Lebesgue spectrum for area preserving flows on the two torus

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Abstract

We study the spectral measures of area preserving mixing flows on the torus having one degenerate singularity. We show that, for a sufficiently strong singularity, the maximal spectral type of these flows is typically Lebesgue measure on the line.

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1 Introduction

How chaotic can the lowest dimensional smooth invertible dynamical systems be? A circle diffeomorphism with irrational rotation number that preserves a smooth measure is smoothly conjugate to a rotation. It is hence rigid in the sense that the iterates along a subsequence of the integers converge uniformly to identity. Rigidity implies the absence of mixing between any two measurable observables. This absence of mixing actually holds for all smooth circle diffeomorphisms with irrational rotation number since, by Denjoy theory, they are topologically conjugated to rotations.

Circle diffeomorphisms with rational rotation number are even farther from mixing, since their non-wandering dynamics are supported on periodic points.

The *lowest dimensional setting* that can be investigated after circle diffeomorphisms is that of smooth flows on surfaces preserving a smooth volume, that are often called multi-valued Hamiltonian flows to emphasize their relation with solid state physics that was pointed out by Novikov [26]. The simplest setting to be examined is then that of smooth area-preserving flows without periodic orbits. On the two-dimensional torus, this setting is reduced to that of reparametrizations of minimal translation flows (see [4] for example).

Kolmogorov showed that such flows are most probably conjugated to translation flows, since it suffices for this that the slope of the translation flow belongs to the full measure set of Diophantine numbers [20]. We remain in this case with the same spectral type as in the one-dimensional context. Kolmogorov also observed that more exotic behaviors should be expected for the reparametrized flows in the case of Liouville slopes. Shklover indeed obtained in [29] examples of real analytic reparametrizations of linear flows on the two-torus that were weak mixing (continuous spectrum). Subsequently, A. Katok showed that sufficiently smooth reparametrizations of linear flows on the torus are actually rigid [14]. In particular, the maximal spectral type of area-preserving flows of the torus without periodic orbits is always purely singular. Kochergin [17] extended the part of Katok's results asserting absence of mixing to all surfaces and to lower regularity. All these results are obtained after the identification of regular reparametrizations of linear flows on $\mathbb{T}^2$ with special flows above irrational rotations of the circle under regular ceiling functions (see Figure 1 and Section 2 for the exact definition of a special flow). This identification follows from the existence of a global Poincaré section for the reparametrized flows.

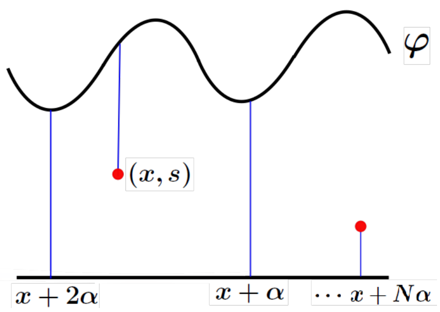

Figure 1: The orbit of a point by the special flow above a rotation of angle α and under a bounded ceiling function φ. Smooth reparametrizations of linear flows on $\mathbb{T}^2$ are equivalent to such flows.

As a consequence of Katok's result, in order to go beyond the purely singular maximal spectral type for smooth area-preserving flows on the torus, one must allow the existence of singularities for the flow. Our main result in this paper is the following

Theorem 1. *There exists a smooth area-preserving flow on $\mathbb{T}^2$ with exactly one singularity, with Lebesgue maximal spectral type.*

Note that area-preserving flows on surfaces have topological entropy zero.[1] Their phase portrait is actually similar to that of a minimal translation flow, apart on one orbit that contains the

[1] The situation is completely different for surface diffeomorphisms. Anosov automorphisms of the torus and their relatives constructed by A. Katok [15] on the sphere and the disc are classical examples of area-preserving Bernoulli surface diffeomorphisms. Later, Bernoulli diffeomorphisms and flows were shown to exist on any compact manifold of dimension larger than 2 and 3 respectively [6, 13].

saddle point which acts as a stopping point (see Figure 2). It is a striking fact that these quasi-minimal flows exhibit the same maximal spectral type as a Bernoulli flow. To give a more detailed description of our result, we start by explaining how mixing appears for area-preserving surface flows.

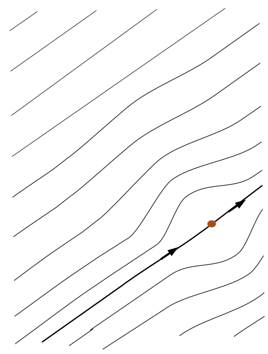

Figure 2: Torus flow with one degenerate saddle acting as a stopping point.

Kochergin gave in [18] the first examples of mixing area-preserving flows on surfaces. These were ergodic flows having at least one degenerate saddle. The examples we study here correspond to the case of the two-torus and a unique singularity (see Figure 3).

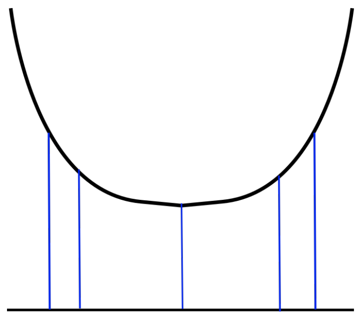

Figure 3: Representation of a two-torus flow with one degenerate saddle as a special flow under a ceiling function (symmetric) power-like singularity.

Considering a section of the flow that is transversal to all orbits and does not contain the saddle, the dynamics can then be viewed as that of a special flow above an irrational rotation of the circle with a return time function (called a *ceiling* or *roof* function) having a power-like singularity. The singularity is precisely the last point where the section intersects the incoming separatrix of the fixed point. The strength of the singularity depends on how abruptly the linear flow is slowed down in the neighborhood of the fixed point (see Remark 1).

In the case of other surfaces and several singularities, the flows obtained by Kochergin are equivalent to special flows above interval exchange transformations (IET) with ceiling functions having power-like singularities at the discontinuity points of the IET.

The mechanism of mixing in Kochergin examples is, in part, the same as in the weak mixing examples of Shklover, namely the stretching of the Birkhoff sums of the ceiling function above the iterates of the ergodic base dynamics. Whenever these sums are uniformly stretched above small intervals, the image of small rectangles by the special flow for large times decomposes into long and thin strips (see Figure 4). These strips are well distributed in the fibers due to uniform stretch, and well distributed in projection on the base because of ergodicity of the base dynamics.

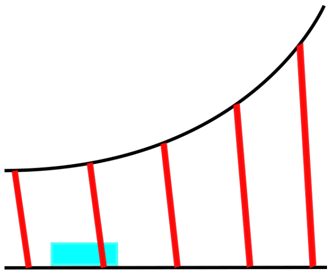

Figure 4: Mixing mechanism for special flows: the image of a rectangle is a union of long narrow strips which fill densely the phase space.

However, the reason behind the uniform stretching is different for Shklover's flows and Kochergin's ones. For the first ones, uniform stretching of the Birkhoff sums of the ceiling function is due to a Liouville phenomenon of accumulation, along a subsequence of time, of the oscillations of the ceiling function due to periodic approximations. In the case of Kochergin's flows, it is the shear between orbits as they get near the fixed points that is responsible for mixing. As a consequence, for the latter uniform stretching holds for *all* large times, while for the former, the existence of Denjoy-Koksma (DK for short) times impedes mixing. Denjoy-Koksma times are integers for which the Birkhoff sums have an *a priori* bounded oscillation around the mean value on all or on a positive measure proportion of the base (see for example the discussion around property DK in [5]). Hence, a key fact behind Kochergin's result is that the Denjoy-Koksma property does not necessarily hold for ceiling functions having infinite asymptotic values at some singularities.

A threshold is given by smooth ceiling functions having logarithmic singularities. When such a singularity is symmetric, it is known that for a typical irrational rotation a Denjoy-Koksma like property holds that prevents mixing of the special flow (see [23] and [5, Section 8]).

This is not true for asymmetric logarithmic singularities and we will now explain why. In [1], V. Arnold showed that multi-valued Hamiltonian flows with non-degenerate saddle points have a phase portrait that decomposes into elliptic islands (topological disks bounded by saddle connections and filled up by periodic orbits) and one open uniquely ergodic component. On this component, the flow can be represented as the special flow over an interval exchange map of the circle and under a ceiling function that is smooth except for some logarithmic singularities. The singularities are typically asymmetric since the coefficient in front of the logarithm is twice as big on one side of the singularity as the one on the other side, due to the existence of homoclinic loops (see Figure 5). Arnold conjectured mixing on the open ergodic component for these flows and Khanin and Sinai proved it in the case of a circular rotation on the base under a certain restriction on the rotation angle [16]. Kochergin [19] later extended their result to all irrational angles.

Being, as we said, a basic model in low dimension dynamics, and being closely related to the theory of translation flows on higher genus surfaces and IET that was recently experiencing spectacular developments, multi-valued Hamiltonian flows on surfaces attracted increasing interest in the last 20 years. The second author studied the deviations of ergodic averages for such flows [10] and proved a substantial part of the conjectures formulated by M. Kontsevich [21] and A. Zorich [35], [36], [37] on their deviation spectrum. The proof of the Kontsevich–Zorich

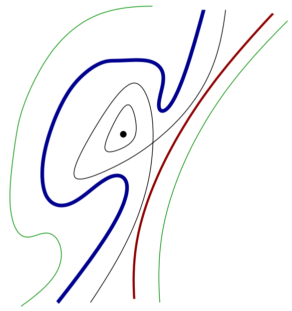

Figure 5: Multivalued Hamiltonian flow. Note that the orbits passing to the left of the saddle spend approximately twice longer time comparing to the orbits passing to the right of the saddle and starting at the same distance from the separatrix since they pass near the saddle twice.

conjectures was later completed by A. Avila and M. Viana [2]. Ergodic properties of multi-valued Hamiltonian flows on higher genus surfaces with non-degenerate saddle singularities were studied by C. Ulcigrai who established that such flows are generically weak mixing [33] and not mixing [34] (see also [28]). Recently, J. Chaika and A. Wright [3] gave mixing examples with finitely many non-degenerate fixed points and no saddle connections on a closed surface of genus 5 (the cancellations in the Birkhoff sums of a symmetric log function above a circle rotation do not happen for all IET and this is why mixing is possible even without asymmetry in the ceiling function as in Arnold examples). The first and third author of this paper established multiple mixing in many cases in the context of a single degenerate or non degenerate saddle [9].

However not much was known, beyond weak mixing or mixing features, about the spectral properties of multi-valued Hamiltonian flows on surfaces. The nature of the spectrum, be its multiplicity or its maximal spectral measure, remained quite obscure. The question of whether these flows may have an absolutely continuous spectrum was often raised (see for example the related discussion in [22] or [5]) in connection with the question whether there exist flows with simple Lebesgue spectrum. This is the flow version of the famous Banach's problem on the existence of a measure preserving transformation having simple Lebesgue spectrum. We believe however that our examples most likely display a countable Lebesgue spectrum. A positive answer to Banach's problem in the flow context was given by A. Prikhodko [27]. However, Prikhodko's constructions are purely measurable while the examples presented in this paper include real analytic flows.

In this paper, we will show that Kochergin flows with a sufficiently strong degenerate singularity typically have a Lebesgue maximal spectral type. We now formulate our results more precisely. The flows which we will consider are special flows given by a base dynamics that is an irrational rotation by $\alpha \in \mathbb{T}$, and a ceiling function $\varphi \in C^2(\mathbb{T} \setminus \{0\})$, $\varphi > 0$, with the following properties:

$$\lim_{\theta \to 0^+} \frac{\varphi(\theta)}{\theta^{-(1-\eta)}} = M_1 \text{ and } \lim_{\theta \to 0^-} \frac{\varphi(\theta)}{\theta^{-(1-\eta)}} = M_1 \tag{1}$$

$$\lim_{\theta \to 0^+} \frac{\varphi'(\theta)}{\theta^{-(2-\eta)}} = -N_1 \text{ and } \lim_{\theta \to 0^-} \frac{\varphi'(\theta)}{\theta^{-(2-\eta)}} = N_1 \tag{2}$$

$$\lim_{\theta \to 0^+} \frac{\varphi''(\theta)}{\theta^{-(3-\eta)}} = R_1 \text{ and } \lim_{\theta \to 0^-} \frac{\varphi''(\theta)}{\theta^{-(3-\eta)}} = R_1 \tag{3}$$

where η is a small number, $\eta \in (0, \frac{1}{1000})$, and $+\infty > M_1, N_1, R_1 > 0$. We refer to the beginning of Section 2 for an exact definition of special flows. We assume that $\int_{\mathbb{T}} \varphi(\theta) d\theta = 1$. We let $M = \{(\theta, s) \in \mathbb{T} \times \mathbb{R} \, : \, s \leqslant \varphi(\theta)\}$ and denote by μ the measure equal to the restriction to M of the product of the Haar measures $\lambda_{\mathbb{T}}$ on the circle $\mathbb{T}$ and $\lambda_{\mathbb{R}}$ on the real line $\mathbb{R}$. This measure is the unique invariant measure for the special flow $T^t_{\alpha,\varphi}$ given by (α, φ). Our main result is the following. For $\xi > 0$, we will say that $\alpha \in D_{\log,\xi}$ if and only if there exists a constant $C(\alpha) > 0$ such that for any $p \in \mathbb{Z}, q \in \mathbb{Z}^*$,

$$|\alpha - \frac{p}{q}| \geqslant \frac{C}{q^2 \log^{1+\xi} q}.$$

It is a classical and easy to prove fact that for any $\xi > 0$, $D_{\log,\xi}$ has full Haar measure in $\mathbb{T}$.

Theorem 2. *For $\alpha \in D_{\log,\xi}$, $\xi < \frac{1}{10}$, the dynamical system $(T^t_{\alpha,\varphi}, M, \mu)$ has Lebesgue maximal spectral type.*

Remark 1. In [18], the following method is adopted to obtain area preserving flows on the torus with a degenerate saddle-node fixed point as in (1)–(3). Consider first some Hamiltonian flow on $\mathbb{R}^2$ with the $x-$axis invariant and with a unique singularity at the origin. In the neighborhood of the origin, the orbits of such a flow are as described in Figure 2. It is then possible to cut a small neighborhood of the origin and paste it smoothly inside the phase portrait of a linear flow of $\mathbb{T}^2$ with any given slope. As a result, one gets a multi-valued Hamiltonian flow that has a unique singularity of saddle-node type. An easy calculation shows that if we consider the Hamiltonian given by $H_l(x,y) = y(x^2 + y^2)^l$ then the corresponding special flow has a unique symmetric power-like singularity as in (1)–(3) with η arbitrarily close to 0 as $l \to \infty$.

One can also obtain analytic examples with one fixed point as in (1)–(3). To do so, one starts with the smooth construction of a multi-valued Hamiltonian described above. Then, for an arbitrary $k > 2l + 4$, one considers a real analytic approximation of the smooth multi-valued Hamiltonian that continues to have the same slope and a unique singularity at $(0,0)$, with the same jets of order k at $(0,0)$ (that is, those of H_l). From there it follows that the corresponding flow has a special flow representation with a ceiling function having a unique symmetric power-like singularity as in (1)–(3).

To prove the absolute continuity of the spectrum in Theorem 2 it is natural to look for a control on the decay of correlations by the flow.

The only result in the direction of getting power-like estimates for the decay of correlations of surface flows was obtained in [7], where the first author proved a polynomial bound $t^{-\eta}$ on the decay of correlations (as functions of time $t > 0$) for Kochergin flows with one power singularity and for the characteristic functions of rectangles. However, in that paper, $\eta \leqslant \frac{1}{4}$, so it is not possible to deduce from the decay anything about the spectral type of the corresponding flow.

However, and as it is often the case, characteristic functions of nice sets do not give the best rate of decay of correlations between observables. A better way of approaching the problem is through the choice of a different dense class of observables. Such an approach was used by the second author and C. Ulcigrai in [11], where it is proved that smooth time changes of horocycle flows for compact hyperbolic surfaces have Lebesgue maximal spectral type. The dense set of observables they considered was the set of smooth coboundaries for the reparametrized flow, that is, of coboundaries with smooth transfer function. We will also consider correlations between smooth coboundary functions and show that they are square summable.

For time-changes of horocycle flows, the decay of correlations for coboundaries exploited in [11] is based on the *uniform shear* of geodesic arcs, linear with respect to time, as in B. Marcus proof of mixing [25]. Such a shear can be readily derived from the commutation relations for the horocycle and the geodesic flows, and the unique ergodicity of the horocycle flow (and hence of all of its time-changes), first established by H. Furstenberg [12] (see also [24]).

In the case of suspension flows above rotations, there is no such fast mixing normalizing action as the geodesic flow, and the shear of horizontal arcs is provided by the stretching of the Birkhoff sums of a roof function with a singularity (see Figure 4). This *non-uniform shear* near the singularity, has a strength that depends on the asymptotics of the roof function at the singular point. It is crucial for our argument that the singularity be chosen strong enough so that, over most of the phase space, the inverse of the stretching is a square integrable function of time. This means that our power singularity must be chosen with exponent in the interval $(1/2,1)$. For *asymmetric* power singularities, the set where the stretching of Birkhoff sums is small, that is, not square-integrable, has small measure and can be neglected in the argument. However, such suspension flows cannot be realized as smooth flows on a surface. For *symmetric* power singularities of exponent close to 1, which indeed can be realized as smooth flows (see Remark 1), the set of insufficient stretching is not negligible anymore, and we have to deal with it in the argument. This is a significant difficulty which, to the authors' best knowledge, does not arise in any of the proofs of absolutely continuous spectrum available in the literature (see [11], [31], [32], [30]).

