$\{(m,p)\in M\times P,\ f(p)=g(m)\}$

$$\begin{tikzcd} & P \arrow[d, "f"] & \\ M \arrow[r, "g"'] & N \arrow[r] & 0 \end{tikzcd}$$

alors on considère

$$\begin{tikzcd} M\times_N P \arrow[r, "\bar{g}"] \arrow[d, "\bar{f}"'] & P \arrow[r, dashed] \arrow[d, "f"] & 0 \\ M \arrow[r, "g"'] & N \arrow[r] & 0 \end{tikzcd}$$

the pullback of f and g. On has $\bar{g}$ surjective (*) hence the upper part splits and we get a map from P to M that makes the diag commute.

* The pullback goes into a short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \ker g \arrow[r] & M \arrow[r] & N \arrow[r] & 0 \\ 0 \arrow[r] & \ker g \arrow[r] \arrow[u, equal] & M\times_N P \arrow[r] \arrow[u] & P \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

+ apply snake lemma $\square$.

Thm 5.20 Let A be an associative ring with 1. Then $\mathrm{Mod}A$ and $A\mathrm{Mod}$ have enough projectives and injectives

Proof (1) Every module is quotient of a free module: $m\in M$; $f_m : A \to M$, $1 \mapsto m$

then $\coprod_{m\in M} A \xrightarrow{(f_m)} M$ is surjective
$\quad\nwarrow$ projective

Moreover A is projective since $\mathrm{Hom}_A(A,-)\simeq \mathrm{Id}_{\mathrm{Mod}A}$.

(2) Statement for injective modules is not dual: $(\mathrm{Mod}(A))^{op} \not\simeq \mathrm{Mod}\,B$!

claim 1 $\mathbb{Q}/\mathbb{Z}$ is an injective abelian gp (see TD)

2 $D = \mathrm{Hom}_{\mathbb{Z}}(-,\mathbb{Q}/\mathbb{Z}) : \mathrm{Mod}A \to A\mathrm{Mod}$ is exact

$M_A \longmapsto \mathrm{Hom}_{\mathbb{Z}}(M_A,\mathbb{Q}/\mathbb{Z})$

since $\mathbb{Q}/\mathbb{Z}$ is injective

and $M \hookrightarrow D(D(M))$ is injective
$m \longmapsto ev_m$

for $m\in M$ if $o(m)=n$ $\quad \phi_m : M' \to \mathbb{Q}/\mathbb{Z}$ gp homo
$M' = \langle m\rangle \subseteq M$ $\quad m \longmapsto [\frac{1}{n}]$

$o(m)=\infty \quad \phi_m : M' \to \mathbb{Q}/\mathbb{Z}$
$m \longmapsto [\frac{1}{2}]$

in both cases

$$\begin{tikzcd} M' \arrow[r, hook] \arrow[d, "\phi_m"'] & M \arrow[dl, dashed, "\exists\phi"] \\ \mathbb{Q}/\mathbb{Z} & \end{tikzcd}$$

and $\underset{ev_m(\phi)}{\phi(m)} \neq 0$ so $ev_m = 0 \Rightarrow m=0$.

claim 3 $M \in \mathrm{Mod}A$, $D(M) \in A\mathrm{Mod}$ $P \twoheadrightarrow D(M)$ by (1)

$\hookrightarrow M \hookrightarrow D(D(M)) \longrightarrow D(P)$

$D(P)$ is injective: can choose P to be $\bigsqcup A$, then $D(A) = \prod D(A)$ and $D(A)$ is injective since

$$\mathrm{Hom}_A(-, D(A)) \simeq \mathrm{Hom}_A(-, \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z}))$$
$$\simeq \mathrm{Hom}_{\mathbb{Z}}(A \otimes_A -, \mathbb{Q}/\mathbb{Z}) \text{ : composite of exact functors. } \triangleright$$

exact

Rem Here we see that $- \otimes_A A$ is exact since it is the identity functor on $\mathrm{Mod}A$. More generally if P is a projective A-module then $- \otimes_A P$ is exact and $P \otimes_A -$ is exact.

Indeed this is true for free modules and if P is a direct summand of a free module, then $P \otimes_A - \mid F \otimes_A - \Rightarrow$ exactitude by additivity of the tensor functor

Def 5.21 Let A be a ring. A Module M is *flat* if $M \otimes_A -$ is exact.

$\hookrightarrow$ $\boxed{\text{Proj} \Rightarrow \text{flat}}$

4. Resolutions and derived functors

Def 5.22 (1) Let $\mathcal{A}$ be an abelian category and $M \in \mathrm{ob}(\mathcal{A})$. A projective resolution of M is a complex $(P_n)_{n \geq 0}$ of projective objects together with an homomorphism $\varepsilon : P_0 \to M$ s.t. the "augmented" complex $\cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} M \to 0$

is exact.

(2) An injective (co)resolution of M is a complex of cochains $(I^n)_{n \geq 0}$ of injective objects with a morphism $\varepsilon : M \to I^0$ s.t. the complex $0 \to M \xrightarrow{\varepsilon} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} \cdots$ is exact

Prop 5.23 If $\mathcal{A}$ has enough projectives (resp. injectives) then every object has a projective resolution (resp injective coresolution)

Proof
$$\begin{tikzcd} 0 \arrow[r] & \ker \varepsilon \arrow[r, hook] & P_0 \arrow[r, two heads, "\varepsilon"] & M \arrow[r] & 0 \\ & & P_1 \arrow[ul, two heads, "\pi_1"] \arrow[u, "d_1"'] & & \\ P_2 \arrow[r, two heads] \arrow[urr, "d_2"] & \ker(\pi_1) \arrow[ur] & & & \end{tikzcd}$$

+ induction + dual for injective $\square$

Thm 5.24 Let $\mathcal{A}$ be an abelian category with enough projective. Let $M, M' \in \mathrm{Ob}(\mathcal{A})$ and $f : M \to M'$. $(P_\bullet)$ be a proj resolution of M
$(Q_\bullet)$ — — M'

Then f can be extended as a map $(f_n)_{n \geq 0}$ from $P_\bullet \to Q_\bullet$.
And given two such maps $(f_n)_{n\geq 0}$ and $(f'_n)_{n \geq 0}$ there is a homotopy $(s_n)_n$ s.t. $f - f' = ds + sd$.

