PRODUITS TENSORIELS, CATÉGORIES ADDITIVES ET COMPLEXES DE CHAINES

Les questions et exercices $*$ peuvent être ignorées.

1. Produit tensoriel

Exercice 1 -

(1) Montrer que $\mathbb{Z}/m\mathbb{Z}\otimes\mathbb{Z}/n\mathbb{Z}$ est isomorphe à $\mathbb{Z}/\mathrm{pgcd}(n,m)\mathbb{Z}$. Que se passe-t-il si m et n sont premiers entre eux ?

(2) Plus généralement, soient R un anneau unitaire (pas nécessairement commutatif) et I et J deux idéaux bilatères. Démontrer que $R/I\otimes_R R/J\cong R/(I+J)$.

(3) Démontrer que le produit tensoriel est associatif.

Correction

(1) Let $d=gcd(m,n)$. One has to check that $\alpha:\mathbb{Z}/m\otimes\mathbb{Z}/n\to\mathbb{Z}/d$ defined by $\alpha(x\otimes y)=xy$ is well defined (by hand or using the universal property). Conversely the map $\beta:\mathbb{Z}/d\to\mathbb{Z}/m\otimes\mathbb{Z}/n$ defined by $\beta(x)=x\otimes 1$ is well defined and then check that they are two isomorphisms inverse of each other. You will need to use $xy\otimes 1=x\otimes y$.

(2) This is similar.

(3) One has to verify that the obvious maps are well defined..

Exercice 2 - Groupe de Picard

(1) Soit k un corps et n un entier. Démontrer que la catégorie des modules (à gauche) sur l'algèbre $M_n(k)$ est équivalente à la catégorie des k-espaces vectoriels. Indication, on pourra considérer $V=k^{n,1}$ les vecteurs colonnes de taille n dans k et $W=k^{1,n}$ les vecteurs lignes de taille n dans k et considérer les foncteurs $V\otimes_k -$ et $W\otimes_{M_n(k)} -$.

(2) Soient A et B des k-algèbres telles que $A\,\mathrm{Mod}$ et $B\,\mathrm{Mod}$ sont équivalentes. Déduire de la première question que A et B n'ont pas nécessairement des groupes d'automorphismes isomorphes. De même leurs groupes des unités ne sont pas nécessairement isomorphes.

(3) Soit k un anneau commutatif et A une k-algèbre. Si X est un A-A-bimodule, on note par $[X]$ sa classe d'isomorphisme en tant que A-A-bimodule et on dit que X est *inversible* s'il existe un A-A-bimodule Y tel que $[X\otimes_A Y]=[Y\otimes_A X]=[A]$. Justifier que le produit tensoriel au dessus de A induit une structure de groupe sur l'ensemble des classes d'isomorphismes de A-bimodules inversibles. On l'appelle le *groupe de Picard* de A et on le note $\mathrm{Pic}(A)$.

(4) Soit M un bimodule inversible. Démontrer que $\mathrm{End}_A(M)\cong A^{op}$.

(5) Soit $\alpha\in\mathrm{Aut}_k(A)$. Justifier brièvement que l'application $-\cdot-\cdot- : A\times A\times A\to A$ définie par $a\cdot x\cdot b=ax\alpha(b)$ munit A d'une structure de A-bimodule. On note par ${}_1A_\alpha$ ce bimodule.

(6) Démontrer que $\alpha\mapsto {}_1A_\alpha$ induit un homomorphisme Φ de groupes de $\mathrm{Aut}_k(A)$ vers $\mathrm{Pic}(A)$ dont le noyau est $\mathrm{Inn}(A)$ l'ensemble des automorphismes intérieurs de A. On note $\mathrm{Out}_k(A)$ le groupe $\mathrm{Aut}_k(A)/\mathrm{Inn}(A)$.

(7) * Soient M et N deux bimodules inversibles. Démontrer que M et N sont isomorphes en tant que A-modules à gauche si et seulement s'il existe $\alpha\in\mathrm{Aut}_k(A)$ tel que $M\cong N\otimes_A {}_1A_\alpha$ en tant que A-bimodules. (On pourra commencer par supposer que $N=A$, puis on s'y ramène en utilisant l'inversibilité de N).

