q-BINOMIAL AND REPRESENTATIONS OF HOPF ALGEBRAS

Exercice 1 - q-binomial numbers Let q be an indeterminate, and $k, n \in \mathbb{N}$.

(1) We define $(n)_q = 1 + q + \cdots + q^{n-1} = \frac{q^n-1}{q-1}$, the *$q$-analogue* of n. If $q = p^n$, show that $(n)_q$ is the number of lines in $\mathbb{F}_q^n$.

(2) We define $(n)!_q = (n)_q \cdot (n-1)_q \cdots (2)_q \cdot (1)_q$, the *$q$-analogue* of $n!$. If $q = p^n$, show that $(n)!_q$ is the number of strictly increasing sequences $0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = \mathbb{F}_q^n$ of vector subspaces of $\mathbb{F}_q^n$.

(3) Check that the number of strictly increasing sequences of subsets $= S_0 \subset S_1 \subset \cdots \subset S_n = [n]$ is $n!$.

(4) We define $\binom{n}{k}_q = \frac{(n)!_q}{(k)!_q (n-k)!_q}$, the q-analogue of the binomial coefficient $\binom{n}{k}$. When $q = p^n$, show that $\binom{n}{k}_q$ is the number of k-dimensional subspaces of $\mathbb{F}_q^n$.

(5) State and prove a q-analogue version of Pascal triangle.

(6) Check that $\binom{n}{k}_q \in \mathbb{Z}[q]$.

(7) Let x, y be two variables such that $yx = qxy$. Prove that $(x+y)^n = \sum_{k=0}^n \binom{n}{k}_q x^k y^{n-k}$.

(8) If q is a d-root of 1 and x, y are two variables such that $yx = qxy$. Check that $(x+y)^d = x^d + y^d$.

(9) Let $w \in S_n$ be a permutation. A pair (i, j) with $i < j$ is an inversion of w if $w(i) > w(j)$. We denote by $inv(w)$ the number of inversions of w. Show that $\sum_{w \in S_n} q^{inv(w)} = (n)!_q$. We could use a bijection $S_{n-1} \times [n] \cong S_n$ and do an inductive proof.

(10) If $S \subseteq [n]$, we denote by $Sum(S)$ the sum of the elements of $[n] = \{1, 2, \ldots, n\}$. Prove

$$\sum_{S \subseteq [n]} q^{Sum(S)} z^{|S|} = \sum_{k=0}^{n} q^{\frac{k(k+1)}{2}} \binom{n}{k}_q z^k.$$

Hint : denote by $B_n(k)$ the subsets of $[n]$ of size k. Construct a bijection $B_n(k) \times S_k \times S_{n-k} \to S_n$ and apply the previous question.

(11) For $a, b \in \mathbb{N}$, we denote by $L_{a,b}$ the set of lattice paths from $(0,0)$ to (a,b) made of east steps $(0,1)$ and north steps $(1,0)$. If $P \in L_{a,b}$ the area of P is the number $area(P)$ of 1×1 boxes underneath the steps of P and above the x-axis. Prove that

$$\sum_{P \in L_{a,b}} q^{area(P)} = \binom{a+b}{a}_q.$$

Hint : construct a bijection $L_{a,b} \to B_{a+b}(a)$ and relate to the previous question.

Exercice 2 - Representations of S_3

(1) Let G be a finite group and k be a field. Recall the usual Hopf algebra structure on the group algebra kG.

(2) If V and W are two kG-modules, recall the natural action of kG on $V \otimes W$.

(3) We recall that $\mathbb{C}G$ is a semisimple algebra and that the number of simple representations is the number of conjugacy classes. Find all simple representations of $\mathbb{C}S_3$. Hint : signature and S_3 has a natural action on $\mathbb{C}^3$.

(4) Compute all the $S_1 \otimes S_2$.

Notes : The Grothendieck ring of $\mathbb{C}S_3$ (and more generally of the group algebra of a finite group G is isomorphic to the ring of 'virtual characters'). For another interesting ring consider the category of finite G-sets. It is monoidal for the disjoint union and cartesian product. The 'split' Grothendieck ring of this category is called the *Burnside group* of G.

Exercice 3 - Representations of C_2 in characteristic 2 We let $C_2 = \{1, g\}$ with $g^2 = 1$ be a cyclic group with two elements. Let k a field of characteristic 2.

(1) Show that the regular representation is indecomposable but not simple.

(2) Check that there are only two indecomposable representations for G over a field of characteristic 2. Hint : Jordan canonical form.

(3) What is the image of the regular representation in the Grothendieck group ?

Note : We could consider the 'Green ring' of the category of kG-modules. This is the 'split' Grothendieck group of the category of kG-modules.

Note : Instead of groups we could consider the ring $k[X]/(X^2)$.

Exercice 4 - Complex representations of $\mathfrak{sl}_2$

(1) Recall the usual Hopf algebra structure on $U(\mathfrak{sl}_2)$.

(2) Recall the classification of the finite dimensional simple modules over $U(\mathfrak{sl}_2)$.

(3) The Clebsch-Gordan Formula. Let $m, n \in \mathbb{N}$ such that $m \leq n$. We denote by $V(m)$ and $V(n)$ the simple modules with respective heighest weight m and n. Show that

$$V(m) \otimes V(n) \cong \bigoplus_{k=0}^{m} V(n+m-2k).$$

(4) Show that the Grothendieck ring of the category of finite dimensional $U(\mathfrak{sl}_2)$-modules is isomorphic to $\mathbb{Z}[X]$ the polynomial ring in the indeterminate X.

Exercice 5 - Representations of H_4 Let k be a field of characteristic $\neq 2$ and let

$$H_4 = k<g,x>/<g^2 = 1, x^2 = 0, gx = -xg>$$

the *Sweedler* 4 dimensional Hopf algebra.

(1) Let V be a finite dimensional H_4-module. Check that the action of g on V is diagonalizable.

(2) Show that a finite dimensional H_4-module is nothing but two linear maps $\alpha : V_+ \to V_-$ and $\beta : V_- \to V_+$ such that $\alpha \circ \beta = 0$ and $\beta \circ \alpha = 0$.

(3) Find two non-isomorphic modules of dimension 1 and two non-isomorphic indecomposable modules of dimension 2.

(4) Show that these are the only indecomposable H_4-modules up to isomorphism. Hint : We could decompose V_- and V_+ using $\ker(\alpha)$ and $\ker(\beta)$ and try to find basis adapted to our situation.

(5) Is the algebra H_4 semisimple ?

(6) Let $K = K_0(H_4)$ the Grothendieck group of the category of finite dimensional H_4-modules. What are the images of the two modules of dimension 2 in this group ?

(7) Show that K is isomorphic to the group algebra of a cyclic group of size 2 over $\mathbb{Z}$.

(8) For $\alpha \in k$, let $R_\alpha = \frac{1}{2}(1 \otimes 1 + 1 \otimes g + g \otimes 1 - g \otimes g) + \frac{\alpha}{2}(x \otimes x + x \otimes gx + gx \otimes gx - gx \otimes x)$. Check that R_α is a universal R-matrix for H_4.

(9) What are the corresponding natural isomorphisms between $M \otimes N$ and $N \otimes M$.