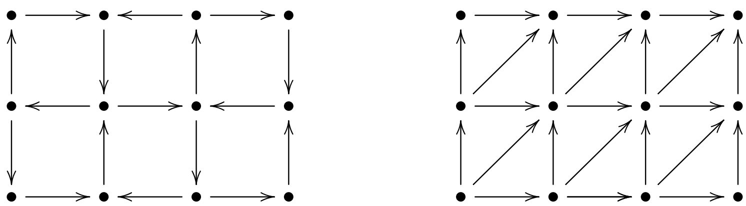
CLUSTER ALGEBRAS

Exercice 1 - Mutations of quivers

(1) Show that all orientations of a tree (with no frozen vertices) are mutation equivalent to each other.

(2) Is the mutation equivalence class of a tree T equal to all the orientations of T ?

(3) Show that the two following quivers are mutation equivalent. You could use the applet of B. Keller `https://webusers.imj-prg.fr/~bernhard.keller/quivermutation/`.



Exercice 2 - Examples of cluster algebras

(1) For the following quivers, prove that the associated cluster algebra is of finite type, describe all the seeds, the cluster variables and the exchange graph.

$$1 \qquad\qquad 1 \longrightarrow \boxed{2} \qquad\qquad \boxed{2} \longleftarrow 1 \longrightarrow \boxed{3} \qquad\qquad \boxed{2} \longrightarrow 1 \longrightarrow \boxed{3}$$

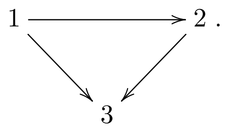
(2) Same question for

$$1 \longrightarrow 2 \longrightarrow \boxed{3}.$$

(3) Same question for the cluster algebra with initial exchange matrix $\begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}$.

(4) Check that the Kronecker quiver $1 \rightrightarrows 2$ is mutation finite but has an infinite number of cluster variables. One could specialize the initial variables at $1, 1$ and find a relation with Fibonacci numbers.

(5) Let $Q =$



   (a) Compute the mutation class of the quiver Q.

   (b) Show that there are infinitely many cluster variables for the corresponding cluster algebra. One could specialize the initial variables at $1, 1, 1$ and look for a nice sequence of mutations.

Exercice 3 - Laurent phenomenon

(1) We consider the cluster algebra associated to the initial seed (x_1, x_2, x_3) and $B = \begin{pmatrix} 0 & 4 \\ -1 & 0 \\ 1 & -3 \end{pmatrix}$. The variable x_3 is frozen.

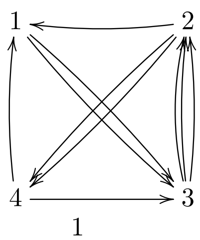
   (a) We consider the specialization $(x_1, x_2, x_3) = (3, -1, 16)$. What are the seeds we get after doing μ_1 and $\mu_2 \circ \mu_1$.

   (b) Show that any cluster variable specializes to an integer. We can use without proof the fact that the frozen variables do not appear in the denominators of the Laurent polynomials expressing the cluster variables in terms of an initial extended cluster.

   (c) Deduce that $2^{32} + 1$ is not a prime number.

(2) Consider the following quiver

$Q =$



(a) What quiver do you obtain after mutating at the vertex 1.

(b) Find a sequence of mutations which generates a sequence of cluster variables satisfying the recursive relation $z_{n+2}z_{n-2} = z_{n+1}z_{n-1} + z_n^2$.

(c) The somos-4 sequence is the sequence satisfying this recurrence and the initial conditions $z_0 = z_1 = z_2 = z_3 = 1$. Prove that all the terms of this sequence are integers.

(3) Using a similar argument one can prove that the sequence of Exercice 2 (4) consists of integers.

(4) There are more examples in Introduction to Cluster Algebras Chapters 1–3 by Fomin, Williams and Zelevinsky.

Exercice 4 - An invitation to Catalan land Let P_m be a convex m-gon with a triangulation T. We label the sides of the m-gon by frozen vertices and the diagonals of the triangulation by non frozen vertices. If two diagonals or sides belong to the same triangle, we connect the corresponding vertices by an arrow whose orientation is determined by the clockwise orientation of the boundary of the triangle. We denote by $Q(T)$ the quiver obtained this way.

(1) Let T' be the triangulation obtained by flipping the diagonal k in the triangulation T. Check that $\mu_k(Q(T)) = Q(T')$. Conversely, check that $\mu_k(Q(T))$ is the quiver of the triangulation obtained by flipping the diagonal k in T.

(2) Find a triangulation of P_m such that the quiver on the non frozen vertices is a quiver of type A with linear orientation.

(3) Show that any two triangulations of P_m are related by a sequence of flips.

(4) Conclude that the mutation class of a quiver of type An (without frozen vertices) is in bijection with the triangulations of an $(n+3)$-gone.

(5) Let $(x_1, \dots, x_{2m-3})$ be an initial variable for the cluster algebra associated to $Q(T)$.

(a) When $m = 4$. what do we obtain if we specialize each variable to the length of the side it labels ?

(b) When $m = 5$ describe all the seeds, the cluster variables and the exchange graph.

(6) We admit that the cluster and coefficients variables are in bijection with the diagonals and sides of the polygon, and the clusters are in bijection with triangulations of the polygon. For more information see Exercise 2.18 of `https://arxiv.org/pdf/1212.6263`.

Prove that the number of triangulations (hence of cluster variables) of an $(n+1)$-gon is the the Catalan number $\frac{1}{n+2}\binom{2n+2}{n+1}$, the famous sequence starting by $1, 1, 2, 5, 14, 42, 132, \dots$.