(8) * En déduire que l'image de Φ est l'ensemble des classes d'isomorphismes de A-A-bimodules qui sont libres de rang 1 en tant que A-module à gauche.

(9) *** On peut montrer que
- $\mathrm{Pic}(A)$ est invariant par équivalence de catégories : si A et B sont deux k-algèbres telles que $\mathrm{Mod}\,A\cong\mathrm{Mod}\,B$, alors $\mathrm{Pic}(A)\cong\mathrm{Pic}(B)$.
- Il y a un homomorphisme de groupe Γ de $\mathrm{Pic}(A)$ vers $\mathrm{Aut}_k(Z(A))$. On note $\mathrm{Picent}(A)$ le noyau de Γ. Si A est une algèbre commutative, alors $\mathrm{Pic}(A)=\mathrm{Picent}(A)\rtimes\mathrm{Aut}_k(A)$.
- Dans certains cas le groupe de Picard de A est réduit à $\mathrm{Out}_k(A)$, c'est par exemple le cas si A est une algèbre de dimension finie dont tous les modules simples sont de dimension 1, comme les algèbres de chemins sur un carquois acyclique.
- Pour plus de détails : The Picard group of noncommutative rings, in particular of orders. A. Fröhlich, Transactions of the American Mathematical Society Volume 180, June 1973.

Correction

(1) We show the following

  (a) If A is a ring, then we have $A \otimes - \cong Id_{\text{Mod }A}$.

  (b) Let A, B be two rings and M and N be two A-B-bimodules. An isomorphism of bimodules induces an isomorphism of functors $M \otimes_B - \cong N \otimes_B -$.

  (c) The bimodule $V \otimes_k W$ is isomorphic to $M_n(k)$.

  (d) The bimodule $W \otimes_{M_n(k)} V$ is isomorphic to k.

We only sketch the ideas for (a) and (b). There is an obvious map from $A \otimes_A M$ to M sending $a \otimes m$ to $a \cdot m$. This is a natural transformation and the map sending m to $1 \otimes m$ is the inverse isomorphism. If ϕ is an isomorphism of bimodules, then it induces a map $\phi \otimes id_W : M \otimes W \to N \otimes W$ and this is a natural transformation.

Let us look at (c). If $c = (c_1, \cdots, c_n)^t$ is a column vector and $a = (a_1, \cdots, a_n)$ is a row vector we defined $\phi(c \otimes a) = (c_i a_j)_{i,j}$. One has to check that this is an isomorphism of $M_n(k)$-bimodules. Conversely if A is a matrix we have $A = \sum_{i,j} a_{i,j} E_{i,j}$ where $E_{i,j}$ is the usual basis of $M_n(k)$. We let $\psi(A) = \sum_{i,j} a_{i,j} e_i \otimes f_j$ where e_i is the ith element of the canonical basis of V and f_j the jth of the canonical basis of W. It remains to prove the ψ is an isomorphism inverse to ϕ.

Question (d) is probably the most intriguing part since we are dealing with a tensor product over $M_n(k)$! (in particular we tensor a vector space of dimension n with one of dimension n and the result is of dimension 1). If $a = (a_1, \cdots, a_n) \in W$ and $b = (b_1, \cdots, b_n)^t \in W$, then we set $\phi(a, b) = a \cdot b = \sum_{i=1}^n a_i b_i \in k$. It is clear that ϕ is k-linear on both variable and if M is a matrix we have $\phi(aM, b) = (aM) \cdot b = a \cdot (Mb) = \phi(a, Mb)$ by associativity of the matrix product. Hence ϕ induces a well-defined linear map from $W \otimes_{M_n(k)} V$ to k.

Before giving the answer, let us think about the map ψ on the other direction. Since we are looking for a linear map we just need to decide the image of 1. The 'canonical' choice, in my opinion, is $\psi(1) = \sum f_i \otimes e_i$. However, such a choice gives $\phi \circ \psi(1) = n$. Which in characteristic zero could be fixed by dividing by n but not for arbitrary characteristic. So no!

