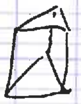
$n=3$  split as 3 2 simplices

More generally $h_n(\sigma) = \sum_{i=0}^{n} (-1)^i \, h \circ (\sigma \times \mathrm{Id})_{|[v_0, \dots, v_i, w_i, \dots, w_n]}$ □

Lemma 4.20 $f : X_\bullet \to Y_\bullet$, $g : Y_\bullet \to Z_\bullet$ morphisms of $Ch_\bullet(\mathcal{E})$. Then $f \sim 0$
$\Rightarrow g \circ f \sim 0$

Proof

$$\begin{tikzcd}
X_i \arrow[r, "d_i^X"] \arrow[d, "f_i"'] & X_{i-1} \arrow[d, "f_{i-1}"] \arrow[dl, "h_i"] \\
Y_i \arrow[r, "d_i^Y"'] \arrow[d, "g_i"'] & Y_{i-1} \arrow[d, "g_{i-1}"] \\
Z_i \arrow[r, "d_i^Z"'] & Z_{i-1}
\end{tikzcd}$$

$\tilde{h}_i := g_i \circ h_i$ □

Prop/Def 4.21 $\mathcal{E}$ be an additive category. The homotopy category $K(\mathcal{E})$ is the category
$$ob(K(\mathcal{E})) = ob(Ch_\bullet(\mathcal{E}))$$
$$Hom_{K(\mathcal{E})}(X,Y) = Hom_{Ch(\mathcal{E})}(X,Y)/_{\sim}$$

Proof one has to check that this is a category using 4.20 □

Rem (1) $K(\mathcal{E})$ is an additive category
(2) ⚠ in general this is a complicated object: it is a triangulated category.

Chap 5 Abelian categories

Def 5.1 $\mathcal{E}$ be an additive category. A kernel of $f \in Mor(\mathcal{E})$ is an equalizer of $(f,0)$. Dually a cokernel of f is a coequalizer of $(f,0)$.

Concretely

$$\begin{tikzcd}
 & Z \arrow[d, "\alpha"] \arrow[dl, dashed, bend right] & \\
\ker f \arrow[r, hook, "c"'] & X \arrow[r, "f"] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[r, "\pi"] \arrow[d, "\alpha"] & \mathrm{coker}(f) \arrow[dl, dashed, bend left] \\
 & Z &
\end{tikzcd}$$

We assume that every morphism in $\mathcal{E}$ has a kernel and a cokernel

Then

$$\begin{tikzcd}
\ker f \arrow[r, "i"] & X \arrow[r, "f"] \arrow[d, "p"'] & Y \arrow[r, "\pi"] & \mathrm{Coker}(f) \\
& \mathrm{Coker}(\ker f) & \ker(\mathrm{coker} f) \arrow[u] &
\end{tikzcd}$$

$f \circ i = 0 \Rightarrow$

$$\begin{tikzcd}
\ker f \arrow[r] & X \arrow[r, "f"] \arrow[d] & Y \arrow[r, "\pi"] & \mathrm{Coker} f \\
& \mathrm{Coker}(\ker f) \arrow[ur, dashed, "\exists \tilde{f}"'] & &
\end{tikzcd}$$

so $\pi \circ \tilde{f} \circ p = \pi \circ f = 0$ ⊛

Lem 5.2 kernels are monomorphism and cokernels are epimorphisms

proof

$$\begin{tikzcd}
\ker f \arrow[r, "i"] & X \arrow[r, "f"] & Y \\
W \arrow[u, shift left, "a"] \arrow[u, shift right, "b"'] & \cdot \arrow[u] &
\end{tikzcd}$$

so $\boxed{i \circ (b-a) = 0}$

$$\begin{tikzcd}
\ker f \arrow[r, "i"] & X \arrow[r, "f"] & Y \\
& W \arrow[ul, bend left, "b-a"] \arrow[ul, "0"'] \arrow[u, "0"'] &
\end{tikzcd}$$

are two solutions for the universal problem so $b - a = 0$ □

hence (*) $\pi \circ \tilde{f} \circ p = \pi f = 0 \underset{p \text{ epi}}{\Rightarrow} \pi \circ \tilde{f}$ ~~[illegible]~~ $= 0$ so $\tilde{f}$ factorizes through $\ker(\mathrm{coker} f)$

hence setting $\mathrm{Coim} f = \mathrm{Coker}(\ker f)$ and $\mathrm{Im} f = \ker(\mathrm{Coker} f)$

every morphism has a canonical factorisation

$$\begin{tikzcd}
\ker f \arrow[r] & X \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "G"] & Y \arrow[r] & \mathrm{Coker} f \\
& \mathrm{Coim}(f) \arrow[r, "\exists ! \bar{f}"'] & \mathrm{Im} f \arrow[u] &
\end{tikzcd}$$

Example $\mathcal{E} = A\text{Mod}$

$$\begin{tikzcd}
\ker f \arrow[r] & X \arrow[r, "f"] \arrow[d] & Y \arrow[r, two heads] & Y/\text{Im} f \\
& X/\ker f \arrow[r, dashed, "\exists \bar{f}"] & \text{Im} f \arrow[u] &
\end{tikzcd}$$

First isomorphism theorem $\bar{f}$ is an isomorphism.

Def 5.3 Let $\mathcal{E}$ be an additive category. Then $\mathcal{E}$ is abelian if

(1) Every morphism has a kernel and a cokernel in $\mathcal{E}$
(2) $\forall$ $f : X \to Y$ the canonical morphism $\bar{f}$: $\text{Coim} f \to \text{Im} f$ is an isomorphism.

