Indeed $0 \to B \to I \to \Omega^{-1}B \to 0$ exact

long exact sequence:

$$\mathrm{Hom}(X, \Omega^{-1}B) \to \mathrm{Ext}^1(X,B) \to \underset{\substack{\| \\ 0}}{\mathrm{Ext}^1(X,I)} \to \mathrm{Ext}^1(X, \Omega^{-1}B) \overset{\simeq}{\to} \mathrm{Ext}^2(X,B) \to \underset{\substack{\| \\ 0}}{\mathrm{Ext}^2(X,I)}$$

so $\mathrm{Ext}^2(X,B) \simeq \mathrm{Ext}^1(X, \Omega^{-1}B)$ and more generally

$\mathrm{Ext}^n(X,B) \simeq \mathrm{Ext}^{n-1}(X, \Omega^{-1}B) \quad \forall n \geq 2.$ $\square$

(3) Example of Tor functor

Def 6.13 A k-alg over commutative ring k. Then $\mathrm{Tor}_i^A(M,N) := L_i(- \otimes_A N)(M)$

Thm 6.14 $L_i(-\otimes N)[M] \simeq L_i(M \otimes -)[N]$

Example B abelian gp $\mathrm{Tor}(\mathbb{Z}/p\mathbb{Z}, B)$?

$0 \to \mathbb{Z} \xrightarrow{\times p} \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$ exact sequence hence $0 \to \mathbb{Z} \xrightarrow{\times p} \mathbb{Z} \to 0$

is a projective resolution of $\mathbb{Z}/p\mathbb{Z}$ so

$\mathrm{Tor}_i^{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, B)$ is the ith homology of

$$\begin{array}{ccccccc} 0 & \to & \mathbb{Z}\otimes B & \xrightarrow{\times p} & \mathbb{Z}\otimes B & \to & 0 \\ & & \wr\| & & \wr\| & & \\ 0 & \to & B & \xrightarrow{\times p} & B & \to & 0 \end{array}$$

So $\mathrm{Tor}_i^{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, B) = \begin{cases} B/pB & \text{if } i=0 \\ \{b \in B \,;\, pb = 0\} & \text{if } i=1 \\ 0 & \text{otw.} \end{cases}$

Prop 6.15 Let B be an A-module. TFAE

(1) B is flat

(2) $- \otimes_A B$ is exact

(3) $\mathrm{Tor}_i^A(-,B) = 0 \ \forall i \geq 1$

(4) $\mathrm{Tor}_1^A(-,B) = 0$

(4) Extensions à la Yoneda

In this section $\mathcal{A}$ is an abelian category. $A, B \in \mathrm{Ob}(\mathcal{A})$

An *extension* E of B by A is an exact sequence

$$0 \to A \to E \to B \to 0$$

two extensions of B by A are *equivalent* if:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "1_A"'] \arrow[dr, phantom, "="] & E \arrow[r] \arrow[d, "\exists\varphi"] \arrow[dr, phantom, "="] & B \arrow[r] \arrow[d, "1_B"] & 0 \\ 0 \arrow[r] & A \arrow[r] & E' \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

by short five lemma φ is an isomorphism. ($\Rightarrow$ equivalence relation)

Def C.17 $\mathcal{E}xt(A,B) = \{\text{extensions of } B \text{ by } A\}/\sim$ is called the (Yoneda)-Ext group

↳ Make it a $\boxed{\text{bi-functor}}$: $\mathcal{A}^{op} \times \mathcal{A} \to \mathrm{Set}$

$f : B_1 \to B_2$ $\qquad \mathcal{E}xt(A, B_1) \xrightarrow{f_\bullet -} \mathcal{E}xt(A, B_2)$

$$\begin{tikzcd} 0 \arrow[r] & B_1 \arrow[r, "\alpha"] \arrow[d, "f"] & E \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & B_2 \arrow[r] & E' \arrow[r] & A \arrow[r] & 0 \end{tikzcd} \longmapsto \text{Pushout of } f \text{ and } \alpha$$

$g : A_1 \to A_2$ $\qquad \mathcal{E}xt(A_2, B) \xrightarrow{-\bullet g} \mathcal{E}xt(A_1, B)$

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] & E \arrow[r] & A_2 \arrow[r] & 0 \\ 0 \arrow[r] & B \arrow[r] \arrow[u, equal] & E' \arrow[r] \arrow[u] & A_1 \arrow[r] \arrow[u] & 0 \end{tikzcd} \longmapsto \text{Pull back}$$

Many small things have to be checked eg $f_1 \cdot - \circ f_2 \cdot - = f_1 f_2 \cdot -$ etc good to use

Lem C.18 Let $\begin{tikzcd} E \arrow[r, "\alpha"] & A_2 \arrow[r] & 0 \\ & A_1 \arrow[u, "\beta"] & \end{tikzcd}$ be two morphisms with α epi.

Then $\begin{tikzcd} E \arrow[r] & A_2 \\ E' \arrow[r] \arrow[u] & A_1 \arrow[u] \end{tikzcd}$ is a pull back iff
$$\begin{tikzcd} 0 \arrow[r] & \ker\alpha \arrow[r] \arrow[dr, phantom, "="] & E \arrow[r] \arrow[dr, phantom, "="] & A_2 \arrow[r] & 0 \\ 0 \arrow[r] & \ker\alpha \arrow[r] \arrow[u, equal] & E' \arrow[r] \arrow[u] & A_1 \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

↳ <u>Make it an additive functor</u> $\mathcal{A}^{op} \times \mathcal{A} \to Ab$

$E_i : 0 \to A \to E_i \to B \to 0$

$E_1 + E_2$ is defined as <u>follows</u>:

$$\begin{tikzcd}
0 \arrow[r] & A \oplus A \arrow[r] \arrow[d, "\nabla"] & E_1 \oplus E_2 \arrow[r] \arrow[d, dashed] & B \oplus B \arrow[r] \arrow[d, "1_{B \oplus B}"] & 0 \\
0 \arrow[r] & A \arrow[r] & E \arrow[r] & B \oplus B \arrow[r] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[u, "1_A"] & E_1 + E_2 \arrow[r] \arrow[u] & B \arrow[r] \arrow[u, "\Delta"] & 0
\end{tikzcd}$$

<u>Def 6.19</u> $E_1 + E_2$ is the <u>Baer sum</u> of E_1 and E_2 : $\nabla \circ E_1 \oplus E_2 \circ \Delta$.