The other 'natural' choice is $\psi(1) = f_1 \otimes e_1$ (or $f_i \otimes e_i$ for your favorite i). With this choice we have $\phi \circ \psi(1) = 1$. However $\psi \circ \phi(f_i \otimes e_i) = \psi(1) = f_1 \otimes e_1$. So, it is time to look in more detail at the equivalence class of $f_1 \otimes e_1$. For every invertible matrix A we have $f_1 \otimes e_1 = f_1 A \otimes A^{-1} e_1$. If we take A to be the permutation matrix associated to the transposition $(1, i)$ we get $f_1 \otimes e_1 = f_i \otimes e_i$. Hence our starting choice is not a choice.

Now lets us look at $f_i \otimes e_j$ with $i \neq j$, playing with transposition we see that this is equal to $f_1 \otimes e_2 = f_2 \otimes e_1$. Now consider the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ (extending as a block diagonal matrix where the other diagonal block is the identity). Then $f_1 A = f_1 + f_2$ and $A^{-1} e_2 = -e_1 + e_2$. Hence we have :

$$f_1 \otimes e_2 = (f_1 + f_2) \otimes (e_2 - e_1) = f_1 \otimes e_2 + f_2 \otimes e_2 - f_1 \otimes e_1 - f_2 \otimes e_1 = 0,$$

since $f_2 \otimes e_2 = f_1 \otimes e_1$ and $f_1 \otimes e_2 = f_2 \otimes e_1$.

Using these remarks and the linearity of the both maps, we see that $\psi \circ \phi(v \otimes w) = v \otimes w$.

(2) Clear consequence of the first question. The idea is now to find a good replacement which is indeed preserved by equivalences of categories.

(3) This is also clear from question 1 and associativity of tensor product. The identity of the group is the class of A with its two natural actions.

(4) As consequence of question 1, if a bimodule M is invertible, then $M \otimes -$ is an equivalence of categories, with quasi-inverse $N \otimes -$. Hence, $M \otimes -$ is fully-faithful. So $\text{Hom}_A(A, A) \cong \text{Hom}_A(M \otimes A, M \otimes A) \cong \text{Hom}_A(M, M)$. It remains to see that $A^{op} \cong \text{Hom}_A(A, A)$.

(5) Clear.

(6) We have many small results to check : the tensor product of A_α and A_β is isomorphic as bimodule to $A_{\alpha \circ \beta}$ and $A_\alpha \cong A$ as bimodule if and only if α is an inner automorphism.

(7) Let $f : M \to A$ be an isomorphism of left A-modules. For $a \in A$, the right multiplication by a, denoted by $r_a : M \to M$ is a morphism of left modules. Hence $f \circ r_a \circ f^{-1} \in End_A(A)$. So, there is an $\alpha(a) \in A$ such that this endomorphism is equal to $r_{\alpha(a)}$. We have to check that α is an algebra endomorphism of a. This is done by computing $r_{\alpha(a)} \circ r_{\alpha(b)}$ and $r_{\alpha(1)}$. To see that it is an algebra automorphism, we can compute $f^{-1} \circ r_a \circ f \in \text{End}(M) \cong A^{op}$. Hence this is given by $r_{\gamma(a)}$ for a $\gamma(a) \in A$. Now check that this is the inverse of α, by computing $r_{\gamma \circ \alpha(a)}$ and $r_{\alpha \circ \gamma(a)}$. Finally, $f : M \to A_\alpha$ is an isomorphism of A-bimodules.

To conclude the proof, if M and N are isomorphic as left modules, let N' be an inverse of N, we have $N' \otimes M \cong N' \otimes N \cong A$. So by the previous argument $N' \otimes M \cong A_\alpha$ as bimodules, multiplying on the left by N gives $N \otimes N' \otimes M \cong M \cong N \otimes A_\alpha$.

(8) This is clear from question (7).

