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Reducibility Without KAM

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Abstract: We prove rotations-reducibility for close to constant quasi-periodic $SL(2, \mathbb{R})$ cocycles in one frequency in the finite regularity and smooth cases, and derive some applications to quasi-periodic Schrödinger operators.

Introduction

In this paper we will study smooth quasi-periodic $SL(2, \mathbb{R})$ cocycles in one frequency. These are skew products of the form

$$\begin{aligned}(\alpha, A) : \mathbb{T} \times \mathbb{R}^2 &\to \mathbb{T} \times \mathbb{R}^2\\ (x, y) &\to (x + \alpha, A(x)y),\end{aligned}$$

$A \in C^\infty(\mathbb{T}, SL(2, \mathbb{R}))$.

We will be interested in the case A is close to a constant matrix. A classical problem is to see if (α, A) is reducible, that is, conjugated to a constant matrix cocycle. We say that (α, A) is C^∞*-reducible* if there exists $B \in C^\infty(\mathbb{R}/2\mathbb{Z}, SL(2, \mathbb{R}))$ and $A_* \in SL(2, \mathbb{R})$ such that

$$B(\theta + \alpha)A(\theta)B(\theta)^{-1} = A_*, \qquad \forall \theta \in \mathbb{R}/2\mathbb{Z}.$$

Reducibility is an important question in the study of quasi-periodic cocycles and its applications to the spectral theory of Schrödinger operators.

When α satisfies a Diophantine condition, many reducibility results were obtained by the KAM (Kolmogorov-Arnold-Moser) technique since the seminal paper by Dinaburg and Sinai [6]. Most of the results were obtained for real analytic cocycles, but KAM methods also yield reducibility results in the smooth category, see [10, Section 2.4] for example.

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If α is just irrational, reducibility does not hold in general. Indeed, the rotations-valued cocycle (α, A), $A(\cdot) = R_{\phi(\cdot)}$, with $\phi \in C^\infty(\mathbb{T}, \mathbb{R})$ and

$$R_{\phi(x)} := \begin{pmatrix} \cos(2\pi\phi(x)) & -\sin(2\pi\phi(x)) \\ \sin(2\pi\phi(x)) & \cos(2\pi\phi(x)) \end{pmatrix},$$

is smoothly reducible if and only if the cohomological equation

$$\phi(x) - \int_{\mathbb{T}} \phi(\theta)d\theta = h(x+\alpha) - h(x) \qquad (\mathcal{E})$$

has a smooth solution. It is known that the cohomological equation does not have smooth solutions in general when α is not Diophantine.

However, one can still ask about reducibility to a rotation-valued cocycle, that is, the existence of a smooth conjugacy B such that $B(x+\alpha)A(x)B^{-1}(x) \in SO(2, \mathbb{R})$. We then say that the cocycle is rotations-reducible. Rotations-reducibility is important in the global theory of 1 dimensional quasi-periodic cocycles, and has many applications for quasi-periodic Schrödinger operators, some of which will be mentioned below.

Rotations-reducibility for close to constant cocycles, irrespective of any arithmetic condition on the irrational base frequency $\alpha \in \mathbb{T}$, was obtained in [2] in the real analytic category. The aim of this paper is to extend the results of [2] to the finite regularity and smooth case. Rotations-reducibility was obtained in some ultradifferentiable classes (for example Gevrey) by Cheng, Ge, You, Zhou in [7], while our rotations-reducibility result holds in finite regularity (see also [11] for questions about reducibility with low regularity).

Before stating our main theorem, we recall the definition of the fibered rotation number $\rho = \rho(\alpha, A)$. If A is homotopic to the identity, G is the projective cocycle associated with (α, A), that is:

$$G : \mathbb{T} \times \mathbb{S}^1 \to \mathbb{T} \times \mathbb{S}^1$$
$$(x, y) \to \left(x+\alpha, \frac{A(x)y}{\|A(x)y\|}\right),$$

and $\tilde{G}(\theta, y) = (\theta+\alpha, y + f(\theta, y))$ is a lift of G in $\mathbb{T} \times \mathbb{R}$, then:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(\tilde{G}^i(\theta, y))$$

exists and it is independent of (θ, y). The class of this number in $\mathbb{T}$ is the fibered rotation number.

We will prove the following

Theorem 1. *Let* $\epsilon > 0$, $r_0 \geq 200$. *There exists* $\epsilon_0 > 0$ *such that, for* $\alpha \in \mathbb{R} - \mathbb{Q}$ *there exists a set* $Q(\alpha) \subseteq \mathbb{T}$ *with* $m(Q(\alpha)^c) < \epsilon$ *such that the following holds:*
if $A_0 \in C^\infty(\mathbb{T}, SL(2, \mathbb{R}))$, $R \in SO(2, \mathbb{R})$, $\rho(\alpha, A_0) \in Q(\alpha)$ *and:*

$$\|A_0 - R\|_{50 r_0} < \epsilon_0,$$

then, there exist $B \in C^\infty(\mathbb{T}, SL(2, \mathbb{R}))$, $\phi \in C^\infty(\mathbb{T}, \mathbb{R})$ *such that,*

$$B(x+\alpha)A_0(x)B^{-1}(x) = R_{\phi(x)},$$

with:

$$\|\phi - \rho\|_{r_0}, \|B - Id.\|_{r_0} < \sqrt{\epsilon_0}.$$

Finite regularity. Observe that in the proof of Theorem 1, in order to show the convergence of the scheme in norm C^{r_0} we will use only derivatives of A up to order $50r_0$. In particular, the same statement holds in class C^{r_0} if A is only assumed to be in $C^{50r_0}(\mathbb{T}, SL(2,\mathbb{R}))$. Our assumption that $r_0 \geq 200$ is not optimal, neither is the closeness condition in class C^{50r_0}. In fact, we will explain later why the more α is Liouville, the more r_0 can be taken small.

Reducibility in the Diophantine case. When the base frequency α is Diophantine, our result implies smooth reducibility since the cohomological equation ($\mathcal{E}$) has a smooth solution h. Finite regularity reducibility results, depending on the Diophantine exponent of α, can also be obtained from the proof of Theorem 1 and the study of ($\mathcal{E}$) in finite regularity.

Full measure rotations-reducibility. Eliasson's theory. Eliasson developed a non-standard KAM scheme which enabled him to prove a much stronger version of Dinaburg-Sinai theorem. He showed that for every Diophantine α, there exists a full Lebesgue measure set $Q(\alpha)$ such that if the cocycle is sufficiently close to constant (depending on α) and if $\rho(\alpha, A) \in Q(\alpha)$ then (α, A) is reducible [8].

Note that for arbitrary irrational frequency α, unconditional almost reducibility and therefore full measure rotations-reducibility, for analytic close to constant cocycles, should follows from Avila's global theory [1].

In [12], Hou and You proved a continuous time version of [2] with a different method. They also proved that almost reducibility always holds in the close to constants regime in the analytic case, which gives rotations-reducibility for a full measure set of fibered rotation numbers.

In the smooth category, full measure rotations-reducibility does not hold, since counter-examples were found in Gevrey class by Avila and Krikorian [5]. In this respect, the result of Theorem 1 is optimal.

The case of general matrix cocycles. Reducibility results for quasi-periodic cocycles valued in $GL(d,\mathbb{C})$ above a one-dimensional Liouvillean rotations, extending the results of [2,12], were obtained in [15] in the real analytic setting. The scheme developed in the proof of Theorem 1 should be useful to address similar extensions in the smooth setting.

Applications to one-dimensional quasi-periodic Schrödinger operators. The main source of examples of cocycles that we are considering are the Schrödinger cocycles

$$A_{v,E}(x) = \begin{pmatrix} E - v(x) & -1 \\ 1 & 0 \end{pmatrix},$$

where $v \in C^\infty(\mathbb{T},\mathbb{R})$ and $E \in \mathbb{R}$ are related to the spectral study of one-dimensional quasi-periodic Schrödinger operators:

$$(Hu)_n = u_{n+1} + u_{n-1} + v(\theta + n\alpha)u_n. \tag{1}$$

An application of our main result is the following.

Theorem 2. *If $v \in C^\infty(\mathbb{T},\mathbb{R})$ is close to a constant, $\alpha \in \mathbb{R}$, then there is a positive measure set of E such that $(\alpha, A_{v,E})$ is rotations-reducible. In particular, for v close to a constant there exists some absolutely continuous part in the spectrum of the corresponding Schrödinger operator in* (1).

Another application concerns Schrödinger's conjecture stating that for general discrete Schrödinger operators over uniquely ergodic base dynamics, all generalized eigenfunctions are bounded for almost every energy in the support of the absolutely continuous part of the spectral measure. This conjecture was disproved by Avila (see [3]). In [13], Marx and Jitomirskaya pointed out that the Schrödinger's conjecture in the quasiperiodic setting is still an open problem in the smooth category. The following result, implies that the Schrödinger conjecture is true for the one-dimensional smooth quasi-periodic Schrödinger operators over irrational circle rotations.

Theorem 3. *Let* $v \in C^\infty(\mathbb{T}, \mathbb{R})$. *Then, for almost every* E *in the spectrum,* $(\alpha, A_{v,E})$ *is either non uniformly hyperbolic or smoothly rotations-reducible.*

The proof of Theorem 3, starts from Kotani theory, and then uses the work of Avila and Krikorian in [4] where they prove the convergence to constants of the renormalizations of cocycles that are L^2-conjugated to rotations, in order pass from global to local. The implication of Theorems 2 and 3 from Theorem 1 can be obtained in exactly the same way as Theorem 1.1 is obtained from Theorem 1.3 in [2], with.

Main novelties and outline of the proof. The proof follows an inductive conjugation scheme based on the so-called cheap trick introduced in [10] and further developed in [2].

One main novelty in our approach is that in finite regularity it is possible to fully use the strength of the cheap trick in which no loss of derivatives is incurred in finding the conjugacy (see (2) below), since no cohomological equation is solved. As a consequence, the convergence of our conjugation scheme is quite different from that of KAM smooth schemes, and is much simpler. In particular, it does not require any approximation of smooth functions by analytic ones.

The main other novelty is in the new choice of the subsequence of the sequence (q_n) of denominators of the best rational approximations along which the cheap trick is applied. Given the sequence (q_n) of denominators of the best rational approximations of an irrational number, the notion of Diophantine bridges, a sufficiently long succession of q_n where q_{n+1} is bounded by a fixed power of q_n, was explicitly introduced in [9] to settle the global smooth conjugacy problem of commuting circle diffeomorphisms.

In [2], a pattern of Diophantine bridges that alternate with big power jumps in the sequence (q_n) was proved to exist for every irrational α. A subsequence (q_{n_h}) (called (Q_h) in [2]) of (q_n) is then chosen using the endpoints of the Diophantine bridges and the q_n with a big jump from q_n to q_{n+1}. This subsequence is used to apply the cheap trick and prove rotations-reducibility in the real analytic context.

