ATTRACTED BY AN ELLIPTIC FIXED POINT

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Abstract. We give examples of symplectic diffeomorphisms of $\mathbb{R}^6$ for which the origin is a non-resonant elliptic fixed point which attracts an orbit.

1. Introduction

Consider a symplectic diffeomorphism of $\mathbb{R}^{2n}$ (for the canonical symplectic form) with a fixed point at the origin. We say that the fixed point is elliptic of frequency vector $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{R}^n$ if the linear part of the diffeomorphism at the fixed point is conjugate to the rotation map

$$S_\omega : (\mathbb{R}^2)^n \circlearrowleft, \qquad S_\omega(s_1, \ldots, s_n) := (R_{\omega_1}(s_1), \ldots, R_{\omega_n}(s_n)).$$

Here, for $\beta \in \mathbb{R}$, R_β stands for the rigid rotation around the origin in $\mathbb{R}^2$ with rotation number β. We say that the frequency vector ω is non-resonant if for any $k \in \mathbb{Z}^n - \{0\}$ we have $(k, \omega) \notin \mathbb{Z}$, where $(\cdot, \cdot)$ stands for the Euclidean scalar product.

It is easy to construct symplectic diffeomorphisms with orbits attracted by a resonant elliptic fixed point. For instance, the time-1 map of the flow generated by the Hamiltonian function $H(x, y) = y(x^2 + y^2)$ in $\mathbb{R}^2$ has a saddle-node type fixed point, at which the linear part is zero, which attracts all the points on the negative part of the x-axis. The situation is much subtler in the non-resonant case.

The Anosov-Katok construction [AK70] of ergodic diffeomorphisms by successive conjugations of periodic rotations of the disc gives examples of smooth area preserving diffeomorphisms with non-resonant elliptic fixed points at the origin that are Lyapunov unstable. The method also yields examples of ergodic symplectomorphisms with non-resonant elliptic fixed points in higher dimensions.

These constructions obtained by the successive conjugation technique have totally degenerate fixed points since they are C^∞-tangent to a rotation S_ω at the origin.

In the non-degenerate case, R. Douady gave examples in [D88] of Lyapunov unstable elliptic points for smooth symplectic diffeomorphisms for any $n \geqslant 2$, for which the Birkhoff normal form has non-degenerate Hessian at the fixed point but is otherwise arbitrary. Prior examples for $n = 2$ were obtained in [DLC83] (note that by KAM theory, a non-resonant elliptic fixed point of a smooth area preserving surface diffeomorphism that has a non zero Birkhoff normal form is accumulated by invariant quasi-periodic smooth curves (see [Mo73]). Hence in the one dimensional case, non-degeneracy implies that the point is Lyapunov stable).

In both of the above examples, no orbit distinct from the origin converges to it. Indeed, in the Anosov-Katok examples, a sequence of iterates of the diffeomorphism converges uniformly to Identity, hence every orbit is recurrent and no orbit can converge to the origin, besides the origin itself. As for the non-degenerate examples of Douady and Le Calvez, their Lyapunov instability is deduced from the existence of a sequence of points that converge to the fixed point, and whose orbits travel, along a simple resonance, away from the fixed point. By construction, these examples do not have a single orbit besides the origin that converges to it.

Our goal in this paper is to construct an example of a Lyapunov unstable fixed point for a Gevrey diffeomorphism with an orbit converging to it. Recall that, given a real $\alpha \geqslant 1$, Gevrey-α regularity is defined by the requirement that the partial derivatives exist at all (multi)orders ℓ and are bounded by $CM^{|\ell|}\,|\ell|!^{\alpha}$ for some C and M (when $\alpha = 1$, this simply means analyticity); upon fixing a real $L > 0$ which essentially stands for the inverse of the previous M, one can define a Banach algebra $\big(G^{\alpha,L}(\mathbb{R}^{2n}), \|\,.\,\|_{\alpha,L}\big)$.

