Incidence algebras of finite poset as quasi-hereditary algebras

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Abstract

In this note we describe the combinatorics of quasi-hereditary structures for the incidence algebras of a finite poset.

1 Introduction

In this note $P = (P, \leq)$ denotes a finite poset. The incidence algebra of P over the field $\mathbf{k}$ is denoted by $A_{\mathbf{k}}(P)$. It has a basis consisting of intervals (a, b) in the poset P and the multiplication if given by

$$(a, b) \cdot (c, d) = \begin{cases} (a, d) & \text{if } b = c, \\ 0 & \text{otherwise.} \end{cases}$$

A module over the incidence algebra is a *right finitely generated* module. The canonical complete set of primitive idempotents of $A_{\mathbf{k}}(P)$ is given by $\{e_x := (x, x) \mid x \in P\}$. The projective indecomposable modules are given by $P_x = e_x A_{\mathbf{k}}(X)$ and P_x has for basis the set of (x, y) such that $x \leq y$ in P. We say that it is supported by the elements larger than x in P. It has a simple top $S(x)$. Dually the injective indecomposable module I_x with simple socle $S(x)$ is supported by the elements smaller than x in P. If $i \leq j \in P$, we denote by $[i, j]$ the interval (i, j) that we may view as an indecomposable module over the incidence algebra of P. The notation (i, j) is reserved for the corresponding basis element of the incidence algebra. Moreover if we want to emphasize the partial order $\leq$, we write $[i, j]_{\leq}$.

A quasi-hereditary algebra is an algebra together with a partial ordering on the isomorphism classes of its simple modules which satisfies some conditions that we recall below. In the particular case of an incidence algebra the isomorphism classes of simple modules are in bijection with the elements of the posets, so understanding the quasi-hereditary structures of the incidence algebras of a finite poset leads to the following (classical) definition.

Definition 1.1. Let P be a finite set. A *double poset* is a triple $(P, \leq, \lhd)$ where $(P, \leq)$ and $(P, \lhd)$ are two posets.

A morphism f of double posets from $(P_1, \leq_1, \lhd_1)$ to $(P_2, \leq_2, \lhd_2)$ is a mapping f such that $\forall x, y \in P$, $x \leq_1 y \Rightarrow f(x) \leq_2 f(y)$ and $x \lhd_1 y \Rightarrow f(x) \lhd_2 f(y)$.

Given a partial ordering $\lhd$ on the set of isomorphism classes of simple modules, we construct the so-called standard modules $\Delta(i)$ as the largest quotient of $P(i)$ whose composition factors $S(j)$ are such that $j \lhd i$. Dually, we construct the so-called co-standard modules $\nabla(i)$ as the largest submodule of $I(j)$ whose composition factors $S(j)$ are such that $j \lhd i$.

For a set of modules Θ we denote by $\mathcal{F}(\Theta)$ the category of modules which have a Θ-filtration. That is the modules M for which there exists a chain of submodules

$$0 = M_0 \subset M_1 \subset \cdots \subset M_{n-1} \subset M_n = M,$$

such that $M_i/M_{i-1} \in \Theta$ for $i = 1, 2, \cdots, n$.

Definition 1.2. Let A be an artinian algebra and $\lhd$ be a partial ordering on its set I of isomorphism classes of simple modules. Let Δ be the corresponding set of standard modules. The pair $(A, \lhd)$ is a quasi-hereditary algebra if for all $i \in I$:

1. $[\Delta(i) : S(i)] = 1$,
2. $P(i) \in \mathcal{F}(\Delta)$,
3. $\big(P(i) : \Delta(i)\big) = 1$ and $\big(P(i) : \Delta(j)\big) \neq 0$ implies $i \lhd j$.

The notation $\big(P(i) : \Delta(j)\big)$ denotes the number of times that $\Delta(j)$ appears in a Δ-filtration of $P(i)$. It can be proved that this number does not depend on the choice of a filtration. From the definition, we see that the property of quasi-hereditary does not depend on the poset $\lhd$ but only on the set of standard modules Δ. This makes the next definition natural.

Definition 1.3. Let A be an artinian algebra and $\lhd_1$ and $\lhd_2$ be two partial orderings on its set of isomorphism classes of simples modules. We say that $\lhd_1$ and $\lhd_2$ are equivalent if and only if their set of standard modules are equal and their set of co-standard modules are equal. If moreover $(A, \lhd_1)$ is a quasi-hereditary algebra, the equivalence class of $\lhd_1$ is called a *quasi-hereditary structure* for A.

It is clear that not all orderings appear as a quasi-hereditary structure on a given algebra A. For example, the equality relation induces a structure of quasi-hereditary algebra on A if and only if A is semi-simple. In order to understand the quasi-hereditary structures we have a first restriction on the possible orderings.

