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INDUCING COUNTABLE LEBESGUE SPECTRUM*

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Dedicated to Ya. G. Sinai on the occasion of his 90th birthday

Abstract. We show that any ergodic dynamical system generates a system with pure Lebesgue spectrum of infinite multiplicity.

Key words. dynamical system, spectral theory of dynamical systems, Koopman operator, ergodic theory, induced transformation (first return map), infinite spectral multiplicity

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1. Introduction.

Given a dynamical system (T, X, μ), we define for a measurable set $A \subset X$ with $\mu(A) > 0$, the induced dynamical system (T_A, A, μ_A), with T_A being the first return map to the set A, and, for any measurable set $B \subset A$, $\mu_A(B) = \mu(B)/\mu(A)$.

Recall that, given a dynamical system (T, X, μ), the corresponding Koopman operator U_T acts on the space $L_0^2(X, \mu)$ of complex square integrable zero mean functions as $U_T f = f \circ T$. As each unitary operator acting on a separable Hilbert space, U_T is determined by its maximal spectral type σ_T measure (a spectral measure on the circle $\mathbf{T}$ which dominates all other spectral measures) and by the multiplicity function $M_T : \mathbf{T} \to \{1, 2, \dots\} \cup \{\infty\}$ (defined σ_T-a.e.).

We say (T, X, μ) has a pure Lebesgue spectrum when σ_T is equivalent to the Lebesgue measure on the circle. We say (T, X, μ) has a pure Lebesgue spectrum with infinite multiplicity when, in addition, $M_T(\,\cdot\,) = \infty$ Lebesgue almost surely.

In [1], De La Rue showed that any ergodic dynamical system (T, X, μ) induces an ergodic dynamical system (T_A, A, μ_A) that has a pure Lebesgue spectrum. He asked if it is possible to ensure that the induced system has infinite Lebesgue spectrum. In this paper, we adapt the construction of De La Rue to show that the answer to this question is positive.

THEOREM A. *For any ergodic dynamical system (T, X, μ), there exists an induced system that has a pure Lebesgue spectrum with infinite multiplicity.*

2. Notation, definitions, and preliminaries.

2.1. Weak and strong closeness between measures on the circle.

DEFINITION 1. *For $\alpha > 0$ and $\tau > 0$, a function $\varphi \in C^0(\mathbf{T}, \mathbf{R})$ is said to be (α, τ)-good if $\varphi(\,\cdot\,) \geqslant 0$ and $\varphi(\theta) > \alpha$ for θ outside $(-\tau, \tau)$.*

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DEFINITION 2 (strong closeness of densities). *Suppose* φ *is* (α, τ)*-good. Given* $\varepsilon > 0$*, we say that a function* φ' *is* ε*-strongly close to* φ *and denote this by* $\varphi' \approx_\varepsilon \varphi$ *if*

$$\text{for any } \theta \notin (-\tau, \tau)\colon \frac{1}{1+\varepsilon}\varphi(\theta) \leqslant \varphi'(\theta) \leqslant (1+\varepsilon)\varphi(\theta).$$

DEFINITION 3 (weak closeness of probability measures on the circle). *We equip the set of positive measures on a circle with the topology of weak convergence and denote by* d *the distance that defines it. We denote* $\sigma \sim_\rho \sigma'$ *when* $d(\sigma, \sigma') < \rho$.

For an absolutely continuous measure on a circle of the form $\varphi(\theta)d\theta$, we often abuse notation and denote the measure simply as φ. As a consequence, we often use the notation $\sigma \sim_\rho \varphi$ when the measure σ is ρ-close in the weak topology to the measure $\varphi(\theta)\,d\theta$.

For $f \in L_0^2(X, \mu)$, we denote by $H(f)$ the cyclic space generated by the family $\{f \circ T^k\}_{k \in \mathbf{Z}}$.

LEMMA 1. *Given* $\tau_n \to 0$, $\alpha_n > 0$, $\varepsilon_n < \min(2^{-n}, \alpha_n/2)$, $\rho_n \to 0$, *and a sequence of measures on the circle* σ_n *such that*

$$\sigma_n \sim_{\rho_n} \varphi_n,$$

where φ_n *is a sequence of* (α_n, τ_n)*-good functions such that*

$$\varphi_{n+1} \approx_{\varepsilon_n} \varphi_n,$$

φ_n *converges on* $\mathbf{T}\backslash\{0\}$ *to a strictly positive continuous function* φ_∞*, and the measures* σ_n *converge weakly to the measure* σ *with density* φ_∞.

