{n\mathbb{Z}}$ on a le résultat.

Thm 3.5 [~~Nakam~~ Isomorphisme chez à Cartan / Schapiro Lemma]
Nakayama, Frobenius...

Let A, B two rings, then ${}_A M_B \otimes_B - : B\text{Mod} \to A\text{Mod}$

is left adjoint to $\mathrm{Hom}_A({}_A M_B, -) : A\text{Mod} \to B\text{Mod}$

Rem first we need a B-module structure on $\mathrm{Hom}_A({}_A M_B, {}_A N)$ $\forall N \in A\text{Mod}$

$\phi \in \mathrm{Hom}_A({}_A M_B, N)$ $b \cdot \phi$ is the map $b \cdot \phi(m) = \phi(mb)$

Proof we only need to give unit and counit of the adjunction.

$$\varepsilon_Y : M \otimes_B \mathrm{Hom}(M, \cancel{N}Y) \longrightarrow \cancel{N}Y$$
$$m \otimes_B \phi \longmapsto \phi(m)$$

$$\eta_X : \cancel{N}X \longrightarrow \mathrm{Hom}_A(M, M \otimes_B \cancel{N}X)$$
$$n \longmapsto (m \longmapsto m \otimes n)$$

Exercise check details + try to build the natural isomorphisms
see TD1.

Example: In general $M \otimes_B -$ does not have a left adjoint

$A = B = k$ field. Then $V \otimes_k -$ has a left adjoint iff $\dim V < \infty$

in this case $\mathrm{Hom}_k(V, -)$ is a left and a right adjoint

Idea $\Rightarrow$ if it has a left adjoint $V \otimes_k -$ commutes with arbitrary product one can show that this implies V fin dim

Conversely V finite dim then $V^* \otimes - \simeq \mathrm{Hom}_k(V,-)$

$$\eta_W : \begin{array}{rcl} V^* \otimes W & \longrightarrow & \mathrm{Hom}_k(V,W) \\ w \otimes \phi & \longmapsto & \left(\begin{array}{rcl} V & \longrightarrow & W \\ v & \longmapsto & \phi(v) \cdot w \end{array} \right) \end{array}$$ natural transf.

Since $\dim V < +\infty$ chose (e_i) basis of V and (e_i^*) dual basis

Then $$\begin{array}{rcl} \mathrm{Hom}_k(V,W) & \longrightarrow & V^* \otimes W \\ f & \longmapsto & \sum_{i=1}^{n} e_i^* \otimes f(e_i) \end{array}$$ is the inverse bijection.

by Thm 3.5: $- \otimes V \simeq - \otimes V^{**} \simeq \mathrm{Hom}(V^*,-)$ hence

$V^* \otimes -$ has a left adjoint but $\mathrm{Hom}(V,-) \simeq V^* \otimes -$. So $\mathrm{Hom}(V,-)$ is both left and right adjoint to $V \otimes -$.

Chap IV Additive categories

I_ Preadditive and additive categories

Def 4.1 A zero object in a category $\mathcal{E}$ is an object that is both final and initial

Ex $\{0\}$ in RMod.

Def 4.2 ① Let k be a commutative ring. A k-category is a category $\mathcal{E}$ s.t

(a) $\forall x, y \in \mathcal{E}$ $\mathcal{E}(x,y) \in \mathrm{Mod}\,k$

(b) $-\circ-$ is k-bilinear

② $k = \mathbb{Z}$ we say that $\mathcal{E}$ is preadditive

Rem One speaks about categories enriched over Mod k

Lem 4.3 Let A be a k-category. For $x, y \in \mathrm{ob}(A)$ we have

① The product $x \times y$ exists iff the coproduct exists

② If so, they are isomorphic

Proof. The key is the existence of 0 morphisms

Assume that $(x\times y, \pi_x, \pi_y)$ is a product. Then look at

$$\begin{tikzcd} & X \arrow[d, dashed, "\exists ! i_X"] \arrow[dl, bend right, "1_X"'] \arrow[dr, bend left, "0"] & \\ X & X\times Y \arrow[l, "\pi_X"] \arrow[r, "\pi_Y"'] & Y \end{tikzcd} \qquad \begin{tikzcd} & Y \arrow[d, dashed, "\exists ! i_Y"] \arrow[dl, bend right, "0"'] \arrow[dr, bend left, "1_Y"] & \\ X & X\times Y \arrow[l] \arrow[r] & Y \end{tikzcd}$$

claim $(x\times y, i_x, i_y)$ is a coproduct.

indeed let

$$\begin{tikzcd} X \arrow[dr, "i_X"] \arrow[ddr, bend right, "a"'] & & Y \arrow[dl, "i_Y"'] \arrow[ddl, bend left, "b"] \\ & X\times Y \arrow[d, dashed, "\exists ?"] & \\ & Z & \end{tikzcd}$$

Step 1 Construct a map :

$$\begin{tikzcd} & X\times Y \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ X \arrow[dr, "a"'] & + & Y \arrow[dl, "b"] \\ & {} & \end{tikzcd}$$

$f = a\,\pi_x + b\,\pi_y$

check the triangles commutes

Step 2 unicity: if $g : X \times Y \to Z$ is another map that makes the ~~square com~~ triangles commute:

Also:

$$\begin{tikzcd}[column sep=small] & X \times Y \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ X \arrow[dr, "i_X"'] & + & Y \arrow[dl, "i_Y"] \\ & X \times Y \arrow[d, "g"] & \\ & Z & \end{tikzcd}$$

gives $g \circ i_X \pi_X + g \circ i_Y \pi_Y = a \pi_X + b \pi_Y$
$= f$

So we are left to

Step 3 check that $\beta = i_X \pi_X + i_Y \pi_Y = Id_{X \times Y}$: just looked at

$$\begin{tikzcd} & X \times Y \arrow[d, "\beta"] \arrow[ddl, bend right, "\pi_X"'] \arrow[ddr, bend left, "\pi_Y"] & \\ & X \times Y \arrow[dl, "\pi_X"] \arrow[dr, "\pi_Y"'] & \\ X & & Y \end{tikzcd}$$

but $Id_{X \times Y}$ also makes this commute so by unicity we get $\beta = Id_{X \times Y}$ $\square$

Def 4.5 $X, Y \in \mathcal{E}$. A biproduct of X and Y is the data of an object $X \oplus Y \in \mathcal{E}$ with $X \underset{\pi_X}{\overset{i_X}{\rightleftarrows}} X \oplus Y \underset{i_Y}{\overset{\pi_Y}{\rightleftarrows}} Y$ so that

(1) $i_Y \pi_Y + i_X \pi_X = Id_{X \oplus Y}$ (2) $\pi_i \, i_j = \delta_{ij} \, Id$

Def 4.6 Let k be a commutative ring. A k-additive category (or k-linear) is a k-category with finite products and finite coproducts.

Rem ① Only need to check that it has finite products and zero objects (= empty product)

② One can recover the structure of k-category from $X \oplus Y$.

③ $k = \mathbb{Z}$ speak about additive categories.