Definition 1.4. Let A be an artinian algebra and $\lhd$ be a partial ordering on the set of isomorphism classes of simple A-modules. Then $\lhd$ is *adapted* to A if for every A-module M with top $S(i)$ and socle $S(j)$ either i and j are comparable, or there M has a composition factor $S(k)$ such that $i \lhd k$ and $j \lhd k$.

Lemma 1.5 (Conde [1]). *If $(A,(I,\lhd))$ is a quasi-hereditary algebra, then $(I,\lhd)$ is adapted for A.*

Proof. This is Proposition 1.4.12 of the PhD thesis of Teresa Conde. For the convenience of the reader, we sketch the proof. Let M be a module with simple top $S(i)$ and simple socle $S(j)$. Since A is artinian, M is a quotient of $P(i)$. In particular $S(j)$ is a composition factor of $P(i)$. Because the algebra is quasi-hereditary $P(i)$ has a Δ-filtration. So $S(j)$ must appear in a standard module $\Delta(k)$. If $k = i$ then $j \lhd i$. If $k \neq i$ and $S(j)$ is at the top of $\Delta(k)$ we have $i \lhd j$. And finally if $S(j)$ is not at the top of $\Delta(k)$ we have $i \lhd k$ and $j \lhd k$. □

Using adapted orders, we can remove the third point in the definition of quasi-hereditary algebras:

Proposition 1.6. *Let A be an artinian algebra together with a partial order $\lhd$ on its set of isomorphism classes of simple modules. Then $(A,\lhd)$ is a quasi-hereditary algebra if and only if*

1. *$\lhd$ is adapted to A.*
2. *For every $i \in I$, $[\Delta(i) : S(i)] = 1$.*
3. *$P(i) \in \mathcal{F}(\Delta)$ for $i \in I$.*

Proof. See Theorem 1 of [2]. □

Finally, let us look at the difference between the equivalence class of an adapted order and a quasi-hereditary structure.

Theorem 1.7. *Let A be an artinian algebra and $\lhd$ be poset adapted to A. We denote by Δ and ∇ the respective direct sum of all the standard and costandard modules. Then, the following are equivalent*

1. *$(A,\lhd)$ is a quasi-hereditary algebra.*
2. *$\operatorname{Ext}^2_A(\Delta,\nabla) = 0$.*

Proof. See Theorem 1 of [2]. □

As an immediate corollary we see that for the hereditary algebras, the quasi-hereditary structures coincides with the equivalence classes of adapted orders.

2 Adapted orders and quasi-hereditary structures of incidence algebras

In this section $(P, \leq)$ denotes a finite poset and $A_{\mathbf{k}}(P)$ its incidence algebra over the field $\mathbf{k}$. As explained in the first section, the study of quasi-hereditary structures on $A_{\mathbf{k}}(P)$ naturally leads to the study of all the double posets $(P, \leq, \lhd)$ where $\lhd$ is adapted to $A_{\mathbf{k}}(P)$. First, let us see that the notion of *adapted poset* is purely combinatorial.

Lemma 2.1. *Let M be an $A_{\mathbf{k}}(P)$-module with top $S(i)$ and socle $S(j)$, then M is isomorphic to $[i,j]_{\leq}$.*

Proof. M has simple top $S(i)$, so it is a quotient of $P(i)$ and it is generated by $m_i = \phi(e_i)$. Since M has simple socle $S(j)$, there is an element $n \in M$ such that $n \cdot e_j = n$ and $n \cdot e_k = 0$ for $k \neq j$. There is $a \in A_{\mathbf{k}}(P)$ such that $m_i \cdot a = n$. We have $0 \neq n \cdot e_j = \phi(e_i \cdot a \cdot e_j)$. Since a is a linear combination of intervals, we see that $m_j := \phi\big((i,j)\big) \neq 0$. For $k \in P$, we let $m_k := \phi\big((i,k)\big)$. It is clear that $m_k \neq 0$ implies $i \leq k$. Moreover if k is maximal such that $m_k \neq 0$, then m_k generates a simple submodule of M, so we have $k = j$. Since $0 \neq \phi\big((i,j)\big) = \phi\big((i,k)\cdot(k,j)\big)$ we have that $m_k \neq 0$ if and only if $k \in [i,j]_{\leq}$ and the $\mathbf{k}$-vector space generated by all the m_ks is a submodule of M isomorphic to $[i,j]_{\leq}$. Let $m \in M$, then $m = \sum_{k \in P} m \cdot e_k$. Since M is generated by m_i, there exists $a = \sum_{x \leq y \in P} \lambda_{x,y}(x,y) \in A_{\mathbf{k}}(P)$ such that $m = m_i \cdot a$. So $m = \sum_{k\in P} \phi(e_i a e_k) = \sum_{k \in P} \lambda_{i,k} m_k = \sum_{k \in [i,j]} \lambda_{i,k} m_k$.
So $M \subseteq [i,j]_{\leq}$. □

It follows that the notion of adapted order can be easily characterized in terms of the partial ordering.