Indeed, it is interesting to note that in our situation, in contrast with all the above-mentioned cases, in particular that of time-changes of horocycle flows investigated in [11], even for smooth coboundaries the correlation coefficients will not be of order at most $t^{-1/2-\varepsilon}$ for all times. To the contrary, along the subsequence t_n given by the denominators of the irrational rotation, the correlation coefficients may in fact be as large as $t_n^{-1/2+\varepsilon}$, for some $\varepsilon > 0$, because there is a set of measure of order $t_n^{-1/2+\varepsilon}$ on which the flow at time t_n is almost equal to the identity. This bad set appears due to the cancellations in the stretching of the Birkhoff sums of the ceiling function that are caused by the symmetry at the singularity (a remanant of the Denjoy-Koksma property). The bad set is essentially a union of thin towers that follow in projection the orbit of the translation on the base. Outside the bad set, the correlations are well controlled due to sufficiently strong uniform stretching. A crucial part of our argument, completely absent in the earlier works mentioned above, deals precisely with the bad set. Indeed, we use a bootstrap argument and the regular structure of the bad set, to show that for most of the times, that are in a medium scale neighborhood of the time t_n, there is some decay of correlations on the bad set, that yields square summability of the total correlations (see Figures 3, 6 and 7).

The proof that the spectral type is not just absolutely continuous, but indeed equivalent to the Lebesgue measure, is based on the construction of functions, localized on a thin strip around a long orbit segment, which have a given arbitrary correlation function on a finite, but arbitrarily long, time interval. The outline of the argument comes from the proof of the corresponding result for time changes of horocycle flows [11]. However, again in contrast with the case of time-changes of horocycle flows, whose phase space has dimension 3, for this approach to work in the case of surface flows, that is, in dimension 2, it is crucial that the constant in the estimates on the square integrals of correlations satisfy good bounds in terms of the smooth norms of the functions. For this reason, we will estimate carefully this dependence throughout the paper.

We end this introduction with some of the questions that naturally arise from our result.

Question 1. *Do Kochergin flows always have the Lebesgue maximal spectral type?*

To answer this question, one has to treat singularities with smaller powers as well as general IET on the base.

Question 2. *What is the maximal spectral type in the case of non degenerate saddles?*

Arnold conjectured a power-like decay of correlation in the asymmetric case, but the decay is more likely to be logarithmic, at least between general regular observables or characteristic functions of regular sets such as balls or squares. Even a lower bound on the decay of correlations is not sufficient to preclude absolute continuity of the maximal spectral type. However, an approach based on slowly coalescent periodic approximations as in [8] may be explored in the aim of proving that the spectrum is purely singular.

Plan of the paper.

In Section 2 we first give the formal definition of our special flows and we describe the set of coboundary functions we will be interested in.

The proof that the flow $T^t_{\alpha,\varphi}$ has an absolutely continuous maximal spectral type follows by a standard argument from Theorem 3 that states that the Fourier transforms of the spectral measures of functions in our special dense set are square-integrable.

The proof of Theorem 3 splits in two parts. We consider a time $t \in [q_n, q_{n+1}]$ for some $n \in \mathbb{N}$. We further consider intervals of time of the type $t \in [l^{21/20}, (l+1)^{21/20}] \subset [q_n, q_{n+1}]$.

First, a decay faster than $t^{-1/2-\varepsilon}$ for some $\varepsilon > 0$ is established outside a bad set $\mathscr{B}_l$ of measure comparable to $t^{-1/2+\varepsilon}$. This result is stated as Proposition 2.1. Second, the squared correlations on the bad set $\mathscr{B}_l$ are controlled on average for $t \in [l^{21/20}, (l+1)^{21/20}]$. This is the content of Proposition 2.2.

Section 3 is devoted to the proof of general stretching estimates for the Birkhoff sums of the ceiling function.

In Section 4 the bad set $\mathscr{B}_l$ is constructed and the stretching properties outside this set are stated. This is the content of Propositions 4.2, 4.4 and 4.5.

Section 5 explains the derivation of correlation decay estimates from uniform stretching of Birkhoff sums. The main results of Section 5.1 are Corollary 5.3 that describes the fast decay of order at least $t^{-1/2-\varepsilon}$ on the good intervals that partition the complement of the bad set, and Corollary 5.4 that describes the decay of order $t^{-1/2+\varepsilon}$ on general intervals (with the bad set $\mathscr{B}_l$ included). Corollary 5.3 will directly yield the proof of Proposition 2.1 on fast decay outside $\mathscr{B}_l$, given in Section 5.2, while Corollary 5.4 is crucial in the bootstrap argument that yields the averaged decay on the set $\mathscr{B}_l$ of Proposition 2.2, given in Section 5.3.

Finally, in Section 6, we complete the proof of Theorem 2 and prove that the maximal spectral type of Kochergin flows is Lebesgue.

2 Special flows, smooth coboundaries, and decay of correlations

Let $R_\alpha : \mathbb{T} \to \mathbb{T}$, $R_\alpha(\theta) = \theta + \alpha \bmod 1$, where $\alpha \in \mathbb{T}$ is an irrational number with the sequence of denominators $(q_n)_{n=1}^{+\infty}$ and let $\psi \in L^1(\mathbb{T}, \mathscr{B}, \lambda_{\mathbb{T}})$ be a strictly positive function. We denote by $d_{\mathbb{T}}$ the distance on the circle. We recall that the special flow $T^t := T^t_{\alpha,\psi}$ constructed above R_α and under ψ is given by

$$\begin{array}{rcl} \mathbb{T}\times\mathbb{R}/\sim & \to & \mathbb{T}\times\mathbb{R}/\sim \\ (\theta,s) & \to & (\theta,s+t), \end{array}$$

where $\sim$ is the identification

$$(\theta,s+\psi(\theta))\sim(R_\alpha(\theta),s)\,. \tag{4}$$

Equivalently (see Figure 1), this special flow is defined for $t+s\geqslant 0$ (with a similar definition for negative times) by

$$T^t(\theta,s)=(\theta+N(\theta,s,t)\alpha,t+s-\psi_{N(\theta,s,t)}(\theta)),$$

where $N(\theta,s,t)$ is the unique integer such that

$$0\leqslant t+s-\psi_{N(\theta,s,t)}(\theta)\leqslant\psi(\theta+N(\theta,s,t)\alpha), \tag{5}$$

and

$$\psi_n(\theta)=\begin{cases} \psi(\theta)+\ldots+\psi(R_\alpha^{n-1}\theta) & \text{if} \quad n>0 \\ 0 & \text{if} \quad n=0 \\ -(\psi(R_\alpha^{n}\theta)+\ldots+\psi(R_\alpha^{-1}\theta)) & \text{if} \quad n<0. \end{cases}$$

Let M denote the configuration space, that is,

$$M:=\{(\theta,s)\in\mathbb{T}\times\mathbb{R}\ :s\leqslant\psi(\theta)\}.$$

In our case $\psi=\varphi$, where φ has the properties stated in formulas (1), (2) and (3). For a given $\zeta>0$, let us denote

$$M_\zeta:=\{(\theta,s)\in M\ :\ d_{\mathbb{T}}(\theta,0)>\zeta,\zeta<s<\varphi(\theta)-\zeta\}.$$

We recall that f is a smooth coboundary for the flow $T^t_{\alpha,\varphi}$ if there exists a smooth function ϕ such that, for any $a<b$,

$$\int_a^b f(u,t)dt=\phi(u,b)-\phi(u,a).$$

The space of smooth coboundaries is dense in the subspace $L^2_0(M)\subset L^2(M)$ of zero average functions, provided $T^t_{\alpha,\varphi}$ is ergodic (which is always the case if α is irrational). Moreover, it can be shown that the subspace $\mathscr{F}$ of the space of all smooth coboundaries defined by the conditions that $f\in\mathscr{F}$ if and only if f is a smooth coboundary and there exists $\zeta>0$ with $f(x)=0$ for every $x\in M^c_\zeta$, is also dense in $L^2_0(M)$. Indeed, let us prove that the orthogonal space $\mathscr{F}^\perp\subset L^2_0(M)$ contains only the zero function. In fact, every function $f\in L^2_0(M)$, which belongs to the orthogonal space $\mathscr{F}^\perp\subset L^2_0(M)$, is by definition orthogonal to the Lie derivative along the flow of every smooth function with support contained in M_ζ for some $\zeta>0$. It follows that for every $t>0$ the function $f\circ T^t_{\alpha,\varphi}-f$ is orthogonal to all smooth functions with support in M_ζ, for every $\zeta>0$, hence it is orthogonal to all square-integrable functions, as the space of smooth functions with support contained in M_ζ for some $\zeta>0$ is dense in $L^2(M)$. It follows that for any $t>0$, the function $f\circ T^t_{\alpha,\varphi}-f$ vanishes, hence f is invariant and constant by the ergodicity of the flow. As f has zero average, it is equal to the zero function.

Let $f\in\mathscr{F}$ be a smooth coboundary and $g\in C^1(M)$. By definition, since $f\in\mathscr{F}$, there exists $\zeta>0$ such that $f=0$ on M^c_ζ.

Theorem 3. *Let f be a smooth coboundary for the flow $(T^t_{\alpha,\varphi})$ and let g be a smooth function on M, both vanishing on some neighborhood of the boundary of M. Then the correlation function*

$$\mathscr{C}_{f,g}(t) := \int_M f(T^t_{\alpha,\varphi}(x))g(x)d\mu\,, \quad \textit{for all } t>0\,, \tag{6}$$

belongs to the space $L^2(\mathbb{R}, d\lambda_{\mathbb{R}})$ of square-integrable functions on the real line.

The symbols $C_{f,g}$, $C'_{f,g}$, $C''_{f,g}$ will denote positive constants depending only on the C^1 norms of $f \in \mathscr{F}$ and $g \in C^1(M)$ and on the C^1 norm of the transfer function ϕ for $f \in \mathscr{F}$. Theorem 3 immediately follows from

Theorem 4. *For every $f \in \mathscr{F}$ and $g \in C^1_0(M_\zeta)$ there exists a constant $C_{f,g} > 0$ such for all $l \in \mathbb{N}$, we have*

$$\int_{l^{21/20}}^{(l+1)^{21/20}} \left| \int_M f(T^t_{\alpha,\varphi}(x))g(x)d\mu \right|^2 dt < C_{f,g}\, l^{-1-\frac{\eta}{100}}.$$

For simplicity, we denote $l_0 = l^{21/20}$, $l_1 = (l+1)^{21/20}$. Let $n \in \mathbb{N}$ be unique such that

$$q_n < l_0 < q_{n+1}\,.$$

Theorem 4 can be derived from the propositions stated below.

Proposition 2.1. *There exists a set $\mathscr{B}_l \subset M$, $\mu(\mathscr{B}_l) < q_n^{-1/2+6\eta}$ such that for every $t \in [l_0, l_1]$, we have*

$$\left| \int_{M\setminus\mathscr{B}_l} f(T^t_{\alpha,\varphi}(x))g(x)d\mu \right| < C_{f,g}\, t^{-1/2-\frac{\eta}{6}}.$$

Proposition 2.2. *We have*

$$\int_{l_0}^{l_1} \left| \int_{\mathscr{B}_l} f(T^t_{\alpha,\varphi}(x))g(x)d\mu \right| dt < C_{f,g} \frac{(l_1-l_0)\mu(\mathscr{B}_l)}{q_n^{20\eta}}.$$

The proofs of the two above propositions will be given later, in Sections 5.2 and 5.3, respectively. Let us show how they imply Theorem 4, and therefore Theorem 3 and the first part of Theorem 2 on the absolute continuity of the spectrum.

Proof of Theorem 4. By Proposition 2.2 we have

$$\begin{aligned}\int_{l_0}^{l_1} \left| \int_{\mathscr{B}_l} f(T^t_{\alpha,\varphi}(x))g(x)d\mu \right|^2 dt &\leqslant C_{f,g} \frac{(l_1-l_0)\mu(\mathscr{B}_l)^2}{q_n^{15\eta}} \\ &\leqslant C_{f,g}\frac{(l_1-l_0)}{q_n^{1+\eta}} \leqslant C'_{f,g}\frac{(l_1-l_0)}{q_{n+1}^{1+\eta/2}} \leqslant C'_{f,g}\frac{(l_1-l_0)}{l_0^{1+\eta/2}} \\ &\leqslant 2C'_{f,g}\frac{l^{1/20}}{l^{21/20(1+\eta/2)}} < C''_{f,g} l^{-1-\frac{\eta}{2}}.\end{aligned}$$

Using this and Proposition 2.1, we have

$$\begin{aligned}\int_{l^{21/20}}^{(l+1)^{21/20}} \left| \int_M f(T^t_{\alpha,\varphi}(x))g(x)d\mu \right|^2 dt &\leqslant 2\int_{l_0}^{l_1} t^{-1-\frac{\eta}{5}} dt + \\ & 2\int_{l_0}^{l_1} \left| \int_{\mathscr{B}_l} f(T^t_{\alpha,\varphi}(x))g(x)d\mu \right|^2 dt \leqslant l^{-1-\frac{\eta}{10}},\end{aligned}$$

which finishes the proof of Theorem 4. □

3 Stretching of the Birkhoff sums

We collect in the section the necessary technical facts about the Birkhoff sums of the ceiling function φ above R_α. Some proofs that are not difficult, but probably a bit tedious, will be deferred to the Appendix A.

For simplicity, we will assume that in our main assumptions (1), (2), (3) we have $M_1, N_1, R_1 = 1$ and that $\int_{\mathbb{T}} \varphi d\lambda = 1$. Throughout this section we suppose fixed $l_0 = l^{21/20}$, $l_1 = (l+1)^{21/20}$ and the unique integer n such that $q_n < l_0 < q_{n+1}$.

For every $x \in M$ we will denote by $\bar{x} \in \mathbb{T}$ its first coordinate. In particular, for any $t \in \mathbb{R}$, we will denote the first coordinate of $T^t_{\alpha,\varphi}(x) \in M$ by $\bar{T}^t_{\alpha,\varphi}(x)$. Similarly, for any horizontal interval $I \subset M$, we will denote $\bar{I} \subset \mathbb{T}$ is vertical projection.

Let $q_k \in [q_n \log^{15} q_n, q_n \log^{20} q_n]$ (such q_k exists by the Diophantine assumptions on α) and consider the partition $\mathscr{I}_k$ of $\mathbb{T}$ into intervals with endpoints $\{-i\alpha\}_{i=0}^{q_k-1}$. For any $\bar{I} \in \mathscr{I}_k$ such that $\bar{I} \cap [-\frac{1}{q_n^{3/5}}, \frac{1}{q_n^{3/5}}] = \emptyset$, let $I_\varphi := \{(\theta, s) \in M \,:\, \theta \in \bar{I}, 0 \leqslant s \leqslant \min_{\theta \in \bar{I}} \varphi(\theta)\}$. Define

$$W := \bigcup \{I_\varphi \,:\, \bar{I} \in \mathscr{I}_k, \bar{I} \cap \left[-\frac{1}{q_n^{3/5}}, \frac{1}{q_n^{3/5}}\right] = \emptyset\}. \tag{7}$$

By a slight abuse of notations, we refer to W as a set as well as a partial partition of M into intervals. Define moreover

$$V := W \cup (W^c \cap \{(\theta, s) \in M \,:\, 0 \leqslant s \leqslant q_n^{3/5+1/10}\}). \tag{8}$$

Notice that $M_\zeta \subset W$.

Notice that since $t \leqslant l_1 \leqslant q_{n+2}$ and $\varphi > c > 0$, we have

$$N_t := \sup_{x \in M} N(x,t) \leqslant \frac{q_{n+2}}{c} \ll q_k.$$

Hence by the definition of the partition $\mathscr{I}_k$, for every $I \subset W$

$$0 \notin \bigcup_{i=0}^{N_t} R^i_\alpha(\bar{I}). \tag{9}$$

As a consequence of (9) the Birkhoff sum $\varphi_{N(x,t)}$ is (twice) differentiable on I, for every $x \in I$ and $t \leqslant l_1$. This fact will be used repeatedly in the proofs.

3.1 Denjoy-Koksma estimates

We start with some Denjoy-Koksma type estimates that allow us to give some control on the Birkhoff sums of φ in function of the closest visit to the singularity.