The choice of the sequence (q_{n_h}) is important to have a nice control on the Birkhoff sums and make sure that they are close to their mean (which allows to use the hypothesis on the fibered rotation number), and also to get sufficient gain from applying the cheap trick in terms of decreasing the magnitude of the non-abelian part of the cocycle.

The choice of the sequence (q_{n_h}) in [2] was adapted to the real analytic category, and in the current work it is crucial to choose quite differently the sequence (q_{n_h}) to adapt to the smooth setting, that is needed not only to have a nice control on the Birkhoff sums, but also to have a good control of high norms of the error term.

Finally, to prove the convergence of the scheme it is crucial also to have improved estimates on the derivatives of the iterated cocycle that are adapted to the smooth setting (see Proposition 3), compared to the previously used bounds such as the ones in [10].

We now recall the idea behind the cheap trick and give an outline of the iterative conjugation scheme that is used to prove Theorem 1.

Let q_n be a denominator of a best rational approximation of α. If the cocycle matrix $A(\cdot) = R_{\phi(\cdot)} + F(\cdot)$ is very close to the $SO(2,\mathbb{R})$ cocycle $R_{\phi(\cdot)}$, then $A^{(q_n)}(x) := R_{S_{q_n}\phi}(x)+\xi(x)$ is elliptic for every $x \in \mathbb{T}$. In this case, we can find $B_1 \in C^\infty(\mathbb{T}, SL(2,\mathbb{R}))$, $\phi_1 \in C^\infty(\mathbb{T},\mathbb{R})$ such that

$$B_1(x)A^{(q_n)}(x)B_1^{-1}(x) = R_{\phi_1(x)},$$

and for $t \in \mathbb{N}$

$$\|B_1 - Id.\|_t, \|\phi_1 - \phi\|_t \leq C(t)\|(R_{2S_{q_n}\phi} - Id.)^{-1}\xi R_{-S_{q_n}\phi}\|_t, \tag{2}$$

If in addition the Birkhoff sum $S_{q_n}\phi$ is uniformly close to its average, then this average is close to $q_n\rho(\alpha, A)$, where $\rho(\alpha, A)$ is the fibered rotation number of (α, A). Hence, if $\|\|2q_n\rho(\alpha, A)\|\|$ ($\|\|\cdot\|\|$ being the distance to the closest integer) is not too small, then B_1 is close to the Identity matrix. Finally, the essential point in the cheap trick is to see that the conjugated cocycle $A_1(x) := B_1(x + q_n\alpha)A^{(q_n)}(x)B_1^{-1}(x)$ is of the order of $\|\|q_n\alpha\|\| \sim \frac{1}{q_{n+1}}$ closer to a rotation valued cocycle then $A^{(q_n)}$ was. One can conjugate A_1 now and gain another factor $\frac{1}{q_{n+1}}$. Repeating the procedure $r_0 + 1$ times one obtains a conjugacy $\tilde{B} := B_{r_0+1} \dots B_1$ such that

$$\tilde{B}(x + q_n\alpha)A^{(q_n)}(x)\tilde{B}^{-1}(x) = \tilde{F}(x) + R_{\phi_h(x)}$$

such that, if we suppose $\|\|2q_n\rho(\alpha, A)\|\| \geq \frac{c}{n^2}$, it holds for $0 \leq h \leq r_0 + 1$:

$$\|\tilde{F}\|_t \leq \frac{C(t+h)n^{2r_0(t+1)}}{q_{n+1}^h} \max_{\beta\in\{0,1\}} \|\bar{F}\|_{t+h}^{\beta} \left(\|\bar{F}\|_1\|\bar{\phi}\|_{t+h}\right)^{1-\beta}. \tag{3}$$

One then derives from (3) similar estimates on how close $\tilde{B}$ conjugates the original cocycle (α, A) to a rotation valued cocycle.

An important point of the cheap trick is that we get inverse powers q_{n+1}^{-h} in the control of the new nonlinearity when we compare to $\|F\|_{t+h}$. However as we see in (2), there is no loss of derivatives in the scheme because we never solve a cohomological equation.

Plan of the paper. The main inductive conjugation step is stated in Proposition 1 and the proof of Theorem 1, based on Propositon 1, are given in Sect. 2.

The outcome of the cheap trick is the content of Propositon 2 of Sect. 3, which is the main step of the proof. The proof of Proposition 1 is given in Sect. 5. It goes through the application of the cheap trick to the iterated cocycle $A^{(q_n)}$, explained in Sect. 5.1, and through the estimates for going back to the original cocycle, explained in Sect. 5.2.

The estimates on the iterated cocycle $(q_n\alpha, A^{(q_n)})$ are included in Sect. 4. In Sect. 4.1 we state and prove the estimates on the upper bounds on the non-abelian part of the iterated cocycle, and in Sect. 4.2 we include the crucial estimates on the Birkhoff sums along the selected subsequence of $(q_n(\alpha))$.

Finally, the Appendix A contains the statements and proofs of some basic analysis lemmas that are used throughout the paper.

1. Notations and Definitions

In all the sequel α will be a fixed irrational number. If $f \in C^\infty(\mathbb{T}, \mathbb{R})$ we denote the Birkhoff sums of f above the circle rotation of angle α for $n \in \mathbb{N}$

$$S_n f(x) := \sum_{h=0}^{n-1} f(x + h\alpha).$$

For $A \in C^\infty(\mathbb{T}, SL(2, \mathbb{R}))$, $n \in \mathbb{N}$, we denote

$$A^{(n)}(x) := \prod_{j=n-1}^{0} A(x + j\alpha).$$

For $f \in C^\infty(\mathbb{T}, \mathbb{R})$, $k \in \mathbb{N}$, we denote by $D^k f$ the k-th derivative of f and

$$|f|_0 := \sup_{x\in\mathbb{T}} |f(x)|, \quad \|F\|_k := \sum_{h=0}^{k} |D^h f|_0.$$

We use the notation

$$\|\|x\|\| = \inf_{p\in\mathbb{Z}} |x - p|.$$

We define (q_n) to be the sequence of denominators of the best rational approximations of α. Recall that q_n satisfies $q_0 = 1$ and

$$\forall 1 \le k < q_n, \qquad \|\|k\alpha\|\| \ge \|\|q_{n-1}\alpha\|\|. \tag{4}$$

Recall also that

$$\frac{1}{q_{k+1} + q_k} < \|\|q_k\alpha\|\| < \frac{1}{q_{k+1}}. \tag{5}$$

2. Main Inductive Conjugation Step

In this section we state the main Proposition 1 containing one step of the reducibility scheme and we show how Theorem 1 follows directly from it.

Definition 1. Let α be irrational. We define a subsequence of convergents $(q_{n_h})_{h\in\mathbb{N}}$ in the following way. We let $n_0 = 0$. Now, suppose that we have defined the subsequence up to q_{n_h}. If there exists k such that $q_{n_h+1}^2 \le q_k < q_{n_h+1}^4$, we define $n_{h+1} := k$. Otherwise, we take $n_{h+1} := \max\{k \in \mathbb{N} : q_k \le q_{n_h+1}^2\}$. Note that in the first case $q_{n_{h+1}} \ge q_{n_h+1}^2 > q_{n_h+1}$ so that $k > n_h + 1 > n_h$, and in the second case, for $k = n_h + 1$ we have $q_k < q_{n_h+1}^2$ so that $\max\{k \in \mathbb{N} : q_k \le q_{n_h+1}^2\} \ge n_h + 1 > n_h$. In particular, the sequence $(n_h)_{h\in\mathbb{N}}$ is strictly increasing. For $h \in \mathbb{N}$, we use the notation $s_h := \prod_{l=0}^{h} q_{n_l}$.

Lemma 1. *For $h \in \mathbb{N}$, $q_{n_{h+1}} \le q_{n_h+1}^4$. Moreover, for $h \in \mathbb{N}$: $s_h^6 \le q_{n_{h+1}}^{12}$.*

Proof. It follows by the definition of $(q_{n_h})_{h\in\mathbb{N}}$. □

Proposition 1. *Let $\epsilon > 0$. There exists $\epsilon_0 > 0$ such that the following holds. Let α be irrational, $(q_n)_{n\in\mathbb{N}}$ be the convergents of α, $r_0 \geq 200$ and $A_0 = R_{\phi_0} + F_0 \in C^\infty(\mathbb{T}, SL(2,\mathbb{R}))$, $\phi_0 \in \mathbb{T}$, $R_{\phi_0} \in SO(2,\mathbb{R})$ such that*

$$\|F_0\|_{50r_0} < \epsilon_0.$$

and $\rho := \rho(\alpha, A_0)$ satisfies $\|2q_{n_h}\rho\| \geq \frac{\epsilon}{n_h^2}$ where (n_h) is as in Definition 1.

Then, there exist $B_h, F_h \in C^\infty(\mathbb{T}, SL(2,\mathbb{R}))$, $\phi_h \in C^\infty(\mathbb{T}, SL(2,\mathbb{R}))$ such that for $h \in \mathbb{N}$:

$$B_h(x+\alpha)(R_{\phi_h(x)} + F_h(x))B_h^{-1}(x) = R_{\phi_{h+1}(x)} + F_{h+1}(x), \tag{6}$$

$$\|F_{h+1}\|_1 \leq \frac{\epsilon_0}{q_{n_{h+1}}^{\frac{r_0}{8}}}, \quad \|F_{h+1}\|_{50r_0}, \|\phi_{h+1}\|_{50r_0} \leq s_h^6 \tag{7}$$

where $s_h := \prod_{l=0}^h q_{n_l}$ and, for $t \in \mathbb{N}$:

$$\|B_h - Id.\|_t, \|\phi_{h+1} - \phi_h\|_t, \|F_{h+1}\|_t \leq C(t) n_h^{4tr_0} q_{n_h}^5 \max_{\beta\in\{0,1\}} \|F_h\|_t^\beta (\|F_h\|_1 q_{n_h} \|\phi_h\|_t)^{1-\beta}. \tag{8}$$

Now we show how Theorem 1 follows directly from Proposition 1.

Proof of Theorem 1. We want to show that the following limits exist in the C^∞ category and that they satisfy all the requirements of Theorem 1

$$B(x) := \lim_{h\to+\infty} B_h \dots B_0(x), \quad \phi(x) := \lim_{h\to+\infty} \phi_h(x).$$

We first address the convergence in class C^{r_0}. Proposition 1 implies by convexity that

$$\|F_h\|_{r_0} \leq C\|F_h\|_0^{1-\frac{1}{50}} \|F_h\|_{50r_0}^{\frac{1}{50}} < \frac{\sqrt{\epsilon_0}}{q_{n_h}^6}, \tag{9}$$

where in the last inequality we have used the estimate in (7) for $\|F_h\|_1$, $\|F_h\|_{50r_0}$ and the fact that $r_0 \geq 200$. Next, by (8), (9):

$$\begin{aligned}\|B - Id\|_{r_0}, \|\phi - \rho\|_{r_0} &\leq \prod_{h=0}^{\infty}(1 + C(r_0) n_h^{4r_0^2} q_{n_h}^5 \sup_{\beta\in[0,1]} \|F_h\|_{r_0}^\beta (\|F_h\|_0 q_{n_h} \|\phi_h\|_{r_0})^{1-\beta}) - 1 \\ &\leq \prod_{h=0}^{\infty}\left(1 + \frac{C(r_0) n_h^{4r_0^2}\epsilon_0}{q_{n_h}}\right) - 1 \leq \sqrt{\epsilon_0}\end{aligned}$$

if ϵ_0 is small enough. So, we have proved the convergence in norm C^{r_0} as well as the bounds (1). The conjugacy equation then follows from (6).