Examples (1) A ring $\text{Mod}A$ A noetherian $\text{mod}A$ of fg module

(2) $\mathcal{E}$ abelian so is $\mathcal{E}^{op}$
(3) There are examples of categories with (1) but not (2)
eg Hausdorff topological abelian gps   kernel = usual
coker = quotient by $\overline{\text{Im} f}$

so
$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Q} \arrow[r, hook] \arrow[d] & \mathbb{R} \arrow[r] & 0 \\
& \mathbb{Q} \arrow[r, phantom, "\not\cong"] & \mathbb{R} \arrow[u] &
\end{tikzcd}$$

Prop 5.4 Let $\mathcal{A}$ be an abelian category and $\mathcal{J}$ small category. Then

(1) $\text{Fun}(\mathcal{J}, \mathcal{A})$ is an <u>abelian category</u>.
(2) $\text{Ch}_\bullet(\mathcal{A})$ is an <u>abelian category</u>

Sketch of proof

(1) $\eta : F \Rightarrow G$ $\ker(\eta) : \mathcal{J} \to \mathcal{A}$

$$\begin{tikzcd}
i \arrow[r, mapsto] \arrow[d] & \ker(\eta_i) \arrow[d, dashed, "\alpha"', "?"] \\
j \arrow[r, mapsto] & \ker(\eta_j)
\end{tikzcd}
\qquad
\begin{tikzcd}
\ker \eta_i \arrow[r, "i"] \arrow[d, dashed, "\alpha"] & F(i) \arrow[r, "\eta_i"] \arrow[d, "F(f)"] & G(i) \arrow[d, "G(f)"] \\
\ker \eta_j \arrow[r, "i"] & F(j) \arrow[r, "\eta_j"] & G(j)
\end{tikzcd}$$

we have $\eta_j \circ F(f) \circ i = G(f) \circ \eta_i \circ i = 0$
so $\exists! \alpha$

univ property of kernel ⇒ this is a functor + a kernel
similarly get cokernel + Coimf ⇒ Imf is an iso since it is at every evaluation.

(2) $Ch_\bullet(\mathcal{A}) \subseteq Fon(\mathbb{Z}, \mathcal{A})$ hence $f : C_\bullet \to D_\bullet$ has a kernel + cokernel in $Fon(\mathbb{Z}, \mathcal{A})$

$$\begin{tikzcd}
\ker f_{i+1} \arrow[r, dashed, "\alpha_{i+1}"] \arrow[d] & \ker f_i \arrow[r, dashed, "\alpha_i"] \arrow[d] & \ker f_{i-1} \arrow[d] \\
C_{i+1} \arrow[r, "d^C_{i+1}"'] \arrow[d] & C_i \arrow[r, "d_i"] \arrow[d, "f_i"] & C_{i-1} \arrow[d, "f_{i-1}"] \\
D_{i+1} \arrow[r, "d^D_{i+1}"] \arrow[d] & D_i \arrow[r, "d_i^D"] \arrow[d] & D_{i-1} \arrow[d] \\
\operatorname{cok} f_{i+1} \arrow[r] & \operatorname{cok} f_i \arrow[r, "d_i^D"'] & \operatorname{cok} f_{i-1}
\end{tikzcd}$$

$\alpha_i \circ \alpha_{i-1}$ is the unique application induced by $d_i \circ d_{i-1} = 0$

so $\alpha_i \circ \alpha_{i-1} = 0$. –

□

Rem (0) There is another equivalent definition of abelian categories
abk preabelian + every mono is a kernel and every epi is a cokernel

(1) Abelian categories have finite limits and colimits

(2) $f \in Mor(\mathcal{A})$ $\mathcal{A}$ abelian f <u>mono</u> ⇔ $\ker f = 0$
g <u>epi</u> ⇔ $\operatorname{Coker} g = 0$

Moreover epi + mono ⇒ isomorphism.

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "f"] \arrow[d] & Y \arrow[r] & 0 \\
 & X \arrow[r, "\bar{f}"', "\text{iso}"] & Y \arrow[u] &
\end{tikzcd}$$

If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are two composable morphisms in an abelian category s.t. $gf = 0$

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\pi"'] & Y \arrow[r, "g"] & Z \\
\operatorname{Coim} f \arrow[r, "\bar{f}"'] & \operatorname{Im} f \arrow[u, "\alpha"'] &
\end{tikzcd}$$

$gf = 0 \Rightarrow g \alpha \underbrace{\bar{f} \pi}_{epi} = 0$

$\Rightarrow g\alpha = 0$

So
$$\begin{tikzcd}
\ker g \arrow[r, hook, "i"] & Y \arrow[r, "g"] & Z \\
\exists ! \bar{\alpha} & \operatorname{Im} f \arrow[u, "\alpha"] \arrow[ul, dashed]
\end{tikzcd}$$
$\bar{\alpha} : \operatorname{Im} f \hookrightarrow \ker(g)$

Def 5.5 (1) $X \xrightarrow{f} Y \xrightarrow{g} Z$ s.t $gf = 0$ is exact if the canonical map $\operatorname{Im} f \to \ker(g)$ is an isomorphism

(2) A Complex (C_n, d_n) is exact if $\operatorname{Im}(d_{i+1}) \xrightarrow{\sim} \ker(d_i)$ $\forall i \in \mathbb{Z}$

(3) A short exact sequence is an exact complex of the form

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

Example ① in Mod A

$$X \xrightarrow{f} Y \xrightarrow{g} Z \qquad gf = 0 \Rightarrow \operatorname{Im} f \subseteq \ker(g)$$

so exactness $\Leftrightarrow \operatorname{Im} f = \ker(g)$

(1') $\boxed{0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \quad \text{exact iff} \quad f = \ker g}$

(2) Ses $\Leftrightarrow \begin{cases} f \text{ mono} \\ g \text{ epi} \\ \operatorname{Im} f \xrightarrow{\sim} \ker g \end{cases} \Leftrightarrow \begin{cases} g = \operatorname{Coker}(f) \\ f = \ker(g) \end{cases}$

Rem There is a difficult theorem of Freyd-Mitchel saying that any abelian category can be seen as a full subcategory of Mod A for some ring A in such a way that the abelian structure is induced by the usual one in Mod A.