Proof. From $\sigma_n \sim_{\rho_n} \varphi_n$, it suffices to see that, for any $\tau > 0$, φ_n converges in the strong topology on $[\tau, 1-\tau]$ to a strictly positive continuous function φ_∞. However, the sequence $\varphi_{n|[\tau,1-\tau]}$ is a Cauchy sequence by the facts that $\tau_n \to 0$ and that $\varepsilon_n < \min(2^{-n}, \alpha_n/2)$. Lemma 1 is proved.

LEMMA 2. *Let* (T, A, μ) *be a dynamical system such that there exist a sequence* $\varepsilon_n \to 0$*, a sequence of functions* $\{f_j\}_{j \in \mathbf{N}}$ *in* $L^2(A, \mu)$*, and a sequence of functions* $\{\varphi_j\}_{j \in \mathbf{N}}$ *in* $C^0(\mathbf{T}, \mathbf{R}_+^*)$ *such that, for every* $i \in \mathbf{N}$,

$$\sigma(f_i) = \varphi_i \approx_{\varepsilon_i} 1, \tag{1}$$

and for all $1 \leqslant i < j$ *and for* $\eta \in \{1, i\}$*, there exists* $\varphi_{i,j,\eta} \in C^0(\mathbf{T}, \mathbf{R}_+^*)$ *such that*

$$\sigma(f_i + \eta f_j) = \varphi_{i,j,\eta} \approx_{\varepsilon_i} 2, \tag{2}$$

and then the system (T, A, μ) *has a spectral component that is Lebesgue with infinite multiplicity.*

Proof. Condition (1) implies that the system (T, A, μ) has a spectral component for which the maximal spectral type is Lebesgue. Let $\bigoplus_1^\infty L^2(\mathbf{T}, \mu_k)$ be the spectral decomposition of the latter component, where $\mu_1 = d\theta$, $\mu_1 \gg \mu_2 \gg \cdots$. We can take $\mu_k = \chi_{C_k}(\theta)\,d\theta$, where $\{C_k\}$ is a sequence of nested measurable subsets of the circle.

If the multiplicity of the Lebesgue component is not infinite, there exists K such that $\text{Leb}(C_K) < 1$. We assume that this holds and take the first $K \geqslant 2$ with this property, and we get a contradiction.

Let $\hat{f}_l = f_{N+l}$, $l \in \{1, \dots, K+1\}$ for some $N \gg 1$ to be determined later. The desired result from choosing N large is to have, due to (1) and (2), that the $\hat{f}_l$ are pairwise almost orthogonal while the densities of their spectral measures are almost equal to 1. This shows that by choosing N sufficiently large, we get a contradiction with the assumption that $\mathrm{Leb}(C_K) < 1$.

Note that since the spectral measure of every $\widehat{f}_l$ is absolutely continuous with respect to Lebesgue, every one spectrally belongs to $\bigoplus_1^\infty L^2(\mathbf{T}, \mu_k)$. For every $l \in \{1, \dots, K+1\}$, let $\widehat{f}_l^1, \widehat{f}_l^2, \dots$ denote the successive orthogonal projections of $\widehat{f}_l$ on $\bigoplus L^2(\mathbf{R}, \mu_k)$.

For any $\varepsilon > 0$, if N is chosen sufficiently large, (1) and (2) imply that, for any pair $i \neq j \in \{1, \dots, K+1\}^2$, any $n \in \mathbf{N}$, and $\eta \in \{0, 1, i\}$,

$$\left|\langle \widehat{f}_i + \eta \widehat{f}_j, (\widehat{f}_i + \eta \widehat{f}_j) \circ T^n \rangle\right| < \varepsilon, \tag{3}$$

which gives

$$\left|\langle \widehat{f}_i, \widehat{f}_j \circ T^n \rangle\right| < \varepsilon. \tag{4}$$

From the spectral isomorphism this yields

$$\varepsilon > \left|\langle \widehat{f}_i, \widehat{f}_j \circ T^n \rangle\right| = \left|\sum_{k=1}^{\infty} \widehat{f}_i^k(\theta) \overline{\widehat{f}_j^k}(\theta) e(-n\theta) \phi_k(\theta)\, d\theta\right|. \tag{5}$$

Since ε can be chosen arbitrarily small (after K and C_K are given), this implies that, for more than 99% of $\theta \in C_K^c$, for any pair $i \neq j \in \{1, \dots, K+1\}^2$,

$$\left|\sum_{k=1}^{K} \widehat{f}_i^k(\theta) \overline{\widehat{f}_j^k}(\theta)\right| < \sqrt{\varepsilon}. \tag{6}$$