We will adopt the following notation: for any $x \in M$ and $N \in \mathbb{N}$, we let

$$x^N_{min} = \min_{0 \leqslant j < N} d(\bar{x} + j\alpha, 0).$$

Lemma 3.1. *For every $x \in M$ and every $N \in [q_r, q_{r+1}]$, we have*

$$\varphi\left(x^N_{min}\right) + \frac{1}{3} q_r \leqslant \varphi_N(\bar{x}) \leqslant \varphi\left(x^N_{min}\right) + 3q_{r+1} \tag{10}$$

$$\varphi'\left(x_{min}^N\right) - 8q_{r+1}^{2-\eta} < |\varphi_N'(\bar{x})| < \varphi'\left(x_{min}^N\right) + 8q_{r+1}^{2-\eta} \tag{11}$$

and

$$\varphi''\left(x_{min}^N\right) \leqslant \varphi_N''(\bar{x}) < \varphi''\left(x_{min}^N\right) + 8q_{r+1}^{3-\eta}. \tag{12}$$

Proof of Lemma 3.1. We will give the proof of (10), the proofs of (11) and (12) are analogous. Let χ_r denote the characteristic function of the interval $[-\frac{1}{3q_r}, \frac{1}{3q_r}]$ and define $\bar{\varphi}_r := (1-\chi_r)\varphi$. By Denjoy-Koksma inequality, since $\int_{\mathbb{T}} \varphi d\lambda = 1$, we have

$$(\bar{\varphi}_{r+1})_{q_{r+1}}(\bar{x}) \leqslant q_{r+1}\int_{\mathbb{T}} \bar{\varphi}_{r+1} d\lambda + 4q_{r+1}^{1-\eta} \leqslant 3q_{r+1}.$$

Therefore

$$\varphi_N(\bar{x}) \leqslant \varphi_{q_{r+1}}(\bar{x}) \leqslant \varphi(x_{min}^N) + (\bar{\varphi}_{r+1})_{q_{r+1}}(\bar{x}) \leqslant \varphi(x_{min}^N) + 3q_{r+1}.$$

This gives the upper bound. Analogously (by Denjoy-Koksma inequality for $\bar{\varphi}_r$), we get the lower bound. The proof is thus finished. □

The following lemma is a direct consequence of (10) and (11), (12).

Lemma 3.2. *For every $x \in M$ and $N \in \mathbb{N}$*

$$|\varphi_N'(\bar{x})| < (\varphi_N(\bar{x}))^{2+2\eta}, \tag{13}$$

$$|\varphi_N''(\bar{x})| > (\varphi_N(\bar{x}))^{3-\eta} \log^{-3} N \tag{14}$$

As a consequence, we have that for every $x \in M \cap (\mathbb{T} \times \{s\})$ and every $t \in \mathbb{R}$

$$|\varphi_{N(x,t)}'(\bar{x})| < 3s^{2+2\eta} + 3t^{2+2\eta} \tag{15}$$

and

$$|\varphi_{N(x,t)}''(\bar{x})| > (t + s - \varphi(\bar{x} + N(x,t)\alpha))^{3-\eta} \log^{-3} N(x,t). \tag{16}$$

We have also the following bound on the discrepancies of the base rotation relative to intervals.

Lemma 3.3. *Let $\bar{J} \subset \mathbb{T}$ be an interval. Then for every $N \in \mathbb{N}$ and every $\theta \in \mathbb{T}$*

$$|(\chi_{\bar{J}})_N(\theta) - N\lambda(\bar{J})| \leqslant 2C^{-1}\log^{2+\xi} N.$$

Proof. Notice that by Denjoy-Koksma inequality, for every $j \in \mathbb{N}$ and $\theta \in \mathbb{T}$, we have

$$|(\chi_{\bar{J}})_{q_j}(\theta) - q_j\lambda(\bar{J})| \leqslant 2. \tag{17}$$

To conclude, we write $N = \sum_{j=0}^{r} a_j q_j$, where $0 \leqslant a_j \leqslant \frac{q_{j+1}}{q_j}$ (it is called Ostrowski expansion of N) use the cocycle identity, the bound in (17) for $j = r, r-1, \ldots, 0$ and the fact that by our Diophantine condition $a_j \leqslant C^{-1}(\log q_j)^{1+\xi}$ for all $j \in \mathbb{N}$. □

3.2 Stretching estimates

Uniform stretching of the Birkhoff sums requires a lower bound on the derivatives of the Birkhoff sums and an upper bound on their second derivatives (see for example Definition 4.3 below). For any interval $I \subset W$, we therefore introduce the notation

$$u_I := \sup_{t\in[l_0,l_1]} \sup_{x\in I} |\varphi''_{N(x,t)}(\bar{x})|. \tag{18}$$

Lemma 3.4. *Let $I \subset W$. If $u_I \geqslant q_n \log^9 q_n$, then for every $t \in [l_0, l_1]$ and every $x \in I \cap T^{-t}_{\alpha,\varphi}(W)$, we have*

$$x_{min}^{N(x,t)} \leqslant \frac{1}{q_n \log^2 q_n} \tag{19}$$

and

$$|\varphi'_{N(x,t)}(\bar{x})| \geqslant \left(\frac{1}{2x_{min}^{N(x,t)}}\right)^{2-\eta} \quad \text{and} \quad |\varphi''_{N(x,t)}(\bar{x})| \leqslant \left(\frac{2}{x_{min}^{N(x,t)}}\right)^{3-\eta}. \tag{20}$$

In what follows, for simplicity, we will denote $N(x) := N(x,t)$.

Lemma 3.5. *Let $x_0, x \in I \subset W$ with $|\bar{x} - \bar{x}_0| \geqslant \frac{1}{q_n^{3/2-2\eta}}$ satisfy $T^t_{\alpha,\varphi}(x) \in V$ and let*

$$|\varphi'_{N(x_0)}(\bar{x}_0)| \leqslant q_n^{7/4+\eta} \quad \text{and} \quad |\varphi''_{N(x_0)}(\bar{x}_0)| \leqslant q_n^{3-\eta} \log^{10} q_n.$$

Then for some $A_{x,x_0} \geqslant \frac{q_n^{3-\eta}}{\log^5 q_n}$ we have

$$|\varphi'_{N(x)}(\bar{x}) - \varphi'_{N(x_0)}(\bar{x}_0) - A_{x,x_0}(\bar{x} - \bar{x}_0)| \leqslant \frac{A_{x,x_0}}{10} |\bar{x} - \bar{x}_0|.$$

The proofs of Lemmas 3.4 and 3.5 will be given in Appendix A. Lemma 3.5 has the following straightforward consequence.

Corollary 3.6. *If $|\varphi'_{N(x_0)}(\bar{x}_0)| < 3q_n^{3/2+\eta}$ and $|\varphi''_{N(x_0)}(\bar{x}_0)| < q_n^{3-\eta} \log^{10} q_n$ for some $x_0 \in W$, then for every $x \in I$ such that $|\bar{x} - \bar{x}_0| \geqslant \frac{1}{q_n^{3/2-3\eta}}$ either $T^t_{\alpha,\varphi}(x) \in V^c$ or if x satisfies $T^t_{\alpha,\varphi}(x) \in V$, then*

$$|\varphi'_{N(x)}(\bar{x})| \geqslant \frac{q_n^{3-\eta}}{2\log^5 q_n} |\bar{x} - \bar{x}_0|. \tag{21}$$

4 Mixing rate on intervals, construction of $\mathscr{B}_l$

In what follows $I \subset W$ will be a horizontal interval (such that $\bar{I} \in \mathscr{I}_k$) and $h = q_n^{3/5}$. Then we know that the iterates $R^i_\alpha(\bar{I})$ for $i = 0, \dots, h$ are all disjoint and do not contain 0. Recall the notation

$$u_I := \sup_{t\in[l_0,l_1]} \sup_{x\in I} |\varphi''_{N(x,t)}(\bar{x})|.$$

Moreover whenever $I_t := I \cap T^{-t}_{\alpha,\varphi} W \neq \emptyset$, we define

$$r^t_I = \inf_{x\in I_t} |\varphi'_{N(x,t)}(\bar{x})| \tag{22}$$

(if $I_t = \emptyset$ we may define $r^t_I = +\infty$). We also let

$$r_I = \inf_{t\in[l_0,l_1]} r^t_I. \tag{23}$$

Definition 4.1 (Complete towers)**.** *Fix a horizontal interval $I \subset M \cap (\mathbb{T} \times \{s\})$ centered at z and a number $h > 0$. A complete tower of 'height' h above the interval I is the set:*

$$\bigcup_{i=0}^{N(z,h)} (R_\alpha^i(\bar{I}))_\varphi \setminus \cup_{t=0}^{s} T_{\alpha,\varphi}^t(\bar{I} \times \{0\}).$$

We now describe the bad set for correlations $\mathscr{B}_l$ (see Figure 6).

Proposition 4.2. *There exists a set $\mathscr{B}_l \subset M$ with the following properties:*

(B_1) *$\mathscr{B}_l = U_1 \cup \cdots \cup U_m$ where U_i are disjoint complete towers with heights $h = q_n^{3/5}$ over intervals $B_i \subset W$ with measure $\lambda(\bar{B}_i) = \frac{2}{q_n^{3/2-5\eta}}$;*

(B_2) *$\mu(\mathscr{B}_l) \leqslant q_n^{-1/2+6\eta}$;*

(B_3) *for every interval $I \subset W$, we have $I = J_1 \sqcup J_2 \sqcup I_{bad}$ where either $I \cap \mathscr{B}_l = \emptyset$ and I_{bad} and J_2 are empty, or I_{bad} is a level of some U_i and J_1, J_2 are intervals. When I_{bad} is not empty, we denote by x_{bad} its center.*

(B_4) *for every interval $I \subset W$ and every $t \in [l_0, l_1]$, we have one of the following*

$(B_4.i)$ *$r_I^t \geqslant q_n^{3/2+\eta}$,*

$(B_4.ii)$ *$r_I^t < q_n^{3/2+\eta}$, $I_{bad} \neq \emptyset$, $u_I \leqslant q_n^{3-\eta} \log^9 q_n$ and for every $x \in J_1 \sqcup J_2$ s.t. $T_{\alpha,\varphi}^t x \in W$*

$$|\varphi'_{N(x,t)}(\bar{x})| \geqslant \frac{q_n^{3-\eta}}{\log^6 q_n} |\bar{x} - \bar{x}_{bad}|$$

(B_5) *For every $t \in [l_0, l_1]$, for every $i \in [1, m]$, there exists a complete tower $\mathscr{T}_{t,i}$ over an interval $B_{t,i} = [-\frac{1}{q_n^{3/2-5\eta}} + \theta_{t,i}, \theta_{t,i} + \frac{1}{q_n^{3/2-5\eta}}] \times \{s_{t,i}\} \subset M$ of height $h_{t,i} \geqslant q_n^{3/5-1/50}$ such that*

$$\mu\big(\big(T_{\alpha,\varphi}^{-t}(U_i) \triangle \mathscr{T}_{t,i}\big) \cap M_\zeta\big) \leqslant q_n^{-1+10\eta}.$$

For a horizontal interval $I \subset W$ such that $T_{\alpha,\varphi}^t I \cap W \neq \emptyset$, the quantity that measures uniform stretching on I is the ratio

$$S_I^t := \inf_{x \in I_t} \frac{(\varphi'_{N(x,t)}(\bar{x}))^2}{\varphi''_{N(x,t)}(\bar{x})}, \tag{24}$$

where $I_t = I \cap T_{\alpha,\varphi}^{-t}(W)$ (we set $S_I^t = +\infty$ if $I \cap T_{\alpha,\varphi}^{-t} W = \emptyset$).

We recall that the integer l, hence the integers $l_0 = l^{21/20}$, $l_1 = (l+1)^{21/20}$, and the integer n such that $q_n < l_0 < q_{n+1}$, are fixed throughout this section.

Definition 4.3. *An interval $J = [u, v] \subset I \subset W$ is called* good *if for every $t \in [l_0, l_1]$, at least one of the following holds:*

$$S_J^t \geqslant t^{\frac{1}{2}+2\varepsilon} \tag{25}$$

or for some choice of $x^* \in I$ and for every $x \in J$ such that $T_{\alpha,\varphi}^t x \in W$, we have

$$|\varphi''_{N(x,t)}(\bar{x})| < q_n^{3-\eta} \log^9 q_n \text{ and } |\varphi'_{N(x,t)}(\bar{x})| \geqslant \frac{1}{2} q_n^{3/2+\eta} + \frac{1}{2} \frac{q_n^{3-\eta}}{\log^6 q_n} |\bar{x} - \bar{x}^*|. \tag{26}$$

When we check (25) *or* (26) *for a given t, we say that J is t-good.*

Proposition 4.4. *In the decomposition $I = J_1 \sqcup J_2 \sqcup I_{bad}$ of (B_3), we have that J_1 and J_2 are good.*

Proof of Proposition 4.4. Let $t \in [l_0, l_1]$. If $r_I^t < q_n^{3/2+\eta}$ then (26) holds on J_1 and J_2 (with $x^* = x_{bad}$) due to Lemma 3.4, Proposition 4.2, part $(B_4.ii)$, and the fact that for $x \in J_1 \cup J_2$ we have that $|\bar{x} - \bar{x}_{bad}| \geqslant q_n^{-3/2+5\eta}$.

Now, if $r_I^t \geqslant q_n^{3/2+\eta}$, then we will actually establish that all of I is t-good (which in particular implies the conclusion of Proposition 4.4 in this case):

Lemma 4.5. *For any $t \in [l_0, l_1]$, if $r_I^t \geqslant q_n^{3/2+\eta}$, then I is t-good.*

Proof of Lemma 4.5. **Case 1: $u_I \geqslant q_n^{3-\eta} \log^9 q_n$.**

In this case we do not use the assumption $r_I^t \geqslant q_n^{3/2+\eta}$. We use Lemma 3.4 and get for every $t \in [l_0, l_1]$ and every $x \in I \cap T_{\alpha,\varphi}^{-t}(W)$

$$S_I^t = \inf_{x \in I_t} \frac{\left(\varphi'_{N(x,t)}(\bar{x})\right)^2}{|\varphi''_{N(x,t)}(\bar{x})|} \geqslant \frac{2^{-7}}{(x_{min}^{N(x,t)})^{1-\eta}} \geqslant q_n^{2/3} \geqslant t^{1/2+\varepsilon}.$$

The last inequality holds because of $t < q_{n+2}$ and the Diophantine assumptions on α. This shows that I satisfies (25) and hence finishes the proof of Lemma 4.5 in this case.

Case 2: $u_I < q_n^{3-\eta} \log^9 q_n$.

Notice first that if $r_I^t \geqslant q_n^{7/4+\frac{\eta}{2}}$ (see (22) for the definition of r_I^t), then either $x \in T_{\alpha,\varphi}^{-t}(W^c)$ or

$$S_I^t = \inf_{x \in I_t} \frac{(\varphi'_{N(x,t)}(\bar{x}))^2}{\varphi''_{N(x,t)}(\bar{x})} \geqslant \frac{q_n^{7/2+\eta}}{q_n^{3-\eta} \log^9 q_n} \geqslant q_n^{1/2+\eta} \geqslant t^{1/2+\varepsilon},$$

where the last inequality again holds because of $t < q_{n+2}$ and assumptions on α. Therefore (25) holds for I and the proof is finished in this case .

Let us consider only $x \in I$ such that $T_{\alpha,\varphi}^t(x) \in W$. If $r_I^t < q_n^{7/4+1/2\eta}$, let $x_0 \in I$ be such that $|\varphi'_{N(x_0,t)}(\bar{x}_0)| = r_I^t$. Let us assume WLOG that $\varphi'_{N(x_0,t)}(\bar{x}_0) > 0$. Then by Lemma 3.5, whenever $\bar{x} \geqslant \bar{x}_0 + \frac{1}{q_n^{3/2-2\eta}}$, we have

$$|\varphi'_{N(x,t)}(\bar{x})| \geqslant \frac{q_n^{3-\eta}}{2\log^5 q_n} |\bar{x} - \bar{x}_0| \geqslant \frac{2q_n^{3-\eta}}{\log^6 q_n} |\bar{x} - \bar{x}_0|. \tag{27}$$

If $\bar{x} < \bar{x}_0 - \frac{1}{q_n^{3/2-2\eta}}$, then $\varphi'_{N(x,t)}(\bar{x}) < 0$. Indeed, otherwise by Lemma 3.5 we have

$$0 \leqslant \varphi'_{N(x,t)}(\bar{x}) < \varphi'_{N(x_0,t)}(\bar{x}_0) + \frac{q_n^{3-\eta}}{2\log^5 q_n} (\bar{x} - \bar{x}_0) \leqslant \varphi'_{N(x_0,t)}(\bar{x}_0) - q_n,$$

which is a contradiction with the choice of x_0. Therefore we have $\varphi'_{N(x,t)}(\bar{x}) < 0$ and, by Lemma 3.5 and by the definition of x_0, we derive

$$|\varphi'_{N(x,t)}(\bar{x})| \geqslant \frac{q_n^{3-\eta}}{4\log^5 q_n} |\bar{x} - \bar{x}_0| \geqslant \frac{2q_n^{3-\eta}}{\log^6 q_n} |\bar{x} - \bar{x}_0|. \tag{28}$$

Then by (27) and (28) and since $r_I \geqslant q_n^{3/2+\eta}$, we get that (26) is satisfied with $x^* := x_0$. This finishes the proof in Case 2. and Lemma 4.5 is established. □

The proof Proposition 4.4 is hence finished. □

4.1 Construction of the bad set $\mathscr{B}_l$

Recall that the partition $\mathscr{I}_k$ is given by two towers i.e. disjoint sets of the form $\{B+i\alpha\}_{i=0}^{q_k}$ and $\{C+i\alpha\}_{i=0}^{q_{k-1}}$ where B,C are intervals around 0 of length $\|q_{k-1}\alpha\|, \|q_k\alpha\|$ respectively. Denote $D_1 = B+\alpha, D_2 = C+\alpha$ (the shift comes from the fact that we want to stay away from the singularity). The following construction works for $D = D_1, D_2$. We will present it for the tower above $D = D_1$, the other case being analogous. Consider a complete tower $\mathscr{D}$ of height $H_k = q_k - 1$ over D. Notice that $\mathscr{D} \cap W$ is a union of horizontal intervals of length $\lambda(D)$. Moreover there is a natural order on horizontal intervals in $\mathscr{D} \cap W$ (coming from the order on $\mathscr{D}$): each interval in $\mathscr{D} \cap W$ is of the form $D(h)$ for some $0 \leqslant h \leqslant H_k$ (with $D(0) = D$).