Now we use interpolation and the bounds in (8) to prove the smooth convergence. By (7), (8):

$$\begin{aligned}\|\phi_{h+1}\|_t &\leq \|\phi_{h+1} - \phi_h\|_t + \|\phi_h\|_t \leq C(t) n_h^{4r_0 t} q_{n_h}^5 (\|F_h\|_{r_0} + q_{n_h}\|F_h\|_0 \|\phi_h\|_t) + \|\phi_h\|_t \\ &\leq C(t) n_h^{4r_0 t} q_{n_h}^5 \|F_h\|_{r_0} + (1 + \frac{1}{q_{n_h}})\|\phi_h\|_t.\end{aligned}$$

So, iterating we get:

$$\|\phi_{h+1}\|_t \le C(t) n_h^{4r_0 t} q_{n_h}^5 \sum_{l=0}^{h} \|F_l\|_t. \tag{10}$$

Let $t \in \mathbb{N}$, $t > r_0$. By (8) and (10), there exists $N \in \mathbb{N}$ such that for $h \ge N$:

$$\|F_{h+1}\|_{20t} \le n_h^{4r_0 t} q_{n_h}^5 \sum_{l=0}^{h} \|F_l\|_{20t} \le q_{n_h}^6 \sum_{l=0}^{h} \|F_l\|_{20t},$$

where in the last inequality we have used Lemma 1. In particular there exists $N_1 > N$ such that, for $h > N_1$:

$$\|F_{h+1}\|_{20t} \le n_h \left(\prod_{l=0}^{h-N} q_{n_{N+l}}^4 \right) \sum_{l=0}^{N} \|F_l\|_{20t} \le q_{n_h}^6 \sum_{l=0}^{N} \|F_l\|_{20t} \le C(t) s_h^6 \le C(t) q_{n_{h+1}}^{12},$$

with the last inequality that follows by Lemma 1. So, by convexity and the estimates of $\|F_{h+1}\|_{r_0}$ in (9) we get:

$$\|F_{h+1}\|_t \le C \|F_{h+1}\|_{r_0}^{1-\frac{t-r_0}{20t-r_0}} \|F_{h+1}\|_{20t}^{\frac{t-r_0}{20t-r_0}} \le \frac{1}{q_{n_{h+1}}^{5+\frac{1}{3}}}.$$

In particular, for any $t \in \mathbb{N}$ there exists $\bar{N} \in \mathbb{N}$ such that for $h > \bar{N}$:

$$\|B_h - Id.\|_t, \|\phi_{h+1} - \phi_h\|_t \le \frac{C(t) n_h^{4tr_0}}{q_{n_h}^{\frac{1}{3}}},$$

finishing the proof of smooth convergence of the scheme. □

3. The Cheap Trick

The aim of this section is to prove the following.

Proposition 2. *Let* $\bar{\phi} \in C^\infty(\mathbb{T}, \mathbb{R})$ *and* $\bar{A} = R_{\bar{\phi}} + \bar{F} \in C^\infty(\mathbb{T}, SL(2, \mathbb{R}))$, $r_0 \in \mathbb{N}$, $n \in \mathbb{N}$ *be such that*

$$\|(R_{2\bar{\phi}} - Id.)^{-1}\|_0 \le C n^2, \quad \|\bar{F}\|_0 < \frac{1}{q_n}, \|\bar{F}\|_{r_0}, \|\bar{\phi}\|_{r_0} < 1. \tag{11}$$

Then, there exist $\tilde{B}, \tilde{F} \in C^\infty(\mathbb{T}, SL(2, \mathbb{R}))$, $\tilde{\phi} \in C^\infty(\mathbb{T}, \mathbb{R})$ *with for any* $t \in \mathbb{N}$

$$\|\tilde{B} - Id.\|_t, \|\tilde{\phi} - \bar{\phi}\|_t \le C(t) n^{2(t+1)} \max_{\beta \in \{0,1\}} \|\bar{F}\|_t^\beta \left(\|\bar{F}\|_1 \|\bar{\phi}\|_t \right)^{1-\beta} \tag{12}$$

and

$$\tilde{B}(x + q_n \alpha) \bar{A}(x) \tilde{B}^{-1}(x) = R_{\tilde{\phi}} + \tilde{F}, \tag{13}$$

and, for $0 \le h \le r_0 + 1$*:*

$$\|\tilde{F}\|_t \le \frac{C(t+h) n^{2r_0(t+1)}}{q_{n+1}^h} \max_{\beta \in \{0,1\}} \|\bar{F}\|_{t+h}^\beta \left(\|\bar{F}\|_1 \|\bar{\phi}\|_{t+h} \right)^{1-\beta}. \tag{14}$$

In order to prove Proposition 2 we prove at first some simple lemmas.

Lemma 2. *Let $D > 0$. There exists $\epsilon > 0$ such that, if $\bar{A} = R_{\bar{\phi}} + \bar{F} \in C^\infty(\mathbb{T}, SL(2,\mathbb{R}))$ with $\|\bar{A}\|_0 < D$, $\|\bar{F}\|_0 < \epsilon \min\{1, \inf_{x\in\mathbb{T}} \|R_{2\bar{\phi}(x)} - Id.\|\}$, then there exists $\tilde{B}_1 \in C^\infty(\mathbb{T}, SL(2,\mathbb{R}))$, $\tilde{\phi}_1 \in C^\infty(\mathbb{T},\mathbb{R})$ with:*

$$\|\tilde{B}_1 - Id.\|_t, \|\bar{\phi} - \tilde{\phi}_1\|_t \le C(t)(\inf_{x\in\mathbb{T}} \|R_{2\bar{\phi}(x)} - Id.)\|)^{-(2t+1)}(\|\bar{F}||_t\|\bar{\phi}\|_0 + \|\bar{F}\|_0\|\bar{\phi}\|_t) \tag{15}$$

such that:

$$\tilde{B}_1(x)\bar{A}(x)\tilde{B}_1^{-1}(x) = R_{\tilde{\phi}_1(x)}. \tag{16}$$

Proof. We will sketch the proof following [2]. Let $Y := \bar{F}R_{-\bar{\phi}}$, $G_1 := \log(1+Y)$, $\theta_1 = \bar{\phi}$, so that $\bar{A} = e^{G_1}R_{\theta_1}$. Let:

$$\begin{aligned} G_1 &= \begin{pmatrix} x_1 & y_1 - 2\pi z_1 \\ y_1 + 2\pi z_1 & -x_1 \end{pmatrix}, \\ \bar{G}_1 &:= \begin{pmatrix} x_1 & y_1 \\ y_1 & -x_1 \end{pmatrix}, \\ \begin{pmatrix} \tilde{x}_1 \\ \tilde{y}_1 \end{pmatrix} &:= (R_{2\theta_1} - Id.)^{-1}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \\ v_1 &:= \begin{pmatrix} \tilde{x}_1 & \tilde{y}_1 \\ \tilde{y}_1 & -\tilde{x}_1 \end{pmatrix}, \quad \theta_2 := \theta_1 + z_1, \end{aligned}$$

and $\bar{C} := (\inf_{x\in\mathbb{T}} \|R_{2\bar{\phi}(x)} - Id.)\|)^{-1}$. Then, we want to show that:

$$e^{v_1}e^{G_1}R_{\theta_1}e^{-v_1} = e^{G_2}R_{\theta_2},$$

with:

$$\|e^{v_1} - Id.\|_t \le \|(R_{2\theta_1} - Id.)^{-1}G_1R_{-\theta_1}\|_t, \quad \|\theta_2 - \theta_1\|_t \le \|G_1\|_t, \tag{17}$$

$$\|G_2\|_t \le C(\|v_1^2\|_t + \|G_1v_1\|_t + \|v_1G_1\|_t + \|G_1^2\|_t). \tag{18}$$

Note that (17) follows directly from the definition of v_1, θ_1. So, we just prove (18):

$$e^{v_1}e^{G_1}R_{\theta_1}e^{-v_1} = (Id. + v_1)(Id. + G_1)R_{\theta_1}(Id. - v_1) + O(v_1^2).$$

Then, since by definition of v_1 we have

$$v_1R_{\theta_1} - R_{\theta_1}v_1 + \bar{G}_1R_{\theta_1} = 0, \tag{19}$$

we get

$$\begin{aligned} (Id. + v_1)(Id. + G_1)R_{\theta_1}(Id. - v_1) &= (Id. + G_1 - \bar{G}_1)R_{\theta_1} + v_1R_{\theta_1} \\ &\quad - R_{\theta_2}v_1 + \bar{G}_1R_{\theta_1} + O(v_1^2) \\ &= (Id. + G_1 - \bar{G}_1)R_{\theta_1} + O(v_1^2) \\ &= R_{\theta_2} + O_2(G_1, v_1), \end{aligned}$$

where the last inequality follows from

$$Id. + G_1 - \bar{G}_1 - R_{\theta_2-\theta_1} = O(G_1^2). \tag{20}$$

Next, (18) follows by the first inequality ($\|e^{v_1} - Id.\|_t \leq \|(R_{2\theta_1} - Id.)^{-1} G_1 R_{-\theta_1}\|_t$) and by Lemma 13 of the Appendix. Note that for all $t \in \mathbb{N}$, by (18) and convexity:

$$\begin{aligned} \|G_2\|_t &\leq C(t)(\|v_1\|_t\|v_1\|_0 + \|G_1\|_0\|v_1\|_t + \|G_1\|_t\|v_1\|_0 + \|G_1\|_0\|G_1\|_t) \\ &\leq C(t)\epsilon\|v_1\|_t \leq C(t)\epsilon \max\{\|G_1\|_t\|\theta_1\|_0\bar{C}, \|G_1\|_0\|\theta_1\|_t\bar{C}^{2t+1}\}. \end{aligned} \tag{21}$$

