Chap IV Additive categories

I _ Preadditive and additive categories

Def 4.1 A zero object in a category $\mathcal{E}$ is an object that is both final and ~~terminal~~ initial

Ex $\{0\}$ in Mod A A ring

Def 4.2 ① Let k be a commutative ring. A k-category is a category $\mathcal{E}$ s.t.

(a) $\forall x,y \in Ob(\mathcal{E})$ $Hom_{\mathcal{E}}(x,y)$ is a k-module

(b) $-\circ-$ is k-bilinear

② When $k = \mathbb{Z}$ we say that $\mathcal{E}$ is preadditive

Rem One says that $\mathcal{E}$ is enriched over Mod k

Lem 4.3 Let $\mathcal{E}$ be a k-category. For $x,y \in Ob(\mathcal{E})$ we have

(1) The product exists $x\times y$ iff the coproduct $x \sqcup y$ exists

(2) If so they are isomorphic

proof If $(x\times y, \pi_x, \pi_y)$ is a product then

$$\begin{tikzcd}
& X \arrow[ld, bend right, "1_x"'] \arrow[rd, bend left, "0"] \arrow[d, dashed, "\exists!\, i_x"] & \\
x & x\times y \arrow[l, "\pi_x"] \arrow[r] & y
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
& Y \arrow[ld, bend right, "0"'] \arrow[rd, bend left, "1_y"] \arrow[d, dashed, "\exists!\, i_y"] & \\
x & x\times y \arrow[l] \arrow[r] & y
\end{tikzcd}$$

claim $(x\times y, i_x, i_y)$ is a coproduct

$$\begin{tikzcd}
X \arrow[rd, "i_x"] \arrow[rdd, bend right, "a"'] & & Y \arrow[ld, "i_y"'] \arrow[ldd, bend left, "b"] \\
& x\times y \arrow[d, dashed] & \\
& Z &
\end{tikzcd}$$

then

$$\begin{tikzcd}
& x\times y \arrow[ld, "\pi_x"'] \arrow[rd, "\pi_y"] & \\
x \arrow[r, "a"'] & z & y \arrow[l, "b"]
\end{tikzcd}$$

$a\circ\pi_x : x\times y \to z$ and $b\circ\pi_y : x\times y \to z$

$f = a\pi_x + b\pi_y$ then

$$\begin{tikzcd}
X \arrow[dr, "i_X"] & & Y \arrow[dl, "i_Y"'] \\
& X\times Y \arrow[d, "f"] & \\
& {} &
\end{tikzcd}$$

$$\begin{aligned} f\circ i_X &= (a\pi_X + b\pi_Y)\circ i_X \\ &= a\pi_X i_X + \underset{\overset{\|}{0}}{b\pi_Y i_X} \\ &= a \end{aligned}$$

similarly $f\circ i_Y = b$ so f makes the diagram a commutative diagram.

(•) unicity If g is another map making diag commute

$$\begin{tikzcd}
X \arrow[dr, "i_X"] \arrow[ddr, bend right, "a"'] & & Y \arrow[dl, "i_Y"'] \arrow[ddl, bend left, "b"] \\
& X\times Y \arrow[d, "g"] & \\
& Z &
\end{tikzcd}$$

$g\circ i_X = a$

$g\circ i_Y = b$

$$\begin{aligned} \text{so } g\circ(i_X\pi_X + i_Y\pi_Y) &= g\circ i_X\pi_X + g\circ i_Y\pi_Y \\ &= \cancel{g}\ a\pi_X + b\pi_Y = f \end{aligned}$$

last step we check that $\beta : i_X\pi_X + i_Y\pi_Y = Id_{X\times Y}$

$$\begin{tikzcd}
& X\times Y \arrow[d, "\beta"] \arrow[ddl, bend right, "\pi_X"'] \arrow[ddr, bend left, "\pi_Y"] & \\
& X\times Y \arrow[dl, "\pi_X"] \arrow[dr, "\pi_Y"'] & \\
X & & Y
\end{tikzcd}$$

(*)

$\pi_X\circ\beta = \pi_X$

$\pi_Y\circ\beta = \pi_Y$

So β makes (*) commute but also $Id_{X\times Y}$

by unicity of the universal property we have the result. □

Def 4.4 Let $\mathcal{E}$ be a k-linear category. A biproduct of X and Y is an object $X\oplus Y \in \mathcal{E}$ with
$$\begin{tikzcd} X \arrow[r, shift left, "i_X"] & X\oplus Y \arrow[l, shift left, "\pi_X"] \arrow[r, shift left, "\pi_Y"] & Y \arrow[l, shift left, "i_Y"] \end{tikzcd}$$

s.t. (1) $i_X\pi_X + i_Y\pi_Y = Id_{X\oplus Y}$

(2) $\pi_X\circ i_Y = 0$ $\quad \pi_X\circ i_X = Id_X$

$\pi_Y\circ i_X = 0$ $\quad \pi_Y\circ i_Y = Id_Y$

Def 4.5 Let k be a commutative ring. A k-additive category (or k-linear) is a k-category with finite products and finite coproducts.