2. Catégories additives

Exercice 3 -

Soit $\mathcal{C}$ une category. On dit que $\mathcal{C}$ a *des morphismes nuls* si pour chaque couple d'objet (X,Y) de $\mathcal{C}$ il existe un morphisme $0_{X,Y} : X \to Y$ tel que
(a) $f \circ 0_{X,Y} = 0_{X,Z}$ pour tout $f \in \mathcal{C}(Y,Z)$.
(b) $0_{X,Y} \circ g = 0_{W,Y}$ pour tout $g \in \mathcal{C}(W,X)$.

(1) Justifier que la famille $(0_{X,Y})$ est unique si elle existe.
(2) Supposons que $\mathcal{C}$ possède un objet zero. Montrer qu'elle a des morphismes nuls.
(3) Supposons que $\mathcal{C}$ est k-linéaire. Montrer qu'elle a des morphismes nuls.
(4) En déduire que dans une catégorie k-additive les éléments neutres des k-modules $\mathcal{C}(X,Y)$ ne dépendent pas du choix de l'enrichissement.
(5) Montrer que si $\mathcal{C}$ possède des morphismes nuls et un objet initial, alors cet objet initial est un objet zéro.

Correction

(1) If we have two families 0 and $0'$ we have $0'_{X,Y} = 0_{X,Y} \circ 0'_{X,X} = 0_{X,Y}$ by applying a to the $0'$ and b to 0.
(2) We denote by $\mathbb{O}$ a zero object of $\mathcal{C}$. Then for $X,Y \in \mathcal{C}$ there is a unique map $X \to \mathbb{O}$ and a unique map $\mathbb{O} \to Y$. We define $0_{X,Y} : X \to \mathbb{O} \to Y$. If $f : W \to Y$, we have $0_{X,Y} \circ Z = 0_{W,Y}$ since $W \to X \to \mathbb{O}$ is the unique morphism from W to $\mathbb{O}$. The rest is similar.
(3) The candidate for $0_{X,Y}$ is obviously the 0-morphism of the k-module $\mathcal{C}(X,Y)$. Let $f : W \to X$ we have to see that $0_{X,Y} \circ f = 0_{W,Y}$, we will use the bilinearity of the composition. We have :

$$0_{X,Y} \circ 0_{W_X} + 0_{X,Y} \circ f = 0_{X,Y} \circ (0_{W,X} + f) = 0_{X,Y} \circ f.$$

Subtracting $0_{X,Y} \circ f$ gives $0_{X,Y} \circ 0_{W_X} = 0_{W,Y}$. Then, we have :

$$0_{W,Y} = 0_{X,Y} \circ (f - f) = 0_{X,Y} \circ f - 0_{X,Y} \circ f = (0_{X,Y} - 0_{X,Y}) \circ f = 0_{X,Y} \circ f.$$

The rest is similar.
(4) This is clear from the previous questions.
(5) Si $\mathbb{O}$ is an initial object, then there is a unique morphism $\mathbb{O} \to \mathbb{O}$. It must be the identity of $\mathbb{O}$ and also the zero morphism $0_{\mathbb{O},\mathbb{O}}$. If $f : X \to \mathbb{O}$ is a morphism we have $f = id_{\mathbb{O}} \circ f = 0 \circ f = 0$. Hence there is a unique morphism from X to $\mathbb{O}$.

Exercice 4 - Soit $\mathcal{C}$ une K-catégorie et $f : X \to Y$ un morphisme de $\mathcal{C}$.

(1) Un égalisateur de $(f,0)$ est appelé un noyau de f. Ecrire la propriété universelle du noyau de f.
(2) Même question pour le conoyau de f qui est un co-égalisateur de $(f,0)$.
(3) Décrire les noyaux et conoyaux dans les catégories de modules sur un anneau.
(4) On suppose de plus que $\mathcal{C}$ possède un objet zero. Montrer que f est un monomorphisme si et seulement si $\mathrm{Ker}(f) \cong \mathbb{O}$. Duallement, montrer que f est un épimorphisme si et seulement si $\mathrm{Coker}(f) \cong \mathbb{O}$.

Correction 1,2,3 are easy. For the last question we may use the previous exercise that give a categorical characterization of the zero morphisms. Assume that $\mathrm{Ker}(f) \cong \mathbb{O}$ and let $a, b : Z \to X$ such that $f \circ a = f \circ b$, so $f \circ (b - a) = 0$, so the morphism $b - a$ factorize through $\mathbb{O} \cong \ker(f)$. Hence $b - a$ is the zero morphism, so $b = a$.