<u>Lem 6.20</u> The Baer sum endowed $Ext(A,B)$ of a structure of abelian group and the equivalence class of split extension is the neutral <u>elmt</u>

Moreover with this structure of abelian group $Ext(-,-)$ becomes an additive functor.

<u>Proof</u> we omit

<u>Thm 6.21</u> (1) If $\mathcal{A}$ has enough projectives. Then
$Ext^1_{\mathcal{A}}(A,B) \simeq Ext^1_A(-,B)[A]$

(2) If $\mathcal{A}$ has enough injectives Then
$Ext^1_{\mathcal{A}}(A,B) \simeq Ext^1_A(A,-)[B]$

<u>Proof</u> we only prove (1), (2) is <u>dual</u>.

Fix $P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} A \to 0$ a projective resolution of A

then $0 \to Hom(P_0, B) \to Hom(P_1, B) \to Hom(P_2, B) \to \dots$ is the complex allowing the computation of $Ext^\cdot(A,B)$

$Ext^1(A,B) = \left\{ f : P_1 \to B \ ; \ \underbrace{P_2 \xrightarrow{d_2} P_1 \xrightarrow{f} B}_{f \circ d_2} = 0 \right\} / Im(-\circ d_1)$

If $0 \to B \to E \to A \to 0 \in \mathcal{E}xt(A,B)$

we have the diagram:

$$\begin{tikzcd} \cdots \arrow[r] & P_2 \arrow[r, "d_2"] \arrow[d, "0"] & P_1 \arrow[r, "d_1"] \arrow[d, "f_1"] & P_0 \arrow[r] \arrow[d, "f_0"] & A \arrow[r] \arrow[d, equal] & 0 & \text{projective} \\ & 0 \arrow[r] & B \arrow[r] & E \arrow[r] & A \arrow[r] & 0 & \text{exact} \end{tikzcd}$$

hence by thm 6.3 Id_A lifts as a morphism of complexes

we have $f_1 \circ d_2 = 0$

so we set $\delta(E) = [f_1] \in H^1(Hom(P_\bullet, B))$

check ⊙ does not depend on the lift

↳ another lift is homotopic hence $\exists\ s : P_0 \to B$ s.t. $f_1' - f_1 = s_0 d_1$

$\Rightarrow [f_1] = [f_1']$

⊙ Only depend on iso class of E

↳

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] & E' \arrow[r] & A \arrow[r] & 0 \\ 0 \arrow[r] & B \arrow[r] \arrow[u, equal] & E \arrow[r] \arrow[u, "h", "\simeq"'] & A \arrow[r] \arrow[u, equal] & 0 \\ P_2 \arrow[r] & P_1 \arrow[r] \arrow[u, "f_1"] & P_0 \arrow[r] \arrow[u, "f_0"] & A \arrow[u, equal] & \end{tikzcd}$$

then f_1 is also obtained in a lift of Id_A for E'.

get $\delta : \mathcal{E}xt(A,B) \longrightarrow Ext^1(A,B)$

Conversely let $\gamma \in Ext^1(A,B)$ represented by a morphism $f : P_1 \to B$

since $f \in \ker(d_2^*)$ we have $f \circ d_2 = 0$ so $\ker d_1 = Im\, d_2 \subseteq \ker f$

So f induces a morphism $\bar{f} : P_1/\ker d_1 \simeq Im\, d_1 \to B$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| $
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \ker(\varepsilon)$

So we get

$$\begin{tikzcd} 0 \arrow[r] & \ker(\varepsilon) \arrow[r] \arrow[d, "\bar{f}"] & P_0 \arrow[r] \arrow[d, "f_0"] & A \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & B \arrow[r, "i_0"] & E \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

and we take the pushout in order to have an extension of B by A.

If f' is another representative of γ we have $[f]=[f']$ so $\exists s: P_0 \to$
s.t. $f'=f+s d_1$ after passing through the quotient we get:

$\hookrightarrow$
$$\begin{tikzcd} 0 \arrow[r] & \ker(\varepsilon) \arrow[r, hook, "i"] \arrow[d, "f'"'] & P_0 \arrow[r] \arrow[dl, dashed, "s"] & A \arrow[r] & 0 \\ & B & & & \end{tikzcd}$$
$$f' = f + s i$$

We have a diagram

$$\begin{tikzcd} 0 \arrow[r] & \ker(\varepsilon) \arrow[r, "i"] \arrow[d, "f'"] & P_0 \arrow[r, "\varepsilon"] \arrow[d, "f_0 + i_B s"] & A \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & B \arrow[r, "i_B"'] & E \arrow[r, "\pi"'] & A \arrow[r] & 0 \end{tikzcd}$$

with commutative square

indeed: $i_B \circ f' = i_B \circ (f + s i) = i_B \circ f + i_B s i$
$$= f_0 \circ i + i_B s i = (f_0 + i_B s) i$$

and $\pi(f_0 + i s) = \pi f_0 + \pi i_B s = \pi f_0 = \varepsilon$

hence E is also a pushout of f' and i and we see that the isomorphism class of $0 \to B \to E \to A \to 0$ is independent of the choice of the representative

$\leadsto$ $\theta : \mathrm{Ext}^1(A,B) \longrightarrow \mathcal{E}xt(A,B)$
$$[f] \longmapsto \text{Pushout}(f, i).$$

It is not difficult to check

- θ and δ are two inverse isomorphism
- ~~naturality~~ naturality in B $\square$

Conclusion $\mathrm{Ext}^1(A,B)$ classify ~~[illegible]~~ short exact sequences up to isomorphism.

