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# ANALYTIC REPARAMETRIZATIONS OF TRANSLATION TORAL FLOWS WITH COUNTABLE LEBESGUE SPECTRUM

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Abstract. We give an example of a real analytic reparametrization of a minimal translation flow on $\mathbb{T}^5$ that has a Lebesgue spectrum with infinite multiplicity. As a consequence, we see that the dynamics on a non-Diophantine invariant torus of an almost integrable Hamiltonian system can be spectrally equivalent to a Bernoulli flow.

1. **Introduction.** Inspired by Kolmogorov's 1954 ICM talk [9], the following questions arise: *Can a completely integrable Hamiltonian flow be perturbed in the real analytic category so that the perturbed flow has an invariant torus exhibiting dynamics with a maximal spectral type equivalent to the Lebesgue measure on $\mathbb{R}$? Can a reparametrized translation flow of the torus have a maximal spectral type that is equivalent to the Lebesgue measure on $\mathbb{R}$?* By a classical construction that we recall below, a positive answer to the second question, with reparametrization functions arbitrarily close to 1, immediately yields examples of perturbations that answer positively the first question. The aim of this paper is to prove the following. Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

**Theorem 1.1.** *There exists $\alpha \in \mathbb{T}^4$ such that the translation flow on $\mathbb{T}^5$ of vector $(\alpha, 1)$ is minimal, and there exists a strictly positive real entire function $\Phi$ defined on $\mathbb{T}^5$, such that the reparametrization of the irrational flow $(\alpha, 1)$ by $\Phi$ has a Lebesgue spectrum with infinite multiplicity. Moreover, $\Phi$ can be chosen arbitrarily close to 1 on any bounded (complex) domain around $\mathbb{T}^5$.*

Note that arbitrarily small perturbations of the reparametrized flow, such as changing slightly $\alpha$ or $\Phi$, can render the flow analytically conjugated to a translation flow, who has a pure point spectrum.

Since the reparametrization function can be chosen arbitrarily close to 1, it is a classical observation that this automatically gives examples of perturbations of

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integrable Hamiltonians with invariant tori carrying uniquely ergodic dynamics with infinite Lebesgue spectrum (ILS). Moreover, the integrable Hamiltonian can be chosen to be convex. Indeed, consider the completely integrable system on $\mathbb{T}^5 \times \mathbb{R}^5$ given by

$$H(\theta, r) = \Phi(\theta)(\frac{1}{2}\sum_{j=1}^{5} r_j^2 - 1).$$

Denote $X_H^t$ the corresponding flow. The energy surface $\{H = 0\}$ is foliated by invariant tori for $X_H^t$ on which the restricted flow is a reparametrization by $\Phi$ of the translation flow of frequency vector $r$. One can then pick the vector $r$ and the function $\Phi$ to be as in the theorem and get the invariant torus with ILS.

**Previous results on the spectral properties of reparametrizations of translation flows.** Kolmogorov observed in [8] that any smooth reparametrization of a minimal translation flow on $\mathbb{T}^2$ with Diophantine slope was conjugate to the original translation flow, hence has a discrete spectrum with two independent frequencies. He guessed that the reparametrized flows may exhibit exotic behaviors in the case of Liouville slopes, which was later proved by Shklover [10] who gave examples of weak mixing real analytic reparametrizations in that case. Katok [5] and Kochergin [6] showed the absence of mixing for non-singular conservative $C^1$ flows on the 2-torus. The absence of mixing is based on the Denjoy-Koksma cancellation property (DKP) for Birkhoff sums above an irrational rotation.

For reparametrizations of minimal translation flows on higher dimensional tori the situation is quite different. Yoccoz showed in [11] that the Denjoy-Koksma cancellation property has no counterpart in higher dimensions.

Using the construction of Yoccoz of counterexamples to the Denjoy- Koksma property, the second author constructed in [1] mixing analytic reparametrizations for a class of minimal translation flows on $\mathbb{T}^3$. These flows can be viewed as special flows above a minimal translations $R_\beta$ of the two torus with $\beta \in \mathbb{T}^2$ as in [11], and the mixing mechanism comes from the uniform stretch of the Birkhoff sums of an adequately chosen ceiling function $\varphi$ above $R_\beta$ (see Figure Section 2 and Section 2 for the definition of the uniform stretch measurement $S_J^t$).

Since the mixing mechanism of [1] will be the same one at play in the examples we construct in this work, we explain its heuristics here : The starting point is the disposition of the best approximations of $\beta_1$ and $\beta_2$ as in [11] such that the denominators, $q_n^{(1)}$ and $q_n^{(2)}$ of the convergents of $\beta_1$ and $\beta_2$ are alternated and such that each term in the increasing sequence $\ldots, q_n^{(1)}, q_n^{(2)}, q_{n+1}^{(1)}, q_{n+1}^{(2)}, \ldots$ is exponentially larger than the precedent one. Then, consider a strictly positive function $\varphi$ that is a sum of two functions $\varphi_1(x_1) + \varphi_2(x_2)$ such that the ergodic sums $\varphi_m$ of the function $\varphi$, for any $m$ sufficiently large, will be always stretching (i.e. have big derivatives at most points), in one or in the other of the two directions, $x_1$ or $x_2$, depending on whether $m$ is far from $\{q_n^{(1)}\}$ or far from $\{q_n^{(2)}\}$ and this stretch will increase when $m$ goes to infinity.

As a consequence, the special flow above $R_\beta$ with special function $\varphi$ will be mixing for all $t \to \infty$, because the image of a small typical interval from the base $\mathbb{T}^2$ taken in the $x_1$ or the $x_2$-direction depending on $t$, will consist of a lot of almost vertical curves whose projection on the base lies along a piece of a trajectory under the translation $R_\beta$.

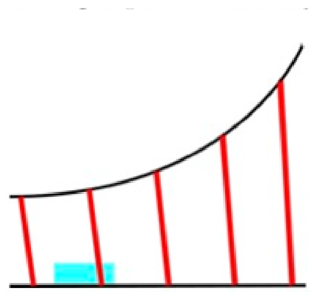

FIGURE 1. Mixing mechanism for special flows: the image of a rectangle is a union of long narrow strips which fill densely the phase space.

**Infinite Lebesgue spectrum for time changed translation flows on $\mathbb{T}^2$ with a rest point, and for other parabolic flows.** Uniform stretch is also responsible for mixing in the conservative surface flows with one degenerate singularity studied by Kochergin in the 1970s [7]. Kochergin flows are special flows under an integrable ceiling function with at least one power singularity (see Figures 2 and 3 and the precise definition of special flows in Section 2). The uniform stretch of the Birkhoff sums in this context comes from the shear between different orbits as they go near the singularity.

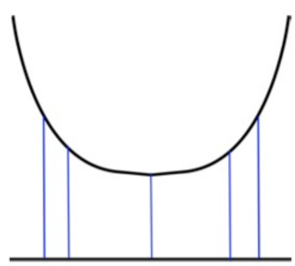

FIGURE 2. Representation of a 2-torus flow with one degenerate saddle as a special flow under a ceiling function with a power-like singularity.

The paper [3] proved ILS for Kochergin flows. To start with, it established square summable decay of correlations for observables that are smooth coboundaries above the flow. This suffices to conclude that the maximal spectral type is absolutely continuous with respect to the Lebesgue measure. The use of smooth coboundaries to get faster decay estimates than for general smooth observables or characteristic functions was inspired from [4] that established Lebesgue maximal spectral type for time changes of horocyclic flows. The crucial point in the use of smooth coboundaries is that it allows to translate all the uniform stretch in speed of mixing.

To establish ILS for Kochergin flows, as well as for time changes of horocyclic flows, the paper [3] introduced a criterion based on the decay of correlations that allows to deduce the ILS property. The criterion exploits the speed of equi-distribution of small sets transversal to the flow direction to build observables that have almost orthogonal cyclic spaces and that have a spectral type close to being Lebesgue. An abstract statement then allows to conclude the ILS property.

The delicate point with Kochergin flows, in comparison with time changes of horocyclic flows for example, is that the shear between orbits is not uniform. Equivalently, the stretch of the Birkhoff sums of the ceiling function can vanish or be very weak on parts of the phase space in a way that is also not uniform in time. The key factor is how close to the singularity different orbits get to before time $t$.

**Infinite Lebesgue spectrum for reparametrized translation flows.** In the mixing reparametrizations of translation flows of [1], the same phenomenon of non uniform shear in space and time appears because the DKP still appears for some special times on part of the phase space. A precise computation shows that, at time $t$, given a direction of uniform stretch, the set with uniform stretch weaker than $t^{1-2\beta}$ is of measure that can be compared to $t^{-\beta+\epsilon}$, for any $\epsilon > 0$ (see Corollary 5.4). This means that for times with only one direction of stretch as it happens for the special flows above $\mathbb{T}^2$, we will get a uniform stretch larger than $t^{1/2}$ away from a set of measure $t^{-1/4+\epsilon}$. The size of the bad set in this case is too large since we seek a square summable decay of correlations. However, if for each $t$ there are 3 independent directions of stretch, the stretch will be stronger than $t^{1/2+\epsilon}$ on a set of measure $t^{-3/4+3\epsilon} = o(t^{-1/2-\epsilon})$.

A simple observation concerning the construction of the frequency vectors as in [11, 1] allows to construct frequencies $\alpha \in \mathbb{R}^4$ and ceiling functions $\varphi$ such that for each time $t$ there are three directions of uniform stretch and gives examples of special flows above $\mathbb{T}^4$ translations with ILS. This corresponds to reparametrizations of minimal flows on $\mathbb{T}^5$. Our method, that insures that for each $t$ the decay is less than $t^{-1/2-\epsilon}$, does not allow to treat the case of $\mathbb{T}^3$ and $\mathbb{T}^4$.

In principle one could apply the strategy adopted in [3] to the case of the mixing reparametrizations on $\mathbb{T}^3$ or $\mathbb{T}^4$. That is, accept the existence of a sequence of special times $(t_n)$ (such as the multiples of the denominators of the convergents) at which the small measure sets that are bad (with no strong uniform stretch) lead to a decay slower than $t_n^{-1/2}$, but still recover square summable correlations due to the fact that for most of the times that are in a medium scale neighborhood of the times $t_n$, there is some small power decay of correlations on the bad set itself. This is what was done for Kochergin flows in [3] because the decay was slower than $t^{-1/2}$ for some special times. We believe the same can be done for reparametrized mixing flows on $\mathbb{T}^4$ and possibly on $\mathbb{T}^3$, but the proof would then be much more technical than the one given here for reparametrizations on $\mathbb{T}^5$.

