Master 2.

Problem

All vector spaces are complex.

We consider the loop algebra $\mathcal{L}sl_2 = sl_2 \otimes \mathbb{C}[t^{\pm 1}]$.

I.1) Let $a \in \mathbb{C}^*$. Recall how to define for V representation of sl_2 an evaluation representation of $\mathcal{L}sl_2$ denoted by $(V)_a$ (by specialization t to a).

I.2) For any $a \in \mathbb{C}^*$, describe the evaluation representation $(V)_a$ of dimension 2 obtained from a simple representation V of sl_2 (that is, give the action on a basis of the representation).

In the next two section we work with these 2-dimensional representations $(V)_a$.

I.3) Prove that for $a \in \mathbb{C}^*$, $(V)_a \otimes (V)_a$ is not simple. Is it semi-simple ?

I.4) Prove that for any $a \neq b$ in $\mathbb{C}^*$, $(V)_a \otimes (V)_b$ is simple.

We consider the quantum analog of these questions. The quantum parameter $q \in \mathbb{C}^*$ is not a root of unity. We recall the $\mathcal{U}_q(sl_2)$ has generators X, Y, $K^{\pm 1}$ with relations $KE = q^2 EK$, $KF = q^{-2} FK$, $[E, F] = (K - K^{-1})(q - q^{-1})^{-1}$.

We admit there exists a quantum loop algebra $\mathcal{U}_q(\mathcal{L}sl_2)$ generated by certain X_1, X_0, Y_1, Y_0, $k^{\pm 1}$ with a coproduct satisfying

$$\Delta(X_i) = X_i \otimes 1 + K_i \otimes X_i \ , \ \Delta(K_i) = K_i \otimes K_i \ , \ \Delta(Y_i) = Y_i \otimes K_i^{-1} + 1 \otimes Y_i,$$

for $i = 0$ or $i = 1$, where $K_0 = k^{-1}$ and $K_1 = k$.

We admit that for any $a \in \mathbb{C}^*$, there is an evaluation morphism

$$ev_a : \mathcal{U}_q(\mathcal{L}sl_2) \to \mathcal{U}_q(sl_2)$$

such that the images of X_1, X_0, Y_1, Y_0, k are respectively X, aY, Y, $a^{-1}X$, K with X, Y, K, K^{-1} the standard generators of $\mathcal{U}_q(sl_2)$. It is an algebra morphism (but not a Hopf algebra morphism).

I.5) For any $a \in \mathbb{C}^*$, describe the evaluation representation $(V_q)_a$ of $\mathcal{U}_q(\mathcal{L}sl_2)$ with V_q simple representation of $\mathcal{U}_q(sl_2)$ of dimension 2.

In the next questions we work with representations of $\mathcal{U}_q(\mathcal{L}sl_2)$ and with the 2-dimensional representations V_q.

I.6) For $a, b \in \mathbb{C}^*$, prove that $(V_q)_a \otimes (V_q)_b$ is simple if $ab^{-1} \notin \{q^2, q^{-2}\}$.

I.7) Prove that if $ab^{-1} = q^2$ or q^{-2}, then $(V_q)_a \otimes (V_q)_b$ is not simple. Is it semi-simple ?

I.8) Are $(V_q)_a \otimes (V_q)_{aq^2}$ and $(V_q)_{aq^2} \otimes (V_q)_a$ isomorphic ?