What about $n \geq 2$

Def 6.22 $A, B \in \mathrm{Ob}(\mathcal{A})$. A degree n Yoneda extension of B by A is an exact sequence
$$E : 0 \to A \to Z_{n-1} \to \cdots \to Z_0 \to B \to 0$$

two exact sequences Z and Z' are equivalent if:

$\exists$:

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r] & Z_{n-1} \arrow[r] & \cdots \arrow[r] & Z_0 \arrow[r] & A \arrow[r] & 0 \\
0 \arrow[r] & B \arrow[r] \arrow[u, equal] \arrow[d, equal] & Z''_{n-1} \arrow[r] \arrow[u] \arrow[d] & \cdots \arrow[r] & Z''_0 \arrow[r] \arrow[u] \arrow[d] & A \arrow[r] \arrow[u, equal] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r] & Z'_{n-1} \arrow[r] & \cdots \arrow[r] & Z'_0 \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

with middle row exact

If $E = 0 \to B \to Z_{n-1} \to \cdots \to Z_0 \to A \to 0$

as before and $P_\bullet$ is a projective resolution of A we get

$$\begin{tikzcd}
P_{n+1} \arrow[r] & P_n \arrow[r] \arrow[d, "f_n"] & P_{n-1} \arrow[d] & \cdots & P_0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r] & Z_{n-1} \arrow[r] & \cdots \arrow[r] & Z_0 \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

$\underbrace{[f_n]}_{=\delta(E)} \in \mathrm{Ext}^n(A,B)$. One can prove that two sequences E, E' are equivalent iff $\delta(E) = \delta(E')$

Thm 6.23. $Y\mathrm{Ext}^n(A,B) = \{\text{long sequences}\}/_{\cong} \xrightarrow{\cong} \mathrm{Ext}^n(A,B)$

- $Y\mathrm{Ext}^n(-,-)$ is an additive bifunctor (taking PB + PO + use PB preserve kernel...)

Sketch we allready have a map from

$Y\mathrm{Ext}^n(A,B) \to \mathrm{Ext}^n(A,B)$

Conversely:

$$\begin{tikzcd}
0 \arrow[r] & P_{n+1} \arrow[r, "d_{n+1}"] & P_n \arrow[r] \arrow[d, "f"] & \cdots \arrow[r] & P_0 \arrow[r] & A \\
& & B & & &
\end{tikzcd}$$

$f \circ d_{n+1} = 0$

$\Rightarrow f$ induces a map φ from

$P_{n+1}/\mathrm{Im}\, d_{n+1} = P_{n+1}/\ker d_n \cong \mathrm{Im}\, d_n = B_{n-1}$

to B.

$$\begin{tikzcd}
0 \arrow[r] & B_{n-1} \arrow[r, hook, "i"] \arrow[d, "\varphi"] & P_{n-1} \arrow[r, two heads] \arrow[d, "x"] \arrow[dr, "B_{n-2}"] & P_{n-2} \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & P_0 \arrow[r] \arrow[d, equal] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r, hook, "\iota"] & E \arrow[r] & P_{n-2} \arrow[r] & \cdots \arrow[r] & P_0 \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

pushout of i, φ

If φ is replaced by $\varphi + si$ then x is replaced by $x + ks$ and the diag is a pushout. So E is independent of the choice.

it is clear that $\mathrm{Ext}^n_{\mathcal{A}}(A,B) \xrightarrow{\theta} \mathcal{Y}\mathrm{Ext}(A,B) \xrightarrow{\theta'} \mathrm{Ext}^n_{\mathcal{A}}(A,B)$ is the identity

conversely: if $E: 0 \to B \to E_{n-1} \to \cdots \to E_0 \to A \to 0 \in \mathcal{Y}\mathrm{Ext}^n(A,B)$

we have

$$\begin{tikzcd}
0 \arrow[r] & B_{n-1} \arrow[r] \arrow[d, "\phi"] \arrow[dr, phantom, "PO"] & P_{n-1} \arrow[r] \arrow[d] \arrow[ddr, bend left, "f_{n-1}"] & P_{n-2} \arrow[r] \arrow[d, equal] & P_1 \arrow[r] \arrow[d, equal] & P_0 \arrow[r] \arrow[d, equal] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\otimes"] & E \arrow[r] \arrow[d, dashed, "\exists!"] & P_{n-2} \arrow[r] \arrow[d, "f_{n-2}"] & \cdots \arrow[r] & P_0 \arrow[r] \arrow[d, "f_0"] & A \arrow[r] \arrow[d, equal] & 0 & \theta' \circ \theta(E) \\
0 \arrow[r] & B \arrow[r] & E_{n-1} \arrow[r] & E_{n-2} \arrow[r] & \cdots \arrow[r] & E_0 \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

$\otimes$ existence follows from pushout property of E. then $\theta' \circ \theta(E)$ is equivalent to E.