Proof
$$\begin{tikzcd} P_1 \arrow[r, "d_1"] \arrow[d, dashed, "h"] & P_0 \arrow[r, "\pi_0"] \arrow[d, dashed, "f_0"] & M \arrow[r] \arrow[d, "f"] & 0 \\ Q_1 \arrow[r, "d'_1"'] & Q_0 \arrow[r, "\pi'_0"'] & M' \arrow[r] & 0 \end{tikzcd}$$
by property of projective objects f lifts as f_0

Now $\pi'_0 f_0 d_1 = f \pi_0 d_1 = 0$ so $\mathrm{Im}(f_0 \circ d_1) \subseteq \ker(\pi'_0) = \mathrm{Im}(d'_1)$

↳ we get a morphism
$$\begin{tikzcd} & P_1 \arrow[d, "d_1"] \arrow[ddl, dashed] \\ & P_0 \arrow[d, "f_0"] \\ Q_2 \arrow[r, two heads, "d'_1"'] & \mathrm{Im}(d'_1) \end{tikzcd}$$
and by projectivity of P_1 $f_0 \circ d_1$ lifts as f_1

+ induction.

Now if we have two lifts
$$\begin{tikzcd} P_1 \arrow[r] \arrow[d, shift right, "f_1"'] \arrow[d, shift left, "f'_1"] & P_0 \arrow[r, "\pi_0"] \arrow[d, shift right, "f'_0"'] \arrow[d, shift left, "f_0"] & M \arrow[d, "f"] \\ Q_1 \arrow[r] & Q_0 \arrow[r, "\pi'_0"'] & M' \end{tikzcd}$$

we have $\pi'_0(f_0 - f'_0) = 0$ so $\mathrm{Im}(f_0 - f'_0) \subseteq \ker(\pi'_0) = \mathrm{Im}(d'_1)$

$$\begin{tikzcd} & P_0 \arrow[d, "f_0 - f'_0"] \arrow[dl, dashed, "s_0"'] \\ Q_1 \arrow[r, two heads, "d'_1"'] & \mathrm{Im}(d'_1) \end{tikzcd}$$
$\to f_0 - f'_0 = d'_1 \circ s_0$

Now $d_1'(f_1 - f_1' - s_0 d_1) = d_1'(f_1 - f_1') - d_1' s_0 d_1$
$$= (f_0 - f_0')d_1 - (f_0 - f_0')d_1 = 0$$

So $\mathrm{Im}(f_1 - f_1' - s_0 d_1) \subseteq \ker(d_1') = \mathrm{Im}(d_2')$

$\hookrightarrow$
$$\begin{tikzcd} & P_1 \arrow[dl, dashed, "s_1"'] \arrow[d, "f_1 - f_1' - s_0 d_1"] \\ Q_2 \arrow[r, "d_2'"'] & \mathrm{Im}(d_2') \end{tikzcd}$$

+ induction $\square$

Rem (1) We only need the upper complex to be projective and bottom complex to be exact (large)

$\hookrightarrow$ See acyclic models for a generalization

(2) Can do the dual for injective coresolution.

Thm 5.25 (1) Let $\mathcal{E}$ be an abelian category with enough projective. Then there exists a functor $P_R : \mathcal{E} \to K_+(\mathrm{Proj}(\mathcal{E}))$ sending $X \in \mathcal{E}$ to one of its projective resolution. Moreover

(a) $H_0 \circ P_R \simeq \mathrm{Id}_{\mathcal{E}}$

(b) $H_i \circ P_R = 0$ if $i \neq 0$

(2) Dual for category with enough injective

Proof by Thm 5.24 the choice of two projective resolutions of M leads to a lift of Id

$$\begin{tikzcd} P_\bullet \arrow[r] \arrow[d, "f"] \arrow[dd, bend right=60, "\mathrm{Id}"'] & M \arrow[d, "\mathrm{Id}"] \\ Q_\bullet \arrow[r] \arrow[d, "g"] & M \arrow[d, "\mathrm{Id}"] \\ P \arrow[r] & M \end{tikzcd}$$

and by unicity of the lift we get $gf \sim \mathrm{Id}$ and $fg \sim \mathrm{Id}$

So P and Q are isomorphic in $K_+(\mathrm{Proj}(\mathcal{E}))$.

Moreover once a choice is made this is functorial

$$\begin{tikzcd} P_\bullet \arrow[r] \arrow[d, "f_\bullet"] \arrow[dd, bend right=60, "g_\bullet f_\bullet"'] & M \arrow[d, "f"] \\ Q_\bullet \arrow[r] \arrow[d, "g_\bullet"] & N \arrow[d, "g"] \\ R \arrow[r] & O \end{tikzcd}$$

$(g_\bullet f_\bullet)$ lifts f and g so it is homotopic to $P_1(g)_\bullet \circ P_1(f)_\bullet$.