Corollary 2.2. *Let $(P, \leq)$ be a finite poset and $A_{\mathbf{k}}(P)$ its incidence algebra. Let $\lhd$ be a partial ordering on P. Then $\lhd$ is adapted to $A_{\mathbf{k}}(P)$ if and only if*

(A) $\forall (i,j) \in P^2 \ : \ i \leq j, \ \exists k \in [i,j]_{\leq}$ *such that* $i \lhd k$ *and* $j \lhd k$.

Remark 2.3.

1. In the corollary, the element k is allowed to be equal to i or j. In this case i and j are comparable for $\lhd$.
2. It follows that the set of adapted orders for the incidence algebra of a finite poset *does not depend* on the choice *of the field.*

3 Minimal adapted orderings for incidence algebras of finite poset

As explained in the first section, for our purpose, we are not interested by adapted orderings but we are interested by equivalence classes of such orderings. It is easy to see that any extension of an adapted ordering gives an equivalent adapted ordering. As a consequence, in the study of quasi-hereditary algebras, one can always assume that the partial ordering on the set of isomorphism classes is a total ordering. However, in the trivial example of a semi-simple algebra with n isomorphism classes of simple modules, there are $n!$ total orderings but only one equivalence class of adapted orderings. Hence, in order to classify the quasi-hereditary structures, it is therefore a lot easier to do the opposite, that is to look at adapted orders with the minimal number of relations.

Lemma 3.1. *Let A be an artinian algebra and I be a set indexing the isomorphism classes of simple A-modules. Let $\lhd_1$ be a poset which is adapted to A. If $\lhd$ is equivalent to $\lhd_1$, then $\lhd$ is adapted to A.*

Partial orders on a finite set I can be ordered by inclusion of their sets of relations. This gives a poset of posets over I where the minimal element is the equality relation on I and the maximal elements are the total orders. The poset obtained by taking the intersection of the relations of two given posets $\lhd_1$ and $\lhd_2$ is called the intersection of $\lhd_1$ and $\lhd_2$.

Lemma 3.2. *Let A be an artinian algebra with I a set indexing a complete set of isomorphism classes of simple A-modules. Let $\lhd_1$ and $\lhd_2$ be two adapted orders in the same equivalence class.*

1. *The intersection of $\lhd_1$ and $\lhd_2$ is an adapted order in the same equivalence class.*

2. *In each equivalence class of adapted poset for A there is a unique minimal poset.*

Proof. It is clear that (1) implies (2). Let $\lhd_1$ and $\lhd_2$ be two posets in the same equivalence class. We denote by $\Delta = \Delta_1 = \Delta_2$ the corresponding set of standard modules. We let $\lhd_I$ be the intersection of $\lhd_1$ and $\lhd_2$ and we denote by Δ_I the corresponding set of standard modules.

Let $i \in I$. By definition $\Delta_I(i)$ is the largest quotient of $P(i)$ whose composition factors are $S(j)$ such that $j \lhd_1 i$ and $j \lhd_2 i$. So $\Delta_1(i)$ surjects onto $\Delta_I(i)$. If they are not isomorphic, at the top of the kernel there is a simple module $S(j)$ such that $j \lhd_1 i$ but j is not smaller that i for $\lhd_2$. This contradicts $\Delta_1(i) = \Delta_2(i)$. As conclusion $\Delta_I = \Delta$ and by a dual argument, we see that $\nabla_I = \nabla$ and the poset $\lhd_I$ is equivalent to the posets $\lhd_1$ and $\lhd_2$. The result follows from Lemma 3.1 □

Let us describe these minimal adapted orderings. For a partial order $\lhd$ on I, let Δ be the set of standard A-modules associated to $\lhd$ and let ∇ be the set of co-standard A-modules.

We define two subsets $\mathrm{Dec}(\lhd)$ and $\mathrm{Inc}(\lhd)$ of I^2 as follows:

$$\mathrm{Dec}(\lhd) := \{(i,j) \in I^2 \mid [\Delta(j) : S(i)] \neq 0\}, \qquad \mathrm{Inc}(\lhd) := \{(i,j) \in I^2 \mid [\nabla(j) : S(i)] \neq 0\}.$$

Clearly, $\mathrm{Dec}(\lhd)$ and $\mathrm{Inc}(\lhd)$ depend on only the equivalence class of $\lhd$. For $i, j \in I$, we write $i \lhd^D j$ if $(i,j) \in \mathrm{Dec}(\lhd)$ and write $i \lhd^I j$ if $(i,j) \in \mathrm{Inc}(\lhd)$. For $i \in I$, we have $i \lhd^D i$ and $i \lhd^I i$.