$\rightsquigarrow$ we will most of the time use this result to simplify the exposition.

Def 5.6 Let $\mathcal{A}$ be an abelian category and $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$
$0 \to D \xrightarrow[h]{} E \xrightarrow[k]{} F \to 0$
two ses. A morphism of ses is the data of 3 morphism α, β, γ

making the diagram a commutative diag

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "m"] \arrow[d, "\alpha"'] & B \arrow[r, "e"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & D \arrow[r, "m'"'] & E \arrow[r, "e'"'] & F \arrow[r] & 0
\end{tikzcd}$$

(*)

Lemma 5.7 [Short five lemma] In (*) $\begin{cases} f + h \text{ mono} \Rightarrow g \text{ mono} \\ f + h \text{ epi} \Rightarrow g \text{ epi} \\ f + h \text{ iso} \Rightarrow g \text{ iso} \end{cases}$

Proof (1) In Mod A $x \in B$; $\beta(x) = 0 \Rightarrow e'\beta(x) = 0 \Rightarrow \gamma \circ e(x) = 0$

γ mono $\Rightarrow e(x) = 0 \Rightarrow$; $x \in \ker(e) \Rightarrow \mathrm{Im}(m)$

$\Rightarrow \exists a;\ x = m(a)$

$0 = \beta m(a) = m' \circ \alpha(a) \Rightarrow \alpha(a) \in \ker m' = \{0\}$

$\Rightarrow \alpha(a) = 0 \Rightarrow a = 0 \Rightarrow x = 0$.

+ dual for the second statement.

(2) In an arbitrary abelian category

$$\begin{tikzcd}
 & & \ker\beta \arrow[d, "k"] \arrow[dl, dashed, "\exists k'"'] & & \\
0 \arrow[r] & A \arrow[r, "m"] \arrow[d, "\alpha"'] & B \arrow[r, "e"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & D \arrow[r, "m'"'] & E \arrow[r, "e'"'] & F \arrow[r] & 0
\end{tikzcd}$$

$\beta k = 0$

give $\gamma e k = e'\beta k = 0$

γ mono $\Rightarrow ek = 0$ so k factorizes

through $\ker(e) = A$

↳ gives k'

we have $\beta k = 0 \Rightarrow 0 = \beta m k' = \underbrace{m'\alpha}_{\text{mono}} k' \Rightarrow k' = 0$ so $k = 0$

so β mono. The rest is dual. □

Thm 5.8 [Splitting lemma] Let $0 \to A \xrightarrow{q} B \xrightarrow{R} C \to 0$ be a short exact sequence in an abelian category $\mathcal{A}$. TFAE

(1) $\exists\, t : B \to A$ s.t $tq = Id_A$

(2) $\exists\, s : C \to B$ s.t $Rs = Id_C$

(3) $\exists\, h : B \xrightarrow{\simeq} A \oplus C$ s.t

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "q"] \arrow[d, equal] & B \arrow[r, "R"] \arrow[d, "h"] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r, "i_A"'] & A\oplus C \arrow[r, "\pi_C"'] & C \arrow[r] & 0
\end{tikzcd}$$

is an isomorphism of ses.

In this case we say that the ses splits.

Proof

$\underline{3 \Rightarrow 2}$ $s : C \xrightarrow[i_C]{} A\oplus C \xrightarrow{h^{-1}} B$ then

$$\begin{tikzcd}
C \arrow[r, "i_C"] \arrow[rrr, bend right, "Id_C"'] & A\oplus C \arrow[r, "h^{-1}"] \arrow[r, phantom, shift right=6, "\circlearrowleft"] & B \arrow[r, "R"] & C
\end{tikzcd}$$

$\pi_C \circ h = R \Rightarrow hh^{-1} = \pi_C$

$\underline{3 \Rightarrow 1}$ $t : B \xrightarrow{h} A\oplus C \xrightarrow{\pi_A} A$

$\underline{1 \Rightarrow 3}$

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, shift right, "q"'] \arrow[d, equal] & B \arrow[l, shift right, "t"'] \arrow[r, "R"] \arrow[d, dashed, "\exists\phi"] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r, "i_A"'] & A\oplus C \arrow[r, "\pi_C"'] & C \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
 & B \arrow[dl, "t"'] \arrow[d, "\exists\phi"] \arrow[dr, "R"] & \\
A & A\oplus C \arrow[l, "\pi_A"] \arrow[r] & C
\end{tikzcd}$$

by construction $\pi_C \circ \phi = R$ and $\pi_A \circ \phi = t$

$$\begin{aligned}
\phi\circ q = Id_{A\oplus C} \circ \phi\circ q &= (i_A\circ\pi_A + i_C\circ\pi_C)\,\phi\circ q \\
&= i_A\, \underbrace{\pi_A\circ\phi}_{t}\circ q + i_C\circ\underbrace{\pi_C\circ\phi}_{R}\circ q \\
&= i_A \quad t + \quad i_C\circ \underbrace{R\circ q}_{\overset{\|}{0}} \\
&= i_A
\end{aligned}$$

by the short five lemma ϕ is an iso $\square$

Def 5.9 Let $\mathcal{E}$ and $\mathcal{D}$ two abelian categories. Let $F : \mathcal{E} \to \mathcal{D}$ a functor.