Rem (1) $k = \mathbb{Z}$ speaks about additive category.

(2) finite product = finite coproducts = finite biproducts

(3) $\mathcal{C}$ k cat. TFAE

(a) $\mathcal{C}$ is k-additive

(b) $\mathcal{C}$ has a zero object and every pair of object has a product

(c) // // // coproduct

(d) // // biproduct

Sketch (b)⇔(c)⇔(d) for (a) we are only missing $\emptyset$ product, $\emptyset$ coproduct
is
0

(4) If A is additive there is a canonical interpretation of the gp structure on $\mathrm{Hom}(-,-)$ using $\oplus$ see TD.

(0) Abelian gps

Ex (1) A ring (k-alg) $\mathrm{Mod}A$, $A\mathrm{Mod}$ + fg versions are additive (k-lin)

(2) $\mathcal{C}$ additive then $\mathcal{C}^{op}$ additive

(3) $\mathcal{C}$ additive I category $\mathrm{Fun}(I, \mathcal{C})$ is additive

(4) A ring $BA = \underset{A}{\bullet}$ is preadditive but not additive

Def 4.5 $F : \mathcal{C} \to \mathcal{C}'$ be a functor between two additive categories (k-lin)

Then F is an additive functor (k-linear functor) if $\forall\, x, y \in \mathrm{Ob}(\mathcal{C})$ $f \mapsto F(f)$ from $\mathrm{Hom}_{\mathcal{C}}(x,y) \to \mathrm{Hom}_{\mathcal{C}'}(Fx, Fy)$ is a group homomorphism (k-lin morph)

$$f \longmapsto F(f)$$

Prop 4.6 $F : \mathcal{C} \to \mathcal{C}'$ is additive iff $F(0) \simeq 0$ and $F(x \oplus y) \simeq F(x) \oplus F(y)$

Proof $\Rightarrow$ (1)
$$\begin{tikzcd} X \arrow[r, shift right, "i_X"'] & X\oplus Y \arrow[l, shift right, "\pi_X"'] \arrow[r, shift left, "\pi_Y"] & Y \arrow[l, shift left, "i_Y"] \end{tikzcd}$$

$$\begin{tikzcd} F(X) \arrow[r, shift left, "F(\pi_X)"] & F(X\oplus Y) \arrow[l, shift left, "F(i_y)"] \arrow[r, shift left, "F(\pi_y)"] & F(Y) \arrow[l, shift left, "F(i_y)"] \end{tikzcd}$$

Fu check that this is a biproduct.

(2) $F(0) = 0$ since

0 object $\Rightarrow Id_0 = 0$ morphism

and $F(0) = 0$
$F(Id) = 0$ □

$\Leftarrow$ requires relation between + and $\oplus$!

Example A, B two rings ${}_A M_B$ bimodule then
$$- \underset{A}{\otimes} M_B : \text{Mod}A \to \text{Mod}B$$
is additive : it is a left adjoint so it preserves coproduct !

II Chain complexes in an additive category

Here all categories are additive

Def 4.7 (•) A chain complex in $\mathcal{E}$ is a collection $C_\bullet = \{C_n \mid n \in \mathbb{Z}\}$ of objects of $\mathcal{E}$ together with $\partial_n : C_n \to C_{n-1}$ morphisms of $\mathcal{E}$ s.t. $\partial_{n-1} \circ \partial_n = 0$

(•) The ∂_n are the differentials of the chain

A cochain complex in $\mathcal{E}$ is a collection $C^\bullet = \{C^n / n \in \mathbb{Z}\}$ of objects of $\mathcal{E}$ together with $\delta^n : C^n \to C^{n+1} \in \text{Mor}(\mathcal{E})$ s.t. $\delta^{n+1} \circ \delta^n = 0$

Rem $C_\bullet$ chain complex $C^n = C_{-n}$ $\delta^n = \partial_{-n}$ is a cochain complex
Formally same mathematical notion But in general chain and cochain complexes represent different objects so it is good to distinguish the two.