For a subset R of I^2, we denote by R^{tc} the transitive closure of R. Then the following lemma is easy.

Lemma 3.3. *Let* $\lhd_m = (\mathrm{Dec}(\lhd) \cup \mathrm{Inc}(\lhd))^{\mathsf{tc}}$.

1. *If* $i \lhd_m j$, *then* $i \lhd j$ *holds.*

2. $\lhd_m$ *is a partial order on* I.

Then we have the following proposition.

Proposition 3.4. *Let* A *be an artin algebra and* I *be a set indexing the isomorphism classes of simple* A*-modules. For an adapted partial order* $\lhd$ *on* I, *let* $\lhd_m = (\mathrm{Dec}(\lhd) \cup \mathrm{Inc}(\lhd))^{\mathsf{tc}}$.

1. *The partial orders* $\lhd$ *and* $\lhd_m$ *are equivalent.*

2. $\lhd_m$ *is the unique minimal partial order among partial orders* $\lhd'$ *on* I *with* $\lhd \sim \lhd'$.

Lemma 3.5. *Let* $(P, \leq)$ *be a finite poset and* $\lhd$ *be an adapted order for* $A_{\mathbf{k}}(P)$. *Then*

1. $j \lhd i \in \mathrm{Dec}(\lhd)$ *if and only if* $i \leq j$ *and* $\forall k \in [i,j]_{\leq}$ *we have* $k \lhd i$.

2. $i \lhd j \in \mathrm{Inc}(\lhd)$ *if and only if* $i \leq j$ *and* $\forall k \in [i,j]_{\leq}$ *we have* $k \lhd j$.

Proof. We only prove the first point, the second is dual. Recall that the projective module P_i is supported by all the elements larger than i in the poset $(P, \leq)$. If $S(j)$ is a composition factor of $\Delta(i)$, then $j \lhd i$ and $i \leq j$. By Lemma 2.1, the interval $[i,j]_{\leq}$ is a quotient of $\Delta(i)$ and we have $k \lhd i$ for all $k \in [i,j]_{\leq}$. Conversely, this condition implies that $[i,j]_{\leq}$ is a quotient of $P(i)$ such that all composition factors $S(k)$ are such that $k \lhd i$. Since $\Delta(i)$ is the largest quotient of $P(i)$ with this property, we conclude that $S(j)$ is a composition factor of $\Delta(i)$. □

We will need these two technical lemmas

Lemma 3.6. *Let* $(P, \leq)$ *be a finite poset and* $\mathbf{k}$ *be a field. Let* $(P, \lhd)$ *be an order adapted to* $A_{\mathbf{k}}(P)$. *Then for* $[i,j]_{\leq}$, *there is* $k \in [i,j]_{\leq}$ *such that* $i \lhd k \in \mathrm{Inc}(\lhd)^{tc}$ *and* $j \lhd k \in \mathrm{Dec}(\lhd)^{tc}$.

Proof. Assume first that $i \lhd j$ and $i \leq j$. If j covers i for $\leq$, then the relation is increasing. If $i \lhd j$ is increasing there is nothing to prove otherwise there exists $k \in]i,j[$ such that $k \not\lhd$. If we choose k to be maximal for this property, then it is easy to check that $j \lhd k$ and this relation is decreasing and by transitivity we have the relation $i \lhd k$. We apply induction to $i \lhd k$ and prove the result. This is similar for relations of the form $j \lhd i$.

Let us look at the case where i and j are not comparable for $\lhd$. Since the poset is adapted, there is $k \in]i,j[$ such that $i \lhd k$ and $j \lhd k$. It is then easy to conclude using the previous argument to replace $i \lhd k$ by $i \lhd a \in \mathrm{Inc}(\lhd)^{tc}$ and $k \lhd a \in \mathrm{Dec}(\lhd)^{tc}$ and apply it one more time to $j \lhd a$.

□

Lemma 3.7. *Let* $(P, \leq)$ *be a finite poset and* $\mathbf{k}$ *be a field. Let* $i, j, h \in P$ *such that* $i < j < k$.