(1) F is left exact if F preserves finite limits
(2) F is right exact if F preserves finite colimits
(3) F is exact if it preserves finite limits and colimits

Lemma 5.10 Let $F : \mathcal{E} \to \mathcal{D}$ be an additive functor between abelian categories. TFAE

(1) F is left exact
(2) F preserves kernels i.e. $F(\ker f) \overset{\text{can}}{\cong} \ker(F(f))$
(3) $\forall$ $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z$ exact sequence in $\mathcal{E}$, the sequence $0 \to F(X) \xrightarrow[F(f)]{} F(Y) \xrightarrow[F(g)]{} F(Z)$ is exact

(4) $\forall$ $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ ses in $\mathcal{E}$ the sequence $0 \to F(X) \xrightarrow[F(f)]{} F(Y) \xrightarrow[F(g)]{} F(Z)$ is exact.

$$\begin{tikzcd} & F(\ker f) \arrow[dl, "\exists !"'] \arrow[d, "F(i)"] & \\ \ker F(f) \arrow[r] & F(X) \arrow[r, "F(f)"] & F(Y) \end{tikzcd}$$

Proof (1) ⇒ (2) clear kernels are limits

(2) ⇒ (3) ~~seq~~ sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z$ exact iff $f = \ker(g)$

so $F(f)$ is a kernel of $F(g)$ ⇒ second sequence is exact

(3) ⇒ (4) $f = \ker g$ ⇒ same

(3) ⇒ (2)
$$\begin{tikzcd} 0 \arrow[r] & \ker f \arrow[r, "i"] & X \arrow[r, "f"] \arrow[d, "p"] & Y \arrow[r] & \mathrm{Coker} f \arrow[r] & 0 \\ & & \mathrm{Coim} f \arrow[r, "\sim"] & \mathrm{Im} f \arrow[u] & & \end{tikzcd}$$
clear

(2) ⇒ (1) also since any finite limits can be expressed using kernels and product

we are missing (4) ⇒ (3) $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z$ then consider

$0 \to X \xrightarrow{f} Y \to \mathrm{Coker}(f) \to 0$

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md

$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z$ exact iff $f = \ker g$

$\Leftarrow$ clear

$\Rightarrow$ exactness $\Rightarrow$ $\mathrm{Im} f \simeq \ker g$ + f mono

f mono gives

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, two heads, "p"'] \arrow[dr, "\tilde{f}"] & Y \arrow[r, "g"] & Z \\
\mathrm{Coim} f \arrow[r, "\simeq", "\bar{f}"'] & \mathrm{Im} f \arrow[u] \arrow[r, "\simeq", "t"'] & \ker g \arrow[ul, "i_g"']
\end{tikzcd}$$

$i \circ \bar{f} \circ p = f$ $\quad f$ mono $\Rightarrow$ p mono so $\tilde{f} = \bar{f} \circ p$ iso

so $\phi := t \circ \tilde{f}$ is an iso s.t $i_g \circ \phi = f \Rightarrow f \circ \phi^{-1} = i_g$

Let us prove that f is $\ker g$ by checking univ property

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \\
\ker g \arrow[u, "\phi^{-1}"] \arrow[ur, "i_g"] & T \arrow[u, "h"'] \arrow[l, dashed, "g"', "\exists ! \bar{h}"]
\end{tikzcd}$$

$gh = 0$

$f \circ \phi^{-1} \circ \bar{h} = i_g \bar{h} = h$

if
$$\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \\
& T \arrow[u, "h"'] \arrow[ul, "\psi", "g"']
\end{tikzcd}$$
is another factorisation then

$$\begin{tikzcd}
\ker g \arrow[r, "i_g"] & Y \arrow[r] & Z \\
X \arrow[u, "\phi"] & & \\
& T \arrow[ul, "\psi"] \arrow[uu, "h"'] &
\end{tikzcd}$$

$i_g \circ \phi \circ \psi = f \circ \psi = h$

so $\phi \circ \psi = \bar{h}$ so $\psi = \phi^{-1} \circ \bar{h}$ □

applying F gives $F(f)$ mono. So F preserves mono.

Moreover we have $0 \to X \xrightarrow{f} Y \xrightarrow{\bar{g}} \mathrm{Im}(g) \to 0$ exact so

$$0 \to F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(\bar{g})} F(\mathrm{Im}(g)) \quad \text{exact}$$

since $i : \mathrm{Im}(g) \hookrightarrow Z$ is mono + F preserves mono we have $F(i) : \mathrm{Im}(g) \to F(Z)$ mono so does not change the kernel!

$$\leadsto \; 0 \to F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \quad \text{exact} \qquad \square$$

Coro 5.11 TFAE for an additive functor between abelian categories

(1) F is exact

(2) $\forall$ ses $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ the sequence $0 \to F(A) \xrightarrow{F(\alpha)} F(B) \xrightarrow{F(\beta)} F(C) \to 0$ is exact.

Prop 5.12 (1) $\mathrm{Hom}_{\mathcal{E}}(-,-) : \mathcal{E}^{op} \times \mathcal{E} \to \mathrm{Mod}\,\mathbb{Z}$ is left exact in each variable

(2) $- \otimes_A - : \mathrm{Mod}\,A \times A\,\mathrm{Mod} \to \mathbb{Z}\,\mathrm{Mod}$ is right exact in each variable

(3) $F \dashv G$, then F is right exact and G is left exact

Proof (3) clear since ~~RAPL~~ ~~and~~ LAP adjoint preserves (co)lim

2_ Chain complexes in abelian categories

Def 5.13 $\mathcal{A}$ be an abelian category $(X_\bullet, d_\bullet) \in \mathrm{Ch}_\bullet(\mathcal{A})$

For $n \in \mathbb{Z}$ we set
$Z_n(X) = \ker(d_n)$ "n-cycles"
$B_n(X) = \mathrm{Im}(d_{n+1})$ "n-boundaries"
$H_n(X) = Z_n(X) / B_n(X)$ "nth homology of X"

⚠ in an arbitrary abelian category! $= \mathrm{coker}$ of $B_n \hookrightarrow Z_n$.

when cochain: speaks about cocycles, coboundaries, cohomology.