Def 4.8 $C_\bullet$ and $D_\bullet$ two chain complexes in $\mathcal{E}$. A morphism of chain complex $f_\bullet : C_\bullet \to D_\bullet$ is a collection $\underset{\forall n \in \mathbb{N}}{f_n} : C_n \to D_n \in \text{Mor}(\mathcal{E})$ s.t.

$$\begin{tikzcd} {} \arrow[r] & C_n \arrow[r, "\partial_n^C"] \arrow[d, "f_n"] \arrow[rd, phantom, "C"] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\ {} \arrow[r] & D_n \arrow[r, "\delta_n^D"'] & D_{n-1} & \end{tikzcd}$$

" $\partial f = f \partial$ "

Def / Prop 4.9 $\mathcal{E}$ is an additive category. $Ch(\mathcal{E})$ is the category with objects the chain complexes in $\mathcal{E}$ and morphisms the morphisms of chain complexes.

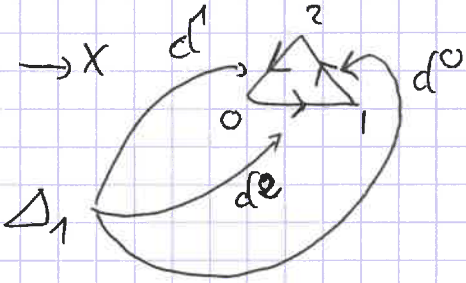
- First differentials

$$\begin{tikzcd}
d_1 : C_1^{sing}(X) \arrow[r] \arrow[d, phantom, "\wr" rotate=90] & C_0^{sing}(X) \arrow[d, phantom, "\wr" rotate=90] \\
\mathbb{Z}[\text{paths in } X] & \mathbb{Z}[X]
\end{tikzcd}$$

$\sigma : \Delta_1 \to X \qquad d_1(\sigma) = \Delta_0 \xrightarrow{d^0 - d^1} \Delta_1 \xrightarrow{\sigma} X$

$= \sigma \circ d^1 - \sigma \circ d^0 \text{ "="} \ \sigma(1) - \sigma(0)$

$\sigma : \Delta_2 \to X$



$d_2(\sigma) = \sigma_{|[1,2]} - \sigma_{|[0,2]} + \sigma_{|[0,1]}$

Prop 4.13 $\quad C^{sing} : Top \to Ch_\bullet(Ab)$ is a functor

proof

$$\begin{tikzcd}
X \arrow[d, "f"'] & \arrow[r] & Hom(\Delta_n, X) \arrow[r, "d_n^X"] \arrow[d, "f\circ -"', "f_n"] & Hom(\Delta_{n-1}, X) \arrow[r] \arrow[d, "f\circ -"', "f_{n-1}"] & {} \\
Y & \arrow[r] & Hom(\Delta_n, Y) \arrow[r, "d_n^Y"'] & Hom(\Delta_{n-1}, Y) \arrow[r] & {}
\end{tikzcd}$$

$\sigma : \Delta_n \to X \qquad f_{n-1}(d_n^X(\sigma)) = f_{n-1}\left(\Sigma(-1)^i \sigma \circ d^i\right)$

$= \sum_i (-1)^i f \circ \sigma \circ d^i = d_n^Y(f \circ \sigma)$ □

Méthodes simpliciales

Δ category - objects $[n] = \{0, 1, \dots, n\}$

- $Hom([n],[m]) = \{f : [n] \to [m] \mid f(i) \leq f(j) \text{ if } i \leq j\}$

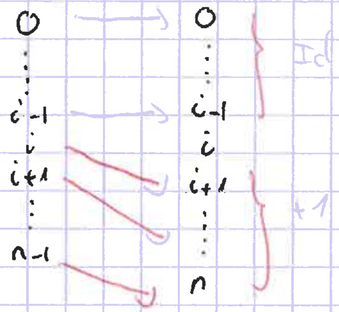
(equivalent to category of total orders with increasing maps as morphisms)

Def 4.14 (1) A simplicial set is a contravariant functor $\Delta \to Set$

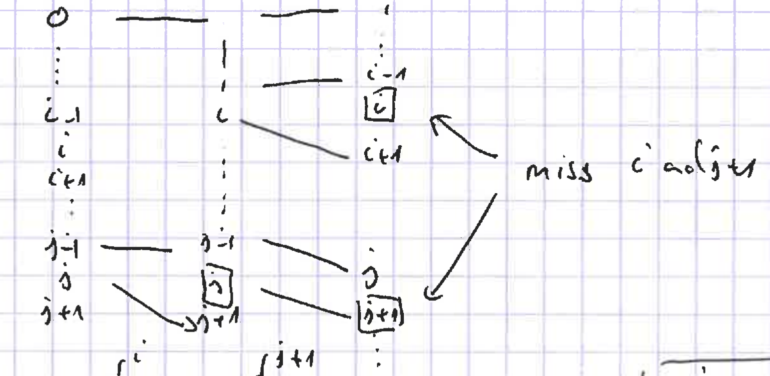
(2) If $\mathcal{E}$ is a category, a simplicial object in $\mathcal{E}$ is a contravariant functor from Δ to $\mathcal{E}$.