1. *If $i \lhd j \in \mathrm{Inc}(\lhd)^{tc}$ and $j \lhd k \in \mathrm{Inc}(\lhd)^{tc}$, then either $i \lhd k \in \mathrm{Inc}(\lhd)$ or there is $j' \in]i,k[$ such that $i \lhd j' \in \mathrm{Inc}(\lhd)$ and $k \lhd j' \in \mathrm{Dec}(\lhd)^{tc}$.*

2. *If $j \lhd i \in \mathrm{Dec}(\lhd)^{tc}$ and $k \lhd j \in \mathrm{Dec}(\lhd)^{tc}$, then either $k \lhd i \in \mathrm{Dec}(\lhd)$ or there is $j' \in]i,k[$ such that $k \lhd j' \in \mathrm{Dec}(\lhd)$ and $i \lhd j' \in \mathrm{Inc}(\lhd)^{tc}$.*

Proof. We only prove the first statement, the second being similar is left to the reader. We prove the result on induction on the size of $[i,j]_{\leq}$. There is nothing to prove when it has size $0,1,2,3$. If $i \lhd k$ is not increasing there is $a \in]i,j[_{\leq}$ such that $a \not\lhd k$ and $a \not\lhd i$. We apply Lemma 3.6 to $i < a$, then there is $a' \in P$ such that $i < a' \leq a$ and $i \lhd a' \in \mathrm{Inc}(\lhd)^{tc}$, $a \lhd a' \in \mathrm{Dec}(\lhd)^{tc}$ and not that $a' \not\lhd k$ because otherwise $a \lhd k$. So we may replace a by a' and assume that we have a relation $i \lhd a$ and that this relation is in $\mathrm{Inc}(\lhd)^{tc}$. Applying Lemma 3.6 to $a < k$, we have $a' \in P$ such that $a \leq a' < k$ and relations $a \lhd a' \in \mathrm{Inc}(\lhd)^{tc}$ and $k \lhd a' \in \mathrm{Dec}(\lhd)^{tc}$. In the end we found $a' \in P$ such that $i < a' < k$ and $i \lhd a' \in \mathrm{Inc}(\lhd)^{tc}$ and $k \lhd a' \in \mathrm{Dec}(\lhd)^{tc}$. If $i \lhd a'$ is increasing, we are done. Otherwise we apply induction to $i < a'$. ☐

Corollary 3.8. *Let $(P,\leq)$ be a finite poset and $\mathbf{k}$ be a field. Let $(P,\lhd)$ be an adapted poset for $A_{\mathbf{k}}(P)$. Then for $[i,j]_{\leq}$, there is $k \in [i,j]_{\leq}$ such that $i \lhd k \in \mathrm{Inc}(\lhd)$ and $j \lhd k \in \mathrm{Dec}(\lhd)$.*

Proof. Let $i < j \in P$. By Lemma 3.6, there is a_0 such that $i < a_0 < j$ and $i \lhd a_0 \in \mathrm{Inc}(\lhd)^{tc}$ and $j \lhd a_0 \in \mathrm{Dec}(\lhd)^{tc}$. Assume that $i \lhd a_0$ is not increasing, then by Lemma 3.7, there is a_1 such that $i < a_1 < a_0$ and $i \lhd a_1 \in \mathrm{Inc}(\lhd)$ and $a_0 \lhd a_1 \in \mathrm{Dec}(\lhd)^{tc}$. Then by transitivity we have $j \lhd a_1 \in \mathrm{Dec}(\lhd)^{tc}$. If this last relation is decreasing we are done. Otherwise we apply Lemma 3.7 to the relation $j \lhd a_1$, then there is $a_2 \in P$ such that $a_1 < a_2 < j$ and $a_1 \lhd a_2 \in \mathrm{Inc}(\lhd)^{tc}$ and $j \lhd a_2 \in \mathrm{Dec}(\lhd)$. Note that $a_2 \neq a_0$ and $a_2 \neq 0_1$ since by construction we have $a_1 < a_2$, $a_0 \lhd a_1$ and $a_0 \lhd a_1 \lhd a_2$. By transitivity, we have the relation $i \lhd a_2$, if it is increasing we are done, otherwise we apply Lemma 3.7 to it. This process of applying successively Lemma 3.7 to the relations $i \lhd a_{2n}$ and $j \lhd a_{2n+1}$ constructs a strictly increasing chain $a_0 \lhd a_1 \lhd a_2 \lhd \cdots$ in the finite poset $(P,\lhd)$ so it must stops. The proof is similar if $j \lhd a_0$ is not decreasing. ☐

Remark 3.9. Note that this Lemma does not imply that any relation for $\lhd$ between two comparable elements for $\leq$ is either increasing or decreasing. This is true for a total order but not in general. It also does not imply that the composition of two increasing relations is increasing.

4 Quasi-hereditary structures on incidence algebras of finite posets

Let us move to the (minimal) adapted order inducing quasi-hereditary structures on the incidence algebra of a finite poset. For this, we use Theorem 1.7 to decide if an adapted order induces a quasi-hereditary structure. Here, we have to be worried: the first extension group between two simple modules is a combinatorial object which is computed by the cover relations in the poset. In particular it does not depend on the choice of the field, but there is no reason for the larger extension groups to be independent of the field. In particular, there are examples of finite posets where the global dimension depends on the characteristic of the field.