## 2. Notations and definitions.

- *Special flows above translations of the torus.* Let $R_\alpha : \mathbb{T}^d \to \mathbb{T}^d$, $R_\alpha(\theta) = \theta + \alpha \bmod 1$, where $\alpha \in [0,1]^d$ is a vector such that $1, \alpha_1, \ldots, \alpha_d$ are independent over $\mathbb{Z}$. Let $\varphi \in L^1(\mathbb{T}^d, \lambda_{\mathbb{T}^d})$ be a strictly positive function. We recall that the special flow $T^t := T^t_{\alpha,\varphi}$ constructed above $R_\alpha$ and under $\varphi$ is the flow defined almost everywhere by

$$\begin{aligned} \mathbb{T}^d \times \mathbb{R}/\sim \quad &\to \quad \mathbb{T}^d \times \mathbb{R}/\sim \\ (\theta, s) \quad &\to \quad (\theta, s+t), \end{aligned}$$

where $\sim$ is the identification, defined on $\mathbb{T}^d \times \mathbb{R}$,

$$(\theta, s + \varphi(\theta)) \sim (R_\alpha(\theta), s)\,.$$

Equivalently (see Figure 3), this special flow is defined for $s \geq 0$ and for all times $t \in \mathbb{R}$ such that $t + s \geq 0$ (with a similar definition for times $t \in \mathbb{R}$ such that $t + s < 0$) by

$$T^t(\theta, s) = (\theta + N(\theta, s, t)\alpha, t + s - S_{N(\theta,s,t)}\varphi(\theta)), \tag{1}$$

where $N(\theta, s, t)$ is the unique integer such that

$$0 \leq t + s - S_{N(\theta,s,t)}\varphi(\theta) \leq \varphi(\theta + N(\theta, s, t)\alpha), \tag{2}$$

and

$$S_n\varphi(\theta) = \begin{cases} \varphi(\theta) + \ldots + \varphi(R_\alpha^{n-1}\theta) & \text{if} \quad n > 0 \\ 0 & \text{if} \quad n = 0 \\ -(\varphi(R_\alpha^n\theta) + \ldots + \varphi(R_\alpha^{-1}\theta)) & \text{if} \quad n < 0. \end{cases}$$

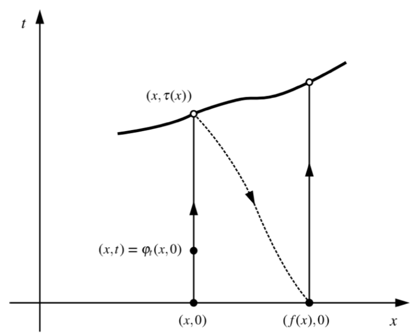


FIGURE 3. The orbit of a point by the special flow above a transformation $f$ and under a ceiling function $\tau$.

- We recall the notations $M = \{(x, s) \in \mathbb{T}^4 \times \mathbb{R} \ : \ 0 \leq s < \varphi(x)\}$ for the configuration space of the flow $T^t_{\alpha,\varphi}$ and $\mu$ for the measure equal to the restriction to $M$ of the product of the Haar measures $\lambda_4 := \lambda_{\mathbb{T}^4}$ on the torus $\mathbb{T}^4$ and $\lambda := \lambda_{\mathbb{R}}$ on the real line $\mathbb{R}$.

  For a given $\zeta > 0$, let us denote

$$M_\zeta := \{(x, s) \in M \ : \ 0 \leq s \leq \varphi(x) - \zeta\}. \tag{3}$$

- For $j \in \{0, \ldots, 4\}$, we denote by $I^j$ intervals in the $j$-direction : $I^j \subset M$ is of the form $\{z = (x_1, x_2, x_3, x_4, s) \in M : x_j \in [a, b]\}$, for some $0 \leq a < b \leq 1$. We also call such intervals $j$-intervals.
- For a point $z \in M$, we denote by $\bar{z}$ its projection on the base $\mathbb{T}^4$. We denote by $\pi_j(z)$ the projection of $z$ on the $j^{\text{th}}$-coordinate of $\mathbb{T}^4$.
- On a $j$-interval $I^j$, we denote by $\lambda$ the Lebesgue measure.
- Given an interval $I \subset M$ (in any direction) and $t \in \mathbb{R}$, we define

$$r_I^t = \inf_{z \in I} |\varphi'_{N(z,t)}(\bar{z})| \tag{4}$$

$$S_I^t = \inf_{z \in I} \frac{(\varphi'_{N(z,t)}(\bar{z}))^2}{\varphi''_{N(z,t)}(\bar{z})}. \tag{5}$$

The quantity $S_I^t$ measures the *uniform stretch* of the Birkhoff sums above the interval $I$. Indeed, when $S_I^t$ is large, this corresponds to almost linear expansion of the interval $I \times \{0\} \subset M$ under the flow at time $t$.

## 3. The construction and precise statements.

### 3.1. The choice of the frequency vector.

Following [11] (see also [1] in the context of special flows), let $Y \subset [0,1]^4$ be the set of vectors $\alpha := (\alpha_1, ..., \alpha_4)$ whose sequences of denominators of best approximations $q_n^{(1)}$,..., $q_n^{(4)}$ satisfy the following for some $n_0 \in \mathbb{N}$ and for all $n \geq n_0$ :

$$q_n^{(2)} \geq e^{nq_n^{(1)}}, \quad q_n^{(3)} \geq e^{nq_n^{(2)}}, \quad q_n^{(4)} \geq e^{nq_n^{(3)}}, \quad q_{n+1}^{(1)} \geq e^{nq_n^{(4)}}. \tag{6}$$

Moreover, we ask that all the $q_n^{(j)}$, $j \in \{1, \ldots, 4\}$, are primes for $n$ sufficiently large.

As in [11], we have that

**Lemma 3.1.** *The set $Y$ is a dense uncountable set in $[0,1]^4$.*

*Proof of Lemma 3.1.* First recall that any irrational number $\alpha \in \mathbb{R} - \mathbb{Q}$ can be written as a continued fraction expansion

$$\alpha = [a_0, a_1, ...] = a_0 + 1/(a_1 + 1/(a_2 + ...1/a_{n-1} + 1/a_n)...),$$

where $\{a_j\}_{j\geq 1}$ is a sequence of integers $\geq 1$ and $a_0 = [\alpha]$. Conversely, any infinite sequence $\{a_j\}_{j\geq 1}$ corresponds to a unique number $\alpha$. The convergents $(p_n, q_n) \in \mathbb{Z} \times \mathbb{Z}^*$of $\alpha$ are defined by $a_j$ in the following way

$$\begin{cases} p_n = a_n p_{n-1} + p_{n-2} & for\ n \geq 2, \quad p_0 = a_0,\ p_1 = a_0 a_1 + 1 \\ q_n = a_n q_{n-1} + q_{n-2} & for\ n \geq 2, \quad q_0 = 1,\ q_1 = a_1. \end{cases} \tag{7}$$

For an arbitrary choice $n_0$ and an arbitrary choice of $a_n^{(j)}$, for $j \in \{0, \ldots, 4\}, n \leq n_0$, it is straightforward from (7) to construct inductively $a_{n_0+1}^{(1)}, a_{n_0+1}^{(2)}, a_{n_0+1}^{(3)}, a_{n_0+1}^{(4)}$, $a_{n_0+2}^{(1)}, \ldots$ such that (6) holds. Indeed, suppose all coefficients are chosen up to $a_n^{(4)}$. Then we just have to take $a_{n+1}^{(1)} > e^{nq_n^{(4)}}$, then $a_{n+1}^{(2)} > e^{(n+1)q_{n+1}^{(1)}}$, then $a_{n+1}^{(3)} > e^{(n+1)q_{n+1}^{(2)}}$, then $a_{n+1}^{(4)} > e^{(n+1)q_{n+1}^{(3)}}$. Since each time we have to pick a coefficient we can choose it in an infinite semi-interval of the integers, the set $Y$ is clearly uncountable. The fact that the coefficients can be chosen arbitrarily for $n \leq n_0$ with $n_0$ arbitrarily large, implies the density of the set $Y$.

Finally, to see that the $q_n^{(j)}$ can be taken prime for sufficiently large $n$, we rely as in Proposition 3 of [2] on the fact that $q_n^{(j)}$ is always relatively prime with $q_{n-1}^{(j)}$ and that by Dirichlet principle the arithmetic sequence $a_{n+1}^{(j)} q_n^{(j)} + q_{n-1}^{(j)}$ contains infinitely many primes so that one can choose $a_{n+1}^{(j)}$ so that in addition to the growth condition, we can guarantee that $q_{n+1}^{(j)}$ is prime. □

Define for $j \in \{0, \ldots, 4\}$

$$\Gamma_n^j = [e^{\frac{n}{2} q_n^{(j)}}, \frac{q_{n+1}^{(j)}}{n+1}] \tag{8}$$

The important consequence from our definition of the set of $Y$ is the following

**Lemma 3.2.** *Every $t \geq 0$ belongs to at least three intervals $\Gamma_{n_j}^j$. More precisely one of the following holds*

1. $t \in \Gamma_n^1 \bigcap \Gamma_n^2 \bigcap \Gamma_n^3$
2. $t \in \Gamma_n^2 \bigcap \Gamma_n^3 \bigcap \Gamma_n^4$
3. $t \in \Gamma_n^3 \bigcap \Gamma_n^4 \bigcap \Gamma_{n+1}^1$
4. $t \in \Gamma_n^4 \bigcap \Gamma_{n+1}^1 \bigcap \Gamma_{n+1}^2$

*Proof of Lemma 3.2.* Let $n$ be such that $t \in [e^{\frac{n}{2}q_n^{(1)}}, e^{(n+1)q_{n+1}^{(1)}}]$. If $t \leq q_n^{(2)}/n$, then $t \in \Gamma_{n-1}^j$ for every $j = 2, 3, 4$. If $t \in [q_n^{(2)}/n, q_n^{(3)}/n]$, then $t \in \Gamma_{n-1}^j$, for $j = 3, 4$ and $t \in \Gamma_n^{(1)}$. If $t \in [q_n^{(3)}/n, q_n^{(4)}/n]$, then $t \in \Gamma_{n-1}^4$, $t \in \Gamma_n^{(j)}$, for $j = 1, 2$. □

3.2. **The choice of the ceiling function.** Following [11, 1], let $\varphi$ be the following strictly positive real analytic function on $\mathbb{T}^4$

$$\varphi(x_1, ..., x_4) = 1 + \sum_{j=1}^{4} \sum_{n \geq n_0} \frac{\cos(2\pi q_n^{(j)} x_j)}{e^{q_n^{(j)}}},$$

where $n_0$ is chosen sufficiently large so that $\varphi$ is strictly positive.