Remains all naturality ... ☐

↳ Now there is an obvious way to "compose" extensions:

$$\left(0 \to B \to E_{n-1} \cdots \to E_0 \to A\right) \circ \left(0 \to A \to F_{m-1} \to \cdots \to F_0 \to C \to 0\right)$$

$$= \quad 0 \to B \to E_{n-1} \to \cdots \to E_0 \to F_{m-1} \to \cdots \to F_0 \to C \to 0$$

$$\begin{tikzcd}
E_0 \arrow[dr] \arrow[rr] & & F_{m-1} \\
& A \arrow[ur] \arrow[dr] & \\
0 \arrow[ur] & & 0
\end{tikzcd}$$

⇝ bilinear map $\mathcal{Y}\mathrm{Ext}^m(C,A) \times \mathcal{Y}\mathrm{Ext}^n(A,B) \longrightarrow \mathcal{Y}\mathrm{Ext}^{m+n}(C,B)$

endowed $\mathrm{Ext}^*(M,M) = \bigoplus_n \mathrm{Ext}^n(M,M)$ of a ring structure

Example: $\mathcal{A} = k[x]/(x^2)$ $\quad 0 \to k \to k[x]/(x^2) \xrightarrow{ev_0} k \to 0$ short exact

gives a long exact sequence

$$\cdots \to k[x]/(x^2) \to k[x]/(x^2) \to \cdots \xrightarrow{\cdot x} k[x]/(x^2) \xrightarrow{\cdot x} k[x]/(x^2) \xrightarrow{ev_0} k \to 0$$

Since $\mathrm{Hom}(k[x]/(x^2), k) \simeq k$
$\phi \longmapsto \phi(x)$

we see that $\mathrm{Hom}(P_\bullet, k) \simeq 0 \to k \xrightarrow{0} k \xrightarrow{0} \cdots \xrightarrow{0} k \xrightarrow{0} k$

Hence $\mathrm{Ext}^*(k,k) \simeq \bigoplus_{n \geq 0} k$ as vector spaces.

it is not difficult to check that $0 \to k \to k[t]/(t^2) \to k \to 0$ is the short exact sequence corresponding to $1 \in H^1(\mathrm{Hom}(P, k))$

More generally $0 \to k \to k[t]/(t^2) \to \cdots \to k[t]/(t^2) \to k \to 0$ correspond to $1 \in H^n(\mathrm{Hom}(P^\bullet, k))$

Hence ~~[illegible]~~ if we denote by y^n the short exact sequence associated to $1 \in \mathrm{Ext}^n(k,k)$ we have

$$y^n \bullet y^m = y^{n+m}$$

↳ $\boxed{\mathrm{Ext}^*_{k[t]/(t^2)}(k,k) \simeq k[y] \text{ as algebra}}$

5) Universal coefficients

Let $(C_\bullet, d_\bullet)$, $(D_\bullet, d'_\bullet)$ be two chain complexes of right and left A-modules

we construct a double complex

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & \\
{} \arrow[r] & C_i \otimes D_j \arrow[r, "(-1)^i 1\otimes d'_j"] \arrow[d, "d_i \otimes 1"'] & C_i \otimes D_{j-1} \arrow[r] \arrow[d, "d_i \otimes 1"] & {} \\
{} \arrow[r] & C_{i-1} \otimes D_j \arrow[r] \arrow[d, "(-1)^{i-1} 1\otimes d'_j"'] & C_{i-1} \otimes D_{j-1} \arrow[d] & \\
 & \vdots & {} &
\end{tikzcd}$$

want to make a classical complex out of it $\quad \mathrm{Tot}(C \otimes D)_n = \bigoplus_{i+j=n} C_i \otimes D_j$

↳ squares of total complex need to anticommute

Koszul sign convention: if an object of degree n "jump" above an object of degree k multiply by $(-1)^{mk}$

here $\quad C_i \otimes D_j \longrightarrow C_i \otimes D_{j-1}$

$\quad\quad x \otimes y \longmapsto x \otimes d'_j(y)$

d' jumps above x of degree i

d' degree -1

So sign is $(-1)^i$

$\partial_n \; \mathrm{Tot}(C \otimes D)_n \to \mathrm{Tot}(C \otimes D)_{n-1}$

$x \otimes y \in C_i \otimes D_j \mapsto d_i(x) \otimes y + (-1)^i x \otimes d'_j(y)$

special case if D is concentrated in degree zero: $\cdots \to C_i \otimes D \to C_{i-1} \otimes D \to \cdots$

Facts (1) $(\mathrm{Tot}(C \otimes D), \partial)$ is a chain complex of abelian groups

(2) There is a canonical map $\bigoplus_{i+j=n} H_i(C) \otimes H_j(D) \xrightarrow{*} H_n(C \otimes D)$

$[x] \otimes [y] \longrightarrow [x \otimes y]$

$$\begin{tikzcd} C_i \otimes D_{j+1} \arrow[r] & C_i \otimes D_j \arrow[r, "1 \otimes d_j"] \arrow[d, "d_i \otimes 1"] & {} \\ & {} & \end{tikzcd}$$

$x \otimes y$

$x \in \ker d_i, \; y \in \ker d_j$

then $x \otimes y \in \ker(\partial_n)$

[$y = d_{j+1}(z)$ then $\partial_{n+1}(x \otimes z) = x \otimes d_{j+1}(z) + \overset{=0}{d_i(x)} \otimes z$ so $(*)$ is well defined]

Thm 6.23 [Künneth formula] If $Z_n(C)$ and $B_n(C)$ are flat $\forall n \in \mathbb{Z}$, there is a short exact sequence

$$0 \to \bigoplus_{i+j=n} H_i(C) \otimes H_j(D) \xrightarrow{*} H_n(C \otimes D) \to \bigoplus_{i+j=n-1} \mathrm{Tor}_1^A(H_i(C), H_j(D)) \to$$

so if $H_i(C)$ and $H_j(D)$ are flat/proj/free the canonical map $*$ is an isomorphism!

Coro 6.24 (1) $(C_\bullet, d_\bullet)$ chain complex of right A-modules such that $Z_n(C)$ and $B_n(C)$ are flat. Let M be a left A-module. Then we have a short exact sequence of abelian groups

$$0 \to H_n(C) \otimes M \to H_n(C \otimes_A M) \to \mathrm{Tor}_1^A(H_{n-1}(C), M) \to 0$$

(2) If C is a chain complex of free abelian gps then (1) applies and moreover the short exact sequence splits.