In particular, for ϵ small enough the C^0 norm of the new error term gets smaller, and $\|G_2\|_0 \leq C\epsilon\|G_1\|_0$. Finally, the Lemma follows by iterating the scheme, with $\tilde{B}_1 := \lim e^{v_n} \dots e^{v_1}$, $\tilde{\phi}_1 = \lim \theta_n$. We prove at first the estimates of the Lemma in norm C^0 and then for higher norms. Note that if for $i = 1, \dots n$:

$$\|G_i\|_0 \leq C(\bar{C}\epsilon)^{i-1}\|G_1\|_0, \|v_{i-1}\|_0 \leq \bar{C}\|G_{i-2}\|_0.$$

then:

$$\|v_n\|_0 \leq C\bar{C}\|G_{n-1}\|_0. \tag{22}$$

Moreover, as in (17) we have for all $n \in \mathbb{N}$:

$$\|\theta_n - \theta_{n-1}\|_0 \leq \|G_{n-1}\|_0 \tag{23}$$

and:

$$\|G_{n+1}\|_0 \leq C(\|v_n\|_0^2 + \|G_{n-1}\|_0\|v_{n-1}\|_0 + \|G_1\|_0^2) \leq C(\bar{C}\epsilon)^n\|G_1\|_0. \tag{24}$$

In particular:

$$\|\tilde{\phi} - \tilde{\phi}_1\|_0 \leq \sum_{n\in\mathbb{N}} \|\theta_n - \theta_{n-1}\|_0 \leq C\|G_1\|_0 \sum_{n\in\mathbb{N}} (\bar{C}\epsilon)^{n-1} \leq C\|G_1\|_0.$$

and:

$$\|B_1 - Id.\|_0 \leq \sum_{n\in\mathbb{N}} \|v_n\|_0 \leq C\bar{C}\|G_1\|_0 \sum_{n\in\mathbb{N}} (\bar{C}\epsilon)^{n-1} \leq C\bar{C}\|G_1\|_0.$$

In particular, the estimates of the lemma hold for $t = 0$. Now let $t > 0$. Then:

$$\begin{aligned} \|v_n\|_t &\leq \|(R_{2\theta_n} - Id)^{-1} G_n R_{-\theta_n}\|_t \\ &\leq C(t)(\|G_n\|_t\|\theta_n\|_0\bar{C} + \|G_n\|_0\|\theta_n\|_t\bar{C}^{2t+1}) \\ &\leq C(t)(\|G_n\|_t\bar{C} + \bar{C}^{2t+1}(\bar{C}\epsilon)^{n-1}\|G_1\|_0\|\theta_n\|_t). \end{aligned} \tag{25}$$

Moreover, as in (17):

$$\|\theta_n - \theta_{n-1}\|_t \leq \|G_{n-1}\|_t. \tag{26}$$

Finally, as in (21), by (22) and (24) we have:

$$\begin{aligned} \|G_n\|_t &\leq C(t)(\|v_{n-1}\|_t\|v_{n-1}\|_0 + \|G_{n-1}\|_0\|v_{n-1}\|_t \\ &\quad + \|G_{n-1}\|_t\|v_{n-1}\|_0 + \|G_{n-1}\|_0\|G_{n-1}\|_t) \\ &\leq C(t)\bar{C}(\bar{C}\epsilon)^{n-1}(\|G_{n-1}\|_t + \bar{C}^{2t}(\bar{C}\epsilon)^{n-2}\|\theta_{n-1}\|_t), \end{aligned} \tag{27}$$

So, combining (27), (26) and (25) we get:

$$\begin{aligned}\|B_1 - Id.\|_t &\le C(t)\sum_{n\in\mathbb{N}}\|v_n\|_t \\ &\le C(t)\bar{C}^{2t+1}(\|G_1\|_t\|\theta_1\|_0 + \|G_1\|_0\|\theta_1\|_t)\sum_{n\in\mathbb{N}}(\bar{C}\epsilon)^{n-1} \\ &\le C(t)\bar{C}^{2t+1}(\|G_1\|_t\|\theta_1\|_0 + \|G_1\|_0\|\theta_1\|_t).\end{aligned}$$

In the same way, by (25), (26) and (27) we have:

$$\|\tilde{\phi} - \tilde{\phi}_1\|_t \le \sum_{n\in\mathbb{N}}\|\theta_{n+1} - \theta_n\|_t \le \sum_{n\in\mathbb{N}}\|G_n\|_t \le C(t)\bar{C}^{2t+1}(\|G_1\|_t\|\theta_1\|_0 + \|G_1\|_0\|\theta_1\|_t).$$

□

Lemma 3. *Let* $\phi \in C^\infty(\mathbb{T}, \mathbb{R})$ *such that* $\inf_{x\in\mathbb{T}}|R_{2\phi(x)} - Id.| \ge \frac{C}{n^2}$. *For* $t \in \mathbb{N}$*:*

$$|D^t(R_{2\phi} - Id.)^{-1}|_0 \le C(t)n^{2(t+1)}\|\phi\|_t,$$

Proof. It follows by Lemma 15 of the Appendix by applying Hadamard's inequality to each term of the homogeneous polynomials. □

Proof of Proposition 2. By (11), we have that $\|A\|_0 < D$, $\|\bar{F}\| < \epsilon\min\{1, \|R_{2\bar{\phi}} - Id.\|_0\}$. Hence, Lemma 2 gives $\tilde{B}_1, \tilde{\phi}_1$ such that (15) and (16) hold. Then, by (15) of Lemma 2, and Lemma 3 and Leibnitz formula, we get for $h \in \mathbb{N}$

$$|D^h(\tilde{B}_1 - Id)|_0, \le C(h)n^{2(h+1)}\sum_{h_1+h_2+h_3=h}|D^{h_1}\bar{F}|_0\|\bar{\phi}\|_{h_2}\|\bar{\phi}\|_{h_3}.$$

Hence, by interpolation inequalities (see Lemma 16 in the Appendix), we get for $t \in \mathbb{N}$:

$$|D^t(\tilde{B}_1 - Id.)|_0 \le C(t)n^{2(t+1)}\max_{\beta\in\{0,1\}}\|\bar{F}\|_t^\beta\left(\|\bar{F}\|_0\|\bar{\phi}\|_t\right)^{1-\beta}. \tag{28}$$

In the same way, (15) implies

$$\|\bar{\phi} - \phi_1\|_t \le C(t)n^{2(t+1)}\max_{\beta\in\{0,1\}}\|\bar{F}\|_t^\beta\left(\|\bar{F}\|_0\|\bar{\phi}\|_t\right)^{1-\beta}.$$

Now, let

$$F_1(x) = \tilde{B}_1(x + q_n\alpha)\bar{A}(x)\tilde{B}_1^{-1}(x) - R_{\phi_1(x)} = (\tilde{B}_1(x + q_n\alpha) - \tilde{B}_1(x))\bar{A}(x)\tilde{B}_1^{-1}(x).$$

We want to show that, for $l = 0, 1$:

$$\|F_1\|_t \le \frac{C(t)n^{2(t+1+l)}}{q_{n+1}^l}\max_{\beta\in\{0,1\}}\|\bar{F}\|_{t+l}^\beta\left(\|\bar{F}\|_0\|\bar{\phi}\|_{t+l}\right)^{1-\beta} \tag{29}$$

We prove at first (29) for $l = 1$.

Note that: $|D^t(\tilde{B}_1(x+q_n\alpha)-\tilde{B}_1(x))|_0 \le \frac{1}{q_{n+1}}|D^{t+1}(\tilde{B}_1-Id.)|_0$. Then:

$$\begin{aligned}
|D^t F_1|_0 &\le \sum_{t_1+t_2+t_3=t} |D^{t_1}(\tilde{B}_1(x+q_n\alpha)-\tilde{B}_1(x))|_0 |D^{t_2}\bar{A}|_0 |D^{t_3}\tilde{B}_1^{-1}|_0 \\
&\le \frac{1}{q_{n+1}} \sum_{t_1+t_2+t_3=t} |D^{t_1+1}(\tilde{B}_1-Id.)|_0 |D^{t_2}\bar{A}|_0 |D^{t_3}\tilde{B}_1^{-1}|_0 \\
&\le \frac{C(t)n^{2(t+2)}}{q_{n+1}} \max_{\beta\in\{0,1\}} \|\bar{F}\|_{t+1}^{\beta} \left(\|\bar{F}\|_0\|\bar{\phi}\|_{t+1}\right)^{1-\beta},
\end{aligned}$$

with the last inequality that follows from (28).

Now we prove (29) for $l=0$:

$$\begin{aligned}
|D^t F_1|_0 \le &\sum_{t_1+t_2+t_3=t, t_1>0} |D^{t_1}(\tilde{B}_1(x+q_n\alpha)-\tilde{B}_1(x))|_0 |D^{t_2}\bar{A}|_0 |D^{t_3}\tilde{B}_1^{-1}|_0 \\
&+ \sum_{t_2+t_3=t} |(\tilde{B}_1(x+q_n\alpha)-\tilde{B}_1(x))|_0 |D^{t_2}\bar{A}|_0 |D^{t_3}\tilde{B}_1^{-1}|_0.
\end{aligned}$$

Note that, from the fact that B_1 is close to the identity, if $Y_1 := B_1 - Id.$, we have:

$$\|B_1^{-1}-Id\|_t \le C(t)\|B_1-Id.\| + C(t)\|Y_1^2\|_t$$

and:

$$\|Y_1^2\|_t \le C(t)\|Y_1\|_0\|Y_1\|_t.$$

In particular:

$$\|B_1^{-1}-Id.\|_t \le C(t)\|B_1-Id.\|_t.$$

Now, for the first term we get:

$$\begin{aligned}
&\sum_{t_1+t_2+t_3=t, t_1>0} |D^{t_1}(\tilde{B}_1(x+q_n\alpha)-\tilde{B}_1(x))|_0 |D^{t_2}\bar{A}|_0 |D^{t_3}\tilde{B}_1^{-1}|_0 \\
&\le 2\sum_{t_1+t_2+t_3=t} |D^{t_1}\tilde{B}_1|_0 |D^{t_2}\bar{A}|_0 |D^{t_3}\tilde{B}_1^{-1}|_0 \le C(t)n^{2(t+2)} \max_{\beta\in\{0,1\}} \|\bar{F}\|_t^{\beta}\left(\|\bar{F}\|_0\|\bar{\phi}\|_t\right)^{1-\beta}.
\end{aligned}$$

For the second term we get:

$$\begin{aligned}
&\sum_{t_2+t_3=t} |(\tilde{B}_1(x+q_n\alpha)-\tilde{B}_1(x))|_0 |D^{t_2}\bar{A}|_0 |D^{t_3}\tilde{B}_1^{-1}|_0 \\
&\le \frac{1}{q_{n+1}} \sum_{t_2+t_3=t} |D(\tilde{B}_1-Id.)|_0 |D^{t_2}\bar{A}|_0 |D^{t_3}\tilde{B}_1^{-1}|_0 \\
&\le C(t)n^{2(t+2)} \max_{\beta\in\{0,1\}} \|\bar{F}\|_t^{\beta}\left(\|\bar{F}\|_1\|\bar{\phi}\|_t\right)^{1-\beta}.
\end{aligned}$$