Let $0 \leqslant h_1 \leqslant H_k$ be the smallest real number such that $D(h_1) \subset \mathscr{D} \cap W$ and $r_{D(h_1)} \leqslant 2q_n^{3/2+\eta}$. Let $t_1 \in [l_0, l_1]$ and $x_1 := (\theta_1, s_1) \in D(h_1)$ be such that

$$T_{\alpha,\varphi}^{t_1} x_1 \in W \text{ and } \varphi'_{N(\theta_1,t_1)}(\theta_1) \leqslant 2q_n^{3/2+\eta}.$$

Let U_1 be the complete tower of height $q_n^{3/5}$ over $B_1 := \left([-\frac{1}{q_n^{3/2-5\eta}} + \theta_1, \theta_1 + \frac{1}{q_n^{3/2-5\eta}}] \times \{s_1\} \right) \cap \mathscr{D}$. Let k_2 be the largest number such that $D(k_2) \subset \mathscr{D} \cap W$.

Now inductively let $H_k \geqslant h_i \geqslant k_i$ be the smallest real number such that $D(h_i) \subset \mathscr{D} \cap W$ and $r_{D(h_i)} \leqslant 2q_n^{3/2+\eta}$. Let $t_i \in [l_0, l_1]$ and $x_i := (\theta_i, s_i) \in D(h_i)$ be such that

$$T_{\alpha,\varphi}^{t_i} x_i \in W \text{ and } \varphi'_{N(\theta_i,t_i)}(\theta_i) \leqslant 2q_n^{3/2+\eta}. \tag{29}$$

We define U_i to be the complete tower of height $q_n^{3/5}$ over $B_i := \left([-\frac{1}{q_n^{3/2-5\eta}} + \theta_i, \theta_i + \frac{1}{q_n^{3/2-5\eta}}] \times \{s_i\} \right) \cap \mathscr{D}$.

We continue this procedure until the last possible $h_m \leqslant H_k$ is defined.

Let us define

$$\mathscr{B}_l := \bigcup_{1 \leqslant i \leqslant m} U_i. \tag{30}$$

Now, (B_1) and (B_3) follow by construction (notice that the top of U_i is below the base of U_{i+1}). Moreover, by Lemma 3.1 we get that $\varphi_{q_k - 1}(\alpha) \leqslant c q_{k+1}$, hence (B_2) follows from

$$\mu(\mathscr{B}_l) \leqslant \varphi_{q_k}(\alpha) \lambda(\bar{B}_i) \leqslant \frac{1}{q_n^{1/2-6\eta}}.$$

It remains to prove (B_4) and (B_5), which will be the subject of the next subsection.

4.2 Proving the properties of the bad set

In this section we give the proofs of (B_4) and (B_5) in Proposition 4.2

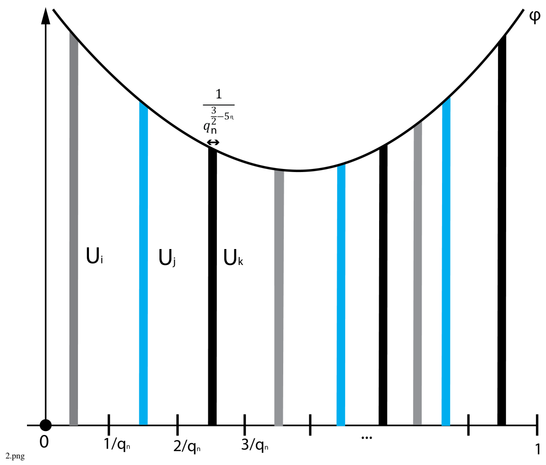

Figure 6: The set $\mathscr{B}_l$ is a union of complete towers U_i.

Proof of (B_4)**.** Fix $t \in [l_0, l_1]$. By the construction of $\mathscr{B}_l$, whenever for a partition interval $I \subset W$ we have $r_I^t \leqslant q_n^{3/2+\eta}$, then

$$I \cap \mathscr{B}_l = I_{bad},$$

where I_{bad} is a level of some U_i. In fact, otherwise $I \cap \mathscr{B}_l = \emptyset$ and by construction $r_I > 2q_n^{3/2+\eta}$. Therefore we need to show $(B_4.ii)$ for $I \subset W$ such that $I_{bad} \neq \emptyset$ and $r_I^t < q_n^{3/2+\eta}$. Then, by definition, there exists $x_I^t \in I$ such that

$$T_{\alpha,\varphi}^t(x_I^t) \in W \subset V \text{ and } \varphi'_{N(x_I^t,t)}(\bar{x}_I^t) \leqslant q_n^{3/2+\eta}. \tag{31}$$

Notice that we have

$$u_I < q_n^{3-\eta} \log^9 q_n. \tag{32}$$

Indeed, if not, then by Lemma 3.4 we would get by (19) and (20) that $\varphi'_{N(x_I^t,t)}(\bar{x}_I^t) \geqslant q_n^{2-\eta}$, which is a contradiction with (31).

Notice that by (31) and (32), the assumptions of Corollary 3.6 are satisfied with $x_0 = x_I^t$. Therefore, for every $x \in I$ such that $T_{\alpha,\varphi}^t(x) \in V$ and $|x - x_I^t| \geqslant \frac{1}{q_n^{3/2-3\eta}}$, we have

$$|\varphi'_{N(x,t)}(\bar{x})| \geqslant \frac{q_n^{3-\eta}}{2\log^5 q_n} |\bar{x} - \bar{x}_I^t|. \tag{33}$$

We claim that

$$|\bar{x}_I^t - \bar{x}_{bad}| \leqslant q_n^{-3/2+4\eta}. \tag{34}$$

Now, (33), (32) and (34) will finish the proof of $(B_4.ii)$ since for $x \in J_1 \sqcup J_2 = I \setminus I_{bad}$, we have that $|\bar{x} - \bar{x}_I^t| \geqslant |\bar{x} - \bar{x}_{bad}| - |\bar{x}_I^t - \bar{x}_{bad}| \geqslant q_n^{-3/2+3\eta}$.

Thus it only remains to show our claim (34). By construction of the U_i's, for some $h > 0$, we can write

$$x_{bad} = T_{\alpha,\varphi}^h x_i.$$

Moreover, since U_i is a complete tower of height $q_n^{3/5}$ and $T_{\alpha,\varphi}^h x_i \in U_i$, we have that

$$h \leqslant \varphi_{N(x_i, q_n^{3/5})}(\bar{x}_i) + \varphi(\bar{x}_i + N(x_i, h)\alpha).$$

Since $x_i \in W$, we get by the definition of special flow

$$\varphi_{N(x_i,q_n^{3/5})}(\bar{x}_i) \leqslant 2q_n^{3/5}.$$

Moreover, since $T^h_{\alpha,\varphi}x_i \in W$, we have

$$\varphi(\bar{x}_i + N(x_i,h)\alpha) \leqslant 2q_n^{3/5}.$$

By putting together the above bounds, we get

$$h < 2q_n^{3/5+1/50}. \tag{35}$$

Let $m_i := \max(t_i,t)$. We will show that

a. $T^{m_i}_{\alpha,\varphi}(T^h_{\alpha,\varphi}x_i), T^{m_i}_{\alpha,\varphi}(x^t_I) \in V$;

b. $|\varphi'_{N(x^t_I,m_i)}(\bar{x}^t_I)| \leqslant 2q_n^{3/2+\eta}$;

c. $|\varphi'_{N(T^h_{\alpha,\varphi}x_i,m_i)}(\bar{T}^h_{\alpha,\varphi}x_i)| \leqslant 5q_n^{3/2+\eta}$.

The above properties will give (34) (and hence $(B4.ii)$), since if $|\bar{T}^h_{\alpha,\varphi}x_i - \bar{x}^t_I| \geqslant q_n^{-3/2+4\eta}$ then by (32) and a., b., the assumptions of Corollary 3.6 are satisfied with $x_0 = x^t_I$, $x = T^h_{\alpha,\varphi}x_i$ but then c. is in contradiction with estimate (21) stated there. It remains then to show a., b.,c.

For a. we notice that by (29) and (31) we have $T^{t_i}_{\alpha,\varphi}x_i, T^t_{\alpha,\varphi}x^t_I \in W$. Moreover, by the immediate bound $|m_i - t| \leqslant l_1 - l_0 < q_n^{1/10}$ and by (35), we have the estimate

$$0 \leqslant m_i - t, m_i - t_i + h \leqslant 2q_n^{3/5+1/50} + q_n^{1/10} \leqslant 3q_n^{3/5+1/50}, \tag{36}$$

from which we derive that

$$\{T^{m_i}_{\alpha,\varphi}(T^h_{\alpha,\varphi}x_i), T^{m_i}_{\alpha,\varphi}(x^t_I)\} = \{T^{m_i-t_i+h}_{\alpha,\varphi}(T^{t_i}_{\alpha,\varphi}x_i), T^{m_i-t}_{\alpha,\varphi}(T^t_{\alpha,\varphi}x^t_I)\} \subset V.$$

This gives a.

For b. we first notice that since $T^t_{\alpha,\varphi}(x^t_I) \in W$ and $|m_i - t| \leqslant l_1 - l_0 < q_n^{1/10}$, by (15), we have

$$\varphi'_{N(x^t_I,m_i-t)}(\bar{T}^t_{\alpha,\varphi}(x^t_I))| \leqslant q_n^{3/2+\eta}$$

and by (31), $|\varphi'_{N(x^t_I,t)}(x^t_I)| \leqslant q_n^{3/2+\eta}$. By the cocycle identity, we then have

$$|\varphi'_{N(x^t_I,m_i)}(x^t_I)| \leqslant |\varphi'_{N(x^t_I,t)}(\bar{x}^t_I)| + |\varphi'_{N(x^t_I,m_i-t)}(\bar{T}^t_{\alpha,\varphi}(x^t_I))| \leqslant 2q_n^{3/2+\eta}.$$

This gives b.

For c., by cocycle identity, (29), (36) and (15) (for $T^{t_i}_{\alpha,\varphi}(x_i) \in W$), we get

$$|\varphi'_{N(x_i,m_i+h)}(\bar{x}_i)| \leqslant |\varphi'_{N(x_i,t_i)}(\bar{x}_i)| + |\varphi'_{N(T^{t_i}_{\alpha,\varphi}(x_i),m_i+h-t_i)}(\bar{T}^{t_i}_{\alpha,\varphi}(x_i))| \leqslant 2q_n^{3/2+\eta}. \tag{37}$$

Since $x_i \in W$, by (35) and (15), we have

$$|\varphi'_{N(x_i,h)}(\bar{x}_i)| \leqslant 2q_n^{3/2+\eta}. \tag{38}$$

Finally from the cocycle identity, (37) and (38) we conclude that

$$|\varphi'_{N(T^h_{\alpha,\varphi}x_i,m_i)}(\bar{T}^h_{\alpha,\varphi}x_i)| \leqslant |\varphi'_{N(x_i,m_i+h)}(\bar{x}_i)| + |\varphi'_{N(x_i,h)}(\bar{x}_i)| \leqslant 5q_n^{3/2+\eta}.$$

This finishes the proof of c. and hence also $(B4.ii)$.

Proof of (B_5)**.**

Let s_i be such that $x_i \in D(h_i) \subset \mathbb{T} \times \{s_i\}$ ($D(h_i)$ is the base of U_i). Let $t^* \in [-t, -t+1]$ be such that for $z_{t,i} = (\theta_{t,i}, s_{t,i}) := T^{t^*}_{\alpha,\varphi}x_i$ we have

$$B_{t,i} := [-\frac{1}{q_n^{3/2-5\eta}} + \theta_{t,i}, \theta_{t,i} + \frac{1}{q_n^{3/2-5\eta}}] \times \{s_{t,i}\} \subset M,$$

Let $h_{t,i} := \varphi_{N(x_i,q_n^{3/5})}(x_i) - s_i - (t - t^*)$ and let $\mathscr{T}_{t,i}$ be the complete tower of height $h_{t,i}$ over $B_{t,i}$. Notice that $s_i \leqslant q_n^{3/5(1-\eta)}$ and $\varphi_{N(x_i,q_n^{3/5})}(x_i) \geqslant q_n^{3/5}\log^{-10} q_n$ (by (10)), hence $h_{t,i} \geqslant q_n^{3/5-1/50}$.

The difference between $U_i \cap M_\zeta$ and $T^t_{\alpha,\varphi}(\mathscr{T}_{t,i}) \cap M_\zeta$ will come from the stretching of Birkhoff sums of the top and at the base of $\mathscr{T}_{t,i}$ and from the difference $|t^* - t| \leqslant 1$. The measure of the symmetric difference between the two sets is twice the maximal stretching times the measure of the base of $\mathscr{T}_{t,i}$. First let us estimate the maximal stretch.

For any $z \in B_{t,i}$ there exists $\xi_i \in [\bar{z}, \theta_{t,i}]$ such that

$$|\varphi_{N(\theta_{t,i},t)}(\bar{z}) - \varphi_{N(\theta_{t,i},t)}(\theta_{i,t})| \leqslant |\varphi'_{N(\theta_{t,i},t)}(\xi_i)||\bar{z} - \theta_{t,i}| \tag{39}$$

Since $t < q_{n+1}$, it follows that for $j = 0, \ldots, N(\theta_{t,i},t) - 1$, we have $\theta_{t,i} + j\alpha \notin [-\frac{1}{q_n \log^{100} q_n}, \frac{1}{q_n \log^{100} q_n}]$ and since $|\xi_i - \theta_{t,i}| < \frac{1}{q_n^{3/2-5\eta}}$, it follows that for $j = 0, \ldots, N(\theta_{t,i},t) - 1$, we have

$$\xi_i + j\alpha \notin \left[-\frac{1}{2q_n \log^{100} q_n}, \frac{1}{2q_n \log^{100} q_n}\right].$$

By the above condition and by (10), we derive from (39) the bound

$$|\varphi_{N(\theta_{t,i},t)}(\bar{z}) - \varphi_{N(\theta_{t,i},t)}(\theta_{t,i})| \leqslant q_n^{1/2+3\eta}.$$

Therefore,

$$\mu((T^{-t}_{\alpha,\varphi}(U_i) \triangle \mathscr{T}_{t,i}) \cap M_\zeta) \leqslant \lambda(\bar{B}_i)(4q_n^{1/2+3\eta} + |t^* - t|) \leqslant q_n^{-1+10\eta}.$$

This finishes the proof of $(B5)$ and hence also of Proposition 4.2.

5 From uniform stretching of Birkhoff sums to decay of correlations

5.1 Uniform stretching of Birkhoff sums and correlations

We will adopt below the following notation.

For all $f \in \mathscr{F}$ with transfer function ϕ and $g \in C^1(M)$, let

$$\mathscr{N}_0(f,g) := \|\phi\|_0 \|g\|_0 \quad \text{and} \quad \mathscr{N}_1(f,g) := (\|f\|_0 + \|\phi\|_0)\|g\|_1 + (\|f\|_1 + \|\phi\|_1)\|g\|_0\,,$$

where $\|\cdot\|_0$ and $\|\cdot\|_1$ respectively denote the C^0 and the C^1 norm. Moreover, we will denote by the letter C a generic constant which depends only on on the rotation number α and on the ceiling function φ.

In all what follows I denotes any interval of the partition of W defined in Section 4.

Our main result in this section is the following relation between uniform stretching of the Birkhoff sums and decay of correlations.

Let us recall the following notation (see (22)). For any interval $J \subset I$ denote

$$r_J^t := \inf_{x \in J_t} |\varphi'_{N(x,t)}(\bar{x})|, \tag{40}$$

where $J_t := J \cap T_{\alpha,\varphi}^{-t} W$ ($r_J^t = +\infty$ if $J_t = \emptyset$).

Proposition 5.1. *For any interval $J = [z,w] \times \{s\} \subset I$, we have the following estimate:*

$$\left| \int_{\bar{J}} f(T_{\alpha,\varphi}^t(\theta,s)) g(\theta,s) d\theta - p(z,w) \right| \leqslant C \left\{ \mathscr{N}_0(f,g) \frac{\lambda(\bar{J})}{S_J^t} + \mathscr{N}_1(f,g) \frac{\lambda(\bar{J})}{r_J^t} \right\},$$

where $p(z,w) = \frac{g(z,s)\phi(T_{\alpha,\varphi}^t(z,s))}{\varphi'_{N(z,t)}(z)} - \frac{g(w,s)\phi(T_{\alpha,\varphi}^t(w,s))}{\varphi'_{N(w,t)}(w)}$.

To prove Proposition 5.1, we will need the following lemma that encloses the main estimate on the correlation of coboundaries based on the stretching of the Birkhoff sums of the roof function.

Let $J_* := [u,v] \times \{s\} \subset J$ be such that $v - u \leqslant t^{-10}$.