Proof we only prove the corollary since it is less technical:

We have $0 \to Z_n(C) \hookrightarrow C_n \xrightarrow{d_{n-1}} B_{n-1}(C) \to 0$ exact

tensor with M gives

$$\underset{\substack{\| \\ 0}}{\mathrm{Tor}_1(B_{n-1}, C)} \to Z_n(C) \otimes M \to C_n \otimes M \to B_{n-1}(C) \otimes M \to 0$$

So $\forall n \in \mathbb{Z}$ we have a short exact sequence

$$0 \to Z_n(C) \otimes M \hookrightarrow C_n \otimes M \to B_{n-1}(C) \otimes M \to 0 \quad (*)$$

View $\left(Z_n(C)\right)_{n \in \mathbb{Z}}$ as a complex with 0 differential

$\left(B_n(C)\right)_n$ as a complex with 0 differential then $(*)$ gives

$$\begin{tikzcd}
 & \arrow[d] & \arrow[d] & \arrow[d] & \\
0 \arrow[r] & Z_n(C) \otimes M \arrow[r] \arrow[d, "0"] & C_n \otimes M \arrow[r, "d_n \otimes M"] \arrow[d, "d_n \otimes M"] & B_{n-1}(C) \otimes M \arrow[r] \arrow[d, "0"] & {} \\
0 \arrow[r] & Z_{n-1}(C) \otimes M \arrow[r] \arrow[d] & C_{n-1} \otimes M \arrow[r, "d_{n-1} \otimes M"'] \arrow[d] & B_{n-2}(C) \otimes M \arrow[r] \arrow[d] & 0 \\
 & {} & {} & {} &
\end{tikzcd}$$

$\leadsto$ a short exact sequence of complexes:

$$0 \to Z_\bullet(C) \otimes M \to C_\bullet \otimes M \to B(C)[1] \otimes M \to 0$$

so a long exact sequence

$$\cdots \to H_n(Z_\bullet(C) \otimes M) \to H_n(C \otimes M) \to H_n(B(C)[1] \otimes M) \xrightarrow{\delta_n} H_{n-1}(Z(C) \otimes M) \to \cdots$$

since $Z_\bullet$ and $B_\bullet$ have zero differential $H_n(Z_\bullet(C) \otimes M) = Z_n(C) \otimes M$

$$H_n(B(C)[1] \otimes M) = B_{n-1}(C) \otimes M$$

Moreover let us compute the connecting homomorphism

$\delta_n = \iota \otimes M$ where $\iota : B_{n-1}(C) \hookrightarrow Z_n(C)$ is the canonical inclusion

So we have $B_n(C)\otimes M \xrightarrow{i\otimes M} Z_n(C)\otimes M \longrightarrow H_n(C\otimes M) \longrightarrow B_{n-1}(C)\otimes M \xrightarrow{\delta_n} \dots$

with $\beta_n = [i\otimes M]$ and $\alpha_n = [d_n \otimes M]$.

So we have:

$$0 \to \operatorname{Ker}(\alpha_n) \longrightarrow H_n(C\otimes M) \longrightarrow \operatorname{Im}(\alpha_n) \to 0$$

$\operatorname{Ker}(\alpha_n) = \operatorname{Im}(\beta_n) = \operatorname{CoKer}(i\otimes M) = \delta_{n+1}$, $\operatorname{Im}(\alpha_n) = \operatorname{Ker}(\delta_n)$

So $0 \to \operatorname{CoKer}(\delta_{n+1}) \to H_n(C\otimes M) \to \operatorname{Ker}(\delta_n) \to 0$

<u>Finally</u> we have a ses:

$0 \to B_n(C) \to Z_n(C) \to H_n(C) \to 0$ if we tensor by M we have

$$0 \to \operatorname{Tor}_1(H_n(C), M) \longrightarrow B_n(C)\otimes M \xrightarrow{i\otimes M} Z_n(C)\otimes M \to H_n(C)\otimes M \to 0$$

So $\operatorname{CoKer}(\delta_n) = H_n(C)\otimes M$

and $\operatorname{Ker}(\delta_n) = \operatorname{Tor}_1(H_n(C), M)$

(2) C complex of free abelian groups. Then $B_n(C)$ and $Z_n(C)$ are free so 1 apply.

Moreover $0 \to Z_n(C) \overset{R}{\hookrightarrow} C_n \to B_{n-1}(C) \to 0$ splits

then $H_n(C\otimes M) \to H_n(C)\otimes M$ is well defined and splits the canonical map. check the details! □

$[x\otimes y] \longmapsto [R(x)]\otimes y$

<u>Similarly we have</u>:

<u>Thm 6.25</u> Let C_n be a chain complex of A-modules, $M \in \operatorname{Mod} A$ and assume that $Z_n(C)$ and $B_n(C)$ are projective. Then there is an exact sequence

$$0 \to \operatorname{Ext}^1_A(H_{n-1}(C), A) \longrightarrow H^n \operatorname{Hom}_A(C, M) \longrightarrow \operatorname{Hom}_A(H_n(C), A) \to 0$$

Terminology: $(C_n)_n$ chain complex of A-modules $M \in A\text{Mod}$

$H_n(C \underset{A}{\otimes} M)$ is the homology of $C_\bullet$ with coef in M

$H^n(C, M)$ is " cohomology of $C_\bullet$ with coef in M

Basically enough to understand $H_n(C)$ + $\mathrm{Ext}^1(-, M)$ + $\mathrm{Tor}_1(M, -)$ to understand the homology with coefs.

- If M and N are two complexes of A-modules (or in an abelian category)

we have a bicomplex $\mathrm{Hom}(M_\bullet, N_\bullet)$

$$\begin{array}{ccc} \mathrm{Hom}(M_i, N_j) & \longrightarrow & \mathrm{Hom}(M_i, N_{j-1}) \\ \uparrow & & \uparrow \\ \mathrm{Hom}(M_{i-1}, N_j) & \longrightarrow & \mathrm{Hom}(M_{i-1}, N_{j-1}) \end{array}$$

get a complex: $\prod_{i-j=n} \mathrm{Hom}(M_i, N_j)$ with $d(f) = d \circ f_N + (-1)^{|f|} f \circ d_M$

Example see next chapter.