Notation $X : \Delta^{op} \to \mathcal{C}$ $X_n := X([n])$. The n-simplices of X

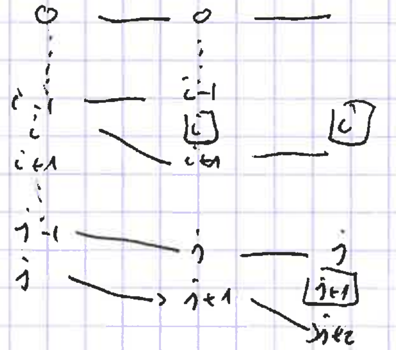
in Δ we have $d^i : [n-1] \longrightarrow [n]$ injective map that misses "i"



we have $[n-1] \xrightarrow{d^j} [n] \xrightarrow{d^i} [n+1]$ $i \leq j$



$[n-1] \xrightarrow{d^i} [n] \xrightarrow{d^{j+1}} [n+1]$ so $\boxed{d^i \circ d^j = d^{j+1} \circ d^i \ \forall i \leq j}$



So if $X : \Delta^{op} \to Ab$ is a simplicial abelian group, then can define $(X_\bullet, \partial)$ with $X_n = X([n])$ and $\partial_n : X_n \to X_{n-1}$

$$x \longmapsto \sum_{i=0}^{n} (-1)^i X(d^i)(x)$$

Prop 4.15 (1) If $X \in sAb$, then $(X_\bullet, d) \in ch_\bullet(Ab)$

(2) $X \longmapsto X_\bullet$ is a functor from sAb to $ch_\bullet(A)$

$s^i : [n+1] \to [n]$ "i is hit twice"

$\left.\begin{matrix} i \\ i+1 \end{matrix}\right\} \mapsto i$

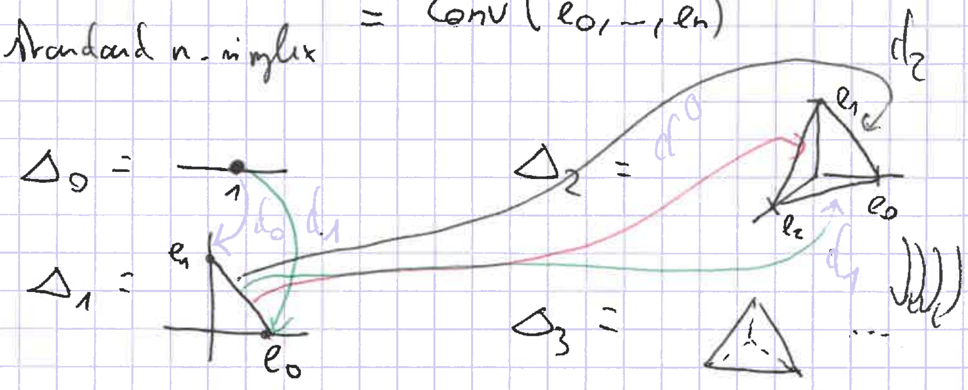
Rem On can easily check that $Ch^{\cdot}(\mathcal{C})$ is an additive category.

Exemple : Singular chain complex

$n \in \mathbb{N}$ $\quad \Delta_n = \{ (x_0,\dots,x_n) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n} x_i = 1 \text{ and } x_i \geq 0 \}$

the standard n-simplex $\quad = \mathrm{Conv}(e_0,\dots,e_n)$

Ex



Rem Δ_n appears $n+1$ times as face of standard $n+1$ simplex

$d^i : \Delta_n \to \Delta_{n+1}$ $\qquad$ ith face map

$(x_0,\dots,x_n) \mapsto (x_0,\dots,x_{i-1},\underset{\overset{\uparrow}{i}}{0},x_{i+1},\dots,x_n)$

Δ_n is a topological space so when X is a topological space we can consider

$\mathrm{Hom}_{Top}(\Delta_n,X) = \{ f : \Delta_n \to X \mid f \text{ continuous} \}$

and we get $d_i : \mathrm{Hom}_{Top}(\Delta_{n+1},X) \to \mathrm{Hom}_{Top}(\Delta_n,X)$