3.3. **The special flows.** Now, for $\alpha \in Y$ and the function $\varphi$, we denote by $\{T_{\alpha,\varphi}^t\}$ the special flow above the translation $R_\alpha$ on $\mathbb{T}^4$ and under the ceiling function $\varphi$ (see Section 2 for the definitions). Our main result is the following

**Theorem 3.3.** *For any $\alpha \in Y$, the special flow constructed over the translation $R_\alpha$ on $\mathbb{T}^4$ and under the ceiling function $\varphi$ has countable Lebesgue spectrum.*

From Theorem 3.3 and the correspondence between special flows above translations and reparametrizations of translation flows on the torus, we derive the following result that gives a precise example of a flow as in Theorem 1.1.

**Corollary 3.4.** *For any $\alpha \in Y$, there exists a strictly positive real analytic function $\Phi$ defined on $\mathbb{T}^5$, such that the reparametrization of the irrational flow $(\alpha, 1)$ by $1/\Phi$ has countable Lebesgue spectrum.*

*Proof.* We sketch the proof and refer to [1, Proposition 6] for the details. Define

$$\Phi(x_1, \ldots, x_5) = 1 + \Re \left( \sum_{n=n_0}^{\infty} d_n^{(j)} e^{i2\pi(q_n^{(j)} x_j + l_n^{(j)} x_5)} \right)$$

where we choose $l_n^{(j)}$ to be the closest relative integer to $-q_n^{(j)}\alpha_j$, and

$$d_n^{(j)} = \frac{i2\pi(q_n^{(j)}\alpha_j + l_n^{(j)})}{e^{i2\pi(q_n^{(j)}\alpha_j + l_n^{(j)})} - 1} e^{-q_n^{(j)}}.$$

Then, if $n_0$ is sufficiently large, $\Phi$ is a real analytic strictly positive function on $\mathbb{T}^5$, that satisfies

$$\varphi(x_1, \ldots, x_4) = \int_0^1 \Phi(x_1 + s\alpha_1, \ldots, x_4 + s\alpha_4, s) ds.$$

Hence the reparametrization of the flow of frequency $(\alpha, 1)$ with the function $1/\Phi$ can be viewed as the special flow above $R_\alpha$ on $\mathbb{T}^4$ with the ceiling function $\varphi$. Hence the function $\Phi$ satisfies the conclusion of Corollary 3.4. □

3.4. **Square summable decay for smooth coboundaries.** We recall that $f$ is called a smooth coboundary over the flow $T_{\alpha,\varphi}^t$ if there exists a smooth function $\phi$ such that, for any $a < b$,

$$\int_a^b f(T^t(x_0, t_0)) dt = \int_a^b f(x_0, t_0 + t) dt = \phi(x_0, t_0 + b) - \phi(x_0, t_0 + a).$$

The function $\phi$ is called the *transfer function* of $f$.

**Definition 3.5.** For $\zeta > 0$, we denote by $\mathcal{F}_\zeta$ the set of smooth coboundaries $f$ over $(T^t_{\alpha,\varphi}, M)$ such that the transfer function $\phi$ is supported inside $M_\zeta$. Let $\mathcal{F} = \cup_{\zeta>0}\mathcal{F}_\zeta$.

**Proposition 3.6** (Proposition 3.1 of [3])**.** *The subspace $\mathcal{F}$ is dense in $L^2_0(M, \mu)$.*

We recall the proof of [3].

*Proof.* Every function $g \in L^2_0(M, \mu)$, which belongs to the orthogonal space $\mathcal{F}^\perp \subset L^2_0(M, \mu)$, is by definition orthogonal to the Lie derivative along the flow of every smooth function with support contained in $M_\zeta$ for some $\zeta > 0$. It follows that for every $t > 0$ the function $g \circ T^t_{\alpha,\varphi} - g$ is orthogonal to all smooth functions with support in $M_\zeta$, for every $\zeta > 0$, hence it is orthogonal to all square-integrable functions, as the space of smooth functions with support contained in $M_\zeta$ for some $\zeta > 0$ is dense in $L^2(M, \mu)$. It follows that for any $t > 0$, the function $g \circ T^t_{\alpha,\varphi} - g$ vanishes, hence $g$ is invariant and constant by the ergodicity of the flow. As $g$ has zero average, it is equal to the zero function. □

Hence to prove that the maximal spectral type of $T^t_{\alpha,\varphi}$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$, it suffices to show the following

**Theorem 3.7.** *For every $\zeta > 0$, for every $f \in \mathcal{F}_\zeta$, there exists a constant $C(\alpha, \varphi, \zeta, f) > 0$ such that, for every $t$, it holds that*

$$\left|\int_M f(T^t_{\alpha,\varphi} z) f(z) d\mu\right| \le C(\alpha, \varphi, \zeta, f) t^{-1/2-\epsilon}.$$

The square summability of the correlations in Theorem 3.7 implies that the spectral measure of $f \in \mathcal{F}_\zeta$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$. The absolute continuity of the maximal spectral type of $T^t_{\alpha,\varphi}$ follows then from Proposition 3.6.

4. **Birkhoff sums and their uniform stretch on good intervals.** First of all we state a standard result on the uniform behavior of $N(x, t)$ due to the unique ergodicity of $R_\alpha$ and continuity of $\varphi$.

**Lemma 4.1.** *For any $t$ sufficiently large, for any $x \in \mathbb{T}^4$*

$$N(x, t) \in [\frac{t}{2}, 2t].$$

*Proof.* By the definition of $N(x, t)$,

$$0 \le t - \varphi_{N(x,t)}(x) \le \varphi(R_\alpha^{N(x,t)} x) \le \|\varphi\|.$$

This shows that $N(x, t) \to \infty$ uniformly as $t \to \infty$. Since by unique ergodicity of $R_\alpha$ we have that $\varphi_{N(x,t)}/N(x, t) \to \int_{\mathbb{T}^4} \varphi(x)dx = 1$, we get the bounds of the lemma. □

For $n \in \mathbb{N}$ and $\theta > 0$ we define for every $j \in \{1, \dots, 4\}$

$$\mathcal{W}(n, \theta, j) = \left\{x \in \mathbb{T}^1 / \{q_n^{(j)} x\} \in [2\theta, \frac{1}{2} - 2\theta] \cup [\frac{1}{2} + 2\theta, 1 - 2\theta]\right\}$$

**Definition 4.2** (Good intervals)**.** We say that a $j$-interval $I^j \subset M$ is $(n, \theta, j)$-good if $\pi_j(I^j) \bigcap \mathcal{W}(n, \theta, j) \neq \emptyset$.

Recall the definition of $\Gamma_n^j = [e^{\frac{n}{2}q_n^{(j)}}, \frac{q_{n+1}^{(j)}}{n+1}]$ given in (8). Let

$$\bar{\Gamma}_n^j := [e^{\frac{n}{2}q_n^{(j)}}/10, 10\frac{q_{n+1}^{(j)}}{n+1}], \quad j = 1, ..., 4.$$

We will need the following estimate on uniform stretch that is similar to the one in [1]. For the convenience of the reader, we sketch the proof below.

**Proposition 4.3.** *For any $m \in \bar{\Gamma}_n^j$, for $\theta > 0$ such that $\theta \geq m^{-1/2}$, if $I^j$ is a $j$-interval that is $(n, \theta, j)$-good, then the following holds*

1. $\inf\limits_{x \in I^j} \mid \frac{\partial \varphi_m}{\partial x} \mid \geq \frac{\theta m q_n^{(j)}}{e^{q_n^{(j)}}}$,
2. $\sup\limits_{x \in I^j} \mid \frac{\partial^2 \varphi_m}{\partial x^2} \mid \leq Cm$.

*Proof.* The bound on the second derivatives is immediate since $\varphi$ is smooth. Observe that for any $m \in \mathbb{N}$, we have

$$\partial_{x_j}\varphi_m(x) = \mathrm{Re}\big(i2\pi q_n^{(j)}\frac{X_j(m,n)}{e^{q_n^{(j)}}}e^{i2\pi q_n^{(j)}x_j}\big) + I + II$$

where

$$X_j(m,l) = \frac{1 - e^{i2\pi m q_l^{(j)}\alpha_j}}{1 - e^{i2\pi q_l^{(j)}\alpha_j}},$$

and

$$I = \mathrm{Re}\big(\sum_{l=1}^{n-1} i2\pi q_l^{(j)}\frac{X_j(m,l)}{e^{q_l^{(j)}}}e^{i2\pi q_l^{(j)}x_j}\big)$$

$$II = \mathrm{Re}\big(\sum_{l=n+1}^{\infty} i2\pi q_l^{(j)}\frac{X_j(m,j)}{e^{q_l^{(j)}}}e^{i2\pi q_l^{(j)}x_1}\big)$$

The proof of the lower bound of 4.3 is based on general upper bounds of $X_j(m,l)$, and on a lower bound of $X_j(m,l)$ for $m \in \bar{\Gamma}_n^j$. The assumption that $I^j$ is an $(n,\theta,j)$-good with $\theta \geq m^{-1/2}$ plays a crucial role to avoid the small values of $\sin(2\pi q_n^{(j)}x_j)$ in the main term of $\partial_{x_j}\varphi_m(x)$ (see the key equation (11)).