$f : X_\bullet \to Y_\bullet$ chain morphism:

(•)
$$\begin{tikzcd}
\ker d_n^X \arrow[r, "i"] \arrow[d, dashed, "Z(f_n)"'] \arrow[rd, phantom, "\circlearrowright"] & X_n \arrow[r, "d_n^X"] \arrow[d, "f_n"] & X_{n-1} \arrow[d, "f_{n-1}"] \\
\ker d_n^Y \arrow[r] & Y_n \arrow[r, "d_n^Y"'] & Y_{n-1}
\end{tikzcd}$$

$0 = f_{n-1}\, d_n^X i = d_n^Y \circ f_n \circ i$

so $\exists ! Z(f_n)$

(•) similarly $X_{n-1} \xrightarrow{f_{n-1}} Y_{n-1}$ induces

$$\begin{tikzcd}
\mathrm{Im}(d_n^X) \arrow[r, "i"] \arrow[d, dashed, "\exists ! B_n(f)"] & X_{n-1} \arrow[r, "\pi"] \arrow[d, "f_{n-1}"] & \mathrm{Cok}(d_n^X) \arrow[d, "f_{n-1}"] \\
\mathrm{Im}(d_n^Y) \arrow[r, "i"'] & Y_{n-1} \arrow[r, "\pi"'] & \mathrm{Cok}(d_n^Y)
\end{tikzcd}$$

so $\pi f_{n-1} i = \overline{f_{n-1}}\, \pi i = 0$

So we have
$$\begin{tikzcd}
\mathrm{Im}\, d_{n+1}^X \arrow[r, hook, "i"] \arrow[d, "B_n(f)"] \arrow[rd, phantom, "\circlearrowright"] & \ker d_n \arrow[r] \arrow[d, "Z_n(f)"] & \mathrm{Cok}(c) \arrow[d, dashed, "H_n(f)"] \\
\mathrm{Im}\, d_{n+1}^Y \arrow[r, hook] & \ker(d_n) \arrow[r, "\pi"'] & \mathrm{Cok}(c)
\end{tikzcd}$$

In Mod A $\quad H_n(X) = \ker d_n / \mathrm{Im}\, d_{n+1}$ $\quad H_n(f)(\,[x]) = [f_n(x)]$.

We get $H_n : \mathrm{Ch}_\bullet(\mathcal{A}) \to \mathcal{A}$ a functor called the nth homology functor. Moreover it is an additive functor.

Def 5.14 $f : X_\bullet \to Y_\bullet$ is a quasi isomorphism if $H_n(f)$ is an isomorphism $\forall n \in \mathbb{Z}$.

Prop 5.15 $f : X_\bullet \to Y_\bullet \in \mathrm{Mor}(\mathrm{Ch}_\bullet(\mathcal{A}))$ for $\mathcal{A}$ an abelian category.

(1) If $f \sim g$ then $H_n(f) = H_n(g)$

(2) If f is a homotopy equivalence, then it is a quasi isomorphism

Proof: (1)

$$\begin{tikzcd} & & \boxed{f \sim 0} \\ & C_{n+1} \arrow[r] & C_n \arrow[r] \arrow[d, "f_n"] \arrow[dl, "s_n"'] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] \arrow[dl, "s_{n-1}"'] & {} \\ & D_{n+1} \arrow[r] & D_n \arrow[r] & D_{n-1} \arrow[r] & {} \end{tikzcd}$$

$f_n = s_{n-1} d_n^C + d_{n+1}^D s_n$

$H_n(f)$: $H_n(C) = \frac{\ker(d_n)}{\mathrm{Im}(d_{n+1})} \longrightarrow \frac{\ker(d_n^D)}{\mathrm{Im}(d_{n+1})}$ $\quad H_n(f) = H_n(\underbrace{d_{n+1}^D s_n}_{\in \mathrm{Im}\, d_{n+1}})$

$= 0$

Since $f \sim g \Rightarrow f - g \sim 0 \Rightarrow H_n(f) = H_n(g)$.

(2) $fg \sim Id \Rightarrow H_n(f) H_n(g) = H_n(Id) = Id$ $\quad \square$

Def 5.16 (1) $C_\bullet$ is contractible if C is homotopy equivalent to 0

($Id_C \sim 0$)

(2) $C_\bullet$ is acyclic if C is qis to 0

↳ Contractible ⇒ acyclic

Thm 5.16 [Long exact sequence] A short exact sequence

$0 \to C' \xrightarrow{\alpha} C \xrightarrow{\beta} C'' \to 0$ gives rise to a long exact sequence

$$\to H_n(C') \xrightarrow{H_n(\alpha)} H_n(C) \xrightarrow{H_n(\beta)} H_n(C'') \xrightarrow{\delta_n} H_{n-1}(C') \to \cdots$$

where the δ_n (to be defined) are called the "connecting" homomorphisms.