Now, suppose that we iterated the scheme m times, with $m \le r_0$. For $k \le m$ let:

$$\begin{aligned}
&\tilde{\tilde{B}}_k = \tilde{B}_k \dots \tilde{B}_1,\\
&R_{\tilde{\phi}_k(x)} = \tilde{\tilde{B}}_k(x)\bar{A}(x)\tilde{\tilde{B}}_k^{-1}(x),\\
&\tilde{F}_k(x) = (\tilde{\tilde{B}}_k(x+q_n\alpha) - \tilde{\tilde{B}}_k(x))\bar{A}(x)\tilde{\tilde{B}}_k^{-1}(x).
\end{aligned}$$

Then, in the same way as above, for $l = 0, 1$ and $k \le m$:

$$\|\tilde{F}_k\|_t, \|\tilde{\phi}_k - \tilde{\phi}_{k-1}\|_t \le \frac{C(t)n^{2(t+1+l)}}{q_{n+1}^l} \max_{\beta\in\{0,1\}} \|\tilde{F}_{k-1}\|_{t+l}^{\beta} \left(\|\tilde{F}_{k-1}\|_1 \|\tilde{\phi}_{k-1}\|_{t+l}\right)^{1-\beta}, \tag{30}$$

In particular, from (30) applied with $l = 0$ and (11) we get:

$$\|\tilde{F}_k\|_{r_0}, \|\tilde{\phi}_k\|_{r_0} \le C(r_0)n^{4(r_0+1)k},$$

$$\begin{aligned}
\|\tilde{F}_k\|_1, \|\tilde{\phi}_k - \tilde{\phi}_{k-1}\|_1 &\le \frac{n^{4r_0}}{q_{n+1}} \max_{\beta\in\{0,1\}} \|\tilde{F}_{k-1}\|_{r_0}^{\beta} \left(\|\tilde{F}_{k-1}\|_1 \|\tilde{\phi}_{k-1}\|_{r_0}\right)^{1-\beta}\\
&\le \frac{C(r_0)n^{4(r_0+1)(k+1)}}{q_{n+1}} < 1.
\end{aligned}$$

So, because $\tilde{B}_m(x+q_n\alpha)A(x)\tilde{B}_m(x)^{-1} = R_{\tilde{\phi}_m(x)} + \tilde{F}_m(x)$ satisfies:

$$\|R_{\tilde{\phi}_m} + \tilde{F}_m\|_0 < 2 < D, \|\tilde{F}_m\|_0 \le \epsilon \min\{1, \|R_{2\tilde{\phi}_m} - Id.\|_0\},$$

we can iterate the scheme one more time. Finally, Proposition 2 follows with:

$$\tilde{B} := \tilde{B}_{r_0+1} \dots \tilde{B}_1, \tilde{\phi} := \tilde{\phi}_{r_0+1}, \tilde{F} := \tilde{F}_{r_0+1}.$$

Indeed, in order to prove (12), we have:

$$\|\tilde{B} - Id.\|_t \le C(t) \sum_{k=1}^{r_0+1} \|\tilde{B}_k - Id.\|_t,$$

and for $k = 1, \dots, r_0 + 1$, by (30) with $l = 0$ we get:

$$\begin{aligned}
\|B_k - Id\|_t &\le C(t)n^{2(t+1)} \max_{\beta\in\{0,1\}} \|\tilde{F}_{k-1}\|_t^{\beta} \left(\|\tilde{F}_{k-1}\|_1 \|\tilde{\phi}_{k-1}\|_t\right)^{1-\beta}\\
&\le C(t)n^{2(t+1)r_0} \max_{\beta\in\{0,1\}} \|\bar{F}\|_t^{\beta} \left(\|\bar{F}\|_1 \|\bar{\phi}\|_t\right)^{1-\beta}.
\end{aligned}$$

In particular, we get:

$$\|\tilde{B} - Id.\|_t \le C(t)n^{2(t+1)r_0} \max_{\beta\in\{0,1\}} \|\bar{F}\|_t^{\beta} \left(\|\bar{F}\|_1 \|\bar{\phi}\|_t\right)^{1-\beta}.$$

Finally, (14) follows for any $0 \le h \le r_0 + 1$ by iterating (30) for h times with $l = 1$ and for $r_0 + 1 - h$ times with $l = 0$. □

4. Estimates for the Iterated Cocycle

Now we start by giving some bounds for the derivatives of the iterates of the cocycle. In the following proposition we will show that if $A = R_\phi + F$ is quite close to a rotation valued cocycle (F small), then also $A^{(q_n)}$ remains close to the rotation valued cocycle $R_{S_{q_n}\phi}$. In Sect. 4.2, we will see that $S_{q_n}\phi$ is close to its average (that is close to $q_n\rho$ with $\rho = \rho(\alpha, A)$).

4.1. Estimates of the derivatives of the iterated cocycle . The goal of this section is to prove the following

Proposition 3. *Let $A = R_\phi + F \in C^\infty(\mathbb{T}, SL(2,\mathbb{R}))$ be such that:*

$$\|F\|_0 < \frac{1}{q_n^6}, \quad \|\phi - \hat{\phi}(0)\|_0 < 1. \tag{31}$$

Let ξ be defined such that $A^{(q_n)} = R_{S_{q_n}\phi} + \xi$. Then, for $t \in \mathbb{N}$:

$$\|\xi\|_t \leq C(t) q_n^5 \left(\|F\|_t + \|F\|_0 \max_{\beta\in\{0,1\}} \|F\|_t^\beta \|\phi\|_t^{1-\beta} \right). \tag{32}$$

Proof of Proposition 3. Write $\xi := A^{(q_n)} - R_{S_{q_n}\phi}$ as:

$$\xi(x) = \sum_{h=1}^{q_n} \sum_{q_n-1\geq i_1>i_2>\ldots>i_h\geq 0} \left(\prod_{j=q_n-1}^{i_1+1} R_{\phi(x+j\alpha)} \right) F(x+i_1\alpha)\ldots$$
$$F(x+i_h\alpha) \left(\prod_{j=i_h-1}^{0} R_{\phi(x+j\alpha)} \right),$$

Then:

$$\xi(x) = \sum_{h=1}^{q_n} \sum_{q_n-1\geq i_1>i_2>\ldots>i_h\geq 0} R_{S_{(q_n-i_1-1)}\phi(x+(i_1+1)\alpha)} F(x+i_1\alpha)$$
$$R_{S_{(i_1-i_2-1)}\phi(x+(i_2+1)\alpha)} F(x+i_2\alpha)\ldots R_{S_{(i_h-1)}\phi(x)}.$$

So, let $t \in \mathbb{N}$. For each $1 \leq h \leq q_n$ we have the sum of $\binom{q_n}{h}$ terms on the form:

$$R_{S_{j_1}\phi(x+(i_1+1)\alpha)} F(x+i_1\alpha)\ldots F(x+i_h\alpha) R_{S_{j_{h+1}}\phi(x)},$$

for some $0 \leq j_l \leq q_n$ and for $1 \leq l \leq h+1$ (and with the same convention $R_{S_0\phi} := Id$). In particular, by Leibnitz formula, the t-h derivative of each of these terms is the sum of at most $(2h+1)^t$ terms of the form:

$$D^{t_1} R_{S_{j_1}\phi(x+(i_1+1)\alpha)} D^{t_2} F(x+i_1\alpha)\ldots D^{t_{2h}} F(x+i_h\alpha) D^{t_{2h+1}} R_{S_{j_{h+1}}\phi(x+(i_h-1)\alpha)},$$

with $t_1 + \ldots t_{2h+1} = t$. If $t_{2l} \neq 0$, by the interpolation Lemma 16 of the Appendix, we get

$$|D^{t_{2l}} F|_0 \leq C\|F\|_0^{1-\frac{t_{2l}}{t}} \|F\|_t^{\frac{t_{2l}}{t}}.$$

In the same way, if $t_{2l+1} \neq 0$, then by Lemma 17:

$$|D^{t_{2l+1}} R_{S_{j_{2l+1}}\phi}|_0 \leq C(t) q_n \|\phi\|_{t_{2l+1}} \leq C(t) q_n \|\phi\|_0^{1-\frac{t_{2l+1}}{t}} \|\phi\|_t^{\frac{t_{2l+1}}{t}}.$$

Moreover, by the assumptions we have $\|\phi - \hat{\phi}(0)\|_0 < 1$. So, it follows that:

$$\begin{aligned}|D^t \xi|_0 &\leq C(t) q_n^3 \|\phi\|_0 \|F\|_t + \sum_{h=2}^{q_n} \binom{q_n}{h} (2h+1)^t q_n^{h+1} C(t)^{2h+1} \|F\|_0^{h-1} \sup_{\alpha \in [0,1]} \|F\|_t^{\alpha} \|\phi\|_t^{1-\alpha} \\ &\leq C(t) q_n^3 \|F\|_t + \|F\|_0 \sup_{\alpha \in [0,1]} \|F\|_t^{\alpha} \|\phi\|_t^{1-\alpha} \sum_{h=2}^{q_n} \frac{q_n^h}{h!} (2h+1)^t q_n^{h+1} C(t)^{2h+1} \|F\|_0^{h-2} \\ &\leq C(t) q_n^3 \|F\|_t + \|F\|_0 \sup_{\alpha \in [0,1]} \|F\|_t^{\alpha} \|\phi\|_t^{1-\alpha} \sum_{h=2}^{q_n} \frac{q_n^h}{h!} (2h+1)^t q_n^{h+1} C(t)^{2h+1} \|F\|_0^{h-2}.\end{aligned}$$

Now, because of $\|F\|_0 < \frac{1}{q_n^6}$, for $h \geq 2$ we get:

$$q_n^{2h+1} \|F\|_0^{h-2} \leq \frac{q_n^7}{q_n^h} = \frac{q_n^5}{q_n^{h-2}}. \tag{33}$$

So, by (33):

$$\begin{aligned}\sum_{h=2}^{q_n} \frac{q_n^h}{h!} (2h+1)^t q_n^{h+1} C(t)^{2h+1} \|F\|_0^{h-2} &\leq \sum_{h=2}^{q_n} \frac{q_n^h}{h!} (2h+1)^t q_n^{h+1} C(t)^{2h+1} ||F||_0^{h-2} \\ &\leq q_n^5 \||\phi\|_t \sum_{h=0}^{+\infty} \left(\frac{C(t)}{q_n}\right)^h \frac{1}{h!} \leq C(t) q_n^5.\end{aligned} \tag{34}$$

□

4.2. Estimates of the Birkhoff sums. The goal of this section is to prove the following

Proposition 4. *Let* $\xi := A^{(q_n)} - R_{S_{q_n}\phi}$, ϵ *as in the statement of Theorem 1, and suppose that* $\|2q_n\rho\| \geq \frac{\epsilon}{n^2}$, $\|\phi\|_7 \leq 1$, $\|\xi\|_0 < \frac{1}{q_n}$. *If* $q_{n+1} > q_n^2$ *or there exists* k *such that* $q_k^2 < q_n < q_k^4$ *then, for* $t \geq 0$*:*

$$|D^t (R_{2S_{q_n}\phi} - Id.)^{-1}|_0 \leq C(t) n^{2(t+1)} \|S_{q_n}\phi\|_t.$$

We first show in Lemmas 4 and 5 that, under certain conditions on q_n, the Birkhoff sum $S_{q_n}\phi$ is close to its average. In the proofs, we will need Fourier truncation and rest operators that we now define.