**Lemma 4.4** ([1]). *We have the following inequalities:*

(a) *For all $m \in \mathbb{N}$, $\mid X_j(m,l) \mid \leq m$.*
(b) *For all $m \in \mathbb{N}$,$\mid X_j(m,l) \mid \leq q_l^{(j)}$.*
(c) *For any $m \leq \frac{q_{l+1}^{(j)}}{2}$, $\mid X_j(m,l) \mid \geq \frac{2m}{\pi}$, and $|\arg(X_j(m,l))| \leq \frac{\pi(m-1)}{q_{l+1}^{(j)}}$.*

*Proof of Lemma 4.4.* The proof of the first inequality is obvious. The others follow from the fact that

$$\mid X_j(m,l) \mid = \left|\frac{\sin(\pi m q_l^{(j)}\alpha_j)}{\sin(\pi q_l^{(j)}\alpha_j)}\right|, \quad \arg(X_j(m,l)) = \pi(m-1)\|q_l^{(j)}\alpha_j\|,$$

and the fact that by the definition of denominators of best approximations, $q_l^{(j)}$ satisfies

$$\|q_{l-1}^{(j)}\alpha_j\| < \|l\alpha_j\|, \quad \forall l < q_l^{(j)}, l \neq q_{l-1}^{(j)},$$

as well as

$$\frac{1}{q_l^{(j)} + q_{l+1}^{(j)}} \leq \|q_l^{(j)}\alpha_j\| < \frac{1}{q_{l+1}^{(j)}}.$$

□

To prove 1., observe that (b) of Lemma 4.4 implies that for some constant $C > 0$

$$|I| \leq \sum_{l \leq n-1} 2\pi q_l^{(1)} \frac{q_n^{(1)}}{e^{q_l^{(1)}}} \leq C q_n^{(1)}. \tag{9}$$

Next, (a) of Lemma 4.4 implies that for some $C > 0$

$$|II| \leq \sum_{l \geq n+1} 2\pi q_l^{(1)} \frac{m}{e^{q_l^{(1)}}} \leq C. \tag{10}$$

Finally, (c) implies that for $x$ in an $(n, \theta, 1)$-good interval

$$\left|\mathrm{Re}\big(i2\pi q_n^{(1)} \frac{X_1(m,n)}{e^{q_n^{(1)}}} e^{i2\pi q_n^{(1)} x_1}\big)\right| \geq 2\frac{\theta m q_n^{(1)}}{e^{q_n^{(1)}}}. \tag{11}$$

Putting together (9)–(11), and the fact that $\theta m \geq m^{\frac{1}{2}} \geq e^{\frac{n}{4} q_n^{(j)}}$, we get the lower bound 1. of Proposition 4.3. □

As an immediate corollary of Proposition 4.3 and the definitions (4) and (5) and Lemma 4.1, we get that

**Corollary 4.5.** *For any $t \in \Gamma_n^j$, for $\theta > 0$ such that $\theta \geq t^{-1/2}$, if $I^j$ is a $j$-interval that is $(n, \theta, j)$-good, then the following holds*

1. $r_{I^j}^t \geq \theta t^{1-\epsilon/100}$
2. $S_{I^j}^t \geq \theta^2 t^{1-\epsilon/50}$

## 5. **From uniform stretch to absolutely continuous spectrum. Proof of Theorem 3.7.**

### 5.1. **Decay of correlations on good intervals due to uniform stretch.**

In this section, we consider a pair of functions $(f, g)$ satisfying the following criteria: $f$ is a coboundary function $f \in \mathcal{F}_\zeta$ for some $\zeta > 0$ (see Definition 3.5), and $g \in C^1(M)$ with support restricted to $M_\zeta$. We define for every $j \in \{1, \ldots, 4\}$

$$\mathcal{N}_0(f, g) := \|\psi\|_0 \|g\|_0 \quad \text{and} \tag{12}$$

$$\mathcal{N}_1(f, g, j) := (\|f\|_0 + \|\psi\|_0)\|g\|_{1,j} + (\|f\|_{1,j} + \|\psi\|_{1,j})\|g\|_0\,, \tag{13}$$

where $\|f\|_0$ denotes the $C^0$ norm of a function $f$ and $\|f\|_{1,j} = \|f\|_0 + \|\partial_{x_j} f\|_0$. We also define $\mathcal{N}_1(f, g) = \max_{j \in \{0, \ldots, 4\}} \mathcal{N}_1(f, g, j)$.

In [1] the uniform stretch estimates above good intervals as in Corollary 4.5 (with $\theta$ bounded from below) are used to prove equi-distribution of good intervals and then mixing by a Fubini argument. For the proof of countable Lebesgue spectrum, we look for square summable estimates on the decay of correlations, and this is why we use observables that are coboundaries. The following proposition that relates uniform stretching of the Birkhoff sums and the decay of correlations is taken from [3, Proposition 6.1].

**Proposition 5.1.** *For $\zeta > 0$, there exists a constant $C = C(\zeta) > 0$, such that for any $j$-interval $I^j \in M$ with endpoints $(a, s)$ and $(b, s)$ and for any $t \in \Gamma_n^j$, we have*

$$\left| \int_{I^j} f(T^t_{\alpha,\varphi} z) g(z) d\lambda(z) - \Delta(I^j) \right| \leq C \left( \mathcal{N}_0(f, g) \frac{\lambda(I^j)}{S^t_{I^j}} + \mathcal{N}_1(f, g, j) \frac{\lambda(I^j)}{r^t_{I^j}} \right),$$

*where* $\Delta(I^j) = \frac{g(a,s)\psi(T^t(a,s))}{\partial_j \varphi_{N(a,s)}(a)} - \frac{g(b,s)\psi(T^t(b,s))}{\partial_j \varphi_{N(b,s)}(b)}$.

**Remark 5.2.** In the criterion of Theorem 3.7, absolutely continuous spectrum follows from the control of the decay of correlations for coboundary functions. However, when computing the correlations of a pair of different functions, it will be sufficient for the control of the decay to have one of the functions being a coboundary.

*Proof.* The proof of the proposition is the same as that of Proposition 6.1 of [3]. In [3] the intervals were all in the same direction since the base was one dimensional, while here the intervals are taken in turns in one of the 4 base directions. This is why the norms involved in the upper bound of Proposition 5.1 are the ones from (13) that take into account the direction of the intervals on which the correlations are measured.

The main step in the proof of Proposition 5.1, is the following lemma that estimates the correlation of coboundaries based on the stretching of the Birkhoff sums of the roof function.

Let $J := [u, v] \times \{s\} \subset I_j$ be such that $|v - u| \leq t^{-10}$.

**Lemma 5.3.** *For $\zeta > 0$, there exists a constant $C = C(\zeta) > 0$, such that for all $t > 0$ we have*

$$\left| \int_J f(T^t_{\alpha,\varphi}(z)) g(z) dz - \Delta(J) \right| \leq C \mathcal{N}_1(f, g, j) \frac{\lambda(J)}{r^t_{I_j}}, \tag{14}$$

*where $\mathcal{N}_1(f, g, j)$ is as in (13) and $r^t_u := -\partial_j \varphi_{N(u,t)}(u)$ and*

$$\Delta(J) := \frac{1}{r^t_u} \left[ g(v, s) \phi(T^t_{\alpha,\varphi}(v, s)) - g(u, s) \phi(T^t_{\alpha,\varphi}(u, s)) \right].$$

The proof of the lemma relies on a change of variable that transforms the correlations integral along a small interval in the $j$-direction into an integral on a long stretched interval. The fact that one of the observables in the integral is a coboundary plays a fundamental role in the argument. The proof follows very closely the lines of the proof of Lemma 6.2 of [3]. For the convenience of the reader, we include it below.

*Proof.* We use the notation

$$T^{t(u,s)}_{\alpha,\varphi} = (\tilde{u}, \tilde{s}) = (u + N(u, t)\alpha, t + s - \varphi_{N(u,t)}(u)),$$

where $0 \leq \tilde{s} \leq \varphi(u + N(u, t)\alpha)$. We also denote $\tilde{v} = v + N(u, t)\alpha$.

In the remainder of this proof we will denote for simplicity the integer $N(u, t)$ by $N$. We will suppose that $r^t_u = -\partial_j \varphi_N(u) \geq r^t_{I_j} \geq 0$, the case where $r^t_u < 0$ being similar. Let us also denote

$$B^t_I := \sup_{\theta \in I} \partial^2_j \varphi_N(\theta).$$

We will use the notation $X = O(Y)$ if there exists a constant $C > 0$ such that $X \leq CY$.

We have for $\theta \in [0, \lambda(J)]$ that $T^t_{\alpha,\varphi}(u+\theta, s) = (\tilde{u}+\theta, \tilde{s}+\varphi_N(u)-\varphi_N(u+\theta))$. By the intermediate value theorem, since $r^t_{I_j} \ll \lambda(J)^{-1}$, we have

$$\begin{aligned}&\int_J f(T^t_{\alpha,\varphi}(\theta,s))g(\theta,s)d\theta\\ =&\int_0^{\lambda(J)} f(\tilde{u}+\theta, \tilde{s}+\varphi_N(u)-\varphi_N(u+\theta))g(u+\theta,s)d\theta\\ =&g(u,s)\int_0^{\bar{\lambda}(J)} f(\tilde{u}+\theta, \tilde{s}+\varphi_N(u)-\varphi_N(u+\theta))d\theta + O(\|f\|_0\|g\|_{1,j}\frac{\lambda(J)}{r^t_{I_j}}).\end{aligned}$$

Now, since $\varphi_N(u)-\varphi_N(u+\theta) \ll 1$ we also have

$$\begin{aligned}&\int_0^{\lambda(J)} f(\tilde{u}+\theta, \tilde{s}+\varphi_N(u)-\varphi_N(u+\theta))d\theta\\ &\qquad =\int_0^{\lambda(J)} f(\tilde{v}, \tilde{s}+\varphi_N(u)-\varphi_N(u+\theta))d\theta + O(\|f\|_1\frac{\lambda(J)}{r^t_{I_j}}),\end{aligned}$$

and by the definition of $B^t_I$, we have $|\varphi_N(u)-\varphi_N(u+\theta)-r^t_u\theta| \leq B^t_{I_j}\theta^2$. Therefore,

$$\begin{aligned}\int_J f(T^t_{\alpha,\varphi}(\theta,s))g(\theta,s)d\theta &= g(u,s)\int_0^{\lambda(J)} f(\tilde{v}, \tilde{s}+r^t_u\theta)d\theta\\ &\qquad + O(\|f\|_0\|g\|_{1,j}\frac{\lambda(J)}{r^t_I}) + O(\|f\|_{1,j}\|g\|_0\frac{\lambda(J)}{r^t_I}).\end{aligned}$$