We need to understand under which condition $\mathrm{Ext}^2(\Delta,\nabla)=0$. Since this is a bi-additive functor, it is enough to look at $\mathrm{Ext}^2(\Delta(i),\nabla(j))$ for $i,j\in P$. We denote by $N(i)$ the kernel of the natural projection $P(i)\to\Delta(i)$ and we denote by $\iota : N(i)\to P(i)$. Similarly, we let $C(j)$ be the cokernel of the natural embedding of $\nabla(j)$ in $I(j)$. We keep all these notation for the rest of the section. Applying $\mathrm{Hom}(\Delta(i),-)$ to the short exact sequence $0\to\nabla(j)\to I(j)\to C(j)\to 0$, we get $\mathrm{Ext}^2(\Delta(i),\nabla(j))\cong Ext^1(\Delta(i),C(j))$. Then, applying $\mathrm{Hom}(-,C(j))$ to the exact sequence $0\to N(i)\to P(i)\to\Delta(i)\to 0$, we get

$$\mathrm{Ext}^2(\Delta(i),\nabla(j))\cong \mathrm{Hom}(N(i),C(j))/\iota^*(\mathrm{Hom}(P(i),C(j))),$$

where $\iota^*(f)=f\circ i$.

If $C(j)e_x\neq 0$ and $N(i)e_x\neq 0$, we say that x is in the support of $N(i)$ and $C(j)$. An s-path of length n between a_1 and a_n is a sequence $(a_1,a_2,\cdots,a_n)$ where all the a_is are in the support of $C(j)$ and $N(i)$ and such that a_i is comparable for $\leq$ with a_{i+1}. By convention, we assume that a_i and a_{i+2} are not comparable, this means that we have either $a_i\leq a_{i+1}\geq a_{i+2}$ or $a_i\geq a_{i+1}\leq a_{i+2}$. If x and y are two elements in the support of $N(i)$ and $C(j)$ connected by an s-path, the minimal length of an s-path between x and y is called the s-distance between x and y. An s-component is a subset of the support of $N(i)$ and $C(j)$ in which any two elements are connected by an s-path and which is maximal (for the inclusion) for this property.

Lemma 4.1. *The dimension of* $\mathrm{Hom}(N(i),C(j))$ *is equal to the number of s-components in the support of* $N(i)\cap C(j)$.

Proof. The module $N(i)$ has basis the set of intervals (i,a) such that a is in the support of $N(i)$. Similarly, the module $C(j)$ has basis the set of linear forms $(a,j)^*$ such that a is in the support of $C(j)$.

Moreover, if $a\leq b$ and a is in the support of $N(i)$, then b is also in the support of $N(i)$ and we have $(i,a)\cdot(a,b)=(i,b)$. Similarly, if a is in the support of $C(j)$ we have $(a,j)^*\cdot(a,b)=(b,j)^*$ if b is in the support of $C(j)$ and 0 otherwise. If b is in the support of $C(j)$ and $a\leq b$, then a is also in the support of $C(j)$.

The first step is to show that a morphism between $N(i)$ and $C(j)$ is 'constant' along the s-paths. More precisely if x and y are both in the support of $N(i)$ and $C(j)$, the value of $\phi\big((i,x)\big)$ is determined by the value of $\phi\big((i,y)\big)$. It is enough to prove it when x and y are comparable. Since ϕ is a morphism of $A_{\mathbf{k}}(P)$-module, we have $\phi\big((i,x)\big)\cdot(x,x)=\phi\big((i,x)\big)$, so there is $\lambda_x\in\mathbf{k}$ such that $\phi\big((i,x)\big)=\lambda_x(x,j)^*$. Now, if $x\leq y$, then we have

$$\lambda_y(y,j)^*=\phi\big((i,y)\big)=\phi\big((i,x)\cdot(x,y)\big)=\lambda_x(x,j)^*\cdot(x,y)=\lambda_x(y,j)^*.$$

So $\lambda_x = \lambda_y$.