Lemma 5.2. *Let $r_u^t := -\varphi'_{N(u,t)}(u)$. For all $f \in \mathscr{F}$ and for all $g \in C_0^1(M)$ and for all $t > 0$ we have*

$$\left| \int_{\bar{J}_*} f(T_{\alpha,\varphi}^t(\theta,s)) g(\theta,s) d\theta - \Delta(J_*,t) \right| \leqslant C \mathscr{N}_1(f,g) \frac{\lambda(\bar{J}_*)}{r_I^t}, \tag{41}$$

where $\mathscr{N}_1(f,g) = (\|f\|_0 + \|\phi\|_0)\|g\|_1 + (\|f\|_1 + \|\phi\|_1)\|g\|_0$ *and*

$$\Delta(J_*,t) := \frac{1}{r_u^t} \left[g(v,s)\phi(T_{\alpha,\varphi}^t(v,s)) - g(u,s)\phi(T_{\alpha,\varphi}^t(u,s)) \right].$$

Proof. Let $I \subset W \cap (\mathbb{T} \times \{s\})$ be a horizontal interval as in Section 4. Let $J_* = [u,v] \subset I$ such that $v_0 - u_0 \leqslant t^{-10}$. If $T_{\alpha,\varphi}^{-t} J_* \subset W^c$ then Lemma 5.2 holds trivially. We use the notation

$$T_{\alpha,\varphi}^t(u,s) = (\tilde{u},\tilde{s}) = (u + N(u,t)\alpha, t + s - \varphi_{N(u,t)}(u))\,,$$

where $0 \leqslant \tilde{s} \leqslant \varphi(u + N(u,t)\alpha)$. We also denote $\tilde{v} = v + N(u,t)\alpha$.

In the remainder of this proof we will denote for simplicity the integer $N(u,t)$ by N. We will suppose that $r_u^t = -\varphi'_N(u) \geqslant r_I^t \geqslant 0$, the case where $r_I^t < 0$ being similar. Let us also denote

$$B_I^t := \sup_{\theta \in I} \varphi''_N(\theta).$$

We will use the notation $X = O(Y)$ if there exists a constant $C > 0$ such that $X \leqslant CY$.

We have for $\theta \in [0, \lambda(\bar{J}_*)]$ that $T^t_{\alpha,\varphi}(u+\theta, s) = (\tilde{u}+\theta, \tilde{s}+\varphi_N(u)-\varphi_N(u+\theta))$. By the intermediate value theorem, since $r^t_I \ll \lambda(\bar{J}_*)^{-1}$, we have

$$\begin{aligned}\int_{\bar{J}_*} f(T^t_{\alpha,\varphi}(\theta,s))g(\theta,s)d\theta &= \int_0^{\lambda(\bar{J}_*)} f(\tilde{u}+\theta, \tilde{s}+\varphi_N(u)-\varphi_N(u+\theta))g(u+\theta,s)d\theta \\ &= g(u,s)\int_0^{\bar{\lambda}(\bar{J}_*)} f(\tilde{u}+\theta, \tilde{s}+\varphi_N(u)-\varphi_N(u+\theta))d\theta + O(\|f\|_0\|g\|_1\frac{\lambda(\bar{J}_*)}{r^t_I}).\end{aligned}$$

Now, since $\varphi_N(u)-\varphi_N(u+\theta) \ll 1$ we also have

$$\begin{aligned}&\int_0^{\lambda(\bar{J}_*)} f(\tilde{u}+\theta, \tilde{s}+\varphi_N(u)-\varphi_N(u+\theta))d\theta \\ &\quad = \int_0^{\lambda(\bar{J}_*)} f(\tilde{v}, \tilde{s}+\varphi_N(u)-\varphi_N(u+\theta))d\theta + O(\|f\|_1\frac{\lambda(\bar{J}_*)}{r^t_I}),\end{aligned}$$

and by the definition of B^t_I, we have $|\varphi_N(u)-\varphi_N(u+\theta)-r^t_u\theta| \leqslant B^t_I\theta^2$. Therefore,

$$\begin{aligned}\int_{\bar{J}_*} f(T^t_{\alpha,\varphi}(\theta,s))g(\theta,s)d\theta &= g(u,s)\int_0^{\lambda(\bar{J}_*)} f(\tilde{v}, \tilde{s}+r^t_u\theta)d\theta \\ &\quad + O(\|f\|_0\|g\|_1\frac{\lambda(\bar{J}_*)}{r^t_I}) + O(\|f\|_1\|g\|_0\frac{\lambda(\bar{J}_*)}{r^t_I}).\end{aligned}$$

For simplicity let us denote $w(f,g) := \|f\|_0\|g\|_1 + \|f\|_1\|g\|_0$. A change of variable then gives

$$\begin{aligned}\int_{\bar{J}_*} f(T^t_{\alpha,\varphi}(\theta,s))g(\theta,s)d\theta &= \frac{1}{r^t_u}g(u,s)\int_0^{r^t_u\lambda(\bar{J}_*)} f(\tilde{v}, \tilde{s}+\theta)d\theta + O(w(f,g)\frac{\lambda(\bar{J}_*)}{r^t_I}) \\ &= \frac{1}{r^t_u}g(u,s)\left[\phi(\tilde{v}, \tilde{s}+r^t_u\lambda(\bar{J}_*)) - \phi(\tilde{v},\tilde{s})\right] + O(w(f,g)\frac{\lambda(\bar{J}_*)}{r^t_I})\end{aligned}$$

but $T^t_{\alpha,\varphi}(v,s) = (\tilde{v}, \tilde{s}+\varphi_N(u)-\varphi_N(v)) = (\tilde{v}, \tilde{s}+r^t_u\lambda(\bar{J}_*)+\mathscr{E})$ with $\mathscr{E} \leqslant B^t_I\lambda(\bar{J}_*)^2$, hence

$$\begin{aligned}\int_{\bar{J}_*} f(T^t_{\alpha,\varphi}(\theta,s))g(\theta,s)d\theta &= \frac{1}{r^t_u}g(u,s)\left[\phi(T^t_{\alpha,\varphi}(v,s)) - \phi(\tilde{v},\tilde{s})\right] \\ &\quad + O(w(f,g)\frac{\lambda(\bar{J}_*)}{r^t_I} + \|g\|_0\|\phi\|_1\frac{\lambda(\bar{J}_*)}{r^t_I}) \\ &= \frac{1}{r^t_u}\left[g(v,s)\phi(T^t_{\alpha,\varphi}(v,s)) - g(u,s)\phi(T^t_{\alpha,\varphi}(u,s))\right]\check{H} \\ &\quad + O(\mathscr{N}_1(f,g)\frac{\lambda(\bar{J}_*)}{r^t_I}),\end{aligned}$$

which is precisely formula (41). ☐

Proof of Proposition 5.1. Since the proof is symmetric for $t > 0$ and $t < 0$, from now on we will assume that $t > 0$. If $T^{-t}_{\alpha,\varphi}(J) \subset W^c$, then Proposition 5.1 holds trivially. We assume for definiteness that $-\varphi'_{N(u,t)}(u) \geqslant r^t_J$ on J. Let us decompose J into finitely many subintervals $J = \bigcup_{i=1}^m J_i$ such that $J_i = [u_i, u_{i+1}) \times \{s\}$ with $|u_{i+1}-u_i| \leqslant t^{-10}$, and so that $N(\cdot,t)$ is constant on each J_i.

Then

$$\int_{\bar{J}} f(T^t_{\alpha,\varphi}(\theta,s))g(\theta,s)d\theta = \sum_{i=1}^{m}\int_{J_i} f(T^t_{\alpha,\varphi}(\theta,s))g(\theta,s)d\theta = \sum_{i=1}^{m}\Delta(J_i,t)+\mathscr{E}, \qquad (42)$$

where, by (41)

$$\mathscr{E} \leqslant \mathscr{N}_1(f,g)\frac{\lambda(\bar{J})}{r^t_J}.$$

Notice that if $T^{-t}_{\alpha,\varphi}(J_i) \subset W^c$ then the corresponding integral in (42) is 0. Therefore we only have to consider those J_i for which $T^{-t}_{\alpha,\varphi}(J_i) \not\subseteq W^c$. By enumeration let us assume that this is the case for all J_i.

Let us denote $r^t_i := -\varphi'_{N(u_i,t)}(u_i)$ and $\Theta_i := g(u_i,s)\phi(T^t_{\alpha,\varphi}(u_i,s))$. We then have

$$\begin{aligned}
|\sum_{i=1}^{m}\Delta(J_i,t)-p(z,w)| &= |\sum_{i=1}^{m}\frac{1}{r^t_i}(\Theta_{i+1}-\Theta_i)-p(z,w)| \\
&= |\frac{1}{r^t_m}\Theta_{m+1}-\frac{1}{r^t_1}\Theta_1+\sum_{i=1}^{m-1}\left(\frac{1}{r^t_i}-\frac{1}{r^t_{i+1}}\right)\Theta_{i+1}-p(z,w)| \\
&= |\sum_{i=1}^{m-1}\left(\frac{1}{r^t_i}-\frac{1}{r^t_{i+1}}\right)\Theta_{i+1}| \leqslant \|\phi\|_0\|g\|_0\left(\frac{1}{r^t_J}+\sum_{i=1}^{m-1}\frac{\left|r^t_{i+1}-r^t_i\right|}{r^t_{i+1}r^t_i}\right).
\end{aligned}$$

To estimate the quantity $\sum_{i=1}^{m-1}\frac{\left|r^t_{i+1}-r^t_i\right|}{r^t_{i+1}r^t_i}$, by the choice of $(u_i)_{i=1}^m$ (since $N(\cdot,t)$ is constant on J_i) and $u_{i+1}-u_i \leqslant t^{-10}$, we get

$$|r^t_{i+1}-r^t_i| \leqslant 2B^t_i\lambda(\bar{J}_i)$$

where $B^t_i := \varphi''_{N(u_i,t)}(u_i)$. To conclude the argument, we notice that (since $u_{i+1}\sim u_i$)

$$\sum_{i=1}^{m-1}\frac{B^t_i\lambda(\bar{J}_i)}{r^t_{i+1}r^t_i} \leqslant \frac{\lambda(\bar{J})}{S^t_J}. \qquad (43)$$

This, by (42), finishes the proof of Proposition 5.1

□

Proposition 5.1 has the following corollaries that allow us to deal with the decay of correlations on good intervals. In the corollaries below C again denotes a global positive constant which depends only on the rotation number α and on the ceiling function φ. It may be different in each corollary.

Corollary 5.3. *For every good interval J, we have*

$$\left|\int_{\bar{J}} f(T^t_{\alpha,\varphi}(\theta,s))g(\theta,s)d\theta\right| \leqslant C(\mathscr{N}_0(f,g)q_n^{-1}+\mathscr{N}_1(f,g)q_n^{-2})t^{-1/2-\frac{\eta}{4}}. \qquad (44)$$

Proof. Assume $J\cap T^{-t}_{\alpha,\varphi}W \neq \emptyset$ (otherwise the LHS is 0) and let first (25) hold in the definition 4.3 of a good interval. Notice that for $x\in T^{-t}_{\alpha,\varphi}(W)$, $\varphi''_{N(x,t)}(\bar{x}) \geqslant q_n^{3-10\eta}$ (see (16)) and hence by (25), $1/r^t_J \leqslant q_n^{-3/2-4\varepsilon} \leqslant t^{-1/2-2\varepsilon}\lambda(\bar{J})$. Moreover, $p(z,w) \leqslant C\mathscr{N}_0(f,g)/r^t_J \leqslant C\mathscr{N}_0(f,g)t^{-1/2-\varepsilon}\lambda(\bar{J})$.

An application of Proposition 5.1 for J finishes the proof in this case. If (26) holds, define $J_{weak} := [-\frac{1}{q_n^{3/2-2\eta}} + x^*, x^* + \frac{1}{q_n^{3/2-2\eta}}] \cap J$. Notice that by (26),

$$r^t_{J_{weak}} \geqslant q_n^{3/2+\eta} \quad \text{and} \quad S^t_{J_{weak}} \geqslant q_n^{\frac{5\eta}{2}}.$$

So by Proposition 5.1 for J_{weak}, we have

$$\left| \int_{\bar{J}_{weak}} f(T^t_{\alpha,\varphi}(\theta,s))g(\theta,s)d\theta \right| \leqslant C(\mathscr{N}_0(f,g)q_n^{-1} + \mathscr{N}_1(f,g)q_n^{-2})t^{-1/2-\frac{\eta}{4}}. \tag{45}$$

Therefore it remains to show (44) with $J \setminus J_{weak}$. Let $J = [z,w] \times \{s\}$ and let $J \setminus J_{weak} = J' \cup J''$, so that $z \in J'$ (unless $J' = \emptyset$) and $w \in J''$ (unless $J'' = \emptyset$). We will show (44) for J' and J''. We will apply the same procedure to both J' and J'', therefore we will explain the argument only in the case of J''. Let $m \in \mathbb{N}$ be the unique positive integer s.t. $2^m \leqslant q_n^{3/2-2\eta}(w - x^*) \leqslant 2^{m+1}$. Let us consider the intervals $J''_i := [w_i, w_{i+1}] \times \{s\} = [x^* + \frac{w-x^*}{2^{i+1}}, x^* + \frac{w-x^*}{2^i}] \times \{s\} \cap J''$, where $i = 0, \dots, m$. Then $J'' = \bigcup_{i=0}^m J''_i$ (notice that J_m may be degenerated). Consider only those J''_i for which $T^{-t}_{\alpha,\varphi}(J''_i) \nsubseteq W^c$. By enumeration assume this is the case for all $i = 0, \dots, m$. By (26) we have

$$r^t_{J''} \geqslant q_n^{3/2+\frac{\eta}{2}}. \tag{46}$$

Moreover by (26), for every J''_i, we have

$$\sup_{x \in J''_i} |\varphi''_{N(x)}(\bar{x})| \leqslant q_n^{3-\eta} \log^9 q_n \quad \text{and} \quad \inf_{x \in J''_i} \varphi'_{N(x)}(\bar{x}) \geqslant \frac{q_n^{3-\eta}(w - x^*)}{2^{i+2} \log^5 q_n}.$$

Therefore, we have the following estimate:

$$\sum_{i=0}^m \frac{\lambda(\bar{J}''_i)}{S^t_{J''_i}} \leqslant \frac{\log^{20} q_n}{q_n^{3-\eta}} \sum_{i=0}^m \frac{2^{2i+4}}{(w-x^*)^2} \lambda(\bar{J}''_i) \leqslant \frac{8 \log^{20} q_n}{(w-x^*)q_n^{3-\eta}} 2^{m+1} \leqslant \frac{1}{q_n^{3/2+\frac{\eta}{2}}} \leqslant t^{-1/2-\frac{\eta}{3}} \lambda(\bar{J}). \tag{47}$$

Notice that by the definition of the function $p(z,w)$ (see Proposition 5.1), we have $p(w_0, w_{m+1}) = \sum_{i=0}^m p(w_i, w_{i+1})$. By Proposition 5.1 for J''_i, $i = 0, \dots, m$ and by (46), (47), we derive

$$\begin{aligned} \left| \int_{\bar{J}''} f(T^t_{\alpha,\varphi}(\theta,s))g(\theta,s)d\theta \right| &\leqslant |p(w_0, w_{m+1})| + \left| \int_{\bigcup J''_i} f(T^t_{\alpha,\varphi}(\theta,s))g(\theta,s)d\theta - p(w_0, w_{m+1}) \right| \\ &\leqslant C\{\mathscr{N}_0(f,g)\lambda(\bar{J}) + \mathscr{N}_1(f,g)\lambda(\bar{J})^2\} t^{-1/2-\frac{\eta}{4}}. \end{aligned}$$

The same estimate is true for J'. This completes the proof of Corollary 5.3. □

Moreover we also have the following crucial corollary for the bootstrap argument in Subsection5.3. Recall that l, l_0, l_1, n and W are chosen as in Section 4.

Corollary 5.4. *For every interval $\bar{I} \in \mathscr{I}_k$ and for all $s \in \mathbb{R}^+$ such that $I := \bar{I} \times \{s\} \subset M$, for all $t \in [l_0, l_1]$, we have*

$$\left| \int_{\bar{I}} f(T^t_{\alpha,\varphi}(\theta,s))g(\theta,s)d\theta \right| \leqslant C\{\mathscr{N}_0(f,g)\lambda(\bar{I}) + \mathscr{N}_1(f,g)\lambda(\bar{I})^2\} t^{-1/2+6\eta}. \tag{48}$$

Proof. If $I \cap W^c \neq \emptyset$, then $I \subset M_\zeta^c$ hence (LHS) is 0. If $I \subset W$ then let $I = J_1 \sqcup J_2 \sqcup I_{bad}$ as in Proposition 4.4. We apply Corollary 5.3 to J_1 and J_2 together with the estimates

$$\lambda(\bar{I}) \geqslant q_n \log^{-20} q_n \quad \text{and} \quad \lambda(\bar{I}_{bad}) < \frac{1}{q_n^{3/2-2\eta}}.$$

For the interval I_{bad} we estimate the integral by the uniform norm of the integrand times the measure $\lambda(\bar{I}_{bad})$ of the domain of integration. □

5.2 Summable decay on good intervals. Proof of Proposition 2.1

We now explain how the results of Section 5.1 imply Proposition 2.1.