For simplicity let us denote $w(f,g) := \|f\|_0\|g\|_{1,j} + \|f\|_{1,j}\|g\|_0$. A change of variable then gives

$$\begin{aligned}\int_J f(T^t_{\alpha,\varphi}(\theta,s))g(\theta,s)d\theta &= \frac{1}{r^t_u}g(u,s)\int_0^{r^t_u\lambda(J)} f(\tilde{v},\tilde{s}+\theta)d\theta + O(w(f,g)\frac{\lambda(J)}{r^t_{I_j}})\\ &= \frac{1}{r^t_u}g(u,s)\left[\phi(\tilde{v},\tilde{s}+r^t_u\lambda(J)) - \phi(\tilde{v},\tilde{s})\right] + O(w(f,g)\frac{\lambda(J)}{r^t_{I_j}})\end{aligned}$$

but $T^t_{\alpha,\varphi}(v,s) = (\tilde{v}, \tilde{s}+\varphi_N(u)-\varphi_N(v)) = (\tilde{v}, \tilde{s}+r^t_u\lambda(J)+\mathcal{E})$ with $\mathcal{E} \leq B^t_{I_j}\lambda(J)^2$, hence

$$\begin{aligned}\int_{\bar{J}} f(T^t_{\alpha,\varphi}(\theta,s))g(\theta,s)d\theta &= \frac{1}{r^t_u}g(u,s)\left[\phi(T^t_{\alpha,\varphi}(v,s)) - \phi(\tilde{v},\tilde{s})\right]\\ &\qquad + O(w(f,g)\frac{\lambda(J)}{r^t_{I_j}} + \|g\|_0\|\phi\|_1\frac{\lambda(J)}{r^t_{I_j}})\\ &= \frac{1}{r^t_u}\left[g(v,s)\phi(T^t_{\alpha,\varphi}(v,s)) - g(u,s)\phi(T^t_{\alpha,\varphi}(u,s))\right]\\ &\qquad + O(\mathcal{N}_1(f,g,j)\frac{\lambda(J)}{r^t_I}),\end{aligned}$$

which is precisely formula (14). □

The rest of the proof of Proposition 5.1 can now follow exactly the same lines as the proof of Proposition 6.1 of [3] and consists of partitioning $I_j$ into finitely many intervals of size less than $t^{-10}$ and applying (14) to every one of them. □

As a direct consequence of Propositions 4.3 and 5.1 we have the following crucial estimate on the decay of correlations of the flow $T^t_{\alpha,\varphi}$ (when, as assumed in all this section, one of the functions is a smooth coboundary):

**Corollary 5.4.** *For $\zeta > 0$, there exists a constant $C = C(\zeta) > 0$, such that if $I^j$ is a $j$-interval that is $(n, t^{-1/4+\epsilon}, j)$-good, and if $t \in \Gamma^j_n$, it holds that*

$$\left|\int_{I^j} f(T^t_{\alpha,\varphi} z) g(z) d\lambda(z)\right| \le C\mathcal{N}_1(f, g, j)\lambda(I^j)t^{-1/2-\epsilon}.$$

5.2. **Partial partitions into good intervals.** From Lemma 3.2, every $t \in \mathbb{R}$ belongs to at least three intervals $\Gamma^j_{n_j}$. Fix now $t$ such that $t \in \Gamma^1_n \bigcap \Gamma^2_n \bigcap \Gamma^3_n$, the other cases being treatable in exactly a similar fashion.

**Proposition 5.5.** *For any $\zeta > 0$, if $t$ is sufficiently large, there exists a partial partition of $M$, $\mathcal{G}_t$ such that the atoms of $\mathcal{G}_t$ are $j$-intervals that are $(n, t^{-1/4+\epsilon}, j)$-good, with $j \in \{1, 2, 3\}$ and $\mu(\mathcal{G}^c_t \cap M_\zeta) \le t^{-1/2-\epsilon}$.*

Note that we cannot take $j$ to be the same for all the intervals in $\mathcal{G}_t$.

*Proof.* We first consider the partial partition $\mathcal{P}_1$ of $\mathbb{T}$ into intervals that are the connected components of

$$\begin{aligned}&\mathcal{W}(n, t^{-1/4+\epsilon}, 1)\\ &= \left\{x \in \mathbb{T}^1/\{q^{(1)}_n x\} \in [t^{-1/4+\epsilon}, \frac{1}{2} - t^{-1/4+\epsilon}] \cup [\frac{1}{2} + t^{-1/4+\epsilon}, 1 - t^{-1/4+\epsilon}]\right\}\end{aligned}$$

. Next we consider on $\mathbb{T}^4$ the partial partition $\mathcal{G}^{(1)}$ consisting of 1-intervals of the form $I \times (x_2, x_3, x_4)$ where $I$ ranges over all the intervals of $\mathcal{P}_1$ and where $(x_2, x_3, x_4)$ ranges over all points in $\mathbb{T}^3$. The union of the atoms of $\mathcal{G}^{(1)}$ covers all $\mathbb{T}^4$ except for a set $\mathcal{E}_1$ that is the union of the bands $x_1 \in \Delta^{(1)}_{n,k}$ and $x_1 \in \bar{\Delta}^{(1)}_{n,k}$, where $\Delta^{(1)}_{n,k} = [\frac{k}{q^{(1)}_n} - \frac{t^{-1/4+\epsilon}}{q^{(1)}_n}, \frac{k}{q^{(1)}_n} + \frac{t^{-1/4+\epsilon}}{q^{(1)}_n}]$ and $\bar{\Delta}^{(1)}_{n,k} = \frac{1}{2q^{(1)}_n} + \Delta_{n,k}$, and $k \in \{0, \dots, q^{(1)}_n - 1\}$.

Next, we consider the partial partition $\mathcal{P}_2$ of $\mathbb{T}$ into intervals that are the connected components of $\mathcal{W}(n, t^{-1/4+\epsilon}, 2)$. Next, we take a partial partition $\mathcal{G}^{(2)}$ of the set $\mathcal{E}_1$ into 2-intervals that are of the form $x_1 \times I \times (x_3, x_4)$ where $I$ ranges over all intervals of $\mathcal{P}_2$ and $(x_3, x_4)$ over all points in $\mathbb{T}^2$ and $x_1 \in \Delta$ where $\Delta$ ranges over all bands $\Delta^{(1)}_{n,k}$ and $\bar{\Delta}^{(1)}_{n,k}$, for $k \in \{0, \dots, q^{(1)}_n - 1\}$.

The union of the atoms of $\mathcal{G}^{(1)}$ and $\mathcal{G}^{(2)}$ covers all $\mathbb{T}^4$ except for a set $\mathcal{E}_2$ that is the union of the bands $(x_1, x_2) \in \Delta \times \Delta'$ where $\Delta$ ranges over all the sets $\Delta^{(1)}_{n,k}$ and $\bar{\Delta}^{(1)}_{n,k}$, for $k \in \{0, \dots, q^{(1)}_n - 1\}$ and $\Delta'$ ranges over all the sets $\Delta^{(2)}_{n,k}$ and $\bar{\Delta}^{(2)}_{n,k}$, for $k \in \{0, \dots, q^{(2)}_n - 1\}$.

Finally, we consider the partial partition $\mathcal{P}_3$ of $\mathbb{T}$ into intervals that are the connected components of $\mathcal{W}(n, t^{-1/4+\epsilon}, 3)$. Next, we take a partial partition $\mathcal{G}^{(3)}$ of the set $\mathcal{E}_2$ into 3-intervals that are of the form $(x_1, x_2) \times I \times x_4$ where $I$ ranges over all intervals of $\mathcal{P}_3$ and $x_4$ over all points in $\mathbb{T}$ and $(x_1, x_2)$ over all points in the projection of $\mathcal{E}_2$ on the first and second coordinates of $\mathbb{T}^4$.

We let $\mathcal{G}$ be the partial partition of $\mathbb{T}^4$ consisting of the atoms of $\mathcal{G}^{(1)}$ and $\mathcal{G}^{(2)}$ and $\mathcal{G}^{(3)}$. The union of the atoms of $\mathcal{G}_t$ covers all $\mathbb{T}^4$ except for a set $\mathcal{E}_3$ that is the union of the one dimensional bands $(x_1, x_2, x_3) \in \Delta \times \Delta' \times \Delta''$ where $\Delta$ ranges over all the sets $\Delta^{(1)}_{n,k}$ and $\bar{\Delta}^{(1)}_{n,k}$, for $k \in \{0, \dots, q^{(1)}_n - 1\}$, and $\Delta'$ ranges over all the sets $\Delta^{(2)}_{n,k}$ and $\bar{\Delta}^{(2)}_{n,k}$ for $k \in \{0, \dots, q^{(2)}_n - 1\}$, and $\Delta''$ over all the sets $\Delta^{(3)}_{n,k}$ and $\bar{\Delta}^{(3)}_{n,k}$

for $k \in \{0, \ldots, q_n^{(3)} - 1\}$. Clearly the set $\mathcal{E}_3$ has a Haar measure on $\mathbb{T}^4$ bounded by $4^3 t^{-3/4+3\epsilon} = o(t^{-1/2-\epsilon})$.

To conclude, we want to extend the partial partition $\mathcal{G}$ to the phase space $M$ of the special flow $\{T^t_{\alpha,\varphi}\}$. It is here that we need to use the sets $M_\zeta$. For a fixed $\zeta > 0$, we define for sufficiently large $t$, the partial partition $\mathcal{G}_t$ as follows. For every interval $I \in \mathcal{G}$ we include in $\mathcal{G}_t$ all the intervals of the form $I \times s$ that satisfy $I \times s \cap M_\zeta \neq \emptyset$ and $I \times s \cap M^c_{\zeta/2} = \emptyset$. The latter condition insures that all the intervals we end up including in $\mathcal{G}_t$ are all disjoint since they were disjoint in $\mathcal{G}$. The former condition insures that $\mu(\mathcal{G}^c_t \cap M_\zeta) \leq \lambda_4(\mathcal{G}) = o(t^{-1/2-\epsilon})$. □

5.3. **Square summable decay of correlations. Proof of Theorem 3.7.** Given $f \in \mathcal{F}_\zeta$, the definition of being a coboundary with a transfer function supported in $M_\zeta$, implies that $f$ is supported in $M_\zeta$.