$\sigma \longmapsto \Delta_n \xrightarrow[d^i]{} \Delta_{n+1} \xrightarrow{\sigma} X$

for $0 \leq i \leq n+1$

⊙ $C_n^{Sing}(X) := \mathbb{Z}[\mathrm{Hom}_{Top}(\Delta_i, X)]$ free abelian group with basis $\mathrm{Hom}_{Top}(\Delta_i,X)$

⊙ $d_n : C_n^{Sing}(X) \to C_{n-1}^{Sing}(X)$

$\underset{\Delta_n \to X}{\sigma} \longmapsto \sum (-1)^i d_i(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma \circ d^i$

Prop 4.11 $(C_n^{Sing}(X), d)$ is a complex of chains of abelian groups

$i \leq j$

Proof

$$\begin{tikzcd} \Delta_{n-1} \arrow[r, "d^j"] \arrow[dr, "d^i"'] & \Delta_n \arrow[r, "d^i"] & \Delta_{n+1} \\ & \Delta_n \arrow[ur, "d^{j+1}"'] & \end{tikzcd}$$

$x \longmapsto \overset{0}{\underset{j}{\uparrow}} \longmapsto \overset{0}{\underset{i}{\uparrow}} \quad \overset{0}{\underset{j+1}{\uparrow}}$

$\overset{0}{\underset{i}{\uparrow}}$

So we have $\boxed{d^i \circ d^j = d^{j+1} \circ d^i \quad \forall\, i \leq j}$

Taking $\mathrm{Hom}_{Top}(-,X)$ (contravariant functor we have

$\boxed{d_i \circ d_{j+1} = d_j \circ d_i \quad \forall\, i \leq j}$

$$\mathrm{Hom}(\Delta_{n+1},X) \xrightarrow{d_{n+1}} \mathrm{Hom}(\Delta_n,X) \xrightarrow{d_n} \mathrm{Hom}(\Delta_{n-1},X)$$

$$d_n \circ d_{n+1}(\sigma) : \Delta_{n-1} \xrightarrow{d^{n-1}} \Delta_n \xrightarrow{d^n} \Delta_{n+1} \xrightarrow{\sigma} X$$

$$\overset{||}{\sigma \circ d^n \circ d^{n+1}} = \sum_{i=0}^{n} (-1)^i \sigma \circ d^n \circ d^i = \sum_{i=0}^{n} \sum_{j=0}^{n+1} (-1)^{i+j} \sigma \circ d^j \circ d^i$$

$$= \sum_{0 \leq i < j \leq n+1} (-1)^{i+j} \sigma \circ \underbrace{d^j \circ d^i}_{= d^i \circ d^{j-1}} + \sum_{0 \leq j \leq i \leq n} (-1)^{i+j} \sigma \circ d^j \circ d^i$$

$$= \sum_{0 \leq i < j \leq n+1} (-1)^{i+j} \sigma\, d^i \circ d^{j-1} + \qquad ''$$

$$= \sum_{0 \leq i \leq j \leq n} (-1)^{i+j+1} \sigma \circ d^i \circ d^j + \sum_{0 \leq j \leq i \leq n} (-1)^{i+j} \sigma \circ d^j \circ d^i = 0 \qquad \square$$

sign $(-1)^{i+j+1}$ sign $(-1)^{i+j}$

Def 4.12 $\sigma : \Delta_n \to X$ continuous is called a singular n-simplex

Ex $\Delta_0 = \bullet$ so 0-singular simplex = point in X

↳ $C_0^{sing}(X) \simeq \mathbb{Z}[X]$ free abelian gp on X

$\Delta_1 =$  $\simeq [0,1] =: I$ 1 singular simplex is a path in X

$\sigma : \Delta_2 \to X$ starts to be complicated

can have singularity

⚠ $C_n^{sing}(X)$ is hudge!

Thm 4.16 Every morphism in Δ is a composition of d^i and s^i and the d^i, s^i are subject to the so-called simplicial relations

$$(*) \begin{cases} d^j \circ d^i = d^i \circ d^{j-1} & i<j \quad (1) \\ s^i \circ s^j = s^j \circ s^{i+1} & i>j \quad (2) \\ d^i \circ s^j = \begin{cases} s^{j-1} \circ d^i & i<j \\ Id & i \in \{j, j+1\} \\ s^j \circ d^{i-1} & i>j+1 \end{cases} & (3) \end{cases}$$

and this is a presentation of Δ by generators and relations

↳ To define a functor from Δ to $\mathcal{E}$ it is enough to define $F(d^i)$, $F(s^i)$ and show that (*) hold