On the other hand, by (1) and the spectral isomorphism, we have, for any $l \in \{1, \dots, K+1\}$,

$$\sum_{k=1}^{\infty} \left|\widehat{f}_l^k(\theta)\right|^2 \phi_k(\theta) = \varphi_i(\theta) \approx_\varepsilon 1. \tag{7}$$

Hence, for more than 99% of $\theta \in C_K^c$, for any $l \in \{1, \dots, K+1\}$, it holds that

$$\sum_{k=1}^{K} \left|\widehat{f}_l^k(\theta)\right|^2 \in \left[1 - \sqrt{\varepsilon}, 1 + \sqrt{\varepsilon}\,\right]. \tag{8}$$

In conclusion, there exists $\theta \in C_K^c$ for which both (7) and (8) hold. This is a contradiction because $K+1$ vectors in $\mathbf{C}^K$ cannot all have norm almost 1 and be almost orthogonal. Lemma 2 is proved.

2.2. Simple functions. Throughout this paper, we only consider functions f on the space (X, μ) or the induction spaces (A, μ_A) that are simple in the sense that f is constant on the atoms of a finite measurable partition $\mathcal{P}$ of A and that the average of f is 0. We denote this by $f \in \mathcal{S}(A)$. This is useful for ensuring that when inducing the function f on a subset $A' \subset A$ as $f' := f_{|A}$, we can guarantee that $av_{\mu_A}(f') = 0$ provided A' is independent of the partition $\mathcal{P}$ that defines f.

2.3. Criterion for the convergence of a sequence of induced systems to a system with an infinite Lebesgue component. We always assume that we are given an ergodic dynamical system (T, X, μ). We use the following criterion that allows one to construct a measurable set $A \subset X$ such that the system (T_A, A, μ_A) has a Lebesgue component of infinite multiplicity in its spectrum.

PROPOSITION 1. *Assume we are given* $\tau_n \to 0$, $\alpha_n > 0$, $\varepsilon_n < \min(2^{-n}, \alpha_n/2)$, $\rho_n \to 0$, *and a nested sequence of measurable sets* A_n *such that* $\mu(A_{n-1} \setminus A_n) < \varepsilon_n$.

Assume also we are given arrays $\{\varphi_j^{(n)}\}_{j\in[1,n]}$, $\{\varphi_{i,j,\eta}^{(n)}\}_{1\leqslant i<j\leqslant n,\, \eta\in\{1,i\}}$, $n \geqslant 1$, *of functions in* $C^0(\mathbf{T}, \mathbf{R}_+)$ *such that*

$(\mathrm{A1}_n)$ *for all* $j \in [1, n]$, $\varphi_j^{(n)}$ *is* (α_n, τ_n)*-good;*

$(\mathrm{A2}_n)$ *for all* $j \leqslant n-1$, *for all* $1 \leqslant i < j \leqslant n-1$, *and for* $\eta \in \{1, i\}$,

$$\varphi_j^{(n)} \approx_{\varepsilon_n} \varphi_j^{(n-1)}, \quad \varphi_{i,j,\eta}^{(n)} \approx_{\varepsilon_n} \varphi_{i,j,\eta}^{(n-1)}, \quad \varphi_{j,n,\eta}^{(n)} \approx_{\varepsilon_n} \varphi_j^{(n)} + 1, \quad \varphi_n^{(n)} \approx_{\varepsilon_n} 1;$$

and given an array of functions $\{f_j^{(n)}\}_{j\in[1,n]} \in \mathcal{S}(A_n)$, $n \geqslant 1$, *such that* A_n *is orthogonal to* $\{f_j^{(n-1)}\}_{j\in[1,n-1]}$, *and for every* $j \in [1, n-1]$,

$(\mathrm{A3}_n)$ $\|f_j^{(n)} - {f_j^{(n-1)}}_{|A_n}\|_2 \leqslant \varepsilon_n$;

and such that

$(\mathrm{A4}_n)$ *for the induced system* $(T_{|A_n}, A_n, \mu_{A_n})$, *for all* $j \leqslant n$, *for all* $1 \leqslant i < j \leqslant n$, *and for* $\eta \in \{1, i\}$,

$$\sigma(f_j^{(n)}) \sim_{\rho_n} \varphi_j^{(n)}, \quad \sigma(f_i^{(n)} + \eta f_j^{(n)}) \sim_{\rho_n} \varphi_{i,j,\eta}^{(n)};$$

then the limiting system $T_{|A_\infty}$ *has a spectral component that is Lebesgue with infinite multiplicity.*

Proof. From the fact that $\mu(A_{n-1} \setminus A_n) < \varepsilon_n$ and $(\mathrm{A3}_n)$ we see that the system $(T_{|A_\infty}, A_\infty, \mu_{A_\infty})$ is well defined and that, for every i, $f_i^{(n)}$ converges in $L_0^2(A_\infty)$ to a function $f_i^{(\infty)}$.