Also, recall that by Proposition 5.5 we have a partial partition $\mathcal{G}_t$ in good intervals of various directions. Since $\mu(\mathcal{G}^c_t \cap M_\zeta) \leq t^{-1/2-\epsilon}$, we can thus apply Corollary 5.4 in the various directions depending on the interval of the partial partition that we consider, we get by Fubini the required decay. More precisely, for every atom of $\mathcal{G}_t$ that is a good interval $I$ in some direction $j$, Corollary 5.4 implies that

$$\left| \int_I f(T^t_{\alpha,\varphi}(z)) f(z) d\lambda(z) \right| \leq C\mathcal{N}_1(f,f)\lambda(I) t^{-1/2-\epsilon}.$$

By Fubini, we get

$$\left| \int_{\mathcal{G}_t} f(T^t_{\alpha,\varphi}(z)) f(z) d\mu(z) \right| \leq C\mathcal{N}_1(f,f) t^{-1/2-\epsilon},$$

which implies, since $g$ vanishes on $M^c_\zeta$ that

$$\left| \int_{\mathcal{G}_t \cup M^c_\zeta} f(T^t_{\alpha,\varphi}(z)) f(z) d\mu(z) \right| \leq C\mathcal{N}_1(f,f) t^{-1/2-\epsilon},$$

which implies, since $\mu(\mathcal{G}^c_t \cap M_\zeta) \leq t^{-1/2-\epsilon}$ that

$$\left| \int_M f(T^t_{\alpha,\varphi}(z)) f(z) d\mu(z) \right| \leq C\mathcal{N}_1(f,f) t^{-1/2-\epsilon},$$

as required. □

## 6. Infinite Lebesgue spectrum.

6.1. **Criterion for infinite Lebesgue spectrum.** In this subsection, we state a version adapted to our context of the criterion for countable Lebesgue spectrum that was proved in [3]. For a multi-interval $J = I_1 \times \ldots \times I_4 \subset \mathbb{T}^4$, let $T_J$ be the maximal real number such that $T^t(J,0) \cap (J,0) = \emptyset$ for every $0 < |t| < T_J$.

For $T \leq T_J$ we define the tower above $J$

$$R^T_J := \bigcup_{t \in (-T,T)} T^t(J,0),$$

and define the flow-box $F^T_J$ above $J$ with range $R^T_J$ as

$$F^T_J(x,t) = T^t(x,0)\,, \quad \text{for all } (x,t) \in J \times (-T,T).$$

The flow-box $F_J := F^{T_J}_J$ will be called a *maximal flow-box* over the *base* $J \subset M$.

Given a flow-box $F^T_J$, we define, for any $\zeta > 0$, the set $S^T_\zeta(J) \subset \mathbb{R}$ as follows

$$S^T_\zeta(J) := \{t \in (-T,T) \ : \ T^t(J) \cap M^c_\zeta = \emptyset\}\,.$$

By definition we have that $S_\zeta^T(J)$ is an open subset (which in general may be empty).

We can now define the functions supported on flow-boxes that we will be working with.

**Definition 6.1.** Given a flow-box $F_J^T$ and constants $C, \zeta > 0$, we define $\mathcal{G}(J, T, C, \zeta)$ to be the class of all functions $g \in C^\infty(M)$, that vanish outside $R_J^T$, while on $R_J^T$ they are defined as

$$g\left(F_J^T(x,t)\right) := \chi_J(x)\psi(t)\,, \qquad \text{if } (x,t) \in J \times (-T, T)\,,$$

with $\psi \in C_0^\infty(\mathbb{R}, \mathbb{R})$, $\psi$ vanishes outside $S_\zeta^T(J)$, and has $C^1$ norm bounded above by $C$, and $\chi_J \in C_0^\infty(J)$ such that $\int_J \chi_J^2 d\lambda_J = 1$, and

$$\|\chi_J\|_0 \leq C\lambda_4(J)^{-1/2}, \quad \|\chi_J\|_{1,j} \leq C\lambda_4(J)^{-1/2}|I_j|^{-1}. \tag{15}$$

The class $\mathcal{F}(J, T, C, \zeta)$ is the subset of $\mathcal{G}(J, T, C, \zeta)$ consisting of smooth coboundaries.

The general criterion for countable Lebesgue spectrum stated in [3] implies in our context the following. We use the notation

$$\langle f \circ T^t, g\rangle = \int_M f \circ T_{\alpha,\varphi}^t(z) g(z) d\mu(z)\,.$$

**Theorem 6.2.** *Assume $\{T_{\alpha,\varphi}^t\}$ has an absolutely continuous maximal spectral type. If there exists a sequence of non empty multi-intervals $J_n$ such that $\lim \lambda(J_n) = 0$ and if for any $T > 0$, $C > 0$ and $\zeta > 0$, for any family $\{(f_n, g_n)\}$ of pair of functions such that $f_n, g_n \in \mathcal{F}(J_n, T, C, \zeta)$, we have*

$$\inf_n \int_{\mathbb{R}\setminus[-T_{J_n}, T_{J_n}]} |\langle f_n \circ T^t, g_n\rangle|^2 dt = 0\,. \tag{16}$$

*Then the flow $\{T_{\alpha,\varphi}^t\}$ has countable Lebesgue spectrum.*

The statement of Theorem 6.2 is almost identical to Theorem 6 of [3], with this important difference that in the class of functions that we consider is more general in that we replaced in (15) the condition

$$\|\chi_J\|_1 \leq C\lambda_4(J)^{-1/2-1}$$

that was used in [3] by a less stringent and more precise one that distinguishes between $C^1$ norms according to the direction in which the derivatives are considered. Our condition is the natural one to control the derivatives of a function supported inside $J$ and with $L^2$ norm bounded away from 0 and $\infty$. It is important for us to have a differentiated control of the derivatives of the observables along different directions because when we show that the decay condition (16) holds for the flow $\{T_{\alpha,\varphi}^t\}$, we will use mixing estimates for intervals in various directions, and these estimates naturally involve derivatives along the direction of the interval as stated in Proposition 5.1.

In [3], the criterion for CILS was stated for general flow-boxes above multi-intervals, but it was used for flows for which the decay of correlations did not distinguish between various directions, a more precise statement such as the one given here was not necessary.

**Remark 6.3.** In [3], $M_\zeta$ consisted of points in $M$ that avoid a neighborhood of the ceiling function, but that also avoid the singularity of the flow. For the latter reason if one wanted to have the range of the flow-box above an interval $J$ to spend most of its time in $M_\zeta$, then it was necessary, in addition to taking $\zeta$ small, to start with an interval $J$ whose first iterates avoid the neighborhood of the singularity. In our case, there is no singularity and we do not need any extra assumption on the multi-interval $J$ besides its measure going to 0.

**Remark 6.4.** Compared to [3], we dropped the unnecessary condition on higher derivatives since only the $C^1$ norms of the observables play a role in the decay of correlations estimates.

The proof of Theorem 6.2 will be given in Section 7. As in [3], the proof will be based on an abstract criterion that is very much the same as in Theorem 5 in [3]. For completeness, we do include a proof of the abstract criterion, that is slightly simpler than the one given in [3]. For the rest of the proof of Theorem 6.2, we follow exactly the same steps as in the proof of Theorem 5 in [3], and we only insist on the differences that are imposed by the differentiated control of the derivatives of the observables along different directions.

6.2. **Verification of the criterion for $\{T^t_{\alpha,\varphi}\}$.** We prove below that the hypotheses of Theorem 6.2 are verified for $\{T^t_{\alpha,\varphi}\}$. First of all, we know from Theorem 3.7 that $\{T^t_{\alpha,\varphi}\}$ has an absolutely continuous maximal spectral type.

We consider the following family of maximal flow boxes. For $j \in \{0, \dots, 4\}$, let $J_{j,n} = [1/(8q_n^{(j)}), 1/(4q_n^{(j)})]$ and take $J_n = J_{1,n} \times \dots \times J_{4,n}$. Note that by Lemma 4.1, and because by the definition of the set $Y$ in §3.1, all the $q_n^{(j)}$ are distinct prime numbers for $n$ large, it holds that $T_{J_n} \geq q_n^{(1)} q_n^{(2)} q_n^{(3)} q_n^{(4)}$.

**Theorem 6.5.** *For any $T > 0$, $C > 0$, and $\zeta > 0$, for any sequence of pair of functions $\{(f_n, g_n)\}$ such that $f_n \in \mathcal{F}(J_n, T, C)$ and $g_n \in \mathcal{G}(J_n, T, C, \zeta)$ we have*

$$\lim_{n\to 0} \int_{\mathbb{R}\setminus[-T_{J_n}, T_{J_n}]} |\langle f_n \circ T^t_{\alpha,\varphi}, g_n \rangle|^2 dt = 0 .$$

*Proof.* Having fixed $T > 0$, we have that $T \in (0, T_{J_n}^{1/2})$ for $n$ sufficiently large. Let $|t| \geq T_{J_n}$. WLOG we can assume $t > 0$ since the argument for $t < 0$ is similar.

We will decompose the interval of integration in two parts: $t \leq q_{n+1}^{(1)}/(n+1)$ and $t > q_{n+1}^{(1)}/(n+1)$.

First, we consider the case of $t \in [T_{J_n}, q_{n+1}^{(1)}/(n+1)] \subset T_n^1$. We want to use the estimate in Proposition 5.1. Observe that the function $g_n$ is supported in $F_{J_n}^T \cap M_\zeta$.

We start by including $F_{J_n}^T \cap M_\zeta$ in a union of intervals in the direction 1. By Lemma 4.1, there exists a set $A_n$ that is a disjoint union of sets of the form $(R_\alpha^k J_n, s) \subset M, |k| \leq 10T$ such that

$$F_{J_n}^T \cap M_\zeta \subset A_n, \quad \mu(A_n) \leq 2\mu(F_{J_n}^T). \tag{17}$$

Because $J_{1,n} = [1/(8q_n^{(1)}), 1/(4q_n^{(1)})]$, and since $|k| \leq 10T$ while $T$ is independent of $n$, we see that $A_n$ is a union of disjoint 1-interval $I$ that are $(n, 1/50, 1)$-good.