Proof (1) In Mod A δ is constructed via diagram chasing

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
0 \arrow[r] & C'_n \arrow[r, "\alpha_n"] \arrow[d, "d'_n"] & C_n \arrow[r, "\beta_n"] \arrow[d, "d_n"] & C''_n \arrow[r] \arrow[d, "d''_n"] & 0 \\
0 \arrow[r] & C'_{n-1} \arrow[r, "\alpha'_{n-1}"'] \arrow[d] & C_{n-1} \arrow[r, "\beta_{n-1}"'] \arrow[d] & C''_{n-1} \arrow[r] \arrow[d] & 0 \\
 & {} & {} & {} &
\end{tikzcd}$$

$H_n(C'') = \ker(d''_{n})/\mathrm{Im}(d''_{n+1})$ $\qquad$ $H_{n-1}(c) = \ker(d'_{n-1})/\mathrm{Im}\, d'_n$

$x \in \ker(d''_n)$ $\exists\, y \in C_n ;\ \beta_n(y) = x$ and

$0 = d''_n \beta_n(y) = \beta_{n-1} \circ d_n(y) \Rightarrow d_n(y) \in \ker(\beta_{n-1}) \Rightarrow \exists\, z \in C'_{n-1}$ tq

$d_n(y) = \alpha'_{n-1}(z)$

Set $\delta_n([x]) := [z]$

well defined? - if y' is another lift of x $\qquad$ $y - y' \in \ker \beta_n = \mathrm{Im}\, \alpha_n$ ($\alpha_n(c)$)

$d_n \alpha_n(c) = \alpha_{n-1} d'_n(c)$

so z differs by an elt of $\mathrm{Im}\, d'_n$

- if $x' = x + t \in \mathrm{Im}(d''_{n+1})$ then lift of t can be chosen $\in \mathrm{Im}(d_{n+1})$ so $d_n(y) = 0$

- $z \in \ker(d'_{n-1})$? $\alpha'_{n-2} \circ d'_{n-1}(z) = d_{n-1} \circ \alpha_{n-1}(z)$
$= d_{n-1} d_n(y) = 0$.

(2) Using snake lemma $\boxed{\text{exercise}}$

It remains to see that the long sequence is exact

(•) $H_n(C') \xrightarrow{H_n(\alpha)} H_n(C) \xrightarrow{H_n(\beta)} H_n(C'')$

$[x] \in \ker(H_n(\beta))$ then $\beta_n(x) \in \operatorname{Im} d''_{n+1}$

$$\begin{tikzcd}
0 \arrow[r] & C'_{n+1} \arrow[r, "\alpha_{n+1}"] \arrow[d] & C_{n+1} \arrow[r, "\beta_{n+1}"] \arrow[d, "d_{n+1}"] & C''_{n+1} \arrow[r] \arrow[d, "d''_{n+1}"] & 0 \\
0 \arrow[r] & C'_n \arrow[r, "\alpha_n"'] & C_n \arrow[r, "\beta_n"'] & C''_n \arrow[r] & 0
\end{tikzcd}$$

$\Rightarrow \exists\, c \in C_{n+1}$; $\beta_n(x) = d_{n+1}(c'')$ so $\exists\, c \in C_{n+1}$; $\beta_n(x) = d''_{n+1}\beta_{n+1}(c)$
$= d_{n+1}(c)$

$\Rightarrow$ $x - d_{n+1}(c) \in \ker(\beta_n) = \operatorname{Im}(\alpha_n)$ ~~so ker(βn~~

$\Rightarrow$ $x - d_{n+1}(c) = \alpha_n(c')$

we have $[x] = [x - d_{n+1}(c)] = [\alpha_n(c')]$ and

$d_n(x) = 0 \Rightarrow c \in \ker d_n'$.

So $\ker(H_n(\beta)) \subseteq \operatorname{Im}(H_n(\alpha))$.

Conversely we have $\operatorname{Im}(\alpha) \subseteq \ker(\beta)$ so also $\operatorname{Im} H_n(\alpha) \subseteq \ker H_n(\beta)$.

- $\boxed{\operatorname{Im}(H_n(\beta_n)) = \ker(\delta_n)}$ if $c'' \in \operatorname{Im}(\beta_n)$ then c is a lift of $\beta_n(c)$
$c'' = \beta_n(c)$

so $d_n(c) = 0$ because $c \in \ker(d_n)$

so $\operatorname{Im}(H_n(\beta_n)) \subseteq \ker(\delta_n)$

<u>Conversely</u> if $x \in \ker(\delta_n)$ we have

$$\begin{tikzcd}
 & y \arrow[r, mapsto] \arrow[d] & x \\
\delta(x) & d_n(y) \arrow[l]
\end{tikzcd}$$

~~so $d_n(y) = 0$ so $y \in \ker d_n$~~

~~[illegible] so $x = \alpha_n(y)$ with $y \in \ker(d_n)$~~

$[\delta(x)] = 0 \Rightarrow \boxed{\delta(x) \in \operatorname{Im}(d'_n)}$ so $\exists\, c' \in C'_n$; $\delta(x) = d'_n(c')$

so $\alpha_{n-1} d'_n(c') = \alpha_{n-1}(\delta(x)) = d_n(y)$
$\qquad\qquad \parallel$
$d_n\alpha_n(c')$ so $y - \alpha_n(c') \in \ker d_n$ and $\beta_n(y - \alpha_n(c'))$
$= x$!