Definition 2. Let $f \in C^{\infty}(\mathbb{T}, \mathbb{R})$, $a > 0$.

$$T_a(f) := \sum_{|l| \leq a} \hat{f}(l) e^{2\pi i l x}, \quad R_a(f) := \sum_{|l| > a} \hat{f}(l) e^{2\pi i l x}.$$

Lemma 4. *Suppose that there exists q_k such tha $q_k^2 < q_n < q_k^4$. Then, for $t \in \mathbb{N}$:*

$$|D^t(S_{q_n}\phi - q_n\hat{\phi}(0)|_0 \leq C(t)\min\{q_n\|\phi - \hat{\phi}(0)\|_t, \frac{1}{q_n^{\frac{1}{4}}}\|\phi - \hat{\phi}(0)\|_{t+7}\}.$$

Proof. Let $t \in \mathbb{N}$. The inequality

$$|D^t(S_{q_n}\phi - q_n\hat{\phi}(0)|_0 \leq q_n C(t)\|\phi - \hat{\phi}(0)\|_t$$

is trivial. Then, for the second inequality:

$$|D^t(S_{q_n}\phi - q_n\hat{\phi}(0))|_0 \leq |D^t T_{q_n^{\frac{1}{4}}}(S_{q_n}\phi - q_n\hat{\phi}(0))|_0 + |D^t R_{q_n^{\frac{1}{4}}}(S_{q_n}\phi - q_n\hat{\phi}(0))|_0$$

So:

$$\begin{aligned}|D^t T_{q_n^{\frac{1}{4}}}(S_{q_n}\phi - q_n\hat{\phi}(0))|_0 &\leq \sum_{1\leq|l|\leq q_n^{\frac{1}{4}}} |2\pi l|^t \left|\frac{e^{2\pi i q_n l\alpha} - 1}{e^{2\pi i l\alpha} - 1}\right| |l\hat{\phi}(l)| \\ &\leq \frac{q_k}{q_{n+1}} \sum_{1\leq|l|\leq q_n^{\frac{1}{4}}} |2\pi l|^{t+1}|\hat{\phi}(l)|,\end{aligned}$$

where in the last inequality we have used the fact that for $|l| \leq q_n^{\frac{1}{4}}$, $|e^{2\pi i q_n l\alpha} - 1| \leq \frac{2\pi|l|}{q_{n+1}}$ and $|e^{2\pi i l\alpha} - 1| \geq \frac{1}{q_k}$ (that follows by $|l| \leq q_n^{\frac{1}{4}} < q_k$). Finally:

$$\frac{q_k}{q_{n+1}} \sum_{1\leq|l|\leq q_n^{\frac{1}{4}}} |2\pi l|^{t+1}|\hat{\phi}(l)| \leq \frac{C q_n^{\frac{1}{2}}}{q_{n+1}}\|\phi - \hat{\phi}(0)\|_{t+3},$$

with the last inequality that follows by $q_k \leq q_n^{\frac{1}{2}}$ and $|\hat{\phi}(l)| \leq \frac{\|\phi - \hat{\phi}(0)\|_{t+3}}{|2\pi l|^{t+3}}$. Moreover:

$$|D^t R_{q_n^{\frac{1}{4}}}(S_{q_n}\phi - q_n\hat{\phi}(0))|_0 \leq q_n|D^t R_{q_n^{\frac{1}{4}}}(\phi - \hat{\phi}(0))|_0 \leq \frac{q_n}{q_n^{\frac{5}{4}}}\|\phi - \hat{\phi}(0)\|_{t+7},$$

with the last inequality that follows by Lemma 18. Then:

$$\begin{aligned}|D^t(S_{q_n}\phi - q_n\hat{\phi}(0))|_0 &\leq |D^t T_{q_n^{\frac{1}{4}}}(S_{q_n}\phi - q_n\hat{\phi}(0))|_0 + |D^t R_{q_n^{\frac{1}{4}}}(S_{q_n}\phi - q_n\hat{\phi}(0))|_0 \\ &\leq \frac{C q_n^{\frac{1}{2}}}{q_{n+1}}\|\phi - \hat{\phi}(0)\|_{t+3} + \frac{q_n}{q_n^{\frac{5}{4}}}\|\phi - \hat{\phi}(0)\|_{t+7} \leq \frac{C}{q_n^{\frac{1}{4}}}\|\phi - \hat{\phi}(0)\|_{t+7}.\end{aligned}$$

□

Lemma 5. *If $q_{n+1} > q_n^2$, then for $t \in \mathbb{N}$:*

$$|D^t(S_{q_n}\phi - q_n\hat{\phi}(0))|_0 \leq \frac{C}{q_n}\|\phi - \hat{\phi}(0)\|_{t+4}.$$

Proof. Let $t \in \mathbb{N}$:

$$|D^t(S_{q_n}\phi - \hat{\phi}(0))|_0 \leq |D^t T_{q_n-1}(S_{q_n}\phi - q_n\hat{\phi}(0))|_0 + |D^t R_{q_n-1}(S_{q_n}\phi - q_n\hat{\phi}(0))|_0$$

Then:

$$\begin{aligned}|D^t T_{q_n-1}(S_{q_n}\phi - q_n\hat{\phi}(0))|_0 &\leq \sum_{1\leq|l|<q_n} |2\pi l|^t \left|\frac{e^{2\pi i q_n l\alpha} - 1}{e^{2\pi i l\alpha} - 1}\right| |\hat{\phi}(l)| \\ &\leq \frac{Cq_n}{q_{n+1}} \sum_{1\leq|l|<q_n} |2\pi l|^{t+1} \frac{|D^{t+3}\phi|_0}{|l|^{t+3}} \leq \frac{C}{q_n}\|\phi - \hat{\phi}(0)\|_{t+3}.\end{aligned}$$

Now we estimate the second term:

$$\begin{aligned}|D^t R_{q_n-1}(S_{q_n}\phi - q_n\hat{\phi}(0))|_0 &\leq q_n|D^t R_{q_n-1}(\phi - \hat{\phi}(0))|_0 \\ &\leq \frac{Cq_n}{q_n^2}\|\phi - \hat{\phi}(0)\|_{t+4},\end{aligned}$$

with the last inequality that follows by Lemma 18. □

Proof of Proposition 4. Let $\xi := A^{(q_n)} - R_{S_{q_n}\phi}$. By the definition of the fibered rotation number, we have that

$$|\rho(R_{S_{q_n}\phi} + \xi) - q_n\hat{\phi}(0)| \leq |R_{S_{q_n}\phi} - R_{q_n\hat{\phi}(0)}|_0 + |\xi|_0. \tag{35}$$

Moreover, by Lemmas 4 and 5:

$$|R_{S_{q_n}\phi} - R_{q_n\hat{\phi}(0)}|_0 \leq \frac{C\|\phi\|_7}{q_n^{\frac{1}{4}}} \leq \frac{C}{q_n^{\frac{1}{4}}},$$

Then, the proposition follows by Lemma 3, (35) and the assumptions $\|\xi\|_0 < \frac{1}{q_n}$ and $\|2q_n\rho\| \geq \frac{\epsilon}{n^2}$. □

5. Proof of Proposition 1

5.1. Applying the cheap trick to the iterated cocycle. The estimates in (7) for $h = 0$ are trivially satisfied by the hypothesis in Theorem 1.

Now, suppose that we are at the h-step so that we have the cocycle $A_h = R_{\phi_h} + F_h$ such that the estimates in (7) hold, that is

$$\|F_h\|_1 \leq \frac{\epsilon_0}{q_{n_h}^{\frac{r_0}{8}}}, \quad \|F_h\|_{50r_0}, \|\phi_h\|_{50r_0} \leq s_h^6, \tag{36}$$

with $s_h = \prod_{l=0}^h q_{n_l}$. We want to find $B_h, F_{h+1} \in C^\infty(\mathbb{T}, SL(2,\mathbb{R}))$, $\phi_{h+1} \in C^\infty(\mathbb{T}, \mathbb{R})$ such that (6), (7), (8) hold. We have

Proposition 5. *There exist* $B_h, \tilde{F} \in C^\infty(\mathbb{T}, SL(2,\mathbb{R}))$, $\tilde{\phi} \in C^\infty(\mathbb{T},\mathbb{R})$ *with:*

$$\|B_h - Id.\|_t, \|\tilde{\phi} - \bar{\phi}\|_t \leq C(t) n^{2r_0(t+1)} q_{n_h}^5 \max_{\beta\in\{0,1\}} \|F_h\|_t^\beta \left(\|F_h\|_1 \|\bar{\phi}\|_t\right)^{1-\beta} \tag{37}$$

such that:

$$B_h(x + q_{n_h}\alpha) A^{(q_{n_h})}(x) B_h^{-1}(x) = R_{\tilde{\phi}} + \tilde{F}, \tag{38}$$

and, for $0 \leq l \leq r_0 + 1$*:*

$$\|\tilde{F}\|_t \leq \frac{C(t+l) q_{n_h}^5 n_h^{2r_0(t+2)}}{q_{n_h+1}^l} \max_{\beta\in\{0,1\}} \|F_h\|_{t+l}^\beta \left(\|F_h\|_1 q_{n_h} \|\phi_h\|_{t+l}\right)^{1-\beta}. \tag{39}$$

In particular:

$$\|\tilde{F}\|_1 \leq \frac{C q_{n_h}^5 n_h^{4r_0}}{q_{n_{h+1}}^{r_0}} \max_{\beta\in\{0,1\}} \|F_h\|_{r_0+1}^\beta \left(\|F_h\|_1 q_{n_h} \|\phi_h\|_{r_0+1}\right)^{1-\beta} \leq \frac{C q_{n_h}^5 n_h^{4r_0}}{q_{n_{h+1}}^{r_0}}, \tag{40}$$

and

$$\|\tilde{F}\|_{50r_0}, \|\tilde{\phi} - \bar{\phi}\|_{50r_0} \leq C(r_0) q_{n_h}^5 n_h^{2r_0(50r_0+2)} \max_{\beta\in\{0,1\}} \|F_h\|_{50r_0}^\beta \left(\|F_h\|_1 q_{n_h} \|\phi_h\|_{50r_0}\right)^{1-\beta} \tag{41}$$

Moreover,

$$\|\tilde{F}\|_{50r_0}, \|\tilde{\phi} - \bar{\phi}\|_{50r_0} \leq C(r_0) q_{n_h}^5 s_h^6 n_h^{2r_0(50r_0+2)}. \tag{42}$$

Proof. By Lemmas 4, 5 we get:

$$\|S_{q_{n_h}}\phi_h - q_{n_h}\hat{\phi}_h(0)\|_0 \leq \frac{C}{q_{n_h}^{\frac{1}{4}}} \|\phi_h - \hat{\phi}_h(0)\|_7 \leq \frac{1}{q_{n_h}^{\frac{1}{4}}}. \tag{43}$$

Then, by the arithmetic condition $\|2q_{n_h}\rho\| \geq \frac{C}{n_h^2}$, (43), and Proposition 4 we get:

$$\|(R_{2S_{q_{n_h}}\phi} - Id.)^{-1}\|_0 \leq C n_h^2.$$

Moreover, if $\bar{F} = A_h^{(q_{n_{h+1}})} - R_{S_{q_{n_{h+1}}}\phi}$, by Proposition 3:

$$\|\bar{F}\|_0 \leq \frac{C}{q_{n_h}}.$$

In particular, we can apply Proposition 3 with $\bar{A} := A^{(q_{n_h})}$, $\bar{\phi} = S_{q_{n_h}}\phi$ to get $B_h, \tilde{F} \in C^\infty(\mathbb{T}, SL(2,\mathbb{R}))$, $\tilde{\phi} \in C^\infty(\mathbb{T},\mathbb{R})$ so that (37)–(41) hold.