If c is an s-component, we associate a morphism $\chi_c : N(i) \to C(j)$ by:

$$\chi_c(i,x) = \begin{cases} (x,j)^* & \text{if } x \in c, \\ 0 & \text{otherwise} \end{cases}$$

Then, it is easy to check that this is a morphism of $A_{\mathbf{k}}(P)$-module. The set of χ_c is clearly linearly independent and by the explanation above, it is a generating set of $\mathrm{Hom}(N(i), C(j))$. $\square$

Lemma 4.2. *If* $\dim_{\mathbf{k}} \mathrm{Hom}(N(i), C(j)) \geq 1$*, then* $\iota^*\Big(\mathrm{Hom}(P(i), C(j))\Big)$ *is a one dimensional vector subspace of* $\mathrm{Hom}(N(i), C(j))$*.*

Proof. Clearly if there is a non-zero morphism between $N(i)$ and $C(j)$, then the interesction of the supports of $N(i)$ and $C(j)$ is non-empty. This implies that $i < j$ and that $S(i)$ is a composition factor of $C(j)$. In particular, $\dim_{\mathbf{k}} \mathrm{Hom}(P(i), C(j)) = 1$. Since $P(i)$ is generated by (i,i), a morphism ϕ from $P(i)$ to $C(j))$ is determined by $\phi\big((i,i)\big)$. There is $\lambda_i \in \mathbf{k}$ such that $\phi\big((i,i)\big) = \lambda_i (i,j)^*$. If x is in the support of $N(i)$ and $C(j)$, then $\phi(i,x) = \lambda_i (i,j)^* \cdot (i,x) = \lambda_i (x,j)^*$. And we see that ϕ is determined by its value on any element in the support of $N(i)$ and $C(j)$. This implies that ι^* is injective and the result follows. $\square$

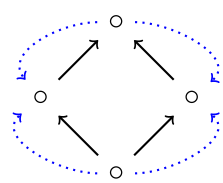

Figure 1: The double square is the double poset $\mathfrak{s} = (\{1,2,3,4\}, \leq, \lhd)$ where the first partial order is drawn in thick black and the second is drawn in dotted blue.

Definition 4.3. Let $(P, \leq, \lhd)$ be a double poset. A double square configuration in P is the data of $i, x, y, j \in P$ such that $i \leq x$, $i \leq y$, $x \leq j$, $y \leq j$, x and y are incomparable and we have $i \lhd x \in \mathrm{Inc}(\lhd)$, $i \lhd y \in \mathrm{Inc}(\lhd)$ and $j \lhd x \in \mathrm{Dec}(\lhd)$ and $j \lhd y \in \mathrm{Dec}(\lhd)$.

Remark 4.4. In other words a double square configuration is the image of an injective morphism of double posets from the double square to P which is full for the first partial order and preserves the increasing and decreasing relations for the second order.

Lemma 4.5. *Let* $(P, \leq)$ *be a finite poset and* $(P, \lhd)$ *be an adapted poset for* $A_{\mathbf{k}}(P)$*. Then* $\dim_{\mathbf{k}}(\mathrm{Hom}(N(i), C(j))) \geq 2$ *if and only if there is a double square configuration with minimal element* i *and maximal element* j *in* P *(with respect to* $\leq$*).*

Proof. Let $\{i, x, y, j\}$ be a double square configuration. We have to show that there are at least two s-components in the support of $N(i)$ and $C(j)$.

The first step is to prove that there is no s-path from x to y. Let us assume that the s-distance between x and y is n. Since x and y are not comparable by assumption, if there is such a path it has length at least 2.

We look first at the case $n = 2$. If there is a path (x, a, y) we have two possibilities: either $x \leq a$ and $y \leq a$ or $a \leq x$ and $a \leq y$. In the first case, since $x < a < j$ and $j \lhd x$ is decreasing, we have a decreasing relation $a \lhd x$ and similarly a decreasing relation $a \lhd y$. Then $\{i, x, y, a\}$ induces a new double-square configuration with $i < a < j$. We have a similar result in the second case.

Let us look at the case $n \geq 3$. We choose a path $(x, a_2, \cdots, a_{n-1}, y)$ of minimal length. We also have two cases, either $a_2 \leq x$ or $x \leq a_2$. Let us look at the first case, the second case being similar is left to the reader.

Since a_2 is in the support of $N(i)$, we have $i \leq a_2$ and there is $b_2 \in [i, a_2]_{\leq}$ such that $i \not\lhd b_2$. As in the proof of Lemma 3.6, if we choose b_2 minimal for this property, then we have $i \lhd b_2$ and this relation is increasing.

1. b_2 and y are not comparable for $\leq$ otherwise the distance between x and y is at most 2.

2. $b_2 \leq a_3$. Indeed we have $b_2 \leq a_2 \leq a_3$.

Similarly, a_3 is in the support of $C(j)$ so we have $a_3 \leq j$ and there is $c_3 \in [j, a_3]_{\leq}$ such that $j \leq a_3$ and $j \lhd c_3$ is decrasing. Then we have:

1. $b_2 \leq c_3$ since $b_2 \leq a_2 \leq a_3 \leq c_3$.

2. By Corollary 3.8, there is a such that $b_2 \leq a \leq c_3$ with $b_2 \lhd a$ increasing and $c_3 \lhd a$ decreasing. By transitivity, $i \lhd a$ is increasing and $j \lhd a$ is decreasing.