In fact, we prove a more general statement that will be relevant in Section 6 to complete the proof that the maximal spectral type is Lebesgue.

Proposition 5.5. *For every set E, measurable with respect to the partition W (see* (7) *for its definition), we have*

$$\left| \int_{E \setminus \mathscr{B}_l} f(T^t_{\alpha,\varphi}(x)) g(x) d\mu \right| < C\{\mathscr{N}_0(f,g)\mu(E) + \mathscr{N}_1(f,g)\mu(E)^2\} t^{-1/2-\frac{\eta}{5}}.$$

Proof. Since $g = 0$ on $M_\zeta^c \supset W^c$, we have

$$\left| \int_{E \setminus \mathscr{B}_l} f(T^t_{\alpha,\varphi}(x)) g(x) d\mu \right| = \left| \int_{(E\cap W) \setminus \mathscr{B}_l} f(T^t_{\alpha,\varphi}(x)) g(x) d\mu \right|.$$

By Fubini, it is enough to show that, for every interval $I \subset W$, we have

$$\left| \int_{\bar{I} \setminus \bar{I}_{bad}} f(T^t_{\alpha,\varphi}(\theta,s)) g(\theta,s) d\theta \right| \leqslant C\{\mathscr{N}_0(f,g)\lambda(\bar{I}) + \mathscr{N}_1(f,g)\lambda(\bar{I})^2\} t^{-1/2-\varepsilon},$$

where the subinterval I_{bad} is as in Proposition 4.4. It is then enough to apply Corollary 5.3 (to the subintervals J_1 and J_2) together with the lower bound $\lambda(\bar{I}) \geqslant q_n \log^{-20} q_n$.

Proposition 5.5 is thus proved, and Proposition 2.1 immediately follows, as among the properties of the bad set (see Proposition 4.2) we have the bound $\mu(\mathscr{B}_l) \leqslant q_n^{-1/2+6\eta}$. □

5.3 Averaged decay on the bad set. Proof of Proposition 2.2

Notice that as the bad set $\mathscr{B}_l$ decomposes by (30) as the union of the towers U_1, ..., U_m, Proposition 2.2 follows by the proposition below.

Let $C_{f,g}$ denote a positive constant which depends on the functions $f \in \mathscr{F}$ and $g \in C_0^1(M)$ only through the quantities $\mathscr{N}_0(f,g)$ and $\mathscr{N}_1(f,g)$.

Proposition 5.6. *For every $i \in \{1,\dots,m\}$, we have*

$$\int_{l_0}^{l_1} \left| \int_{U_i} f(T^t_{\alpha,\varphi}(x)) g(x) d\mu \right| dt < C_{f,g} \frac{(l_1 - l_0)\mu(U_i)}{q_n^{20\eta}}.$$

Proof. Fix $i \in \{1,\dots,m\}$. Let $A := \{t \in [l_0,l_1] \ : \ \int_{U_i} f(T^t_{\alpha,\varphi}(x,s))g(x,s)dx > 0\}$. Let $\rho(t) = 1$ if $t \in A$ and $\rho(t) = -1$ if $t \in [l_0,l_1] \setminus A$. Then, by Cauchy-Schwarz (Hölder) inequality, we have

$$\begin{aligned}\int_{l_0}^{l_1} \left| \int_{U_i} f(T^t_{\alpha,\varphi}(x))g(x)d\mu \right| dt &= \int_{U_i} \left(\int_{l_0}^{l_1} \rho(t) f(T^t_{\alpha,\varphi}(x))dt \right) g(x)d\mu \\ &\leqslant \left(\int_{U_i} \left(\int_{l_0}^{l_1} \rho(t) f(T^t_{\alpha,\varphi}(x))dt \right)^2 d\mu \right)^{1/2} \left(\int_{U_i} g(x)^2 d\mu \right)^{1/2} \\ &\leqslant \|g\|_0 \mu(U_i)^{1/2} \left(\int_{U_i} \left(\int_{l_0}^{l_1} \rho(t) f(T^t_{\alpha,\varphi}(x))dt \right)^2 d\mu \right)^{1/2}.\end{aligned}$$

Moreover we have

$$\begin{aligned}\left(\int_{U_i} \left(\int_{l_0}^{l_1} \rho(t) f(T^t_{\alpha,\varphi}(x))dt \right)^2 d\mu \right) &\leqslant \|f\|_0^2 (l_1 - l_0)^{3/2} \mu(U_i) + \\ &\left(\int_{U_i} \left(\int_{l_0}^{l_1} \left(\int_{r \in [l_0,l_1] \, : \, |r-t| \geqslant (l_1-l_0)^{1/2}} \rho(r)\rho(t) f(T^t_{\alpha,\varphi}(x)) f(T^r_{\alpha,\varphi}(x))dr \right) dt \right) d\mu \right).\end{aligned}$$

Therefore, to finish the proof of Proposition 5.6 it is enough to show that there exists a constant $C > 0$ such that, for every $t \leqslant r$ with $t, r \in [l_0,l_1]$ s.t. $|t-r| \geqslant (l_1 - l_0)^{1/2}$, we have

$$\left| \int_{T^{-t}_{\alpha,\varphi}(U_i)} f(x) f(T^{r-t}_{\alpha,\varphi}(x))d\mu \right| \leqslant C \mathscr{N}_1(f,f) \frac{\mu(U_i)}{q_n^{40\eta}}. \tag{49}$$

Note that $t^* := r - t \in [q_n^{\frac{1}{41}}, q_n^{\frac{1}{19}}]$. Let us then fix such a $t^* \in [q_n^{\frac{1}{41}}, q_n^{\frac{1}{19}}]$. Following the notation of Section 4 we then let $l^* = [t^*]$ and n^* be the unique integer such that $q_{n^*} \leqslant l^* < q_{n^*+1}$.

Let k^* be any integer such that $q_{k^*} \in [q_{n^*} \log^{15} q_{n^*}, q_{n^*} \log^{20} q_{n^*}]$. It follows by construction that we have $q_{k^*} \in [q_n^{\frac{1}{41}}, q_n^{\frac{1}{19}} \log^{20} q_n]$

Observe now that by Corollary 5.4 there exists a constant $C > 0$ such that, for any interval $\bar{I} \in \mathscr{I}_{k^*}$ and for all $s \in \mathbb{R}^+$ such that $I := \bar{I} \times \{s\} \subset M$, we have

$$\left| \int_{\bar{I}} f(T^{t^*}_{\alpha,\varphi}(\theta,s)) f(\theta,s)d\theta \right| \leqslant C\{\mathscr{N}_0(f,f) + \mathscr{N}_1(f,f)\lambda(\bar{I})\} \frac{\lambda(\bar{I})}{q_n^{\frac{1}{100}}}. \tag{50}$$

Thus, it only remains to be seen that the integral in (49) decomposes into integrals over the sets of the form $T^{-t}_{\alpha,\varphi}(U_i) \cap I, \bar{I} \in \mathscr{I}_{k^*}$, and that each is roughly equal to the product of $\frac{\lambda(U_i \cap I)}{\lambda(\bar{I})}$ times the integral in (50). This is what we will now derive from Proposition 4.2, namely from the property that $T^{-t}_{\alpha,\varphi}(U_i)$ is almost equal to the tower $\mathscr{T}_{t,i}$ of (B_5). In fact, by properties (B_1), (B_2) in Proposition 4.2, we have the bound $m \leqslant q_n^{2/5+\eta}$, hence by property (B_5) we conclude that

$$\sum_{i=1}^{m} \mu(\mathscr{T}_{t,i} \triangle T^{-t}_{\alpha,\varphi}(U_i)) \leqslant q_n^{-3/5+15\eta}. \tag{51}$$

The intersection of each tower $\mathscr{T}_{t,i}$ with I is a regular union of equally separated small intervals (see Figure 7). In this situation the interpolation between the integrals is possible. To carry it out, we introduce the following

Definition 5.7. *Let $\nu, \gamma \in (0,1)$. We will say that a collection $\mathscr{S} := K_1 \sqcup \ldots \sqcup K_H \subset \mathbb{T} \times \{s\}$ of pairwise disjoint horizontal intervals of equal lengths is (ν,γ)-uniformly distributed in the interval I if there exists a decomposition of I into a disjoint union of $L \leqslant \gamma H$ intervals $I_1, \ldots, I_L$ of equal length $\ell \in [\nu, 2\nu]$ such that, for all $j \in [1,L]$, we have*

$$\#\{i \in [1,H] : K_i \subset I_j\} \in [(1-\gamma)\frac{H}{L}, (1+\gamma)\frac{H}{L}].$$

This definition is useful in the following straightforward lemma.

Lemma 5.8. *If $\mathscr{S}$ and I are as in Definition 5.7, then for any C^1 real function G defined over the interval $I := \bar{I} \times \{s\}$, we have*

$$\left| \int_{\bar{\mathscr{S}} \cap \bar{I}} G(\theta, s) d\theta - \frac{\lambda(\bar{\mathscr{S}} \cap \bar{I})}{\lambda(\bar{I})} \int_{\bar{I}} G(\theta, s) d\theta \right| \leqslant C(\nu \|G\|_1 + \gamma \|G\|_0)\, \lambda(\bar{\mathscr{S}} \cap \bar{I}).$$

Lemma 5.9. *For any complete tower $\mathscr{T}$ of height $h \geqslant q_n^{3/5-1/50}$ above any horizontal interval of the the form $B_{\mathscr{T}} = [-\frac{1}{q_n^{3/2-5\eta}} + \theta_{\mathscr{T}}, \theta_{\mathscr{T}} + \frac{1}{q_n^{3/2-5\eta}}] \times \{s_{\mathscr{T}}\}$, we have the following:*

- (I_1) *if $N(\theta_{\mathscr{T}}, h) \leqslant q_n^{1/3}$, then $\mu(\mathscr{T} \cap M_\zeta) \leqslant q_n^{1/2-3/5} \mu(\mathscr{T})$;*
- (I_2) *if $N(\theta_{\mathscr{T}}, h) \geqslant q_n^{1/3}$, then for any $\bar{I} \in \mathscr{I}_{k^*}$ such that $I := \bar{I} \times \{s\} \subset M_\zeta$, the set $\mathscr{T} \cap I$ is contained in a collection of disjoint intervals of equal size $(q_n^{-1/4}, q_n^{-1/100})$-uniformly distributed in the interval I.*

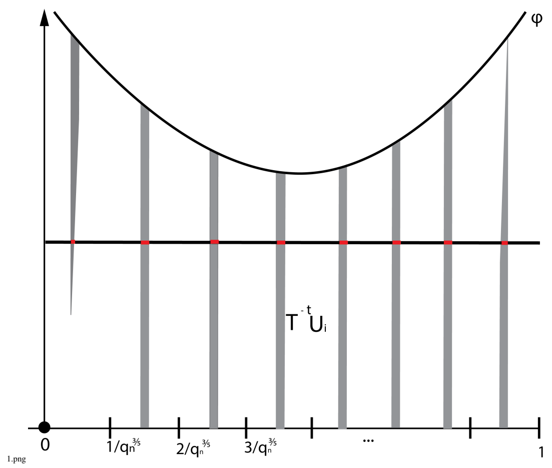

Figure 7: The image of the set U_i under the flow. The intersection with any horizontal interval is a union of equispaced intervals.

Before proving Lemma 5.9, we show how it implies (49). By (51), it suffices to show that there exists a constant $C > 0$ such that

$$\left| \int_{\mathscr{T}_{t,i}} f(x) f(T^{t^*}_{\alpha,\varphi}(x) d\mu \right| \leqslant C \mathscr{N}_1(f,f) \frac{\mu(\mathscr{T}_{t,i})}{q_n^{50\eta}}. \tag{52}$$

If (I_1) holds, then since f is supported on M_ζ we have

$$\left|\int_{\mathscr{T}_{t,i}} f(x)f(T^{t^*}_{\alpha,\varphi}(x))d\mu\right| \leqslant \|f\|_0^2\,\mu(\mathscr{T}_{t,i}\cap M_\zeta) \leqslant \|f\|_0^2\,\frac{\mu(\mathscr{T}_{t,i})}{q_n^{\frac{1}{10}}}, \tag{53}$$

hence the proof is finished in this case. Notice that by Fubini's theorem (52) follows from the following claim: there exists a constant $C>0$ such that, for any $I := \bar{I}\times\{s\}$ with $\bar{I}\in\mathscr{I}_{k^*}$, we have

$$\left|\int_{\overline{\mathscr{T}_{t,i}\cap I}} f(\theta,s)f(T^{t^*}_{\alpha,\varphi}(\theta,s))d\theta\right| \leqslant C\{\mathscr{N}_0(f,f)+\mathscr{N}_1(f,f)\lambda(\bar{I})\}\,\frac{\lambda(\overline{\mathscr{T}_{t,i}\cap I})}{q_n^{50\eta}}. \tag{54}$$

In fact, the above bound is stronger than what we need to prove the absolute continuity of the spectrum. The precise dependence of the constant on the function f and on the interval $I\in\mathscr{I}_{k^*}$ will be crucial in the proof that the maximal spectral type is Lebesgue in Subsection 6.

Now, if $I\subset M_\zeta^c$ then the integral in (54) is zero. Notice that, since $t^*\leqslant q_n^{1/19}$, by Lemma 3.2 the function $G: I\to\mathbb{R}$ defined as $G(\cdot) = f(\cdot)f(T^{t^*}_{\alpha,\varphi}(\cdot))$ satisfies $\|G\|_1\leqslant q_n^{1/8}\|f\|_0\|f\|_1$, thus (I_2) and Lemma 5.9 imply that

$$\left|\int_{\overline{\mathscr{T}_{t,i}\cap I}} G(\theta,s)d\theta - \frac{\lambda(\overline{\mathscr{T}_{t,i}\cap I})}{\lambda(\bar{I})}\int_{\bar{I}} G(\theta,s))d\theta\right| \leqslant C\|f\|_0\{\|f\|_0+\|f\|_1\lambda(\bar{I})\}\frac{\lambda(\overline{\mathscr{T}_{t,i}\cap I})}{q_n^{\frac{1}{200}}},$$

and therefore (54) follows from (50). The proof of the derivation of the bound in (49) from Lemma 5.9 is complete.

It only remains to give the

Proof of Lemma 5.9. Let us first consider the case $N := N(\theta_{\mathscr{T}},h)\geqslant q_n^{1/3}$. Let $\{K_1,\dots,K_H\}$ be the smallest collection of disjoint intervals of equal length such that

$$I\cap\mathscr{T}\subset K_1\sqcup K_2\sqcup\cdots\sqcup K_H.$$

Notice that for every $i\in\{1,\dots,H\}$, the interval $\bar{K}_i$ is centered at the point $\theta_{\mathscr{T}}+k_i\alpha$, for some $k_i\in[0,N]$. In fact, there is an injective map from the set of $k\in[0,N]$ such that $\theta_{\mathscr{T}}+k\alpha\in\bar{I}$ to the collection of intervals $\{K_1,\dots,K_H\}$ which misses at most 2 intervals. By Lemma 3.3 for $\bar{J}=\bar{I}$ and $\theta=\theta_{\mathscr{T}}$, we have

$$|H-N\lambda(\bar{I})|\leqslant 2+2C^{-1}\log N^{2+\xi}. \tag{55}$$

Let us then divide I into equal intervals $I_1,\dots,I_L$ of equal length $\ell\in[q_n^{-1/4},2q_n^{-1/4}]$ and let us consider $I_j\subset I$. The map from the set $\{i\in[1,H]: K_i\subset I_j\}$ to the set of $k\in[0,N]$ such that $\theta_{\mathscr{T}}+k\alpha\in\bar{I}_j$, which sends every interval $\bar{K}_i$ to its center, is injective and misses at most 2 elements. From Lemma 3.3 for $\bar{J}=\bar{I}_j$ and $\theta=\theta_{\mathscr{T}}$, it follows that

$$|\#\{i\in[1,H]: K_i\subset I_j\}-N\lambda(\bar{I}_j)|\leqslant 2+2C^{-1}\log N^{2+\xi}. \tag{56}$$

Notice that since $I\in\mathscr{I}_k$ by the bound (55), it follows that $H\geqslant q_n^{1/3-1/20}$ and by construction we have $L\leqslant q_n^{1/4-1/40}$, hence in particular $H/L\geqslant q_n^{1/12-1/40}$. We then derive the estimate

$$|N\lambda(\bar{I}_j)-\frac{H}{L}| = |\frac{N\lambda(\bar{I})}{L}-\frac{H}{L}|\leqslant\frac{2+C^{-1}\log N^{2+\xi}}{L}\leqslant q_n^{-1/10}\frac{H}{L},$$

which in turn by the bound (56) implies that

$$\#\{i \in [1,H] : K_i \subset I_j\} \in \left[(1-q_n^{-1/100})\frac{H}{L}, (1+q_n^{-1/100})\frac{H}{L}\right].$$

This shows that the collection $\mathscr{S} = K_1 \sqcup \cdots \sqcup K_H$ is $(q_n^{-1/4}, q_n^{-1/100})$-uniformly distributed in I. The proof of Lemma 5.9 is finished in case (I_2).