By (36) and (40):

$$\|F_h\|_1 q_{n_h} \|\phi_h\|_{50r_0} < 1.$$

Then (42) follows from the estimates of $\|F_h\|_{50r_0}$ in (36).

5.2. Going backward from the renormalized cocycle to the starting one. Now let $B_h := \bar{B}_{r_0+1}...\bar{B}_1, \tilde{\phi}$ as in Proposition 5 (with $\bar{A} = A_h^{(q_{n_h})}, \bar{\phi} = S_{q_{n_h}}\phi_h$ and with $\bar{B}_1, ..., \bar{B}_{r_0+1}$ that are defined applying $r_0 + 1$ times the cheap trick). We want to show that, because (α, A_h) commutes with $(q_{n_h}\alpha, A_h^{(q_{n_h})})$, if $B_h(x + q_{n_h}\alpha)A_h^{(q_{n_h})}(x)B_h^{-1}(x)$ is close to a rotation valued cocycle, then also $B_h(x + \alpha)A_h(x)B_h^{-1}(x)$ is close to a rotation valued cocycle. Let also $\bar{B} := \bar{B}_{r_0+1}, \tilde{B} := \bar{B}_{r_0}...\bar{B}_1$, so that $B_h = \bar{B}\tilde{B}$.

Lemma 6.

$$\bar{B}(x)\tilde{B}(x + q_n\alpha)\bar{A}(x)\tilde{B}^{-1}(x)\bar{B}^{-1}(x) = R_{\tilde{\phi}(x)}.$$

Proof. It follows by definition of $\bar{B}, \tilde{B}, \phi_h$. □

Definition 3. Let:

$$J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

For $M \in SL(2, \mathbb{R})$,

$$Q(M) := \frac{M + JMJ}{2}.$$

In the following Lemma we state some properties of $Q(M)$ that are stated also in [2].

Lemma 7. *For $\theta \in \mathbb{R}$, $M \in SL(2, \mathbb{R})$, $R_\theta Q(M) = Q(R_\theta M) = Q(MR_{-\theta})$. Moreover, $M - Q(M)$ is of the form:*

$$M - Q(M) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

In particular, $M - Q(M)$ commutes with rotations.

Definition 4. $\tilde{A}(x) := B_h(x + \alpha)A_h(x)B_h^{-1}(x)$, $L(x) := Q(\tilde{A})$, $L_1(x) := Q(\tilde{A}(x + q_{n_h}\alpha) - \tilde{A}(x))$.

By Lemma 7:

$$Q(R_{\tilde{\phi}(x+\alpha)}\tilde{A}(x) - \tilde{A}(x)R_{\tilde{\phi}(x)}) = (R_{\tilde{\phi}(x+\alpha)} - R_{-\tilde{\phi}(x)})L(x).$$

Lemma 8.

$$R_{\tilde{\phi}(x+\alpha)}\tilde{A}(x) - \tilde{A}(x)R_{\tilde{\phi}(x)} = J_1(x) + J_2(x) + J_3(x),$$

with:

$$\begin{aligned} J_1(x) &:= (\bar{B}(x + \alpha) - \bar{B}(x + \alpha + q_{n_h}\alpha))\tilde{B}(x + q_{n_h}\alpha + \alpha)A_h(x + q_{n_h}\alpha) \\ &\quad B_h^{-1}(x + q_{n_h}\alpha)(R_{\tilde{\phi}(x)} + \tilde{F}(x)), \\ J_2(x) &:= (\tilde{A}(x + q_{n_h}\alpha) - \tilde{A}(x))(R_{\tilde{\phi}(x)} + \tilde{F}(x)), \\ J_3(x) &:= \tilde{A}(x)(\bar{B}(x) - \bar{B}(x + q_{n_h}\alpha))\tilde{B}(x + q_{n_h}\alpha)A_h^{(q_{n_h})}(x)B_h^{-1}(x). \end{aligned}$$

Proof. By Lemma 6

$$
\begin{aligned}
&R_{\tilde{\phi}(x+\alpha)}\tilde{A}(x) - \tilde{A}(x)R_{\tilde{\phi}(x)} \\
&= \bar{B}(x+\alpha)\tilde{B}(x+q_{n_h}\alpha+\alpha)A_h^{(q_{n_h})}(x+\alpha)A_h(x)B_h^{-1}(x) \\
&\quad -\bar{B}(x+\alpha)\tilde{B}(x+\alpha)A_h(x)\tilde{B}^{-1}(x)\tilde{B}(x+q_{n_h}\alpha)A_h^{(q_{n_h})}(x)B_h^{-1}(x) \\
&= \bar{B}(x+\alpha)\tilde{B}(x+q_{n_h}\alpha+\alpha)A_h(x+q_{n_h}\alpha)B_h^{-1}(x+q_{n_h}\alpha) \\
&\quad B_h(x+q_{n_h}\alpha)A_h^{(q_{n_h})}(x)B_h^{-1}(x) \\
&\quad -\bar{B}(x+\alpha)\tilde{B}(x+\alpha)A_h(x)\tilde{B}^{-1}(x)\tilde{B}(x+q_{n_h}\alpha)A_h^{(q_{n_h})}(x)B_h^{-1}(x) \\
&= J_1(x)+J_2(x)+J_3(x).
\end{aligned}
$$

□

Lemma 9. *Let $t \in \mathbb{N}$. Then:*

$$\|(R_{\tilde{\phi}(x+\alpha)} - R_{-\tilde{\phi}(x)})^{-1}R_{-\bar{\phi}(x)}\|_t < C(t)n_h^{2(t+1)}\|\tilde{\phi}\|_t.$$

Proof. By Lemmas 4, 5 and (43):

$$||\tilde{\phi} - q_n\rho||_0 \le \frac{C}{q_n^{\frac{1}{4}}}. \tag{44}$$

In particular:

$$
\begin{aligned}
&\|(R_{\tilde{\phi}(x+\alpha)} - R_{-\tilde{\phi}(x)})^{-1}R_{-\bar{\phi}(x)}\|_t \\
&= \|R_{\tilde{\phi}(x)}(R_{\tilde{\phi}(x+\alpha)+\tilde{\phi}(x)} - Id.)^{-1}R_{-\bar{\phi}(x)}\|_t \\
&\le \sum_{h_1+h_2+h_3=t} ||D^{h_1}R_{\tilde{\phi}(x)}||_0||D^{h_2}(R_{\tilde{\phi}(x+\alpha)+\tilde{\phi}(x)} - Id.)^{-1}||_0||||D^{h_3}R_{\tilde{\phi}(x)}||_0.
\end{aligned}
$$

By (44) and by the arithmetic condition on the fibered rotation number, for all $t \in \mathbb{N}$:

$$||D^t(R_{\tilde{\phi}(x+\alpha)+\tilde{\phi}(x)} - Id.)^{-1}||_0 \le C(t)n_h^{2(t+1)}||\tilde{\phi}||_t. \tag{45}$$

So, by (45) we get:

$$
\begin{aligned}
&\|(R_{\tilde{\phi}(x+\alpha)} - R_{-\tilde{\phi}(x)})^{-1}R_{-\bar{\phi}(x)}\|_t \\
&\le C(t)\sum_{h_1+h_2+h_3=t} C(t)n_h^{2(t+1)}||\tilde{\phi}||_{h_1}||\tilde{\phi}||_{h_2}||\tilde{\phi}||_{h_3} \le C(t)n_h^{2(t+1)}||\tilde{\phi}||_t,
\end{aligned}
$$

with the last inequality that follows by convexity. □

Lemma 10. *For $t \le r_0 - 7$:*

$$
\begin{aligned}
&\|\bar{B}\|_t, \|\tilde{B}\|_t, \|\tilde{\phi}\|_t, \|\tilde{F}\|_t, \|\tilde{A}\|_t, \|A_h^{(q_{n_h}}\|_t \le C, \\
&\|\bar{B}(x) - \bar{B}(x+q_{n_h}\alpha)\|_t \le C\|\tilde{F}\|_t.
\end{aligned}
$$

In particular:

$$\|Q(J_1)\|_t, \|Q(J_3)\|_t \le C\|\tilde{F}\|_t, \|Q(J_2)\|_t \le C\|L_1\|_t.$$

Proof. It follows directly by Proposition 5. □

Let B_h, $\tilde{F}$, $\tilde{\phi}$ defined as in Proposition 4, L, L_1, $\bar{A}$ as in Definition 4. Let:

$$A_{h+1}(x) := B_h(x+\alpha)A_h(x)B_h^{-1}(x), \quad R_{\phi_{h+1}} := \frac{A_{h+1}-L}{(\det(A_{h+1}-L))^{\frac{1}{2}}}, F_{h+1} = A_{h+1} - R_{\phi_{h+1}}.$$

The fact that $R_{\phi_{h+1}}$ is a rotation follows by Lemma 7. The estimates for B_h in (8) follow by Proposition 5. By Lemmas 8, 10 and 9, we get the following Lemma:

Lemma 11. *Let $t \le r_0 - 7$. Then:*

$$\|L\|_t \le Cn_h^{2(t+1)}\|L_1\|_t + Cn_h^{2(t+1)}\|\tilde{F}\|_t \le \frac{Cn_h^{2(t+1)}}{q_{n_h+1}}\|L\|_{t+1} + Cn_h^{2(t+1)}\|\tilde{F}\|_t.$$

We also have

Lemma 12. *For $0 \le j \le r_0 - 7$:*

$$\|\tilde{F}\|_j \le \frac{1}{q_{n_h+1}^{r_0-j}}.$$

Proof. It follows by:

$$\|\tilde{F}\|_j \le C\|\tilde{F}\|_{50r_0}^{\frac{j}{50r_0}}\|\tilde{F}\|_0^{1-\frac{j}{50r_0}},$$

the estimates of $\|\tilde{F}\|_0$, $\|\tilde{F}\|_{50r_0}$ in (40), Lemma 1 and (42). □