By $(\mathrm{A1}_n)$, $(\mathrm{A2}_n)$, and $(\mathrm{A4}_n)$ with $\eta = 0$, Lemma 1 implies that, for every $j \in \mathbf{N}$, $\sigma(f_j^{(\infty)}) = \varphi_j^{(\infty)}$ is equivalent to the Lebesgue measure on the circle, and $\varphi_j^{(\infty)} \approx_{\varepsilon_j} 1$.

Finally, $(\mathrm{A4}_n)$ and $(\mathrm{A2}_n)$ with $\eta \in \{1, i\}$ imply that, for $i < j$,

$$\sigma(f_i^{(\infty)} + \eta f_j^{(\infty)}) = \varphi_{i,j,\eta}^{(\infty)} \approx_{\varepsilon_j} \varphi_{i,j,\eta}^{(j)} \approx_{2\varepsilon_j} \varphi_i^{(j)} + 1 \approx_{3\varepsilon_j} \varphi_i^{(\infty)} + 1 \approx_{3\varepsilon_i} 2,$$

which by Lemma 2 implies that $(T_{|A_\infty}, A_\infty, \mu_{A_\infty})$ has a spectral component that is Lebesgue with infinite multiplicity. Proposition 1 is proved.

2.4. De La Rue's strategy for inducing Lebesgue spectrum. In [1], the following strategy is given to induce a Lebesgue spectrum. First of all, one shows how, starting from a simple function f, it is possible to induce T on a set A_∞ to get a function $f^{(\infty)}$ with spectral measure equivalent to Lebesgue measure.

For this purpose, A_∞ is constructed inductively as a limit of nested sets A_n, and $f^{(\infty)}$ is the limit of the sequence $f^{(n)} := {f^{(n-1)}}_{|A_n}$, $f^{(0)} := f$.

Each step of the inductive procedure relies on two mechanisms. First, spread out the spectral measure of $f^{(n)}$ into a measure that is equivalent to Lebesgue using the Meilijson skew products [2] of T and T^2 above the Bernoulli shift on $\{1, 2\}^{\mathbf{Z}}$. Second, induce T on a set $A_{n+1} \subset A_n$ so that the spectral measure of $f^{(n+1)} = {f^{(n)}}_{|A_{n+1}}$ is as close as desired in the weak distance to the spread-out measure (inducing T can imitate Bernoulli convolution). While doing so inductively, it is possible to ensure that the densities of the spread-out measures at each step n are converging in the strong sense of Lemma 1, which guarantees that the spectral measure of $f^{(\infty)}$ for the system T_{A_∞} is equivalent to Lebesgue measure.

Starting from a dense countable family of functions in $\mathcal{S}(X,\mu)$ and performing the regularizing induction simultaneously for all the functions, one thus gets a limit system with maximal spectral type equivalent to Lebesgue (see section 4, where we come back to this density argument since we need it to conclude our proof of Theorem A).

The two main ingredients of [1] just discussed are summarized in the following two propositions that are also crucial in our proof of Theorem A.

For $f \in L^2_0(X)$, we use the notation $\sigma(f_\delta)$ for the measures $\sigma(f)_\delta$ as in Definition 3 of [1]. These are the spread-out measures coming from the Meilijson skew products. For the convenience of the reader, we include the definition.

DEFINITION 4. *Given a positive measure σ on $\mathbf{T}$ such that $\sigma(\{0\}) = 0$, and given $\delta > 0$, we define a positive measure σ_δ on $\mathbf{T}$ by its Fourier coefficients*

$$\widehat{\sigma_\delta}(0) = \sigma(\mathbf{T}),$$

$$\widehat{\sigma_\delta}(p) = \int_{\mathbf{T}} z_\delta(\tau)^p \, d\sigma(\tau) \quad \forall\, p > 0, \qquad \widehat{\sigma_\delta}(p) = \overline{\widehat{\sigma_\delta}(-p)} \quad \forall\, p < 0,$$

where $z_\delta(\tau) = (1-\delta)e^{-i\tau} + \delta e^{-2i\tau}$.