Chapter 7 Singular homology

1 - Quick recollection:

X topological space, R ring with 1 associative

- $\Delta_n = \left\{ (x_0, \dots, x_n) \in \mathbb{R}^{n+1} \,/\, \sum_{x_i \geq 0} x_i = 1 \right\}$

- $C_n^{\mathrm{Sing}}(X, R) = R \underset{\mathbb{Z}}{\otimes} C_n^{\mathrm{Sing}}(X) = R\left[\mathrm{Hom}_{\mathrm{Top}}(\Delta_n, X)\right]$

- $d_n(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma \circ d^i$ $\quad d^i : \Delta^{n-1} \longrightarrow \Delta^n$
$\quad (x_0, \dots, x_n) \longmapsto (x_0, \dots, x_{i-1}, 0, x_i, \dots, x_n)$

Def 7.1 Singular homology of X with coef in R is $H_\bullet(C_\bullet^{\mathrm{Sing}}(X, R))$

$\varepsilon : C_0(X, R) \to R$, $\sigma \longmapsto 1_R$ augmentation (if $X \neq \emptyset$) and we denote by $\tilde{C}^{\mathrm{sing}}(X, R)$ the augmented complex and $\tilde{H}(X, R)$ its homology

2. Relative homology

<u>Def 7.2</u> (1) Top^2 cat (X, A) X top and $A \subseteq X$

$$f : (X,A) \to (Y,B) = \{ f : X \to Y \text{ continuous s.t. } f(A) \subseteq B \}$$

(2) $f, g \in Hom_{Top^2}((X,A),(Y,B))$ then a homotopy from f to g is a morphism $H \in Hom_{Top^2}((X \times I, A \times I), (Y,B))$ s.t.

$$H(-,0) = f$$
$$H(-,1) = g$$

(i.e. a homotopy from f to g s.t. $\forall a \in A$ $H(a,x) \in B$ $\forall x \in I$)

<u>Lem 7.3</u> $A \subseteq X$ then $C_\bullet^{sing}(A,R) \subseteq C_*^{sing}(X,R)$

<u>Proof</u> $\sigma : \Delta_n \to A \mapsto i \circ \sigma : \Delta_n \xrightarrow{\sigma} A \hookrightarrow X$ clearly injective $\square$

<u>Def 7.4</u> $C_*^{sing}(X,A) := \dfrac{C_*^{sing}(X,R)}{C_*^{sing}(A,R)}$. Its homology is the singular homology of X relatively to A.

<u>Thm 7.5</u> (Eilenberg-Steenrod axioms)

$(H_n)_{n \geq 0}$ is a family of functors from Top^2 to Ab together with natural transformations $\delta_n : H_n(X,A) \to \underset{\overset{\|}{H_{n-1}(A)}}{H_{n-1}(A,\phi)}$ satisfying

(1) [Dimension] $H_n(pt) = 0$ $\forall n > 0$

(2) [Additivity] $X = \coprod_{\alpha \in I} X_\alpha$ disjoint union, then $H_n(\coprod_\alpha X_\alpha) \cong \bigoplus_\alpha H_n(X_\alpha)$

(3) [Long exact sequence] $\forall (X,A) \in Top^2$ have les:

$$\xrightarrow{\delta_{n+1}} H_n(A) \xrightarrow{H_n(i)} H_n(X) \xrightarrow{\pi_n(\pi)} H_n(X,A) \xrightarrow{\delta_n} H_{n-1}(A) \to \cdots$$

$i : A \hookrightarrow X$

$\pi : C_n(X) \twoheadrightarrow C_n(X,A)$

(4) [Homotopy axiom] If $f_0, f_1 : (X,A) \to (Y,B)$ are homotopic then $H_n(f_0) = H_n(f_1)$ $\forall n \in \mathbb{Z}$.

(5) [Excision] $\forall (X,A) \in Top^2$ $\mathcal{U} \subseteq A$ with $\bar{\mathcal{U}} \subseteq \mathring{A}$ then

then homomorphism $H_n(k): H_n(X\setminus\mathcal{U}, A\setminus\mathcal{U}) \to H_n(X,A)$ is an iso where
$k: (X\setminus\mathcal{U}, A\setminus\mathcal{U}) \hookrightarrow (X,A)$ is the inclusion.

Proof

(1) $0 \to C_\bullet^{sing}(A) \to C_\bullet^{sing}(X) \to C_\bullet^{sing}(X,A) \to 0$ ses of complexes
gives (3)

(2) X top $X = \coprod_\alpha X_\alpha$ path connected components. Then $C_n(X) \simeq \bigoplus_\alpha C_n(X_\alpha)$
Since H_n is additive we have (2)

Thm 7.6 (1) $X \neq \phi$ path connected, then $H_0(X) \simeq \mathbb{Z}$
(2) More generally $H_0(X) \simeq \mathbb{Z}^{\text{nb of con comp}}$

Proof : $\varepsilon: \mathbb{Z}[X] \to \mathbb{Z}$, $x \mapsto 1$ $\quad \ker(\varepsilon) = \{\sum_{x\in X} \lambda_x x ; \sum \lambda_x = 0\}$

(1) $\xi \in Im(d_0)$ $\quad \sum_{x\in X} \lambda_x x = \sum_{i=0}^n \lambda_i x_i$; $\sum \lambda_x = 0$ choose $x \in X$ and
$\sigma_i : x \rightsquigarrow x_i$ path

then $\sum \lambda_i \sigma_i \in CH_1(X)$ $\quad d_1(\sum \lambda_i \sigma_i) = \sum \lambda_i \sigma_i(1) - \sum \lambda_i \sigma_i(0)$
$= \sum \lambda_i x_i - \underbrace{\sum_{i=1}^{n} \lambda_i}_{0} x$