Since $g_n$ vanishes on $A_n^c$, it is enough to prove bounds on

$$\int_{A_n} f_n \circ T^t_{\alpha,\varphi}(z) g_n(z) d\mu(z) .$$

Consider any 1-interval $I$ in the decomposition of $A_n$. Observe that since $f_n, g_n \in \mathcal{G}(J_n, T, C)$ (see Definition 6.1) then

$$\mathcal{N}_0(f_n, g_n) \leq C|J_n|^{-1}, \quad \mathcal{N}_1(f, g, 1) \leq C q_n^{(1)} |J_n|^{-1} \leq t^{\epsilon/100} \lambda_4(J)^{-1}.$$

Since $f_n \in \mathcal{F}(J_n, T, C)$, Propositions 4.3 and 5.1 and the fact that the 1-interval $I$ is $(n, 1/50, 1)$-good, imply

$$\left|\int_I f(T^t_{\alpha,\varphi} z) g(z) d\lambda(z)\right| \leq C\{\mathcal{N}_0(f,g))\frac{\lambda(I)}{S^t_I} + \mathcal{N}_1(f,g,1)\frac{\lambda(I)}{r^t_I}\}, \\ \leq Ct^{-1+2\epsilon}|J_n|^{-1}\lambda(I).$$

Integrating over $I$, we get by Fubini

$$\left|\int_M f(T^t_{\alpha,\varphi} z) g(z) d\mu(z)\right| = \left|\int_{A_n} f(T^t_{\alpha,\varphi} z) g(z) d\mu(z)\right| \leq Ct^{-1+2\epsilon}|J_n|^{-1}\mu(A_n) \\ \leq Ct^{-1+2\epsilon}. \tag{18}$$

For $t > q^{(1)}_{n+1}/(n+1)$, we use Corollary 3.7 and the fact that $\mathcal{N}_1(f_n, g_n) \leq (q^{(1)}_{n+1})^{\epsilon/100}$ to see that

$$\left|\int_M f(T^t_{\alpha,\varphi} z) g(z) d\mu\right| \leq t^{-1/2-\epsilon/2}. \tag{19}$$

In conclusion, we have proved that for $|t| \geq T_{J_n}$

$$\left|\int_M f(T^t_{\alpha,\varphi} z) g(z) d\mu\right| \leq Ct^{-1/2-\epsilon/2}.$$

Squaring and integrating and using $T_{J_n} \to \infty$ as $n \to \infty$ we get the desired decay. □

6.3. **Proof of Theorem 3.3.** The proof follows directly from Theorems 3.7, 6.2, and 6.5. □

## 7. The decay of correlations criterion for infinite Lebesgue spectrum.

7.1. **The general abstract criterion.** Our criterion for countable Lebesgue spectrum of smooth flows is based on the following abstract criterion that gives lower bounds on the multiplicity of a strongly continuous one-parameter unitary group $\{\Phi_t\}_{t\in\mathbb{R}}$ on a separable Hilbert space $H$ with absolutely continuous spectrum. The criterion is essentially the same as in Theorem 5 in [3]. The proof that we give of this theorem is slightly simpler than the one in [3].

Before we state the criterion, recall that the spectral theorem asserts that for every $f \in H$, there exists a positive spectral measure $\nu_f$ on $\mathbb{R}$ such that $\langle f \circ \Phi_t, f\rangle = \int_{\mathbb{R}} e(t\theta) d\nu_f(\theta)$, where $e(\cdot) = e^{i2\pi \cdot}$. Since we assume that $\{\Phi_t\}$ has absolutely continuous spectrum, we have that $\nu_f$ is given by a positive density function with respect to the Lebesgue measure.

Given a measurable set $C \subset \mathbb{R}$ and an $L^2$ function $G$ on $\mathbb{R}$, we denote by $\|G\|_{L^2(C)}$ the $L^2$ norm of the restriction of $G$ on $C$. Unless specified, $L^2$ norms will be considered with respect to the Lebesgue measure $\lambda$ on $\mathbb{R}$.

**Theorem 7.1.** *For a fixed $n \in \mathbb{N}$, let us assume that for every bounded set $C \subset \mathbb{R} \setminus \{0\}$ of positive Lebesgue measure there exists $\epsilon_{n,C} > 0$ such that the following holds. For every $\epsilon \in (0, \epsilon_{n,C})$ there exist vectors $f_1, \ldots, f_n \in H$ such that*

$$\|\langle f_i \circ \Phi_t, f_j\rangle\|_{L^2} \leq \delta_{ij} + \epsilon\,, \quad \textit{for all } i,j \in 1,\ldots,n\,; \tag{20}$$

$$\|F_i^2 - 1\|_{L^2(C)} \leq \epsilon, \quad \textit{for all } i \in 1,\ldots,n\,, \tag{21}$$

*where $F_i^2(\cdot)$ is the density of the spectral measure associated to $f_i$.*

*Then the spectral type of $\{\Phi_t\}_{t\in\mathbb{R}}$ is Lebesgue with multiplicity at least $n$.*

*Proof.* Let $\bigoplus_{k\in\mathbb{N}} H_k$ denote the orthogonal decomposition of $H$ into cyclic subspaces of $\Phi_t$ such that for all $i \in \mathbb{N}$, we have $H_k \simeq L^2(\mathbb{R}, \mu_k)$. We can assume that the sequence $(\mu_k)$ is given by $\mu_k = \phi_k(\theta)\frac{1}{1+\theta^2}d\theta$ where $\phi_k$ are the characteristic functions of nested measurable sets $C_k$ on $\mathbb{R}$ since the maximal spectral type is absolutely continuous with respect to the Lebesgue measure.

Let us assume by contradiction that the spectrum is not Lebesgue with multiplicity at least $n$. Then, there exists a compact set $\widehat{C}$ of positive Lebesgue measure such that $\phi_k = 0$ on $\widehat{C}$ for every $k \geq n$.

Take $C = \widehat{C}$, and $f_1, .., f_n \in H$ be as in Theorem 7.1 with $\epsilon \ll 1$ to be specified later.

Now for any $U \in L^\infty(\mathbb{R}, \lambda)$ such that $\|U\|_\infty \leq 1$, take $h = h_{C,U} \in L^2(\mathbb{R})$ to be the inverse Fourier transform of $\chi_C U$:

$$\chi_C(\theta)U(\theta) = \int_{\mathbb{R}} h(s)e(\theta s)ds.$$

We have that $\|h\|_{L^2} \leq \lambda(C)$.

For every $i \in \{1, \dots, n\}$, let $f_i^1, f_i^2, \dots$ denote the successive orthogonal projections of $f_i$ on the spectral decomposition $\bigoplus L^2(\mathbb{R}, \mu_k)$. By the definition of the spectral isomorphism the projections of $f_i \circ \Phi_s$ are given by $e(s\cdot)f_i^1(\cdot), e(s\cdot)f_i^2(\cdot), \dots$

For every $i \in \{1, \dots, n\}$, introduce the function $\nu_i \in H$:

$$\nu_i = \int_{\mathbb{R}} h(s) f_i \circ \Phi_s ds.$$

We apply assumption (20) to $(f_i, f_j)$ for any $(i,j) \in \{1, \dots, n\}^2$ with $i \neq j$. By the Cauchy Schwarz inequality, we obtain

$$\begin{aligned} < \nu_i, f_j > &= \int_{\mathbb{R}} < h(s) f_i \circ \Phi_s, f_j > ds \\ &= \int_{\mathbb{R}} h(s) < f_i \circ \Phi_s, f_j > ds \\ &\leq \|h\|_{L^2} \| < f_i \circ \Phi_s, f_j > \|_{L^2} \\ &\leq \lambda(C)\epsilon. \end{aligned} \tag{22}$$

We compute the same quantity using the spectral identification

$$\begin{aligned} |< \nu_i, f_j >| &= |\int_{\mathbb{R}} < h(s) f_i \circ \Phi_s, f_j > ds \,| \\ &= |\int_{\mathbb{R}} \sum_{k=1}^{\infty} \chi_C(\theta)U(\theta)f_i^k(\theta)f_j^k(\theta)\phi_k(\theta)\frac{1}{1+\theta^2}d\theta \,| \\ &= |\int_C U(\theta)(\sum_{k=1}^{n-1} f_i^k(\theta)f_j^k(\theta))\phi_k(\theta)\frac{1}{1+\theta^2}d\theta \,|, \end{aligned} \tag{23}$$

where the last equality is justified by the fact that $\mu_k(C) = 0$.

By Chebyshev's inequality we get from (22) and (23) that for $\epsilon > 0$ sufficiently small

$$\lambda\left(\left\{\theta \in C : \exists i, j \in \{1, \dots, n\} \text{ with } i \neq j, \ |\sum_{k=1}^{n-1} f_i^k(\theta)f_j^k(\theta)| > \epsilon^{1/10}\right\}\right) < \epsilon^{0.8}. \tag{24}$$

We now observe that

$$F_i^2(\theta) = \sum_{k=1}^{\infty} |f_i^k(\theta)|^2 \phi_k(\theta) \frac{1}{1+\theta^2}.$$

Hence, the second assumption (21) of the theorem implies that

$$\| \sum_{k=1}^{n-1} |f_i^k(\theta)|^2 \phi_k(\theta) \frac{1}{1+\theta^2} - 1 \|_{L^2} \leq \epsilon.$$

This implies that

$$\lambda \left( \left\{ \theta \in C : \exists i \in \{1, \ldots, n\}, \ \sum_{k=1}^{n-1} |f_i^k(\theta)|^2 \notin [1/2, 2] \right\} \right) < \epsilon^{0.9}. \tag{25}$$

Finally, fix some $\epsilon < 10^{-10n}$ and observe that conditions (24) and (25) imply that there exists $\theta_0$ such that the vectors

$$v_i = (f_1^1(\theta_0), .., f_i^{n-1}(\theta_0)), \quad i \in \{1, \ldots, n\},$$

satisfy for all $(i, j) \in \{1, \ldots, n\}^2$, $j \neq i$

$$|(v_i|v_j)| < \epsilon_0^{0.1}, \quad (v_i|v_i) \in [\frac{1}{2}, 2],$$

where $(\cdot|\cdot)$ denotes the Euclidean scalar product on $\mathbb{C}^{n-1}$.