Proof voir annexe □

For using 4.15 only need the d^i s, that generates Δ_{inj} so to construct $F : \Delta_{inj}^{op} \to \mathcal{E}$ only need to define $F(d^i)$ and check (1)

Cd

$\mathcal{E} \leftarrow$ additive

Thm 4.17 If $F : \Delta_{inj}^{op} \to \mathcal{A}b$ is a (semisimplicial abelian gp) functor then $(F([n]), d_\bullet)$ with $d : F([n]) \to F([n-1])$, $x \longmapsto \sum (-1)^i F(d^i)(x)$ is a chain complex of abelian gps (of $\mathcal{E}$)

Examples

(1) $Top \to sSet \to sAb$ recovers example of before

$X \mapsto Hom_{Top}(\Delta(-), X) \mapsto \mathbb{Z}[Hom_{Top}(\Delta(-), X)]$

(2) G be a finite group F_n = free abelian gp on $\{(g_0, \dots, g_n) ; g_i \in G\}$ $\simeq G^{\times(n+1)}$

F_n is a $\mathbb{Z}[G]$-module for $g \bullet (g_0, \dots, g_n) = (gg_0, \dots, gg_n)$

so $F_n \in \mathbb{Z}[G]Mod$.

$\partial_i : F_n \to F_{n-1}$ remove g_i

$(g_0, \dots, g_n) \longmapsto (g_0, \dots, g_{i-1}, g_{i+1}, \dots, g_n)$

For $i<j$

$$\partial_i \circ \partial_j (g_0, \dots, g_n) = \partial_i(\dots, \check{g}_j) = (\dots, \check{g}_i, \dots, \check{g}_j, \dots)$$

$$\partial_{j-1} \circ \partial_i (g_0, \dots, g_n) = \partial_{j-1}(\dots, \check{g}_i, \dots) = (\check{g}_i, \dots, \check{g}_j, \dots)$$

so setting $F([n]) = F_n$ and $F(d^i) = \partial_i$ produces $F : \Delta^{op}_{inj} \to \mathbb{Z}[G]\mathrm{Mod}$

so applying Thm 4.17 we have $(F_n, \partial) \in Ch_\bullet(\mathbb{Z}[G]\mathrm{Mod})$

called the <u>Bar resolution</u> of <u>G</u>

(2) Koszul complex, Hochschild complex, ...

4

<u>Def</u> <u>4.18</u> $\mathcal{E}$ be an additive category, $C_\bullet, D_\bullet \in Ch_\bullet(\mathcal{E})$ $f, g \in \mathrm{Hom}($ ta $(g_\cdot)$

(1) A homotopy H from f to g is the data of $h_i : C_i \to D_{i+1}$ in $\mathcal{E}$ such that $f_n - g_n = h_{n-1} d_n^C + d_{n+1}^D h_n$ "$f - g = hd + dh$"

$$\begin{tikzcd}
C_{n+1} \arrow[r, "d_{n+1}^C"] \arrow[d, "f_{n+1}-g_{n+1}"] & C_n \arrow[r, "d_n^C"] \arrow[d, "f_n-g_n"] \arrow[dl, "h_n"'] & C_{n-1} \arrow[d, "f_{n-1}-g_{n-1}"] \arrow[dl, "h_{n-1}"'] \\
D_{n+1} \arrow[r, "d_{n+1}^D"'] & D_n \arrow[r, "d_n^D"'] & D_{n-1}
\end{tikzcd}$$

we say that f and g are homotopic and write $f \sim g$

(2) $f_\bullet$ and $g_\bullet$ are homotopy equivalences if $fg \sim Id_D$ and $gf \sim Id_C$

<u>Motivation</u> : <u>Topology</u>

<u>Recall</u> $f, g: X \to Y$ continuous maps between topological spaces f and G are homotopic if $\exists\ F: X \times I \to Y$ $\quad I = [0,1]$ continuous s.t. $F(-,0) = f$ and $F(-,1) = g$

F is called a homotopy from f to g

<u>Thm 4.19</u> X, Y be two topological spaces $f, g: X \to Y$ two homotopic maps. Then $C^{sing}(f)$ and $C^{sing}(g): C^{sing}(X) \to C^{sing}(Y)$ are homotopic with respect to definition 4.18

<u>Proof</u>

$$\begin{tikzcd}
& \mathrm{Hom}(\Delta_0, X) \arrow[dl, dashed, "h_0?"'] \arrow[d] \\
\mathrm{Hom}(\Delta_1, Y) \arrow[r, "d"'] & \mathrm{Hom}(\Delta_0, Y)
\end{tikzcd}$$

we look for h_0 s.t. $\boxed{d \circ h_0 = g_- - f_-}$

$$h_0(\sigma): \Delta_1 \simeq \Delta_0 \times \Delta_1 \simeq \Delta_0 \times I \xrightarrow{\sigma \times Id_I} X \times I \xrightarrow{H} Y$$

