TD3 Ex 2

(1) $CH_n(A,M) = M \otimes_k A^{\otimes_k n}$ $\qquad CH_0(A,M) = M \otimes_k A^{\otimes 0} \simeq M$

$$d_i : CH_n(A,M) \longrightarrow CH_{n-1}(A,M)$$

$$m \otimes a_1 \otimes \cdots \otimes a_n \longmapsto \begin{cases} ma_1 \otimes a_2 \otimes \cdots \otimes a_n & i=0 \\ m \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n & i=1,\dots,n-1 \\ a_n m \otimes a_1 \cdots \otimes a_{n-1} & i=n \end{cases}$$

$d_n = \sum_{i=0}^{n} (-1)^i d_n^i$ with obvious notation check ssimplicial relations.

$CH^0(A,M) = \mathrm{Hom}_k(k,M) \simeq M$

(2) $CH^n(A,M) = \mathrm{Hom}_k(A^{\otimes n}, M)$

$$\delta^i : \mathrm{Hom}_k(A^{\otimes n}, M) \longrightarrow \mathrm{Hom}_k(A^{\otimes n+1}, M)$$

$$f \longmapsto \delta^i(f) : A^{\otimes n+1} \longrightarrow M$$

$$a_0 \otimes a_1 \otimes \cdots \otimes a_n \longmapsto \begin{cases} a_0 f(a_1 \otimes \cdots \otimes a_n) & i=0 \\ f(a_0 \otimes \cdots \otimes a_{i-1} a_i \otimes \cdots \otimes a_n) & 0<i<n \\ f(a_0 \otimes \cdots \otimes a_{n-1}) a_n & i=n+1 \end{cases}$$

$$d^n = \sum_{i=0}^{n+1} (-1)^i \delta^i$$

check cosimplicial relations.

(3) (•)

$$M \otimes A \xrightarrow{d_1} M \to 0$$
$$m \otimes a \longmapsto ma - am$$

Hence $HH_0(A,M) = M/[M,M]$ where $[M,M] = \langle \{ ma - am \,;\, m \in M, a \in A \} \rangle$

$\hookrightarrow M = A$ this is the "cocenter" of A.

So $d^0(m) = am - ma$

(•) $M \simeq \mathrm{Hom}_k(k,M) \xrightarrow{d^0} \mathrm{Hom}(A,M)$

$m \mapsto f_m \longrightarrow d^0(f_m) = \delta^0(f) - \delta^1(f)$

So $HH^0(A,M) = \{ m \in M ; \ am = ma \ \forall a \in A \}$

$M = A$ this is the <u>center</u> of A.

(4) $HH^1(A,A) = \dfrac{\operatorname{Ker} d^1}{\operatorname{Im} d^0}$

we already saw that $\operatorname{Im} d^0 = \{ f : A \to A ; \ \exists m \in A \text{ with } f(a) = ma - ma \}$

$= \operatorname{IDer}(A)$

$d^1 : \operatorname{Hom}_k(A,A) \longrightarrow \operatorname{Hom}_k(A \otimes A, A)$

$\phi \longmapsto \delta^0 f - \delta^1 f + \delta^2 f$

$(d^1 \phi)(a \otimes b) = a f(b) - f(ab) + f(a) b$

Hence $\phi \in \operatorname{Ker} d^1$ iff $a f(b) + f(a) b = f(ab)$.

iff $\phi \in \operatorname{Der}(A)$

(5) $HH_1(A,A) = \dfrac{\operatorname{Ker} d_0}{\operatorname{Im} d_1}$

$A \otimes A \otimes A \xrightarrow{d_1} A \otimes A \xrightarrow{d_0} A \longrightarrow 0$

$\operatorname{Ker} d_0 = \{ a \otimes b ; \ ab = ba \}$

$\operatorname{Im} d_1 = \langle d_1(a \otimes b \otimes c) \rangle = \langle ab \otimes c - a \otimes bc + ca \otimes b \rangle$

Hard to say <u>more</u>

(6) $B_n(A) = A^{\otimes_k (n+2)}$

$B_2 \qquad B_1 \qquad B_0$

$A \otimes A \otimes A \otimes A \longrightarrow A \otimes A \otimes A \longrightarrow A \otimes A \longrightarrow 0$

easy to see that it is a complex of chains

(7) $B_n(A)$ is an <u>A-A-bimodule</u> for every n

for the action
$$A \otimes (A \otimes A^{\otimes n} \otimes A) \otimes A \longmapsto A^{\otimes n+2}$$
$$x \otimes (a_0 \otimes \cdots \otimes a_{n+1}) \otimes y \longmapsto x a_0 \otimes \cdots \otimes a_{n+1} y$$

If M is an A-A-bimodule we can tensor M and $A^{\otimes n+2}$ over $A \otimes A^{op}$

$$M \underset{A\otimes A^{op}}{\otimes} B_n(A) \simeq CH_n(A, M)$$
$$m \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes a_{n+1} \longmapsto a_{n+1} m a_0 \otimes a_1 \otimes \cdots \otimes a_n$$

this induces an isomorphism of complexes $M \underset{A\otimes A^{op}}{\otimes} B_\bullet(A) \simeq CH_\bullet(A, M)$

(8)
$$\begin{tikzcd}
\cdots \arrow[r] & A\otimes A\otimes A \arrow[r] \arrow[d] & A\otimes A \arrow[r, "a\otimes b \mapsto ab"] \arrow[d] \arrow[dl] & A \arrow[r] \arrow[d] \arrow[dl, "s_{-1}"'] & 0 \\
\arrow[r] & A\otimes A\otimes A \arrow[r] & A\otimes A \arrow[r, "\Sigma"'] & A \arrow[r] & 0
\end{tikzcd}$$
$1 \otimes a$

$s_{-1}(a) = 1 \otimes a$ then $\Sigma \circ s_{-1}(a) = a$

$s_0(a \otimes b) = 1 \otimes a \otimes b$ $\quad d_0(1 \otimes a \otimes b) = a \otimes b - 1 \otimes ab$

and $s_{-1}\Sigma(a\otimes b) = 1 \otimes ab$

<u>general case</u> $s_n(a_0 \otimes \cdots \otimes a_{n+1}) = 1 \otimes a_0 \otimes \cdots \otimes a_{n+1}$

s_n $\quad = d_{n+1}(1 \otimes a_0 \otimes \cdots \otimes a_{n+1}) = a_0 \otimes \cdots \otimes a_{n+1} + \sum_{i=1}^{n+1} (-1)^i \, 1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots$

d_{n+1} $\quad = \sum_{i=0}^{n} (-1)^i \, 1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n$ $\quad$ cancel