Since this is impossible, we conclude that the spectrum is Lebesgue with multiplicity at least $n$. □

7.2. **The proof of Theorem 6.2.** We want to use the assumption of Theorem 6.2 to check the validity of the criterion of Theorem 7.1. In this section, the proofs will be exactly the same as the ones given in Section 7 of [3], except for (27) where the control on the derivatives of the compactly supported functions that we use in the construction are given in the various directions of the base $J$. Notice that the control that we require in (27) is exactly the one that allows to guarantee the fast decay of correlations that we obtained in Theorem 6.5.

First of all, we recall the following corollary of Theorem 7.1 that was proved in [3].

**Corollary 7.2.** *Let us assume that for every $n \in \mathbb{N}$, for any even functions $\omega_1, \ldots, \omega_n \in \mathcal{S}(\mathbb{R})$ (the Schwartz space), and for any any $\epsilon > 0$, there exist $f_1, \ldots, f_n \in H$ such that, for all $i, j \in \{1, \ldots, n\}$, we have*

$$\|\langle f_i \circ \phi_t, f_j \rangle - \frac{d^2}{dt^2} \omega_i * \omega_i(t) \delta_{ij} \|_{L^2(\mathbb{R})} \leq \epsilon\,.$$

*Then the spectral type of the strongly continuous one-parameter unitary group $\phi_{\mathbb{R}}$ is Lebesgue with countable multiplicity.*

The proof of the fact that Theorem 7.1 implies Corollary 7.2 is exactly the same as that of Corollary 4 from Theorem 5 in [3]. For the convenience of the reader, we give a sketch of it.

*Sketch of the proof of Corollary 7.2 .* Let $C$ be a given compact subset of $\mathbb{R} \setminus \{0\}$ of positive Lebesgue measure. By the Lebesgue density theorem, it is not restrictive to assume that there exists an interval $[a, b]$ with $0 < a < b$ such that $\mathrm{Leb}(C \cap [a, b]) \geq (b-a)/2$. The case when $C \cap \mathbb{R}^+ = \emptyset$ is similar. Let $\chi_C : \mathbb{R} \to [0, 1]$ denote any smooth odd function with compact support in $[-2b, -a/2] \cup [a/2, 2b]$ such that

$\chi_C^2 \equiv 1$ on $[-b,-a]\cup[a,b]$. For all $i \in \{1,\dots,n+1\}$ let $\omega_i$ be the function determined by the identity

$$\mathcal{F}(\omega_i)(\tau) = \frac{\sqrt{-1}}{\tau}\frac{\chi_C(\tau)}{\|\chi_C^2\|_{L^2(\mathbb{R})}^{1/2}}\,, \qquad \text{for all } i \in \{1,\dots,n+1\}\,.$$

The functions $\omega_i$ are all even, and we can take $f_1,\dots,f_{n+1}$ as in the statement of the corollary. We then check that the functions $f_1,\dots,f_{n+1}$ satisfy the conditions of Theorem 7.1.

By Theorem 7.1 it follows that the strongly continuous one-parameter unitary group $\{\phi_{\mathbb{R}}\}$ has Lebesgue spectrum with multiplicity at least $n$. Since $n \in \mathbb{N}$ is arbitrary, it has Lebesgue spectrum with countable multiplicity. □

We will derive Theorem 6.2 from Corollary 7.2. Since we only control the decay of correlations for functions in the classes $\mathcal{F}(J,T,C,\zeta)$ and $\mathcal{G}(J,T,C,\zeta)$, we need a simple approximation lemma to approximate the target functions $\omega_1,\dots,\omega_n$ by functions supported inside sets of the type $S_\zeta^T(J)$, that we take as is from [3].

**Lemma 7.3** ([3, Lemma 7.4])**.** *Let $\Phi = \{F_J\}$ be a family of maximal flow-boxes. For every $\epsilon > 0$, and even function $\omega \in \mathcal{S}(\mathbb{R})$, there exist $\tau := \tau(\epsilon,\omega) > 0$ with the following property. For every $T \geq \tau$, there exist constants $C := C(\epsilon,\omega,T) > 0$ and $\zeta := \zeta(\epsilon,\omega,T) > 0$ such that, for any $F_J \in \Phi$ with $T_J > T$, there exists an even function $\psi \in C_0^\infty(-T,T)$ satisfying*

*(a)* $\frac{d\psi}{dt} \in C_0^\infty(S_\zeta^T(J) \cap (-S_\zeta^T(J)))$;
*(b)* *the $C^2$ norm is bounded above by $C$;*
*(c)*

$$\|\frac{d^2}{dt^2}(\psi * \psi) - \frac{d^2}{dt^2}(\omega * \omega)\|_{L^2(\mathbb{R})} < \epsilon\,.$$

*Proof of Theorem 6.2.* Let us fix $\epsilon > 0$ and any given number $n \in \mathbb{N}\setminus\{0\}$ of even Schwartz functions $\omega_1,\dots,\omega_n \in \mathcal{S}(\mathbb{R})$. Let $\Phi = \{F_J\}$ be a family of maximal flow-boxes.

By Lemma 7.3 there exists $\tau > 0$ and, for all $T > \tau$, there exists $\zeta > 0$ (small) such that, for every multi-interval $J = I_1 \times \dots \times I_4 \subset \mathbb{T}^4$, with $T_J > \tau$, there exist even functions $\psi_i \in C_0^\infty((-T,T))$, $i = 1,\dots,n$, with the property that $\frac{d\psi_i}{dt} \in C_0^\infty(S_\zeta^T(J))$ and with $C^2$ norm uniformly bounded above by a constant $C' := C'(\epsilon,\omega_1,\dots,\omega_n,T) > 0$, such that

$$\|\frac{d^2}{dt^2}(\psi_i * \psi_i) - \frac{d^2}{dt^2}(\omega_i * \omega_i)\|_{L^2(\mathbb{R})} < \epsilon/2\,. \tag{26}$$

We will use the following

**Claim.** *There exists $\chi_J^{(1)},\dots,\chi_J^{(n)} \in C_0^\infty(J)$ such that*

$$\int_J \chi_J^{(i)}\chi_J^{(j)}d\lambda = \delta_{ij}\,, \quad \textit{for all } i,j \in \{1,\dots,n\}\,,$$

*and such that for $j \in \{1,\dots,4\}$*

$$\|\chi_J^{(i)}\|_0 \leq C''\lambda_4(J)^{-1/2}, \quad \|\chi_J^{(i)}\|_{1,j} \leq C''\lambda_4(J)^{-1/2}|I_j|^{-1}, \tag{27}$$

*where $C''$ depends only on $n$.*

*Proof.* Take $\chi_0^{(1)}, \dots, \chi_0^{(n)} \in C_0^\infty([0,1]^4, \mathbb{R})$ such that

$$\int_{[0,1]^4} \chi_0^{(i)} \chi_0^{(j)} d\lambda = \delta_{ij}, \quad \text{for all } i, j \in \{1, \dots, n\}.$$

WLOG, suppose now $J = [0, \lambda_1] \times \ldots \times [0, \lambda_4]$ and let $\chi_J^{(i)} = \lambda_4(J)^{-1/2} \chi_0^{(j)}(\lambda_1^{-1}\cdot, \lambda_2^{-1}\cdot, \lambda_3^{-1}\cdot, \lambda_4^{-1}\cdot)$. These functions satisfy the requirements of the claim. □

Let $C > \max\{C', C''\}$. For every $i \in \{1, \dots, n\}$, let $f_J^{(i)} \in \mathcal{F}(J, T, C, \zeta)$ be the function defined on the range $R_J^T$ of the flow-box map $F_J^T$ as

$$f_J^{(i)} \circ F_J^T(x,t) := \chi_J^{(i)}(x) \frac{d}{dt}\psi_i(t), \quad \text{if } (x,t) \in J \times (-T, T),$$

and defined as $f_J^{(i)} = 0$ on $M \setminus R_J^T$.

We then compute the correlations. Let $T_J/2 > \max\{T, \tau/2\}$. For all $t \in [-T_J, T_J]$ we have (since the functions $\psi_1, \dots, \psi_{n+1}$ are all even)

$$\begin{aligned}\langle f_J^{(i)} \circ T^t, f_J^{(j)} \rangle &= \int_J \int_{-T}^{T} \chi_J^{(i)}(x) \chi_J^{(j)}(x) \frac{d\psi_i}{dt}(\sigma + t) \frac{d\psi_j}{dt}(\sigma) d\sigma dx \\ &= (\frac{d\psi_i}{dt} * \frac{d\psi_j}{dt})(t)\, \delta_{ij} = \frac{d^2}{dt^2}(\psi_i * \psi_j)(t) \delta_{ij}.\end{aligned}$$

By the assumption of Theorem 6.2 that we are proving, if $\lambda_4(J)$ is small enough, for every $i, j \in \{1, \dots, n\}$ we have:

$$\| \langle f_J^{(i)} \circ T^t, f_J^{(j)} \rangle \|_{L^2(\mathbb{R} \setminus [-T_J, T_J])} \le \epsilon/2.$$

Note that, since the functions $\psi_i$ are supported in $[-T, T]$ and $T < T_J/2$, we also have

$$\frac{d^2}{dt^2}(\psi_i * \psi_j)(t) \delta_{ij} = 0, \text{ for } t \in \mathbb{R} \setminus [-T_J, T_J]. \tag{28}$$

By putting together formulas (26)–(28), it follows that if $\lambda_4(J)$ is small enough (hence $T_J$ is large enough), the functions $f_J^{(i)}$, with $i \in \{1, \dots, n\}$, satisfy the assumptions of Corollary 7.2:

$$\| \langle f_J^{(i)} \circ T^t, f_J^{(j)} \rangle - \frac{d^2}{dt^2}(\omega_i * \omega_j) \delta_{ij} \|_{L^2(\mathbb{R})} \le \epsilon.$$

It follows then by Corollary 7.2 that, under the hypotheses of Theorem 6.2, the flow $\{T^t\}$ has countable Lebesgue spectrum, hence the argument is completed. □

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