<u>unformaly</u> $\mathrm{Hom}(\Delta_0, X) \simeq X$

$\Delta_1 \simeq Y$ so $h_0: \left(\begin{array}{l} X \to \mathrm{Hom}(I, Y) \\ x \longmapsto (i \mapsto H(i,x)) \end{array} \right)$ so h_0 is H

and $d_1 h_0(\sigma) = h_0(\sigma)(1) - h_0(\sigma)(0)$

$= h_0(\sigma(0), 1) - h_0(\sigma(0), 0) = g(\sigma(0)) - f(\sigma(0))$

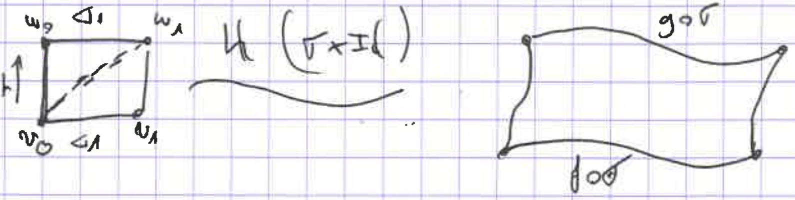
so $\boxed{d_1 \circ s_0 = g_* - f_*}$

$\boxed{n=1}$

$$\begin{tikzcd}
& \mathrm{Hom}(\Delta_1, X) \arrow[r, "d_1"] \arrow[dl, dashed, "?"'] \arrow[d, "g_* - f_*"] & \mathrm{Hom}(\Delta_0, X) \arrow[dl, "h_0"] \\
\mathrm{Hom}(\Delta_2, Y) \arrow[r, "d_2"'] & \mathrm{Hom}(\Delta_1, Y) &
\end{tikzcd}$$

$$\Delta_1 \times I \xrightarrow{\sigma \times Id_I} X \times I \xrightarrow{H} Y \quad \text{for} \quad \sigma : \Delta_1 \to X$$

But $\Delta_1 \times I =$



at $t=0$ $\quad H \circ \sigma \times Id_I|_{t=0} = h(\sigma(-), 0) = f \circ \sigma$

$$H \circ \sigma \times Id_I|_{t=1} = h(\sigma(1), 1) = g \circ \sigma$$

at $\check{v}_1$: i.e $[v_0, \hat{v}_1]$ we have $h(\sigma|_{[v_0, \hat{v}_1]}, -)$

$\check{v}_0$: $[\hat{v}_0, v_1]$ $\quad h(\sigma|_{[\check{v}_0, v_1]}, -)$

these two are **exactly** $h_0 \circ d_1(\sigma)$

Problem $\Delta_1 \times I \neq \Delta_2$ we can actually cut $\Delta_1 \times I$ into 2 copies of Δ_2

and we set

$$h_1(\sigma) = h \circ (\sigma \times Id_I)|_{[v_0, w_0, w_1]} - h \circ (\sigma \times Id_I)|_{[v_0, v_1, w_1]}$$

we identify Δ_2 with $[a, b, c]$

$$e_0 \longmapsto a$$
$$e_1 \longmapsto b$$
$$e_2 \longmapsto c$$

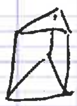
claim $\quad (d_2 \circ h_1 - h_0 \circ d_1)(\sigma) = g\sigma - f\sigma$

indeed $\quad d_2 \circ h_1(\sigma) = d_2 h_1(\sigma)|_{[v_0, w_0, w_1]} - d_2 h_1(\sigma)|_{[v_0, v_1, w_1]}$

$$= h \circ (\sigma \times Id)|_{[\check{v}_0, w_0, w_1]} - \cancel{h \circ (\sigma \times Id)|_{[v_0, \check{w}_0, w_1]}} + h \circ (\sigma \times Id)|_{[v_0, w_0, \check{w}_1]}$$

$$- h \circ (\sigma \times Id)|_{[\check{v}_0, v_1, w_1]} + \cancel{h \circ (\sigma \times Id)|_{[v_0, w_1]}} + h \circ (\sigma \times Id)|_{[v_0, w_1]}$$

$$= g \circ \sigma - f \circ \sigma + \cancel{d_1 h_1}\, h_0 \circ d_1(\sigma)$$

$\underline{n=3}$  split as 3 2 simplices

More generally $h_n(\sigma) = \sum_{i=0}^{n} (-1)^i\, h \circ (\sigma \times Id)_{|[v_0,\dots,v_i,w_i,\dots,w_n]}$ ☐