Conversely if f is a monomorphism we show that $\mathbb{O}$ satisfies the universal property of the kernel of f.

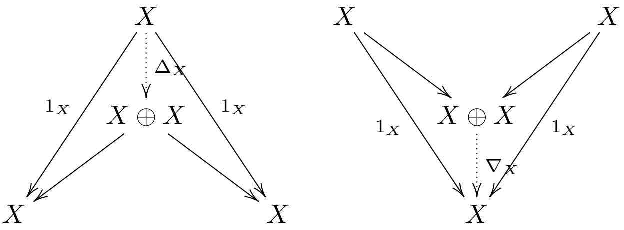
$$\begin{array}{ccccc} & & Z & & \\ & & \downarrow{\scriptstyle h} & & \\ \mathbb{O} & \longrightarrow & X & \xrightarrow{f} & Y. \end{array}$$

We have $f \circ h = 0$ so $f \circ h = f \circ 0$ hence h is the zero morphism hence it factorizes as $Z \to \mathbb{O} \to X$.

Exercice 5 - * Soit $\mathcal{C}$ une catégorie additive. Soient $f, g \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$. En utilisant les propriétés universelles des produits et coproduits construire un morphism $\Delta_X \in \mathrm{Hom}_{\mathcal{C}}(X, X \oplus X)$, un morphisme $f \oplus g : \mathrm{Hom}(X \oplus X, Y \oplus Y)$ et un morphisme $\nabla_Y \in \mathrm{Hom}_{\mathcal{C}}(Y \oplus Y, Y)$ tels que $f + g = \nabla_Y \circ (f \oplus g) \circ \Delta_X$.

Correction

The diagonal and co-diagonal can be build by looking at the following diagrams and using the universal properties :



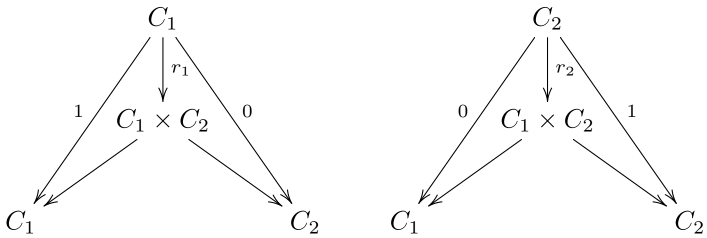
Simillary if $f, g : X \to Y$ are two morphisms between f and g, using the universal property of the product we can construct $f \oplus g : X \oplus X \to Y \oplus Y$. It has the property that $f \circ \pi_X = \pi_Y \circ f \oplus g$ and $g \circ \pi_X = \pi_Y \circ f \oplus g$ where the first X is the most left in $X \oplus X$.

Then we have $\nabla \circ f \oplus g \circ \Delta = \nabla \circ Id_{Y \oplus Y} \circ f \oplus g \circ \Delta$. Then $Id_{Y \oplus Y} = i_y \circ \pi_Y + i_y \circ i_y$. Using the bilinearity of the composition we have : $\nabla \circ f \oplus g \circ \Delta = (\nabla \circ i_Y) \circ \pi_Y \circ f \oplus g \circ \Delta + (\nabla \circ i_Y) \circ \pi_Y \circ f \oplus g \circ \Delta$. Since $\nabla \circ i_Y = Id_Y$ and $\pi_Y f \oplus g = f \circ \pi_X$ or $g \circ \pi_X$ accordingly to which projective we look at, we obtain :

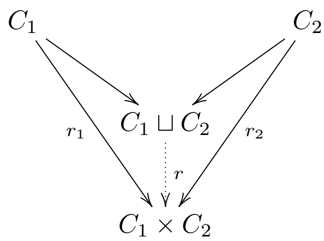
$$\nabla \circ f \oplus g \circ \Delta = f \circ \pi_X \circ \Delta + g \circ \pi_X \circ \Delta = f + g.$$

Hence $f + g$ is completely determined by the biproduct. If we want to see that $f + g$ is part of the structure of the category $\mathcal{C}$ we need a definition of biproduct which does not involve the pre-abelian structure of $\mathcal{C}$.