(•) $\boxed{\operatorname{Ker}(H_{n-1}(\alpha)) = \operatorname{Im}(\delta_n)}$

(•) $[x] \in \operatorname{Ker}(H_{n-1}(\alpha)) \Rightarrow \alpha_{n-1}(x) \in \operatorname{Im}(d_{n+1})$

$\Rightarrow \alpha_{n-1}(x) = d_n(y)$ and $\beta_n(y) \in C_n$

y is a lift so $\delta(\beta_n(y)) = x$

(•) $\alpha_{n-1}(\delta_n(x)) \in \operatorname{Im} d_n$ by contradiction

Remark (1) $\boxed{\delta_n = \alpha_{n-1}^{-1} \circ d_n \circ \beta_n^{-1}}$

(2) δ is natural in the following sens

if

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "f"'] & B \arrow[r, "\beta"] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A' \arrow[r, "\alpha'"'] & B' \arrow[r, "\beta'"'] & C' \arrow[r] & 0
\end{tikzcd}$$

are two ses then

We have a morphism of long exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(A) \arrow[r, "H_n(\alpha)"] \arrow[d, "H_n(f)"'] \arrow[dr, phantom, "\circlearrowright"] & H_n(B) \arrow[r] \arrow[d, "H_n(g)"'] \arrow[dr, phantom, "\circlearrowright"] & H_n(C) \arrow[r, "\delta_n"] \arrow[d, "H_n(h)"] & H_{n-1}(A) \arrow[r] \arrow[d, "H_{n-1}(f)"] \arrow[dr, phantom, "\circlearrowright"] & H_{n-1}(B) \arrow[r] \arrow[d] & \cdots \\
& H_n(A') \arrow[r] & H_n(B') \arrow[r] & H_n(C') \arrow[r, "\delta_n'"'] & H_{n-1}(A) \arrow[r] & H_{n-1}(B) \arrow[r] & {}
\end{tikzcd}$$

just because H_n is a functor

Need a proof

3- Projective and injective objects

Def 5.17 Let $\mathcal{E}$ be an abelian category. An object

(1) $I \in \operatorname{ob}(\mathcal{E})$ is injective if $\operatorname{Hom}_{\mathcal{E}}(\bullet, I) : \mathcal{E}^{op} \to \mathcal{A}b$ is exact

(2) $P \in \operatorname{ob}(\mathcal{E})$ is projective if $\operatorname{Hom}_{\mathcal{E}}(P, \bullet) : \mathcal{E} \to \mathcal{A}b$ is exact

(3) $\mathcal{E}$ has enough projectives (injectives) if $\forall x \in \operatorname{ob}(\mathcal{E})$ $\exists f : P \to x$ an epi with P projective ($g : x \hookrightarrow I$ a mono with I injective).

Prop 5.18 (1) $I \in ob(\mathcal{E})$ is injective iff $\forall$

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "f"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "="] & Y \arrow[dl, dashed, "\exists \alpha'"] \\ & I & \end{tikzcd}$$

(2) $P \in \mathcal{E}$ is projective iff $\forall$

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists \gamma"'] \arrow[d, "f"] \arrow[dl, phantom, shift right=3, "\circlearrowleft"] & \\ M \arrow[r, "g"'] & N \arrow[r] & 0 \end{tikzcd}$$

Proof P is projective iff $Hom_{\mathcal{A}}(P,-)$ is exact

iff $Hom_{\mathcal{A}}(P,-)$ is right exact

iff $\forall$ $M \xrightarrow{g} N \to 0$ exact

the induce sequence $Hom(P,M) \to Hom(P,N) \to 0$

is exact

iff $\forall$

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists \alpha"'] \arrow[d, "f"] \arrow[dl, phantom, shift right=3, "\circlearrowleft"] & \\ M \arrow[r, "g"'] & N \arrow[r] & 0 \end{tikzcd}$$

(1) is dual $\square$

Def 5.19 Let $\mathcal{E}$ be an abelian category. A pullback of $A \xrightarrow{\alpha} C$ $B \xrightarrow{\beta} C$ is an object P together with two morphisms $p_1 : P \to A$
$p_2 : P \to C$

s.t. $\forall$

$$\begin{tikzcd} Q \arrow[dr, dashed, "\exists !"] \arrow[drr, bend left, "q_2"] \arrow[ddr, bend right, "q_1"'] & & \\ & P \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[d, "\beta"] \\ & A \arrow[r, "\alpha"'] & C \end{tikzcd}$$

Dually we have notion of pushout.

Prop 5.20 Let $\mathcal{A}$ be an abelian category

(1) Any pair of morphism has a pullback in $\mathcal{A}$ and a pushout in $\mathcal{A}$

(2) If β is a mono then p_1 is a mono : pullback preserves mono

(3) Pushout preserves epi

(4) Pullback preserves kernels :

$$\begin{tikzcd} \ker p_2 \arrow[r] \arrow[d, dashed, "\simeq"] & P_2 \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d, "g"] \\ \ker f \arrow[r] & X \arrow[r, "f"] & Z \end{tikzcd}$$

(5) Pullback preserves epimorphisms in an abelian category

Proof (1) pullbacks and kernels are finite limits

(2)

$$\begin{tikzcd}
W \arrow[dr, shift left, "b"] \arrow[dr, shift right, "a"'] \arrow[drr, bend left, "p_2 b"] \arrow[ddr, bend right, "p_1 a"'] & & \\
& P \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d, "g"] \\
& X \arrow[r, hook, "f"'] & Z
\end{tikzcd}$$

$p_2 b = p_1 a$

then $f \circ p_1 a = g \circ p_2 a = g \circ p_2 b$
$= f \circ p_1 b$

So $p_1 a = p_1 b$ since f mono and we see that a and b are two solutions for the same universal problem so $a = b$.

(3) dual

Note do not used abelian in (2) and (3).

(4)

$$\begin{tikzcd}
\ker p_2 \arrow[r, "i_2"] \arrow[d, dashed, "\exists ! p"'] & P \arrow[r, "p_2"] \arrow[d, hook, "p_1"] & Y \arrow[d, "g"] \\
\ker f \arrow[r, "i_1"'] & X \arrow[r, "f"'] & Z
\end{tikzcd}$$

and

$$\begin{tikzcd}
& \ker f \arrow[dl, dashed, "\exists ! q"'] \arrow[d, dashed, "\exists ! t"] \arrow[dr, bend left, "0"] \arrow[dd, bend right, "i_1"'] & \\
\ker p_2 \arrow[r, "i_2"] & P \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"'] & Z
\end{tikzcd}$$

(*)

$p_2 \circ t = 0$
$\Rightarrow \exists ! q$

NOW

$$\begin{tikzcd}
\ker f \arrow[r, hook, "i_1"] & X \arrow[r] & Z \\
\ker p_2 \arrow[u, "p"] & & \\
& \ker f \arrow[ul, "q"] \arrow[uu, "i_1"'] &
\end{tikzcd}$$

we have $i_1 p q = p_1 i_2 q$
$= p_1 t = i_1$

So pq and $Id_{\ker f}$ are two solution for same universal problem
↳ they are equal!