Lemma 4.20 $f : X_\bullet \to Y_\bullet$ $g : Y_\bullet \to Z_\bullet$ morphisms of $Ch_\bullet(\mathcal{E})$. Then $f \sim 0$ $\Rightarrow g \circ f \sim 0$

Proof

$$\begin{tikzcd}
X_i \arrow[r, "d_i^X"] \arrow[d, "f_i"'] & X_{i-1} \arrow[dl, "h_i"] \arrow[d, "f_{i-1}"] \\
Y_i \arrow[r, "d_i^Y"'] \arrow[d, "g_i"'] & Y_{i-1} \arrow[d, "g_{i-1}"] \\
Z_i \arrow[r, "d_i^Z"'] & Z_{i-1}
\end{tikzcd}$$

$\tilde{h}_i = g_i \circ h_i$ ☐

Prop/Def 4.21 $\mathcal{E}$ be an additive category. The homotopy category $k(\mathcal{E})$ is the category

$$Ob(k(\mathcal{E})) = Ob(Ch_\bullet(\mathcal{E}))$$
$$Hom_{k(\mathcal{E})}(X,Y) = Hom_{Ch(\mathcal{E})}(X,Y)/_\sim$$

Proof One has to check that this is a category using 4.20 ☐

Rem (1) $k(\mathcal{E})$ is an additive category

(2) ⚠ in general this is a complicated object: it is a triangulated category.

Chap 5 Abelian categories

Def 5.1 $\mathcal{E}$ be an additive category. A kernel of $f \in Mor(\mathcal{E})$ is an equalizer of $(f,0)$. Dually a cokernel of f is a coequalizer of $(f,0)$.

Concretely

$$\begin{tikzcd}
& Z \arrow[dl, dashed] \arrow[d, "\alpha"] & \\
\ker f \arrow[r, hook, "\iota"'] & X \arrow[r, "f"] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[r, "\pi"] \arrow[d, "\alpha"] & \operatorname{coker}(f) \arrow[dl, dashed] \\
& Z &
\end{tikzcd}$$

We assume that every morphism in $\mathcal{E}$ has a kernel and a cokernel

Then
$$\begin{tikzcd}
\ker f \arrow[r, "i"] & X \arrow[r, "f"] \arrow[d, "p"'] & Y \arrow[r, "\pi"] & \mathrm{Coker}(f) \\
& \mathrm{Coker}(\ker f) & \ker(\mathrm{coker} f) \arrow[u] &
\end{tikzcd}$$

$$\begin{tikzcd}
\ker f \arrow[r] & X \arrow[r, "f"] \arrow[d] & Y \arrow[r, "\pi"] & \mathrm{coker} f \\
& \mathrm{Coker}(\ker f) \arrow[ur, dashed, "\tilde f"'] & &
\end{tikzcd}$$

$f\circ i = 0 \Rightarrow$ so $\pi\circ\tilde f\circ p = \pi\circ f = 0$ ⊛

Lem 5.2 kernels are monomorphism and cokernels are epimorphisms

proof
$$\begin{tikzcd}
\ker f \arrow[r, "i"] & X \arrow[r, "f"] & Y \\
W \arrow[u, shift left, "a"] \arrow[u, shift right, "b"'] & \arrow[u] &
\end{tikzcd}$$

$i\circ(b-a)=0$ so
$$\begin{tikzcd}
\ker f \arrow[r, "i"] & X \arrow[r, "f"] & Y \\
& W \arrow[ul, "0"] \arrow[ul, bend left, "b-a"] \arrow[u, "0"'] &
\end{tikzcd}$$

are two solutions for the universal problem so $b-a=0$ □

hence (*) $\pi\circ\tilde f\circ p = \pi f = 0 \underset{p\text{ epi}}{\Rightarrow} \pi\circ\tilde f = 0$ so $\tilde f$ factorizes through $\ker(\mathrm{coker} f)$

hence setting $\mathrm{Coim} f = \mathrm{Coker}(\ker f)$ and $\mathrm{Im} f = \ker(\mathrm{Coker} f)$

every morphism has a canonical factorisation

$$\begin{tikzcd}
\ker f \arrow[r] & X \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[r] & \mathrm{Coker} f \\
& \mathrm{Coim}(f) \arrow[r, "\exists!\,\tilde f"'] & \mathrm{Im} f \arrow[u] &
\end{tikzcd}$$