If a category $\mathcal{C}$ has zero morphisms we have the following observations. If two objects X, Y have a product and a coproduct there is a canonical map $r : C_1 \sqcup C_2 \to C_1 \times C_2$ which is defined by looking at the following universal properties.



and then



If the morphism r is an isomorphism, we say that C_1 and C_2 have a biproduct and $C_1 \times C_2$ and $C_1 \sqcup C_2$ are called the biproduct of C_1 and C_2. It remains to see that this definition coincide with the one of the class when the category $\mathcal{C}$ is pre-additive. As a corollary if a category is additive, it has a zero objects, so it has zero morphisms and we see that the addition of morphism is determined by the categorical property if $\mathcal{C}$. So an additive category is pre-additive in a unique way.

Exercice 6 - Soient $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$ trois petites catégories et $F : \mathcal{D} \to \mathcal{E}$ un foncteur. Montrer que la composition par F induit un foncteur la catégorie $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ des foncteurs de $\mathcal{C}$ vers $\mathcal{D}$ vers la catégorie $\mathrm{Fun}(\mathcal{C}, \mathcal{E})$ des foncteurs de $\mathcal{C}$ vers $\mathcal{E}$.

Correction

This is a good exercise! Give as many, or as little, details as you need.

Exercice 7 - * Soit $\mathcal{C}$ une catégorie preadditive. Montrer qu'il existe une catégorie additive $\mathrm{Add}(\mathcal{C})$ telle que :

(1) Il existe un foncteur additif pleinement fidèle $J : \mathcal{C} \to \mathrm{Add}(\mathcal{C})$.

(2) Si $\mathcal{D}$ est une catégorie additive et $F : \mathcal{C} \to \mathcal{D}$ est un foncteur linéaire, alors il existe un unique foncteur additif $\overline{F} : \mathrm{Add}(\mathcal{C}) \to \mathcal{D}$ tel que $\overline{F} \circ J = F$.

La propriété universelle entraine qu'une telle catégorie sera unique à équivalences de catégorie près. On l'appelle *l'additivisation* de $\mathcal{C}$. Indication : on peut chercher une catégorie dont les objets sont les suites finies d'objets de $\mathcal{C}$. Moralement la suite $(X_1, \cdots, X_n)$ représente la somme directe des X_is.

3. Catégories de complexes de chaines

Exercice 8 - On travaille dans la catégorie des complexes de chaines de A-modules pour un anneau A. On dit qu'un complexe de chaines $(C_\bullet, d)$ est scindé s'il existe une famille d'applications $s_n : C_n \to C_{n+1}$ telle que $d_{n+1} = d_{n+1}s_nd_{n+1}$.

(1) Montrer que $C_\bullet$ est exact et scindé si et seulement si il est contractile (i.e. l'application identité $C_\bullet \to C_\bullet$ est homotope à l'application nulle).

(2) Donner un exemple d'un complexe de chaines exact qui n'est pas contractile.

(3) Considérons une suite exacte de R-modules libres $C_\bullet : \cdots \to C_n \to \cdots \to C_0 \to 0$. Montrer que $C_\bullet$ est scindée.

(4) Trouver un exemple de suite exacte (non bornée) de R-modules libres qui n'est pas scindée.

Correction

(1) If $C_\bullet$ is contractible we denote by (h_n) a homotopy. We have $id_{C_n} = h_{n-1}d_n + d_{n+1}h_n$. So

$$d_n = d_nh_{n-1}d_n + d_nd_{n+1}h_n = d_nh_{n-1}d_n.$$

Hence $C_\bullet$ is split. Moreover if $x \in Ker(d_n)$ we have $x = d_{n+1}h_n(x) \in Im(d_{n+1})$, hence the complex is exact.