(2) Conversely : $y = \sum \lambda_i \sigma_i$ then $d(y) = \sum \lambda_i \sigma_i(1) - \sum \lambda_i \sigma_i(0)$

appear twice
with opposite sign

hence $d(y) \in \ker(\varepsilon)$

so $H_0(X) \simeq \frac{\mathbb{Z}[X]}{Im\, d_1} = \frac{\mathbb{Z}[X]}{\ker(\varepsilon)} \simeq Im(\varepsilon) = \mathbb{Z}$. $\square$

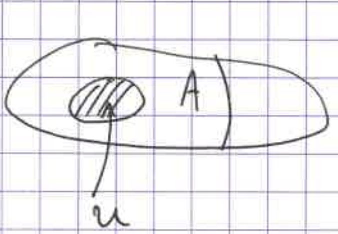
Thm 7.7 [Dimension axiom] $H_n(\{\bullet\}) = 0$ $\forall n \geq 1$
Proof There is only one $\sigma_n : \Delta_n \to \{\bullet\}$ hence $C_n^{sing}(X) = \mathbb{Z}[\sigma_n]$
and $d_n(\sigma_n) = \sum (-1)^n \sigma_{n-1} = \begin{cases} 0 & \text{if } n \text{ odd} \\ \sigma_{n-1} & \text{otw} \end{cases}$

So $\mathbb{Z} \xrightarrow{0} \mathbb{Z} \simeq \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$ no homology except in degree 0 □

Htpy Axiom $f \sim g \Rightarrow f_* \sim g_*$ as morphism of chain complexes

Need relative version for $(f,g) : (X,A) \to (Y,B)$
check that the construction of the Htpy $h : f_* \sim g_*$ sends $C_{n-1}(A)$ to $C_n(B)$

3 - Excision Axiom:



can remove $\mathcal{U}$ without changing relative homology: Moral "$H_n(X,A)$ ignores A so can cut part of A"

Thm 7.8 (Excision II) $\forall X_1, X_2$ s.t. $X = \mathring{X}_1 \cup \mathring{X}_2$ the inclusion

$$(X_1, X_1 \cap X_2) \hookrightarrow (X, X_2) \text{ is a q iso.}$$

Proof:

⊙ Equivalent to Excision I

I $\Rightarrow$ II $\mathcal{U} \subseteq A$ $\overline{\mathcal{U}} \subseteq \mathring{A}$ $\rightsquigarrow$ take $\boxed{\mathcal{U} = X \setminus X_1}$ $\boxed{A = X_2}$ ~~then $X_1 \cap X_2 = X_2 \setminus X_1$~~

then $(X \setminus \mathcal{U}, A \setminus \mathcal{U}) \hookrightarrow (X, A)$ q.i. if $\overline{X \setminus X_1} \subseteq \mathring{A}$

$$\underset{(X_1,\, X_1 \cap X_2)}{\underset{\|}{(X \setminus \mathcal{U}, A \setminus \mathcal{U})}} \hookrightarrow \underset{(X, X_2)}{\underset{\|}{(X, A)}}$$

II $\Rightarrow$ I $\overline{\mathcal{U}} \subseteq \mathring{A}$ set $X_2 = A$, $X_1 = X \setminus \mathcal{U}$ then $(X_1, X_1 \cap X_2) \hookrightarrow (X, X_2)$ q.i.

$$\| $$

$$(X \setminus \mathcal{U}, A \setminus \mathcal{U}) \hookrightarrow (X, A) \text{ q.i.}$$

if can check $X = \mathring{X}_1 \cup \mathring{X}_2$ ...

(•)
$$\begin{tikzcd}
\cdots \arrow[r] & A_n \arrow[r, "a_n"] \arrow[d, "\alpha_n"] & B_n \arrow[r, "b_n"] \arrow[d, "\beta_n"] & C_n \arrow[r, "c_n"] \arrow[d, "\gamma_n"] & A_{n-1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & A_n' \arrow[r, "a_n'"'] & B_n' \arrow[r, "b_n'"'] & C_n' \arrow[r, "c_n'"'] & A_{n-1}' \arrow[r] & \cdots
\end{tikzcd}$$

commutative diagram with exact rows in an abelian category.

If γ_n is an iso $\forall n$

We have a long exact sequence

$$\cdots \to A_n \xrightarrow{(a_n, \alpha_n)} B_n \oplus A'_n \xrightarrow{\binom{\beta_n}{-a'_n}} B'_n \xrightarrow{c_n \partial_n b'_n} A_{n-1} \to \cdots$$

Proof diagram chasing

Thm 7.14 (Mayer - Vietoris) $X_1, X_2 \subseteq X$ with $\mathring{X}_1 \cup \mathring{X}_2$

There is a long exact sequence

$$\cdots \to H_n(X_1 \cap X_2) \to H_n(X_1) \oplus H_n(X_2) \to H_n(X) \to H_{n-1}(X_1 \cap X_2) \to \cdots$$

Proof
$$\begin{tikzcd}
(X_1 \cap X_2, \emptyset) \arrow[r, "i_1"] \arrow[d, "i_2"] & (X_1, \emptyset) \arrow[r, "p"] \arrow[d, "g"] & (X_1, X_1 \cap X_2) \arrow[d, "h \ \text{q is}"] \\
(X_2, \emptyset) \arrow[r, "j"'] & (X, \emptyset) \arrow[r, "q"'] & (X, X_2)
\end{tikzcd}$$
commutative diag in Top2

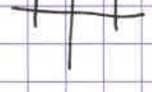
Then apply: Singular chain functor + ~~...~~ Excision + Lem 7.13. □
↳ short exact + les in homology +