Now consider the Pullback diagram:

$$\begin{tikzcd}
\ker p_2 \arrow[dr, "p"] \arrow[ddddrrr, bend right, "p_1 \circ c_2"'] & & & & \\
& \ker f \arrow[dr, "q"] & & & \\
& & \ker p_2 \arrow[dr, "c_2"] & & \\
& & & P \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d, "g"] \arrow[uuullll, bend right, "0"'] \\
& & & X \arrow[r, "f"] & Z
\end{tikzcd}$$

$$\underbrace{p_1 \circ c_2 \circ q}_{c_1 \circledast} \circ p = c_1 \circ p = p_1 \circ c_2$$

Hence
$$\begin{tikzcd}
\ker p_2 \arrow[dr, "c_2"] \arrow[ddr, bend right, "p_1 \circ c_2"'] \arrow[drr, bend left, "0"] & & \\
& P \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d] \\
& X \arrow[r] & Z
\end{tikzcd}$$
are the same Pullback so we have $\boxed{c_2 = c_2 \circ q \circ p}$ so $q \circ p = Id$ since c_2 is a monomorphism.

(5) Let
$$\begin{tikzcd}
P \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner"] & Y \arrow[d, "g"] \\
X \arrow[r, "f"'] & Z
\end{tikzcd}$$
be a pullback. Then we have a exact sequence
$$P \xrightarrow{\binom{p_1}{p_2}} X \oplus Y \xrightarrow{(-f,g)} Z$$

Moreover saying that P is a pullback is equivalent to $\binom{p_1}{p_2} = \ker(-f,g)$ indeed

$$\begin{tikzcd}
P \arrow[r, "\binom{p_1}{p_2}"] & X \oplus Y \arrow[r, "(-f,g)"] & Z \\
& T \arrow[ul, dashed, "\exists! \alpha"] \arrow[u, "\binom{a}{b}"] \arrow[ur, dashed, "0"'] &
\end{tikzcd}$$

$(-f,g)\binom{a}{b} = 0 \iff af = gb$

$\exists \alpha$ si $\binom{p_1}{p_2}\alpha = \binom{a}{b}$ is equivalent to $\begin{cases} \alpha p_1 = a & \text{et} \\ \alpha p_2 = b \end{cases}$

If f is an epimorphism then $(-f,g)$ is an epimorphism so we have a short exact sequence $0 \to P \xrightarrow{\binom{p_1}{p_2}} X \oplus Y \xrightarrow{(-f,g)} Z \to 0$ saying $\mathrm{Coker}\binom{p_1}{p_2} = (-f,g)$.

by an argument dual of above $$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, two heads, "g"] \\ X \arrow[r, two heads, "f"'] & Z \end{tikzcd}$$ is also a pullback square.

~~and~~ It remains to see that p_2 is an epi

$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d, "g"] \arrow[ddr, bend left, "a-b"] & \\ X \arrow[r, two heads, "f"'] \arrow[drr, bend right, "0"'] & Z \arrow[dr, dashed, "\exists! \alpha"] & \\ & & W \end{tikzcd}$$

$(a-b) \circ p_2 = 0$

we have $(a-b) p_2 = 0 = 0 \circ p_1$

so by universal property of pullback

$\exists! \alpha : Z \to W$ s.t. $\begin{cases} \alpha g = a-b \\ \alpha f = 0 \end{cases}$

but f epi $\Rightarrow \alpha = 0 \Rightarrow a-b = 0$ □

Thm 5.21 Let $\mathcal{A}$ be an abelian category

(1) Let $(P_\lambda)_{\lambda \in \Lambda}$ be a family of objects of $\mathcal{A}$. Then $\coprod_{\lambda \in \Lambda} P_\lambda$ is projective iff P_λ is projective $\forall \lambda \in \Lambda$

(2) P is projective iff $\forall X \overset{f}{\twoheadrightarrow} P$ $\exists s : P \to X$; $fs = 1_P$

+ dual for injective

Proof (1) if P_λ is projective $\forall \lambda \in \Lambda$

$$\begin{tikzcd} & & P_\lambda \arrow[d, "i_\lambda"] \arrow[dll, dashed, "f_\lambda"'] & \\ & & \coprod P_\lambda \arrow[d, "f"] \arrow[dl, dashed, "\exists \bar{f}"] & \\ 0 \arrow[r, tail] & M \arrow[r, "\alpha"'] & N \arrow[r] & 0 \end{tikzcd}$$

by projectivity $f \circ i_\lambda$ lifts along α $\exists f_\lambda$ now by univ prop of coproduct

$$\begin{tikzcd} P_\lambda \arrow[r, "i_\lambda"] \arrow[dr, bend right, "f_\lambda"'] & \coprod P_\lambda \arrow[d, dashed, "\exists! \bar{f}"] & P_{\lambda'} \arrow[l, "i_{\lambda'}"'] \arrow[dl, bend left, "f_{\lambda'}"] \\ & M & \end{tikzcd}$$

and $\forall \lambda$ $\alpha \circ \bar{f} \circ i_\lambda = \alpha \circ f_\lambda$
$= f \circ i_\lambda$