Conversely, if the complex is split and exact. We denote by $Z_n = Ker(d_n)$ and $B_n = Im(d_{n+1})$. We have a short exact sequence :

$$0 \to Z_n \to C_n \to B_{n-1} \to 0$$

and $s_{n-1} : B_{n-1} \to C_n$ is a splitting for this exact sequence. Hence we have an isomorphism $h : C_n \to Z_n \oplus B_{n-1}$ that makes the following diagram commute :

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Z_n & \longrightarrow & C_n & \xrightarrow{d_n} & B_{n-1} & \longrightarrow & 0 \\
 & & \| & & \downarrow h & & \| & & \\
0 & \longrightarrow & Z_n & \xrightarrow{i_1} & Z_n \bigoplus B_{n-1} & \xrightarrow{\pi_2} & B_{n-1} & \longrightarrow & 0.
\end{array}$$

The complex is exact hence we have $Z_n = B_n$ and the family of morphisms $h_\bullet$ induces and isomorphism of complexes :

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & C_{n+1} & \xrightarrow{d_{n+1}} & C_n & \xrightarrow{d_n} & C_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & B_{n+1} \oplus B_n & \longrightarrow & B_n \oplus B_{n-1} & \longrightarrow & B_{n-1} \oplus B_{n-2} & \longrightarrow & \cdots
\end{array}$$

The differential of the bottom sends $(x, y) \in B_n \oplus B_{n-1}$ to $(y, 0) \in B_{n-1} \oplus B_{n-2}$. It remains to see that the complex of the bottom is acyclic. There is an obvious candidate for the homotopy $s_n : B_n \oplus B_{n-1} \to B_{n+1} \oplus B_n$ sending (x, y) to $(0, x)$. It remains to check that this is indeed a homotopy.

(2) For example $0 \to Z \to Z \to Z/n \to 0$ where the first map is the multiplication by n.

(3) One can replace free by projective in this question. Since the complex is exact the map $d_1 : C_1 \to C_0$ is surjective with C_0 projective, so it splits. We have $s_0 : C_0 \to C_1$ such that $d_1s_0 = id_{C_0}$. So $d_1s_0d_1 = d_1$. Moreover by looking at the short exact sequence

$$0 \to Ker(d_1) \to C_1 \to C_0 \to 0$$

we see that it is split hence $C_1 \cong Ker(d_1) \oplus C_0$. Since the complex is exact we have $Ker(d_1) = Im(d_2)$. Hence $C_1 = Im(d_2) \oplus s_0(C_0)$ and $Im(d_2)$ is a direct summand of a projective module, so it is projective. Now the short exact sequence

$$0 \to Ker(d_2) \to C_2 \to Im(d_2) \to 0$$

splits : there is $s_1' : Im(d_2) \to C_2$ such that $d_2s_1 = id_{d_2}$. We let $s_1 : C_1 \to C_2$ be the map defined by $s_1(d_2(x), s_0(y)) = s_1'd_2(x)$ and we have

$$d_2s_1d_2(x) = d_2s_1(d_2(x), 0) = d_2s_1'd_2(x) = d_2(x).$$

We have the result by induction.

(4) Consider a complex where all the modules are $\mathbb{Z}/4$ and all the maps are the multiplication by 2.

Exercice 9 - * Soit $f : C \to D$ un morphisme de complexes de chaines de A-modules.
On note $C[1]$ le complexe défini par $C[1]_n = C_{n-1}$, $d = -d^C$.
On pose $\mathrm{Cone(f)_n = C_{n-1} \oplus D_n}$ et $d : \mathrm{Cone(f)_n} \to \mathrm{Cone(f)_{n-1}}$ définie par

$$d(x, y) = (-d^Cx, d^Dy + f(x))$$

(1) Montrer que Cone(f) est un complexe et que $0 \to D \to$ Cone(f) $\to$ C[1] $\to 0$ est une suite exacte courte de complexes.

(2) En déduire que f est un quasi-isomorphisme si et seulement si Cone(f) est exact.

(3) Montrer que si f est une équivalence d'homotopie alors Cone(f) est contractile.

(4) En déduire que Cone($\mathrm{id_C}$) est scindé et exact. On le note Cone(C). On note également $j : C \to$ Cone(C) l'application canonique.