Given an ergodic dynamical system (T, X, μ) and $f \in L^2_0(X)$, and letting σ be the spectral measure on $\mathbf{T}$ associated to f, we define $\sigma(f_\delta) := \sigma_\delta$.

PROPOSITION 2 (weak closeness implies strong closeness for the spread-out densities; see [1, Lemme 8]). *Let (φ_j), $j = 1, \dots, K$, be a finite family of (α, τ)-good functions. For any $\varepsilon > 0$, for all $\delta < \delta(\alpha, \tau, \varepsilon)$ there exists $\rho > 0$ such that if $f_1, \dots, f_K$ are simple functions such that $\sigma(f_j) \sim_\rho \varphi_j$, then the densities $\varphi_{j,\delta}$ of $\sigma(f_{j,\delta})$ are strictly positive continuous functions on $(0,1)$ and satisfy $\varphi_{j,\delta} \approx_\varepsilon \varphi_j$.*

PROPOSITION 3 (approaching the spread densities by inducing; see [1, Proposition 7]). *If $\delta \in (0, 1/2)$, and $f_1, \dots, f_K$ are simple functions, then, for any $\rho > 0$, there exists A such that $\mu(A^c) < 2\delta$, and, for $f'_j = f_{j|A}$ and (T_A, A, μ_A), we have*

$$\sigma(f'_j) \sim_\rho \sigma(f_{j,\delta}).$$

Moreover, A can be chosen to be independent of the functions $f_1, \dots, f_K$ such that the functions f'_j are simple.

2.5. Adding one almost orthogonal function to a family. In our inductive construction for proving Theorem A based on Proposition 1, at each step n we need to introduce an additional function to the almost orthogonal functions already constructed so that at the end we guarantee an infinite multiplicity. For this, we need the following simple lemma.

The following lemma shows that it is possible to import spectral multiplicity in the weak sense. The regularizing lemma transforms it into an actual increase of the spectral multiplicity.

LEMMA 3. *Let (X, T) be an ergodic dynamical system. Given $\rho > 0$ and a family of K functions $\{f_j\}_{j=1,2,\dots,K} \in L^2(X)$, there exists a simple function f_{k+1} such that*

(a) $\sigma(f_{K+1}) \sim_\rho 1$,

(b) *for all $\eta \in \{-1, +1\}$, for all $j = 1, 2, \dots, K$,*

$$\sigma(f_{K+1} + \eta f_j) \sim_\rho \sigma(f_j) + 1.$$

Proof. We consider (Y, B), where B is a Bernoulli shift acting on Y, and we form the product $(X \times Y, T \times B)$. Since B is Bernoulli, there exists a Y-measurable function ϕ (in $L^2(X \times Y)$) such that

(1) the $(T \times B)^i \phi$, $i \in Z$, are orthogonal; and furthermore,

(2) H_ϕ is orthogonal to $L^2(X)$ (in $L^2(X \times Y)$).

By a simple application of Rokhlin's lemma, we know that if two finite partitions P and Q are given in $(X \times Y)$, where P is X-measurable, together with an integer n and $\delta > 0$, there exists a partition $\widetilde{Q}$ which is X-measurable such that two finite partitions $\bigvee_0^n (T \times B)^i (P \bigvee Q)$ and $\bigvee_0^n (T \times B)^i (P \bigvee \widetilde{Q})$ have very close distributions (the closeness is controlled by δ). Applying this to suitable simple functions approximating ϕ and the f_i's (Q being the support of f_{K+1} which approximates ϕ and P being spanned by the supports of the simple approximations of the f_i's) one gets statements (a) and (b) as a direct consequence of (1) and (2). Lemma 3 is proved.

3. Inducing a system with infinite Lebesgue spectral component.

In this section, we see how we can find an induced system of any ergodic system (T, X, μ) that has a spectral component that is Lebesgue with infinite multiplicity. The proof of the main Theorem A that we postpone to the last section is a direct combination of the construction that we propose in this section and the construction in [1] of an induced system with a pure Lebesgue spectrum.

THEOREM 1. *For any ergodic dynamical system (T, X, μ), there exists an induced system that has a spectral component of a pure Lebesgue type and infinite multiplicity.*

3.1. The inductive step.

The proof of Theorem 1 relies on the criterion of Proposition 1 and on the following main inductive step that is based on Propositions 2 and 3 and Lemma 3.