From Lemma 11, it follows that for $t + l \le r_0$:

$$\|L\|_t \le \frac{Cn_h^{2r_0 l(r_0+1)}}{q_{n_h+1}^l}\|L\|_{t+l} + \sum_{j=0}^{l-1}\frac{Cn_h^{2r_0 j(r_0+1)}}{q_{n_h+1}^j}\|\tilde{F}\|_{t+j}. \tag{46}$$

Moreover:

$$\|L\|_{r_0} \le \|\tilde{A}\|_{r_0} \le s_{h+1}^6. \tag{47}$$

Then, by definition of F_{h+1}, Lemma 12, (46), (47) it follows that:

$$\|F_{h+1}\|_0 < \frac{1}{q_{n_h+1}^{\frac{r_0}{2}}}, \|F_{h+1}\|_{50r_0} \le s_{h+1}^6$$

Finally, by Proposition 5, the estimates for L and the definition of ϕ_{h+1} and Lemma 1, the estimate (8) in Proposition 1 for F_{h+1}, ϕ_{h+1} follow. Indeed, note that by Proposition 5, the estimates for B_h hold. Moreover:

$$L(x) = Q(B_h(x+\alpha)(R_{\phi_h(x)} + F_h(x))B_h^{-1}(x)) = Q(R_{\phi_h(x)} + F_h(x))$$
$$+Q((B_h(x+\alpha) - Id.)(R_{\phi_h(x)} + F_h(x))B_h^{-1}(x))$$
$$Q((R_{\phi_h(x)} + F_h(x))(B_h^{-1}(x) - Id.)).$$

Note that $Q(R_{\phi_h(x)}) = 0$. Then, by usual convexity estimates it follows that for all $t \in \mathbb{N}$:

$$\begin{aligned}||Q(A_{h+1})||_t &\leq \sum_{h_1+h_2+h_3=t} ||B_h - Id.||_{h_1}(\max\{||F_h||_{h_2} + ||\phi_h||_{h_2}\})||B_h^{-1}||_{h_3} \\ &\quad + ||F_h||_t + \sum_{h_2+h_3=t}(||F_h||_{h_2} + ||\phi_h||_{h_2})||B_h^{-1} - Id.||_{h_3} \\ &\leq C(t) n_h^{4tr_0} q_{n_h}^5 \max_{\beta\in\{0,1\}} \|F_h\|_t^{\beta}(\|F_h\|_1 q_{n_h}\|\phi_h\|_t)^{1-\beta},\end{aligned}$$

with the last inequality that follows by convexity and the estimates on B_h in Proposition 5. In the same way, from the fact that:

$$\begin{aligned}A_{h+1}(x) &= B_h(x+\alpha)(R_{\phi_h(x)} + F_h(x))B_h^{-1}(x) \\ &\quad + (B_h(x+\alpha) - Id.)(R_{\phi_h(x)} + F_h(x))B_h^{-1}(x)) + (R_{\phi_h(x)} + F_h(x)) \\ &\quad + (R_{\phi_h(x)} + F_h(x))(B_h^{-1}(x)) - Id.),\end{aligned}$$

we get the same estimates for $A_{h+1} - R_{\phi_h}$, that is:

$$||A_{h+1} - R_{\phi_h}||_t \leq C(t) n_h^{4tr_0} q_{n_h}^5 \max_{\beta\in\{0,1\}} \|F_h\|_t^{\beta}(\|F_h\|_1 q_{n_h}\|\phi_h\|_t)^{1-\beta}. \tag{48}$$

Let also $\psi_h(x) := \frac{1}{(\det(A_{h+1}(x)-L(x))^{\frac{1}{2}}}$. Then, from the fact that the C^0 norm of of L is small and from the fact that $\det(A_{h+1}) = 1$, for all $t \in \mathbb{N}$:

$$||\psi_h - 1||_t \leq C(t)||(A_{h+1})^{-1}L||_t \leq C(t) n_h^{4tr_0} q_{n_h}^5 \max_{\beta\in\{0,1\}} \|F_h\|_t^{\beta}(\|F_h\|_1 q_{n_h}\|\phi_h\|_t)^{1-\beta} \tag{49}$$

with the last inequality that follows from the estimates for $||A_{h+1}||_t$, $||L||_t$ and convexity. So:

$$\begin{aligned}||\phi_{h+1} - \phi_h||_t &\leq ||(\psi_h - 1)(A_{h+1} - L)||_t + ||A_{h+1} - L + R_{\phi_h}||_t \\ &\leq C(t)\sum_{t_1+t_2=t} ||(\psi_h - 1)||_{t_1}(||A_{h+1} - R_{\phi_h}||_{t_2} + ||\phi_h||_{t_2} + ||L||_{t_2}) \\ &\quad + ||L||_t + ||A_{h+1} - R_{\phi_h}||_t.\end{aligned}$$

Then, by (48), (49), the estimates on L and convexity we get:

$$||\phi_{h+1} - \phi_h||_t \leq C(t) n_h^{4tr_0} q_{n_h}^5 \max_{\beta\in\{0,1\}} \|F_h\|_t^{\beta}(\|F_h\|_1 q_{n_h}\|\phi_h\|_t)^{1-\beta},$$

and:

$$\begin{aligned}||F_{h+1}||_t &= ||A_{h+1} - \psi_h(A_{h+1} - L)||_t \leq ||(\psi_h - 1)A_{h+1}||_t + ||\psi_h L||_t \\ &\leq C(t) n_h^{4tr_0} q_{n_h}^5 \max_{\beta\in\{0,1\}} \|F_h\|_t^{\beta}(\|F_h\|_1 q_{n_h}\|\phi_h\|_t)^{1-\beta}\end{aligned}$$

with the last inequality that follows by the previous estimates and by convexity. □

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Appendix

Here we state and prove some basic Lemma that we have used in other sections.

Lemma 13. *Let $D > 0$. There exists $\epsilon > 0$ such that, if $\phi \in C^\infty(\mathbb{T}, \mathbb{R})$, $F \in C^\infty(\mathbb{T}, SL(2, \mathbb{R}))$ with $\|\phi\|_0 < D$ and $\|F\|_0 < \epsilon$ and $Y := FR_\phi$, $G := \log(1 + Y)$, then for all $t \in \mathbb{N}$:*

$$\|G\|_t \leq C(t)\|Y\|_t.$$

Proof. Let $t \in \mathbb{N}$, $Y := \bar{F}R_{-\bar{\phi}}$. Then:

$$\|D^t G\|_0 \leq \sum_{h\geq 1} \left|\frac{D^t Y^h}{h}\right|_0.$$

By Leibnitz formula, for $h \geq 1$ $D^t Y^h$ is equal to the sum of h^t terms of the form:

$$D^{t_1}Y \ldots D^{t_h}Y,$$

with $t_1 + \ldots t_h = t$. For each j such that $t_j > 0$, by Hadamard's inequality:

$$\|D^{t_j}Y\|_0 \leq C(t)\|Y\|_0^{1-\frac{t_j}{t}}\|Y\|_t^{\frac{t_j}{t}}.$$

So, because there are at most t terms such that $t_j > 0$, we get:

$$\|D^{t_1}Y \ldots D^{t_h}Y\|_0 \leq C(t)^t\|Y\|_0^{h-1}\|Y\|_t.$$

Then:

$$\|D^t G\|_0 \leq C(t)\sum_{h\geq 1}\frac{h^t\|Y\|_0^{h-1}}{h}\|Y\|_t \leq C(t)\|Y\|_t$$

where in the last inequality we have used that:

$$\|Y\|_0 \leq \|\bar{F}\|_0\|R_{\bar{\phi}}\|_0 \leq C\|\bar{F}\|_0\|\bar{\phi}\|_0 \leq \frac{C}{q_n}.$$

□

Lemma 14. *For $\bar{A} = R_{\bar{\phi}} + \bar{F} \in C^\infty(\mathbb{T}, SL(2, \mathbb{R}))$, let G be such that $\bar{A} = e^G R_{\bar{\phi}}$. Then, for $t \in \mathbb{N}$:*

$$\|D^t G\|_0 \leq C(t)\|\bar{F} R_{-\bar{\phi}}\|_t$$

Lemma 15. *Let $t \in \mathbb{N}$. There exist polynomials $P_{1,t}(X_1, ...X_t), ..., P_{t,t}(X_1, ..., X_t)$ that are homogenous of degree less or equal then t if the variable X_i has weight i for $i = 1, \dots, t$, such that for $g \in C^\infty(\mathbb{T}, \mathbb{R})$:*

$$D^t \left(\frac{1}{g}\right) = \sum_{i=1}^{t} \frac{P_{i,t}(Dg, \dots, D^t g)}{g^{i+1}}.$$

Lemma 16. (See [14]) *There exists $C > 0$ such that, for $0 \leq a < b < c \in \mathbb{N}$, $f \in C^\infty(\mathbb{T}, \mathbb{R})$:*

$$\|D^b f\|_0 \leq C\|F\|_a^{1-\frac{b-a}{c-a}} \|F\|_c^{\frac{b-a}{c-a}}.$$

Lemma 17. *For $t \in \mathbb{N}$:*

$$\|D^t R_\phi\|_0 \leq C(t)\|\phi\|_t.$$

Proof. It follows by Faa Di Bruno's formula and Hadamard's inequality (Lemma 1). □

Lemma 18. *Let $f \in C^\infty(\mathbb{T}, \mathbb{R})$, $a > 0$, $t, h \in \mathbb{N}$. Then:*

$$\|D^{t+h} T_a(f)\|_0 \leq C(t+h)a^{2+h}\|F\|_t, \quad \|D^t R_a(f)\|_0 \leq Ca^{-h}\|F\|_{t+h+2}$$

Proof. Let $t, h \in \mathbb{N}$, $a > 0$. Then:

$$\|D^{t+h} T_a(f)\|_0 \leq \sum_{|l| \leq a} |(2\pi l)|^{t+h} |\hat{f}(l)| \leq |D^t f|_0 |2\pi a|^{h+2},$$

where in the last inequality we have used the fact that:

$$|\hat{f}(l)| \leq \frac{\|D^t f\|_0}{|2\pi \bar{l}|^t},$$

with $\bar{l} := \max\{|l|, 1\}$. Now we prove the second inequality:

$$\begin{aligned}\|D^t R_a(f)\|_0 &\leq \sum_{|l|>a} |\hat{f}(l)\|2\pi l|^t \leq \frac{1}{a^h} \sum_{|l|>a} |\hat{f}(l)\|2\pi l|^{t+h} \\ &\leq \frac{\|D^{t+h+2} f\|_0}{a^h} \sum_{|l|>a} \frac{1}{|2\pi l|^2} \leq \frac{C}{a^h}\|D^{t+h+2} f\|_0.\end{aligned}$$

□

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