Assume now that $N(\theta_{\mathscr{T}}, h) \leqslant q_n^{1/3}$. Notice that, since the height of the complete tower $\mathscr{T}$ is $h \geqslant q_n^{3/5-1/10}$, we have

$$\varphi_{N(\theta_{\mathscr{T}},h)+1}(\theta_{\mathscr{T}}) \geqslant q_n^{3/5-1/50}.$$

But then

$$\mu(\mathscr{T} \cap M_\zeta) \leqslant q_n^{1/3}\zeta^{-1}\lambda(\bar{B}_{\mathscr{T}}) \leqslant q_n^{1/2-3/5} q_n^{3/5-1/50}\lambda(\bar{B}_{\mathscr{T}}) \leqslant q_n^{1/2-3/5}\mu(\mathscr{T}).$$

This finishes the proof of Lemma 5.9. □

□

6 The maximal spectral type

We will now complete the proof of the second part of Theorem 2, which states that the maximal spectral type is the Lebesgue measure on the real line. To achieve this aim we begin with a general consequence of the assumption that the maximal spectral type is not Lebesgue.

Lemma 6.1. *Assume that the maximal spectral type of the flow $\{T^t_{\alpha,\varphi}\}$ is not Lebesgue. It follows that there exists a smooth non-zero function $\omega \in L^2(\mathbb{R}, dt)$ such that for all functions $f, g \in L^2(M)$, we have*

$$\int_{\mathbb{R}} \omega(t) < f \circ T^t_{\alpha,\varphi}, g >_{L^2(M)} dt = 0$$

Proof. If the maximal spectral type is not equivalent to Lebesgue, then the Lebesgue measure is not absolutely continuous with respect to the maximal spectral measure. Thus, there exists a compact set $\Omega \subset \mathbb{R}$ such that Ω has measure zero with respect to the maximal spectral measure of the flow $\{T^t_{\alpha,\varphi}\}$, hence with respect to all its spectral measures, but Ω has strictly positive Lebesgue measure. Let $\omega \in L^2(\mathbb{R})$ be the complex conjugate of the Fourier transform of the characteristic function χ_Ω of $\Omega \subset$. For any pair of functions f, $g \in L^2(M)$, let $\mu_{f,g}$ denote the joint spectral measure. We have

$$\int_{\mathbb{R}} \omega(t) < f \circ T^t_{\alpha,\varphi}, g >_{L^2(M)} dt = \int_{\mathbb{R}} \chi_\Omega(\xi) d\mu_{f,g}(\xi) = 0. \tag{57}$$

This finishes the proof. □

We will construct below functions supported on a thin strip along a long orbit segment of the flow. These functions can be chosen to achieve essentially arbitrary correlation functions on arbitrarily large subintervals of the real line. This construction will contradict Lemma 6.1, hence completes the proof that the maximal spectral type is equivalent to Lebesgue.

For any given horizontal interval $J = \bar{J} \times \{s\} \subset M$, let T_J be the maximal real number $T > 0$ such that the map

$$F^T_J(x,t) = T^t_{\alpha,\varphi}(x), \text{ for all } (x,t) \in J \times (-T,T), \tag{58}$$

is a flow-box for the flow $\{T^t_{\alpha,\varphi}\}$. Let R^T_J denote the range of the flow-box map F^T_J. Since the flow $\{T^t_{\alpha,\varphi}\}$ has no periodic orbits, for any $T > 0$ there exists an interval J such that $T_J > T$. Let $\zeta > 0$ be fixed such that $J \subset M_\zeta$. Let $S^T_\zeta(J) \subset \mathbb{R}$ be the set defined as follows

$$S^T_\zeta(J) := \{t \in [-T,T] \,:\, T^t_{\alpha,\varphi}(J) \cap M^c_\zeta = \emptyset\}\,.$$

By definition we have that $S^T_\zeta(J)$ is an open subset. Let then $\chi_J \in C^\infty_0(J)$ be any positive function such that $\int_J \chi_J^2 = 1$ with C^0 norm bounded above by $C/\lambda(\bar{J})^{1/2}$ and C^1 norm bounded above by $C/\lambda(\bar{J})^{3/2}$ for some constant $C > 0$. For any function $\phi \in C^\infty_0(S^T_\zeta(J))$, let

$$\tilde{\phi}_J(x,t) := \chi_J(\bar{x})\phi(t)\,, \quad \text{ for all } (x,t) \in J \times (-T,T)\,.$$

Let $\phi_J \in C^\infty(M)$ be the function defined as $\phi_J = 0$ on $M \setminus R^T_J$ and as

$$(\phi_J \circ F^T_J)(x,t) := \tilde{\phi}_J(x,t)\,, \ \text{ for all } (x,t) \in J \times (-T,T)\,, \tag{59}$$

on the range R^T_J of the flow-box map F^T_J. By construction the vertical derivative of the function ϕ_J is a smooth coboundary $f_J \in \mathscr{F}$. For any function $\psi \in C^\infty_0(S^T_\zeta(J))$, let

$$\tilde{\psi}_J(x,t) := \chi_J(\bar{x})\psi(t)\,, \quad \text{ for all } (x,t) \in J \times (-T,T)\,.$$

Let $g_J \in C^\infty_0(M_\zeta)$ be the function defined as $g_J = 0$ on $M \setminus R^T_J$ and as

$$(g_J \circ F^T_J)(x,t) := \tilde{\psi}_J(x,t)\,, \ \text{ for all } (x,t) \in J \times (-T,T)\,, \tag{60}$$

on the range R^T_J of the flow-box map F^T_J. Let us also assume that $T_J/2 > T$. By construction, we have

$$\begin{aligned} \int_{\mathbb{R}} \omega(t) < f_J \circ T^t_{\alpha,\varphi}, g_J > dt &= \int_{\mathbb{R}\setminus[-T_J/2,T_J/2]} \omega(t) < f_J \circ T^t_{\alpha,\varphi}, g_J > dt \\ &+ \int_J \chi_J^2(x)[\int_{-T_J}^{T_J} \left(\int_{-T}^{T} \omega(t)\frac{d\phi}{dt}(\sigma+t)dt\right)\psi(\sigma)d\sigma]dx\,. \end{aligned} \tag{61}$$

We will let the width of the flow-box, that is, the length of the horizontal interval J, converge to zero, so that the height $T_J > 0$ of the flow-box diverges. The following result establishes the key property that the contribution to the integral in (61) of the correlation function outside of the large interval $\mathbb{R} \setminus [-T_J/2, T_J/2]$, that is, the first term on the RHS, vanishes in the limit.

Lemma 6.2. *There exists a decreasing sequence of intervals J with common midpoint of length $\lambda(\bar{J})$ converging to zero such that*

$$\limsup_{\lambda(\bar{J})\to 0} \int_{\mathbb{R}\setminus[-T_J/2,T_J/2]} \omega(t) < f_J \circ T^t_{\alpha,\varphi}, g_J > dt = 0\,.$$

Proof. Since the function g_J is supported on the range R^T_J of the flow-box map F^T_J it is enough to prove bounds on

$$\int_{R^T_J} f_J \circ T^t_{\alpha,\varphi}(x) g_J(x) d\mu\,.$$

There exist a constant $C > 0$ and a sequence of intervals J of length $\lambda(\bar{J})$ converging to zero such that

$$T_J \geqslant \frac{C}{\lambda(\bar{J})}\,. \tag{62}$$

For instance, one may take a sequence of intervals $\{J_m\}$ of the form

$$J_m := [\theta_0 - \frac{1}{10q_m}, \theta_0 + \frac{1}{10q_m}] \times \{s_0\}, \qquad \text{for all } m \in \mathbb{N}.$$

Throughout this argument, the symbol $C_{\phi,\psi}$ will denote a generic constant depending only on the the C^1 norms of the functions ϕ and $\psi \in S^T_\zeta(J) \subset \mathbb{R}$ in the definition of $f_J \in \mathscr{F}$ and $g_J \in C^\infty_0(M_\zeta)$.

Let then $t \geqslant T_J/2$. We claim that there exists $C_{\phi,\psi} > 0$ such that for some $\varepsilon > 0$ we have

$$|\int_{R^T_J \setminus \mathscr{B}_l} f_J \circ T^t_{\alpha,\varphi}(x) g_J(x) d\mu| \leqslant C_{\phi,\psi} T\, t^{-1/2-\varepsilon}. \tag{63}$$

In fact, there exists a constant $C' > 0$ such that, since by assumption $t \geqslant C/2\lambda(\bar{J})$ there exists a product set $E^T_{J,k}$ with base $\bar{E}^T_{J,k}$ measurable with respect to the partition $\mathscr{I}_k$, such that $R^T_J \subset E^T_{J,k}$ and we have

$$\mu(E^T_{J,k}) \leqslant C' \mu(R^T_J) = C' T \lambda(\bar{J}).$$

By construction there exists a constant $C'' > 0$ such that

$$\mathscr{N}_0(f_J, g_J) = \|f_J\|_0 \|g_J\|_0 \leqslant \frac{C''}{\lambda(\bar{J})} \|\phi\|_0 \|\psi\|_0;$$

$$\mathscr{N}_1(f_J, g_J) = (\|f_J\|_0 + \|\phi_J\|_0)\|g_J\|_1 + (\|f_J\|_1 + \|\phi_J\|_1)\|g_J\|_0 \leqslant \frac{C''}{\lambda(\bar{J})^2} \|\phi\|_2 \|\psi\|_1.$$

Hence it follows from Proposition 5.5 that

$$\left| \int_{E^T_{J,k} \setminus \mathscr{B}_l} f(T^t_{\alpha,\varphi}(x,s)) g(x,s) d\mu \right| < C'' \left(C'T \|\phi\|_0 \|\psi\|_0 + (C'T)^2 \|\phi\|_2 \|\psi\|_1 \right) t^{-1/2-\varepsilon}.$$

The above claim is therefore proved.

It remains to estimate the integral on the bad set $\mathscr{B}_l \cap R^T_J$. Let U_i be any such tower. We follow the proof of Proposition 5.6 in Subsection 5.3. Let $t \in [l^{21/20}, (l+1)^{21/20}]$. Let us recall the notation $l_0 = l^{21/20}$, $l_1 = (l+1)^{21/20}$ and let $n \in \mathbb{N}$ be the unique natural number such that $q_n < l_0 < q_{n+1}$.

Let $A_J := \{t \in [l_0, l_1] \,:\, \int_{U_i} f_J(T^t_{\alpha,\varphi}(x)) g_J(x) d\mu > 0\}$. Let $\rho_J(t) = 1$ if $t \in A_J$ and $\rho_J(t) = -1$ if $t \in [l_0, l_1] \setminus A_J$. Let F^T_J denote the flow-box map introduced above and let $R^T_J \subset M$ denote its range. Then, by Cauchy-Schwarz (Hölder) inequality, we have

$$\begin{aligned} \int_{l_0}^{l_1} \left| \int_{U_i} f_J(T^t_{\alpha,\varphi}(x)) g_J(x) d\mu \right| dt &= \int_{U_i \cap R^T_J} \left(\int_{l_0}^{l_1} \rho_J(t) f_J(T^t_{\alpha,\varphi}(x)) dt \right) g_J(x) d\mu \\ &\leqslant \left(\int_{U_i \cap R^T_J} \left(\int_{l_0}^{l_1} \rho_J(t) f_J(T^t_{\alpha,\varphi}(x)) dt \right)^2 d\mu \right)^{1/2} \left(\int_{U_i \cap R^T_J} g_J(x)^2 d\mu \right)^{1/2} \\ &\leqslant \|g_J\|_0 \mu(U_i \cap R^T_J)^{1/2} \left(\int_{U_i \cap R^T_J} \left(\int_{l_0}^{l_1} \rho(t) f_J(T^t_{\alpha,\varphi}(x)) dt \right)^2 d\mu \right)^{1/2}. \end{aligned} \tag{64}$$

Finally we have

$$\left(\int_{U_i\cap R_J^T}\left(\int_{l_0}^{l_1}\rho_J(t)f_J(T^t_{\alpha,\varphi}(x))dt\right)^2 d\mu\right)\leqslant \|f_J\|_0^2\,(l_1-l_0)^{3/2}\mu(U_i\cap R_J^T)+$$
$$\left(\int_{U_i\cap R_J^T}\left(\int_{l_0}^{l_1}\left(\int_{r\in[l_0,l_1]\,:\,|r-t|\geqslant(l_1-l_0)^{1/2}}\rho(r)\rho(t)f_J(T^t_{\alpha,\varphi}(x))f_J(T^r_{\alpha,\varphi}(x))dr\right)dt\right)d\mu\right). \quad (65)$$

Let us assume that J is chosen sufficiently small so that $t\geqslant T_J/2$ implies $(l_1-l_0)^{1/2}\geqslant 10T$. Then by construction, whenever $x'_t:=T^t_{\alpha,\varphi}(x)\in R_J^T$ and $(l_1-l_0)^{1/2}\leqslant r-t\leqslant T_J/10$, we have that $T^r_{\alpha,\varphi}(x)=T^{r-t}_{\alpha,\varphi}(x'_t)\notin R_J^T$, hence

$$\int_{U_i\cap R_J^T}\left(\int_{l_0}^{l_1}\left(\int_{r\in[l_0,l_1]\,:\,|r-t|\geqslant(l_1-l_0)^{1/2}}\rho(r)\rho(t)f_J(T^t_{\alpha,\varphi}(x))f_J(T^r_{\alpha,\varphi}(x))dr\right)dt\right)d\mu=0\,.$$

It follows that, under the assumption that $l_0-l_1\leqslant T_J/10$, the self-correlation term in formula (65) vanishes identically, hence from the bound $(l_1-l_0)^{1/2}\gg q_n^{40\eta}$ we derive that there exists a constant $C_{\phi,\psi}>0$ such that

$$\int_{l_0}^{l_1}\left|\int_{U_i\cap R_J^T}f_J(T^t_{\alpha,\varphi}(x,s))g_J(x,s)d\mu\right|dt\leqslant C_{\phi,\psi}\frac{\mu(U_i\cap R_J^T)}{\lambda(\bar J)}\frac{(l_1-l_0)}{q_n^{20\eta}}\,. \quad (66)$$

Next let us assume that $l_1-l_0\geqslant r-t\geqslant T_J/10$ and $r-t\geqslant(l_1-l_0)^{1/2}$. Let then $t^*=r-t$ and recall the notation established in Subsection 5.3: let $l^*=[t^*]$ and n^* to be the unique integer such that $q_{n^*}\leqslant l^*<q_{n^*+1}$. Let k^* be any integer such that $q_{k^*}\in[q_{n^*}\log^{15}q_{n^*},q_{n^*}\log^{20}q_{n^*}]$. We recall that by construction we have $q_{k^*}\in[q_n^{\frac{1}{41}},q_n^{\frac{1}{19}}\log^{20}q_n]$. By the lower bound (62), since $t^*\geqslant T_J/10$, this implies that $\lambda(\bar J)\geqslant 1/q_{k^*}$ and that, for any interval $\bar I\in\mathscr{I}_{k^*}$, we have $\lambda(\bar I)\leqslant 1/q_{k^*}\leqslant 1/q_n^{1/41}$.