PROPOSITION 4. *Suppose that $K \in \mathbf{N}$ and $\{\varphi_j\}_{j=1,\dots,K}$ are (α, τ)-good and that $\{\varphi_{i,j,\eta}\}_{1 \leqslant i<j \leqslant K}$ are in $C^0(\mathbf{T}, \mathbf{R}_+)$. Suppose next that, for any $\varepsilon > 0$, there exist $\rho = \rho(\{\varphi_j\}_{j=1,\dots,K}, \{\varphi_{i,j,\eta}\}_{1 \leqslant i<j \leqslant K}, \varepsilon) > 0$ such that, if $\{f_j\}_{j=1,\dots,K}$ are simple functions such that, for every $(i, j, \eta) \in \{1, \dots, K\}^2 \times \{1, i\}$ with $i < j$,*

$$\sigma(f_i) \sim_\rho \varphi_i, \quad \sigma(f_i + \eta f_j) \sim_\rho \varphi_{i,j,\eta}, \tag{9}$$

then there exist $\{\varphi'_j\}_{j=1,\dots,K}$ such that $\varphi'_j > 0$ on $(0, 1)$ for every j, $\{\varphi_{i,j,\eta}\}_{1 \leqslant i<j \leqslant K}$ in $C^0(\mathbf{T}, \mathbf{R}_+)$, and

$$\varphi'_j \approx_\varepsilon \varphi_j, \quad \varphi'_{i,j,\eta} \approx_\varepsilon \varphi_{i,j,\eta},$$

and for any $\rho' > 0$ one can find $A \subset X$ such that $\mu(X \setminus A) < \varepsilon$, A is orthogonal to $\{f_j\}_{j=1,\dots,K}$, and simple functions f'_j defined on A such that $f'_j = f_{j|A}$, and for the system $(T_{|A}, A, \mu_A)$,

$$\sigma(f'_i) \sim_{\rho'} \varphi'_i, \quad \sigma(f'_i + \eta f'_j) \sim_{\rho'} \varphi'_{i,j,\eta}. \tag{10}$$

Note that since $\varphi'_j > 0$ on $(0, 1)$, there exists $\alpha' > 0$ such that φ'_j is $(\alpha', \tau/2)$-good, which allows us to iterate the above proposition.

3.2. Proof of Theorem 1.

Before proving the proposition, we see how it implies Theorem 1. We fix $\tau_n = 1/2^n$.

From Proposition 1, it suffices to construct inductively $\varepsilon_n < \min(2^{-n}, \alpha_n/2)$, $\alpha_n > 0$, $\rho_n \to 0$, a nested sequence of measurable sets A_n, and an array of simple functions $\{f_j^{(n)}\}_{j \in [1,n]} \in \mathcal{S}(A_n)$, $n \geqslant 1$, such that $\mu(A_{n-1} \setminus A_n) < \varepsilon_n$, A_n is orthogonal to $\{f_j^{(n-1)}\}_{j=1,\dots,n-1}$, and arrays of functions $\{\varphi_j^{(n)}\}_{j \in [1,n]}$, $\{\varphi_{i,j,\eta}^{(n)}\}_{1 \leqslant i<j \leqslant n \in [1,n]}$, $n \geqslant 1$, such that (A1$_n$)–(A4$_n$) hold.

Moreover, we suppose that in the construction, ρ_k is given by

$$\rho_k = \rho\big(\{\varphi_j^{(k)}\}_{j\in[1,k+1]}, \{\varphi_{i,j,\eta}\}_{1\leqslant i<j\leqslant k+1}, \varepsilon_{k+1}\big),$$

where ρ_k is the function from the first part of Proposition 4, and where we took $\varphi_{k+1}^{(k)} := 1$ and $\varphi_{i,k+1,\eta} := 1 + \varphi_i^{(k)}$ for $i \in [1,k]$.

The construction for $n = 1$. We let $\rho_1 = \rho(\{\varphi_1^{(1)}\}, \varepsilon_2)$, where $\rho(\cdot)$ is given by the first part of Proposition 4.

Using Lemma 3, we start with $A_1 = X$, $\varphi_1^{(1)} \equiv 1$, and $f_1^{(1)}$ is simple and such that

$$\sigma(f_1^{(1)}) \sim_{\rho_1} \varphi_1^{(1)}.$$

Inducting from n to $n+1$. Further, we suppose that everything is constructed up to n; that is, A_n such that $\mu(A_{n-1}\setminus A_n) < \varepsilon_n$, an array of simple functions $\{f_j^{(n)}\}_{j\in[1,n]} \in \mathcal{S}(A_n)$, A_n is orthogonal to $\{f_j^{(n-1)}\}_{j=1,\dots,n-1}$, $\{\varphi_j^{(n)}\}_{j\in[1,n]}$ are (α_n,τ_n)-good for some $\alpha_n > 0$, and $\{\varphi_{i,j,\eta}^{(n)}\}_{1\leqslant i<j\leqslant n}$ satisfy (A1$_n$)–(A4$_n$). We define $\varphi_{i,n+1}^{(n)} = 1$ and $\varphi_{i,n+1,\eta}^{(n)} = \varphi_i^{(n)} + 1$ and take $\rho_{n+1} = \rho(\{\varphi_j^{(n)}\}_{j\in[1,n+1]}, \{\varphi_{i,j,\eta}^{(n)}\}_{1\leqslant i<j\leqslant n+1}, \varepsilon_{n+1})$.