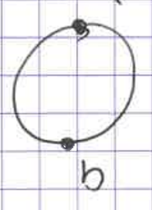
4 - Homology of Spheres:

Thm 7.15 $S^n = \{ x \in \mathbb{R}^{n+1} / \sum x_i^2 = 1 \}$ nth sphere

Rem $H_k(S^0) = \begin{cases} \mathbb{Z}^2 & \text{if } k = 0 \\ 0 & \text{otw} \end{cases}$

$\forall n \geq 1$ $H_k(S^n) = \begin{cases} \mathbb{Z} & \text{if } k = 0 \text{ or } n \\ 0 & \text{otw} \end{cases}$

Proof $\boxed{n = 0}$ $S^0 =$  $S^0 = \{\bullet\} \cup \{\bullet\}$ the result follows

$\boxed{n \geq 1}$ 

a = north pole $\quad X_1 = S^n \setminus \{a\}$ open

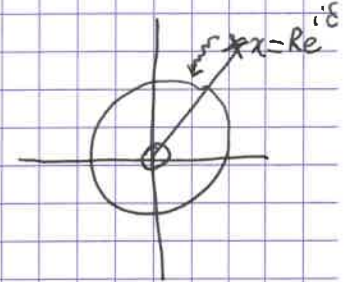
b = south pole $\quad X_2 = S^n \setminus \{b\}$ $\quad X = X_1 \cup X_2$

Then by Mayer-Vietoris we have:

$$H_n(X_1 \cap X_2) \to \underbrace{H_n(X_1) \oplus H_n(X_2)}_{=0} \to H_n(X) \xrightarrow{\cong} H_{n-1}(X_1 \cap X_2) \to \underbrace{H_{n-1}(X_1) \oplus H_{n-1}(X_2)}_{=0}$$

Since $X_i \simeq \mathbb{R}^n$ via projection stereographic + $\mathbb{R}^n$ is contractible

Hence $H_n(X) \simeq H_{n-1}(X_1 \cap X_2)$



$$X_1 \cap X_2 = S^n \setminus \{\text{two points}\} \underset{\text{homéo}}{\simeq} \mathbb{R}^n \setminus \{1\,pt\} \underset{\text{homotopique}}{\simeq} S^{n-1}$$

$$\left[\; S^{n-1} \overset{i}{\hookrightarrow} \mathbb{R}^n \setminus \{pt\} \xrightarrow{\pi} S^{n-1} \qquad x \longmapsto \frac{x}{\|x\|} \right.$$

Then $\left[\begin{array}{l} \pi \circ i = Id_{S^{n-1}} \\ i \circ \pi \sim Id_{\mathbb{R}^n \setminus \{pt\}} \quad \text{via} \quad H(x,t) = \dfrac{x}{\|x\|(1-t)+t} \end{array}\right.$

⊙ So $H_n(S^n) \simeq H_{n-1}(S^{n-1})$ if $n \geq 2$.

(•) $n=1$ look at $0 \to H_1(S^1) \to H_0(S^0) \to H_0(X_1) \oplus H_0(X) \to H_0(S^1) \to 0$

$$0 \to H_1(S^1) \to \mathbb{Z}^2 \to \mathbb{Z}^2 \to \mathbb{Z} \to 0$$

$$\mathbb{Z}^2 \searrow \mathbb{Z} \nearrow \mathbb{Z}^2, \qquad 0 \nearrow \mathbb{Z} \searrow 0$$

$\leadsto H_1(S^1) \simeq \mathbb{Z}$. $\square$

<u>Cor</u> (1) $m \neq n$ $S^m \not\simeq S^n$

(2) $m \neq n$ $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic (otw $\mathbb{R}^n \setminus pt \simeq \mathbb{R}^m \setminus pt$)

(3) $n \geq 0$ S^n is not contractible

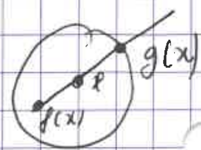
(4) $f: D^n \to D^n$ continuous, $\exists\, x \in D^n;\ f(x) = x$

(4) $n \geq 0$ $S^n \hookrightarrow D^{n+1} \xrightarrow{\pi} S^n$ $\pi \circ i = Id$ does not exist

<u>otw</u> $\underset{\mathbb{Z}}{H^n(S^n)} \to \underset{0}{H^n(D^{n+1})} \to \underset{\mathbb{Z}}{H^n(S^n)}$ $Id_{\mathbb{Z}} = 0$.

$$H^n(S^n) \xrightarrow{Id} H^n(S^n)$$

So if $f(x) \neq x \ \forall x \in D^n$ $\quad (x, f(x))$ is a line in $\mathbb{R}^{n+1}$



so cut ∂D^{n+1} and set $g(x) = \partial D^{n+1} \cap]f(x), x($

Then $\pi : x \to g(x)$ satisfies $S^n \xrightarrow{i} D^{n+1} \xrightarrow{\pi} S^n$ $\quad \pi \circ i = Id$ $\quad$ contrad. □

<u>Thm</u> X, Y two topological spaces. Then

(1) $C_*^{sing}(X \times Y)$ and $C_*^{sing}(X) \otimes C_*^{sing}(Y)$ are homotopy equivalent

(2) $0 \to \bigoplus_{p+q} H_p(X) \otimes H_q(Y) \to H_n(X \times Y) \to \bigoplus_{p+q=n-1} \mathrm{Tor}_1^{\mathbb{Z}}(H_p(X), H_q(Y)) \to 0$

<u>Ex</u> $H_n(S^1 \times S^1) = \begin{cases} \mathbb{Z} & n = 0, 2 \\ \mathbb{Z} \oplus \mathbb{Z} & n = 1 \\ 0 & \text{otw} \end{cases}$

<u>indeed</u> $H_n(\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } n = 0, 1 \\ 0 & \text{otw} \end{cases}$