(5) Montrer que f est homotopiquement nulle si et seulement si il existe s : Cone(C) $\to$ D telle que $f = sj$.

Correction

(1) The first question is clear.

(2) Since we have a short exact sequence of complexes we have a long exact sequence in homology. By diagram chasing we see that the connecting homomorphism if $H_*(f)$. The result now follows from the exactness of the sequence.

(3) This question is harder than it seems : we apply the usual idea : do the what you can ! That is use the inverse of g and the two homotopies to construct a map $S_n : Cone(f)_n \to Cone(f)_{n+1}$. Then check that $\phi = SD + DS(x, y) = (x, y) + (0, \star)$ where $\star$ is a residual element depending on the two homotopies and f. We have $D\phi = DSD = \phi D$ so ϕ is a chain-endomorphism of $Cone(f)$. Moreover, ϕ is invertible (its matrix is triangular with one over the diagonal). Since ϕ is homotopic to zero, we have $Id = \phi^{-1} \circ \phi$ is homotopic to zero and this concludes the proof. You can also deduce the formula for the homotopy using the inverse of ϕ.

The converse is also true : if the mapping cone of f is contractible, then there is a homotopy S such that $SD + DS = Id$. Using matrix notation it is not hard to see that one can construct a homotopy inverse of f.

There is also a much more elegant proof using triangulated categories.

(4) This is a consequence of the previous exercise.

(5) If f factorizes as sj we have $f = s \circ Id_{Cone(C)} \circ j$ and since Id is homotopic to zero, we have f is homotopic to zero. For the converse one can easily construct the map s.

Exercice 10 -

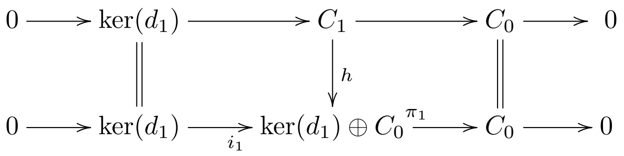
Soit A un anneau et $Ch_+(A)$ la catégorie des complexes de chaines de A-modules concentrés en degrés positifs. Montrer qu'un complex $C_\bullet$ est isomorphe à 0 dans la catégorie homotopique $K_+(A)$ si et seulement si $C_\bullet$ est isomorphe à une somme directe de complexes de la forme $0 \to Z \overset{id}{\to} Z \to 0$ concentrés en deux degrés.

Correction

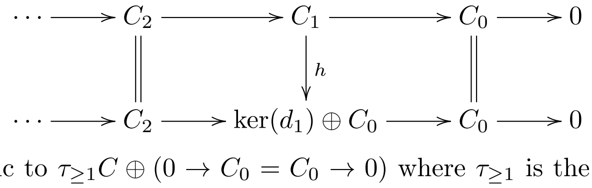
The complexes $0 \to Z \ = Z \to 0$ are homotopic to zero, so isomorphic to zero in the homotopy category. Any direct sum of such complexes is isomorphic to zero. Conversely if a complex is isomorphic to zero in the homotopy category, it is homotopy equivalent to zero, so contractible. By Exercise 8 it is exact and split. If $(C_n, d_n)_{n\geq 0}$ is such a complex by exactness, we have $d_1 : C_1 \to C_0$ surjective. And there is a splitting $s_0 : C_0 \to C_1$ such that $d_1 = d_1 s_0 d_1$. So we have a short exact sequence

$$0 \to \ker(d_1) \to C_1 \to C_0 \to 0$$

and s_0 is such that $d_1 s_0 = Id_{C_0}$. Since $d_1 s_0(x) = d_1 s_0 d_1(y) = d_1(y) = x$. So by the splitting lemma there is an isomorphism $hC_1 \to \ker(d_1) \oplus C_0$ such that the following diagram commutes



Then we get an ismorphism of complexes



and we see that C is isomorphic to $\tau_{\geq 1} C \oplus (0 \to C_0 = C_0 \to 0)$ where $\tau_{\geq 1}$ is the complex obtained by removing C_0 and replacing C_1 by $\ker(d_1)$. This new complex is still exact and split. We continue by induction.