Using Lemma 3, we add to $\{f_j^{(n)}\}_{j\in[1,n]} \in \mathcal{S}(A_n)$ a function $f_{n+1} \in \mathcal{S}(A_n)$ such that

$$\sigma(f_{n+1}) \sim_{\rho_{n+1}} 1, \quad \sigma(f_i^{(n)} \pm f_{n+1}) \sim_{\rho_{n+1}} \varphi_i^{(n)} + 1. \tag{11}$$

Now we apply Proposition 4 to $\{f_1^{(n)},\dots,f_n^{(n)},f_{n+1}\}$, $\{\varphi_j^{(n)}\}_{j\in[1,n+1]}$, and $\{\varphi_{i,j,\eta}^{(n)}\}_{1\leqslant i<j\leqslant n+1}$. So, we get A_{n+1} orthogonal to $\{f_1^{(n)},\dots,f_n^{(n)},f_{n+1}\}$ such that $\mu(A_n\setminus A_{n+1}) < \varepsilon_{n+1}$ and an array of simple functions $\{f_j^{(n+1)}\}_{j\in[1,n+1]} \in \mathcal{S}(A_{n+1})$ and $\{\varphi_j^{(n+1)}\}_{j\in[1,n+1]}$ that are $(\alpha_{n+1},\tau_{n+1})$-good for some $\alpha_{n+1} > 0$, and functions $\{\varphi_{i,j,\eta}^{(n+1)}\}_{1\leqslant i<j\leqslant n+1} \in C^0(\mathbf{T},\mathbf{R}_+)$ that satisfy (A1$_{n+1}$)–(A4$_{n+1}$).

In conclusion, Theorem A now follows from Proposition 1.

3.3. Proof of Proposition 4. The proof of Proposition 4 has two steps, analogous to the main two steps of [1].

Step 1. Spreading out. In the first step, we elaborate on Proposition 2 and get the following lemma.

LEMMA 4. *Let $K \in \mathbf{N}$, $\{\varphi_j\}_{j=1,\dots,K}$ be (α,τ)-good functions, and let $\{\varphi_{i,j,\eta}\}_{1\leqslant i<j\leqslant K}$ be in $C^0(\mathbf{T},\mathbf{R}_+)$. Then, for every $\varepsilon > 0$, there exists*

$$\rho = \rho(\{\varphi_j\}_{j=1,\dots,K}, \{\varphi_{i,j,\eta}\}_{1\leqslant i<j\leqslant K}, \varepsilon) > 0$$

such that if $\{f_j\}_{j=1,\dots,K}$ are simple functions satisfying

$$\sigma(f_i) \sim_\rho \varphi_i, \quad \sigma(f_i + \eta f_j) \sim_\rho \varphi_{i,j,\eta}, \tag{12}$$

for all $1 \leqslant i < j \leqslant K$ and $\eta \in \{1, i\}$, then, for all sufficiently small $\delta > 0$, the spread-out measures $\sigma(f_{j,\delta})$ and $\sigma(f_{i,\delta} + \eta f_{j,\delta})$ have continuous strictly positive densities $\varphi_{j,\delta}$ and $\varphi_{i,j,\eta,\delta}$ satisfying

$$\sigma(f_{j,\delta}) \approx_\varepsilon \varphi_j, \quad \sigma(f_{i,\delta} + \eta f_{j,\delta}) \approx_\varepsilon \varphi_{i,j,\eta}. \tag{13}$$

Proof. The proof is a direct application of Proposition 2 to several functions at the same time.

Step 2. Approaching by induction. Suppose $K \in \mathbf{N}$, α, τ, ε, $\delta > 0$, $\{\varphi_j\}_{j=1,\dots,K}$, $\{\varphi_{i,j,\eta}\}_{1\leqslant i<j\leqslant K}$, ρ, and $\{f_j\}_{j=1,\dots,K}$ are as in Lemma 4.