3. a is not comparable with a_i for $i \geq 4$.

We see that $\{i, a, j, y\}$ is a new configuration of length $n - 1$.

By induction on the length we see that starting with a configuration $\{i, x, y, j\}$ having an s-path between x and y we construct a new configuration $\{i', x', y', j'\}$ such that $[i', j']_{\leq}$ is a strict subset of $[i, j]_{\leq}$. Since $[i, j]$ is finite and in a double square configuration we must have at least 4 elements in $[i, j]_{\leq}$, we will find a configuration in which there is no s-path between x' and y'.

For the reciprocity, if $\dim_{\mathbf{k}} \operatorname{Hom}(N(i), C(j)) \geq 2$, there are at least two s-components in the support of $N(i)$ and $C(j)$. In other words, there are x and y such that $i \leq x$, $i \leq y$, $x \leq j$ and $y \leq j$ and such that x and y are not connected by an s-path. Since x is in the support of $N(i)$ and $C(j)$, there are $a_1, b_1 \in P$, such that $i \leq a_1 \leq x \leq b_1 \leq j$ and $i \lhd a_1$ is increasing and $j \lhd b_1$ is decreasing. By Corollary 3.8 there is x' such that $a_1 \leq x' \leq b_1$ and $a_1 \lhd x'$ is increasing and $b_1 \lhd x'$ is decreasing. Similarly we find y' such that $i \leq y' \leq j$ and

$i \lhd y'$ is increasing and $j \lhd y'$ is decreasing. Moreover since there is no s-path between x and y there is also no s-path between x' and y'. In particular, x' and y' are not comparable and $\{i, x', y', j\}$ is a double-square configuration. □

Theorem 4.6. *Let* $\mathbf{k}$ *be a field and* $(P, \leq)$ *be a finite poset. Let* $(P, \lhd)$ *be an adapted poset for* $A_{\mathbf{k}}(P)$*. Then* $\lhd$ *is* not *a quasi-hereditary structure if and only if there is a double square configuration in* $(P, \leq, \lhd)$*.*

Proof. By Theorem 1.7 the partial order $\lhd$ does not induce a quasi-hereditary structure if and only if there are $i, j \in P$, such that $\mathrm{Ext}^2(\Delta(i)\nabla(j)) \neq 0$. By the discussion at the beginning of the section and Lemma 4.2, this is equivalent to $\dim_{\mathbf{k}} \mathrm{Hom}(N(i), C(j)) \geq 2$. And by Lemma 4.5, this is equivalent to the existence of a double square configuration. □

Corollary 4.7. *The set of quasi-hereditary structures of the incidence algebra of a finite poset* P *over the field* $\mathbf{k}$ *does not depend on the field.*

5 Partial ordering of the set of quasi-hereditary structures

Let $[\lhd_1]$ and $[\lhd_2]$ be two quasi-hereditary structures on A with respective set of standard modules Δ_1 and Δ_2, then we set $[\lhd_1] \preceq [\lhd_2]$ if $\mathcal{F}(\Delta_2) \subseteq \mathcal{F}(\Delta_1)$. By this ordering, we regard $(\mathrm{qhstr}(A), \preceq)$ as a poset.

As we see in the next lemma, this ordering is induced from a partial ordering on tilting modules.

Lemma 5.1. *Let* $[\lhd_1]$ *and* $[\lhd_2]$ *be two quasi-hereditary structures on* A *and* T_i *be the characteristic tilting module of* $(A, \lhd_i)$ *for* $i = 1, 2$*. The following statemetns are equivalent*

1. $[\lhd_1] \preceq [\lhd_2]$.
2. $\mathcal{F}(\nabla_1) \subseteq \mathcal{F}(\nabla_2)$.
3. $T_1^{\perp} \subseteq T_2^{\perp}$.
4. $\mathrm{Ext}^1(\Delta_2(i), \nabla_1(j)) = 0$ *for all* i *and* j *in* I.

We use the point 4 of the previous Lemma, and we obtain the following characterization of this partial ordering.

Proposition 5.2. *Let* $(P, \leq)$ *be a finite poset and* $(P, \lhd_1)$*,* $(P, \lhd_2)$ *be two adapted posets corresponding two quasi-hereditary structures. Then* $\lhd_1 < \lhd_2$ *in the poset of quasi-hereditary structures on* $A_{\mathbf{k}}(P)$ *if and only if*

1. $\mathrm{Inc}(\lhd_2) \subseteq \mathrm{Inc}(\lhd_1)$ *and*
2. *If* $i \lhd j \in \mathrm{Inc}(\lhd_2)$ *and* $j \lhd k \in \mathrm{Inc}(\lhd_1)$*, then* $i \lhd k \in \mathrm{Inc}(\lhd_1)$.

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