By property (B_5) in Proposition 4.2 it suffices to estimate

$$\sum_{i=1}^m\left|\int_{\mathscr{T}_{t,i}\cap R_J^T}f_J(T^{t^*}_{\alpha,\varphi}(x))f_J(x)d\mu\right|.$$

In fact, from the measure bound in (51) we derive that

$$\begin{aligned}\sum_{i=1}^m\left|\int_{\mathscr{T}_{t,i}\triangle T^{-t}_{\alpha,\varphi}(U_i)}f_J(T^{t^*}_{\alpha,\varphi}(x))f_J(x)d\mu\right|&\leqslant\frac{\|f_J\|_0^2}{q_n^{3/5-15\eta}}\\&\leqslant\frac{C_{\phi,\psi}}{\lambda(\bar J)}\frac{1}{q_n^{3/5-15\eta}}\leqslant C_{\phi,\psi}\frac{q_{k^*}}{q_n^{3/5-15\eta}}\leqslant\frac{C_{\phi,\psi}}{q_n^{1/2+50\eta}}\,.\end{aligned} \quad (67)$$

Following Lemma 5.9, we distinguish two cases. In the first case we have $N(\theta_{t,i},h_{t,i})\leqslant q_n^{1/3}$. By the bound in (53) we then have

$$\left|\int_{\mathscr{T}_{t,i}}f_J(x)f_J(T^{t^*}_{\alpha,\varphi}(x))d\mu\right|\leqslant\|f_J\|_0^2\frac{\mu(\mathscr{T}_{t,i})}{q_n^{\frac{1}{10}}}\leqslant\frac{C_{\phi,\psi}}{q_n^{50\eta}}\mu(\mathscr{T}_{t,i}). \quad (68)$$

In the second case we have $N(\theta_{t,i},h_{t,i})\geqslant q_n^{1/3}$. From the bound in (54), for all $I:=\bar I\times\{s\}$ with $\bar I\in\mathscr{I}_{k^*}$ we have

$$\left|\int_{\overline{\mathscr{T}_{t,i}\cap I}}f_J(T^{t^*}_{\alpha,\varphi}(\theta,s))f_J(\theta,s)d\theta\right|\leqslant C\{\mathscr{N}_0(f_J,f_J)+\mathscr{N}_1(f_J,f_J)\lambda(\bar I)\}\frac{\lambda(\overline{\mathscr{T}_{t,i}\cap I})}{q_n^{50\eta}}. \quad (69)$$

By the lower bound (62) it follows that $\lambda(\bar{J}) \geqslant 1/q_{k^*}$, hence there exists a product set E^T_{J,k^*} with base $\bar{E}^T_{J,k^*}$ measurable with respect to the partition $\mathscr{I}_{k^*}$, such that $R^T_J \subset E^T_{J,k^*}$ and we have

$$\mu(E^T_{J,k^*}) \leqslant C\mu(R^T_J) = CT\lambda(\bar{J}).$$

In conclusion, we derive the following estimate:

$$\left|\int_{\mathscr{T}_{t,i}\cap R^T_J} f_J(x) f_J(T^{t^*}_{\alpha,\varphi}(x)) d\mu\right| \leqslant \frac{C_{\phi,\psi}}{q_n^{50\eta}} \frac{\mu(\mathscr{T}_{t,i}\cap R^T_J)}{\lambda(\bar{J})}. \tag{70}$$

By the above bounds on self-correlations and by the bound in (65), the term on the RHS of (64) can therefore be bounded by the product

$$\frac{C_{\phi,\psi}}{q_n^{20\eta}}(l_1-l_0)\frac{\mu(U_i\cap R^T_J)^{1/2}}{\lambda(\bar{J})^{1/2}}\left(\frac{\mu(U_i\cap R^T_J)}{\lambda(\bar{J})}+\frac{\mu(\mathscr{T}_{t,i}\cap R^T_J)}{\lambda(\bar{J})}+\mu(\mathscr{T}_{t,i})\right)^{1/2}.$$

After summing over $i \in \{1,\dots,m\}$, by Cauchy-Schwarz (Hölder) inequality and by taking into account the bound (51) on the error in approximating the towers U_i by the complete towers $\mathscr{T}_{t,i}$, we get a bound by the quantity

$$\frac{C_{\phi,\psi}}{q_n^{20\eta}}(l_1-l_0)\frac{\mu(\mathscr{B}_l\cap R^T_J)^{1/2}}{\lambda(\bar{J})^{1/2}}\left(\frac{\mu(\mathscr{B}_l\cap R^T_J)+q_n^{-3/5+15\eta}}{\lambda(\bar{J})}+\mu(\mathscr{B}_l)+q_n^{-3/5+15\eta}\right)^{1/2}. \tag{71}$$

By the equidistribution properties of the base rotation under the Diophantine assumption on the rotation number, there exists a constant $C>0$ such that

$$\mu(\mathscr{B}_l) \leqslant C\frac{1}{q_n^{1/2-4\eta}} \quad \text{and} \quad \mu(\mathscr{B}_l\cap R^T_J) \leqslant C\mu(R^T_J)\frac{\log^2 q_n}{q_n^{1/2-4\eta}} \leqslant 2CT\frac{\lambda(\bar{J})}{q_n^{1/2-5\eta}}.$$

Thus in all cases by the estimate in (64)-(70) we can conclude that

$$\int_{l_0}^{l_1}\left|\int_{\mathscr{B}_l\cap R^T_J} f_J(T^t_{\alpha,\varphi}(x)) g_J(x) d\mu\right| dt \leqslant C_{\phi,\psi}T\frac{l_1-l_0}{q_n^{1/2+15\eta}},$$

which together with the immediate estimate

$$\left|\int_{\mathscr{B}_l\cap R^T_J} f_J(T^t_{\alpha,\varphi}(x)) g_J(x) d\mu\right| \leqslant C_{\phi,\psi}\frac{\mu(\mathscr{B}_l\cap R^T_J)}{\lambda(\bar{J})}.$$

implies our final estimate on the bad set, that is, as soon as $l_0 \geqslant T_J/2$,

$$\int_{l_0}^{l_1}\left|\int_{\mathscr{B}_l\cap R^T_J} f_J(T^t_{\alpha,\varphi}(x)) g_J(x) d\mu\right|^2 dt \leqslant C_{\phi,\psi}T\frac{l_1-l_0}{q_n^{1+\eta}} \leqslant C_{\phi,\psi}\frac{l_1-l_0}{l_0^{1+\eta/2}} \leqslant \frac{2C_{\phi,\psi}}{l^{1+\eta/3}}. \tag{72}$$

Since $\omega \in L^2(\mathbb{R},dt)$, the statement of the lemma then follows by Cauchy-Schwarz (Hölder) inequality from the estimates in formulas (63) and (72). □

We finally derive the following result.

Lemma 6.3. *Let $w \in L^2(\mathbb{R}, dt)$ be a smooth function. Assume that for all functions $f \in \mathscr{F}$ and for all $g \in C_0^1(M)$ we have*

$$\int_{\mathbb{R}} \omega(t) < f \circ T^t_{\alpha,\varphi}, g >_{L^2(M)} dt = 0\,,$$

then the function w vanishes identically.

Proof. By Lemma 6.2 there exists a decreasing sequence of intervals $\{J\}$ with common midpoint $x_0 \in J$ and length $\lambda(\bar{J})$ converging to zero such that

$$\lim_{\lambda(\bar{J})\to 0} \int_{\mathbb{R}} \omega(t) < f_J \circ T^t_{\alpha,\varphi}, g_J >_{L^2(M)}$$
$$= \lim_{\lambda(\bar{J})\to 0} \int_{\bar{J}} \chi_J^2(\theta)[\int_{-T_J}^{T_J} \left(\int_{-T}^{T} \omega(t) \frac{d\phi}{dt}(\sigma+t)dt \right) \psi(\sigma)d\sigma]d\theta\,.$$

Since $\int_{\bar{J}} \chi_J^2(\theta)d\theta = 1$ for all J and $T_J \to +\infty$, we have

$$\lim_{\lambda(\bar{J})\to 0} \int_{\bar{J}} \chi_J^2(\theta)[\int_{-T_J}^{T_J} \left(\int_{-T}^{T} \omega(t) \frac{d\phi}{dt}(\sigma+t)dt \right) \psi(\sigma)d\sigma]d\theta$$
$$\lim_{\lambda(\bar{J})\to 0} \int_{-T_J}^{T_J} \left(\int_{-T}^{T} \omega(t) \frac{d\phi}{dt}(\sigma+t)dt \right) \psi(\sigma)d\sigma = \int_{\mathbb{R}} \left(\int_{-T}^{T} \omega(t) \frac{d\phi}{dt}(\sigma+t)dt \right) \psi(\sigma)d\sigma\,.$$

Since the function $\psi \in C_0^\infty(S_\zeta^T(J))$ is arbitrary, it follows that

$$\int_{-T}^{T} \omega(t) \frac{d\phi}{dt}(\sigma+t)dt = 0\,, \quad \text{for all } \sigma \in S_\zeta^T(J)\,,$$

and since the function $\phi \in C_0^\infty(S_\zeta^T(J))$, but it is otherwise arbitrary, it follows that the function $\omega \in L^2(\mathbb{R}, dt)$ is constant, hence it vanishes identically. □

By Lemma 6.1 and Lemma 6.3 it follows that under the hypotheses of Theorem 2 the maximal spectral type of the flow $\{T^t_{\alpha,\varphi}\}$ is Lebesgue. The proof of Theorem 2 is therefore complete.

A Birkhoff sums estimates

Proof of Lemma 3.4. By the definition of u_I in (18), we know that there exist $x_0 \in I \cap T^{-t}_{\alpha,\varphi}(W)$ and $t_0 \in [l_0, l_1]$ such that $\varphi''_{N(x_0,t_0)}(\bar{x}_0) \geqslant q_n^{3-\eta} \log^9 q_n$. Since $N(x_0,t_0) < cq_{n+1}$, by (12), we get

$$(q_n)^{3-\eta} \log^9 q_n < \varphi''_{N(x_0,t_0)}(\bar{x}_0) < 2c^3 q_{n+1}^{3-\eta} + \frac{1}{x_{min}^{N(x_0,t_0)}} < q_n^{3-\eta} \log^4 q_n + \frac{1}{x_{min}^{N(x_0,t_0)}},$$

which means that there exists $j \in [0, N(x_0,t_0) - 1]$ s.t.

$$\bar{x}_0 + j\alpha \in [-\frac{1}{q_n \log^3 q_n}, \frac{1}{q_n \log^3 q_n}]. \tag{73}$$

We will show that, for every $t \in [l_0, l_1]$ and every $x \in I \cap T^{-t}_{\alpha,\varphi}(W)$, we have

$$N(x,t) > j. \tag{74}$$

Let us first show how (74) implies (19) and (20). Since $\lambda(\bar{I}) \leqslant \frac{1}{q_n \log^{15} q_n}$ it follows by (74) that for every $t \in [l_0, l_1]$ and every $x \in I \cap T^{-t}_{\alpha,\varphi}(W)$

$$x_{min}^{N(x,t)} \leqslant d(\bar{x}+j\alpha,0) \leqslant d(\bar{x}+j\alpha,0)+\lambda(\bar{I}) \leqslant \frac{1}{2q_n \log^3 q_n}.$$

This gives (19). For (20), we have by (11) and (12)

$$|\varphi'_{N(x,t)}(\bar{x})| \geqslant \left(\frac{2}{3x_{min}^{N(x,t)}}\right)^{2-\eta} - 4q_{n+2}^{2-\eta} \geqslant \left(\frac{1}{2x_{min}^{N(x,t)}}\right)^{2-\eta},$$

and

$$|\varphi''_{N(x,t)}(\bar{x})| \leqslant \left(\frac{3}{2x_{min}^{N(x,t)}}\right)^{3-\eta} + 4q_{n+2}^{3-\eta} \leqslant \left(\frac{2}{x_{min}^{N(x,t)}}\right)^{3-\eta}.$$

This gives (20). Therefore it remains to show (74). Notice that for $x \in I \cap T^{-t}_{\alpha,\varphi}(W)$, (74) is equivalent to

$$N(x,t) \geqslant j \tag{75}$$

(since $T^t_{\alpha,\varphi}(x) = (\bar{x}+N(x,t)\alpha, s') \in W$). Notice also that if the lower bound

$$N(x,t_0) \geqslant j, \tag{76}$$

holds, then (75) follows for all $t \in [l_0, l_1]$. Indeed, otherwise we have

$$(4q_{n+1})^{1-\eta} \geqslant \varphi(\bar{x}+N(x,t)\alpha) \geqslant t+s-\varphi_{N(x,t)}(\bar{x}) \geqslant \\ \varphi_{N(x,t_0)}(\bar{x}) - \varphi_{N(x,t)}(\bar{x}) \geqslant \varphi(\bar{x}+j\alpha) \geqslant q_n^{1-\eta}\log^2 q_n,$$

a contradiction. Hence it remains to show (76). Assume by contradiction that $N(x,t_0) < j$ for some $x \in I \cap T^{-t}_{\alpha,\varphi}(W)$. Then, by the definition of j, we have

$$\bigcup_{i=0}^{N(x,t_0)} R^i_\alpha(\bar{I}) \cap \left[-\frac{1}{5q_{n+2}}, \frac{1}{5q_{n+2}}\right] = \emptyset. \tag{77}$$

Therefore, for every $\theta \in \bar{I}$ by (11) we have

$$|\varphi'_j(\theta)| < 10q_{n+2}^{2-\eta}. \tag{78}$$

Hence, by (73), (77), and (78), for some $\theta \in \bar{I}$, we get

$$(5q_{n+2})^{1-\eta} \geqslant \max(\varphi(\bar{x}+N(x,t_0)\alpha), \varphi(\bar{x}_0+N(x_0,t_0)\alpha)) \geqslant \\ |\varphi_{N(x,t_0)}(\bar{x}) - \varphi_{N(x_0,t_0)}(\bar{x}_0)| \geqslant \varphi(\bar{x}_0+j\alpha) - |\varphi'_j(\theta)|\lambda(\bar{I}) \geqslant 1/2\left(q_n \log^3 q_n\right)^{1-\eta} - (q_{n+1})^{1-\eta},$$

which yields a contradiction since $q_{n+2} < q_n \log^{2+3\xi} q_n$. So (76) holds. This completes the proof of Lemma 3.4. □

Proof of Lemma 3.5. Notice that for some $\theta \in [\bar{x}, \bar{x}_0]$ we have

$$\varphi'_{N(x)}(\bar{x}) - \varphi'_{N(x_0)}(\bar{x}_0) = \varphi''_{N(x_0)}(\theta)(\bar{x} - \bar{x}_0) + \varphi'_{N(x)-N(x_0)}(\bar{x} + N(x_0)\alpha).$$

Since $|\varphi''_{N(x_0)}(\bar{x}_0)| \leqslant q_n^{3-\eta}\log^{10} q_n$, by (12) for $N = N(x_0)$ it follows that

$$\{\bar{x}_0, \ldots, \bar{x}_0 + (N(x_0)-1)\alpha\} \cap [-\frac{1}{q_n \log^4 q_n}, \frac{1}{q_n \log^4 q_n}] = \emptyset. \tag{79}$$

Notice that since $x_0 \in W$, for some constant $c > 0$, we have

$$\varphi_{N(x_0)}(\bar{x}_0) \geqslant t - q_n^{3/4} \geqslant c q_n.$$

So by (79), by (10) for $N = N(x_0)$ and by the Diophantine condition on α, we have $q_{r+1} \geqslant \frac{cq_n}{10}$ (where r is such that $q_r \leqslant N(x_0) \leqslant q_{r+1}$). But then by (10) for $N = N(x_0)$ and $x = \theta$ and again by the Diophantine condition on α, we have

$$\varphi''_{N(x_0)}(\theta) \geqslant \frac{q_n^{3-\eta}}{\log^5 q_n}.$$

Define $A_{x,x_0} := \varphi''_{N(x_0)}(\theta)$. We will show that

$$|\varphi'_{N(x)-N(x_0)}(\bar{x} + N(x_0)\alpha)| \leqslant \frac{A_{x,x_0}}{10}|\bar{x} - \bar{x}_0|. \tag{80}$$

By the definition of $N(x)$, $N(x_0)$ and since $T^t_{\alpha,\varphi}(x) \in V$, for some $z \in [\bar{x}, \bar{x}_0]$ we have

$$2q_n^{3/4(1-\eta)} \geqslant |(t - \varphi_{N(x_0)}(\bar{x}_0)) - (t - \varphi_{N(x)}(\bar{x}))| \geqslant |\varphi_{N(x)}(\bar{x}) - \varphi_{N(x_0)}(\bar{x}_0)| =$$
$$|\varphi'_{N(x_0)}(\bar{x}_0)(\bar{x} - \bar{x}_0) + \varphi''_{N(x_0)}(\bar{z})(\bar{x} - \bar{x}_0)^2 + \varphi_{N(x)-N(x_0)}(\bar{x} + N(x_0)\alpha)|.$$

Moreover, we have the following:

Claim. *If $\varphi''_{N(x)}(\bar{x}) < q_n^{3-\eta}\log^{10} q_n$, then for every $z \in I$*

$$\varphi''_{N(x)}(\bar{z}) < 30 q_n^{3-\eta}\log^{10} q_n.$$

Therefore

$$|\varphi_{N(x)-N(x_0)}(\bar{x} + N(x_0)\alpha)| \leqslant 2q_n^{3/4(1-\eta)} + q_n^{7/4+\eta}|\bar{x} - \bar{x}_0| + q_n^{3-\eta}\log^5 q_N(\bar{x} - \bar{x}_0)^2,$$

so by Lemma 3.2,

$$|\varphi'_{N(x)-N(x_0)}(\bar{x} + N(x_0)\alpha)| \leqslant$$
$$3\left(4q_n^{3/2(1-\eta^2)} + q_n^{(7/2+2\eta)(1+\eta)}|\bar{x} - \bar{x}_0|^2 + q_n^{(6-2\eta)(1+\eta)}\log^{10+2\eta} q_n(\bar{x} - \bar{x}_0)^4\right). \tag{81}$$

Notice however that since $\frac{1}{q_n \log^{15} q_n} \geqslant \frac{1}{q_k} \geqslant \lambda(\bar{I}) \geqslant |\bar{x} - \bar{x}_0| \geqslant \frac{1}{q_n^{3/2-2\eta}}$, we have

$$\frac{q_n^{3-\eta}}{\log^{10} q_n}|\bar{x} - \bar{x}_0| \geqslant$$
$$100 \max\left(q_n^{3/2(1-\eta^2)}, q_n^{(7/2+2\eta)1+\eta}|\bar{x} - \bar{x}_0|^2, q_n^{(6-2\eta)(1+\eta)}\log^{10+2\eta} q_n(\bar{x} - \bar{x}_0)^4\right).$$

Therefore and using (81) we get (80) which completes the proof of Lemma 3.5.

We just have to give the proof of the claim.

Proof of the Claim. We know that $N(x) \leqslant q_{n+2}$. If $\varphi''_{N(x)}(\bar{z}) \geqslant 30 q_n^{3-\eta} \log^{10} q_n$, by (12) it follows that $z_{min}^{N(x)} \leqslant \frac{1}{3q_n \log^{\frac{10}{3-\eta}} q_n}$. But since $x, z \in I$ and $\lambda(\bar{I}) < \frac{1}{q_n \log^{15} q_n}$, we would have $x_{min}^{N(x)} \leqslant \frac{1}{2q_n \log^{\frac{10}{3-\eta}} q_n}$. So by applying (12) for $N(x)$ and x, we would get $\varphi''_{N(x)}(\bar{x}) \geqslant 2 q_n^{3-\eta} \log^{10} q_n$, a contradiction. □

□

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