We can apply Proposition 3 to the family of simple functions $\{f_j\}_{j=1,\dots,K}$ and get, for any $\rho' > 0$, a set A orthogonal to $\{f_j\}_{j=1,\dots,K}$ such that $\mu(X \setminus A) < \varepsilon$, and, for the simple functions $f'_j = f_{j|A}$ and the system (T_A, A, μ_A), it holds that

$$\begin{aligned} \sigma(f'_i) &\sim_{\rho'} \varphi'_i := \sigma(f_{j,\delta}) \approx_\varepsilon \varphi_j, \\ \sigma(f'_i + \eta f'_j) &\sim_{\rho'} \varphi'_{i,j,\eta} := \sigma(f_{i,\delta} + \eta f_{j,\delta}) \approx_\varepsilon \varphi_{i,j,\eta}, \end{aligned} \tag{14}$$

with $\varphi'_j > 0$ on $(0,1)$ for every j and $\{\varphi'_{i,j,\eta}\}_{1\leqslant i<j\leqslant K}$ in $C^0(\mathbf{T}, \mathbf{R}_+)$.

The proof of Proposition 4 is thus completed.

4. Proof of Theorem A. To go from Theorem 1 to Theorem A, we can keep the inductive construction of Theorem 1 essentially as is and add a feature to guarantee that the maximal spectral type is equivalent to Lebesgue. To do so, we just need to make sure that the family of functions we are constructing becomes dense in $L^2_0(X, \mu)$. In fact, we find it simpler to add to the array of simple functions $\{f_j^{(n)}\}_{j\in[1,n]} \in \mathcal{S}(A_n)$ another array of simple functions $\{h_j^{(n)}\}_{j\in[1,n]} \in \mathcal{S}(A_n)$ whose role is to guarantee a pure Lebesgue spectrum for the final induced system. For this, we follow verbatim [1, section 3.3]. We recall first the approach in [1] to guarantee a pure Lebesgue spectrum for the final induced system.

Start with a family of simple functions $\{h_j\}_{j\in\mathbf{N}}$ that is dense in $L^2_0(X, \mu)$. At each step of the construction, we pick a set A_n that works simultaneously for the family $\{h_j^{(n)}\}_{j\in[1,n]}$, where $h_j^{(k)} = h_{j|A_k}^{(k-1)}$, for every $j \leqslant k-1$ and $h_k^{(k)} = h_{k|A_k}$, in the sense that, for every fixed j, $\sigma(h_j^{(n)}) \sim_{\rho_n} \psi_{j,n}$, where the spectral measures are considered with respect to the induced system $T_{|A_n}$, and $\psi_{j,n}$ is a sequence of densities that converge in L^2 in the sense of Lemma 1. To keep the functions simple at each step of the induction, the set A_n is chosen independent of the partitions that define the simple functions $\{h_j^{(n)}\}_{j\in[1,n]}$.

At the end of the construction, the spectral measure of h_j^∞ for the system $T_{|A_\infty}$ is equivalent to Lebesgue for every $j \in \mathbf{N}$.

For every j, $h_j^\infty = h_{j|A_\infty}$ it follows from the density of the family $\{h_j\}_{j\in\mathbf{N}}$ that the family $\{h_j^\infty\}_{j\in\mathbf{N}}$ is dense. Hence the system (T_{A_∞}, A_∞) has a pure Lebesgue spectrum.

We return to our construction and assume, as in the beginning of this proof, that we are given a family of simple functions $\{h_j\}_{j\in\mathbf{N}}$ that is dense in $L^2_0(X, \mu)$.

As in [1], when we carry out the inductive construction in the proof of Theorem 1, it is possible to choose A_{n+1} which works simultaneously for $\{f_j^{(n)}\}_{j\in[1,n]}$ (defined as in subsection 3.2) as well as for $\{h_j^{(n)}\}_{j\in[1,n]}$ (defined as above). Note that in this procedure, at each step n of the induction, the family $\{f_j^{(n)}\}$ and the family $\{h_j^{(n)}\}_{j\in[1,n]}$ are updated by mere induction $h_j^{(n)} = h_{j|A_n}^{(n-1)}$. Hence the density of the family $\{h_j\}_{j\in\mathbf{N}}$ is automatically transmitted to $\{h_j^\infty\}_{j\in\mathbf{N}}$.

In conclusion, from the family $\{f_j^{(\infty)}\}_{j\in\mathbf{N}}$ we get the infinite multiplicity of the Lebesgue component, and from the family $\{h_j^{(\infty)}\}_{j\in\mathbf{N}}$ we get that the spectrum is a pure Lebesgue. The proof of Theorem A is thus complete.

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