We set $X := (\mathbb{R}^2)^3$ and denote by $\mathcal{U}^{\alpha,L}$ the set of all Gevrey-(α, L) symplectic diffeomorphisms of X which fix the origin and are C^∞-tangent to Id at the origin. We refer to Appendix A for the precise definition of $\mathcal{U}^{\alpha,L}$ and of a distance $\operatorname{dist}(\Phi, \Psi) = \|\Phi - \Psi\|_{\alpha,L}$ which makes it a complete metric space. We will prove the following.

Theorem A. *Fix $\alpha > 1$ and $L > 0$. For each $\gamma > 0$, there exist a non-resonant vector $\omega \in \mathbb{R}^3$, a point $z \in X$, and a diffeomorphism $\Psi \in \mathcal{U}^{\alpha,L}$ such that $\|\Psi - \mathrm{Id}\|_{\alpha,L} \leqslant \gamma$ and $T = \Psi \circ S_\omega$ satisfies $T^n(z) \underset{n\to+\infty}{\longrightarrow} 0$.*

We do not know how to produce real analytic examples. Recall that not even one example of a real analytic symplectomorphism with a Lyapunov unstable non-resonant elliptic fixed point is known.

For other instances of the use of Gevrey regularity with symplectic or Hamiltonian dynamical systems, see *e.g.* [Po04], [MS03], [MS04], [MP10], [LMS18], [BF18].

Our construction easily extends to the case where $X = (\mathbb{R}^2)^n$ with $n \geqslant 3$, however we do not know how to adapt the method to the case $n = 2$. As for the case $n = 1$, there may well be no regular examples at all. Indeed if the rotation frequency at the fixed point is Diophantine, then a theorem by Herman (see [FK09]) implies that the fixed point is surrounded by invariant quasi-periodic circles, and thus is Lyapunov stable. The same conclusion holds by Moser's KAM theorem if the Birkhoff normal form at the origin is not degenerate [Mo73]. In the remaining case of a degenerate Birkhoff normal form with a Liouville frequency, there is evidence from [AFLXZ] that the diffeomorphism should then be rigid in the neighborhood of the origin, that is, there exists a sequence of integers along which its iterates converge to Identity near the origin, which clearly precludes the convergence to the origin of an orbit.

Similar problems can be addressed where one searches for Hamiltonian diffeomorphisms (or vector fields) with orbits whose α-limit or ω-limit have large Hausdorff dimension (or positive Lebesgue measure) and in particular contain families of non-resonant invariant Lagrangian tori instead of a single non-resonant fixed point. A specific example for Hamiltonian flows on $(\mathbb{T} \times \mathbb{R})^3$ is displayed in [KS12], while a more generic one has been announced in [KG14]. In these examples, the setting is perturbative and the Hamiltonian flow is non-degenerate in the neighborhood of the tori. The methods involved there are strongly related to Arnold diffusion and are completely different from ours.

2. Preliminaries and outline of the strategy

From now on we fix $\alpha > 1$ and $L > 0$. We also pick an auxiliary $L_1 > L$. For $z \in \mathbb{R}^2$ and $\nu > 0$, we denote by $B(z, \nu)$ the closed ball relative to $\| \, . \, \|_\infty$ centred at z with radius ν. Since $\alpha > 1$, we have

Lemma 2.1. *There is a real $c = c(\alpha, L_1) > 0$ such that, for any $z \in \mathbb{R}^2$ and $\nu > 0$, there exists a function $f_{z,\nu} \in G^{\alpha, L_1}(\mathbb{R}^2)$ which satisfies*

- $0 \leqslant f_{z,\nu} \leqslant 1$,
- $f_{z,\nu} \equiv 1$ *on* $B(z, \nu/2)$,
- $f_{z,\nu} \equiv 0$ *on* $B(z, \nu)^c$,
- $\|f_{z,\nu}\|_{\alpha, L_1} \leqslant \exp(c \, \nu^{-\frac{1}{\alpha - 1}})$.

Proof. Use Lemma 3.3 of [MS04]. □

We now fix an arbitrary real $R > 0$ and pick an auxiliary function $\eta_R \in G^{\alpha,L_1}(\mathbb{R})$ which is identically 1 on the interval $[-2R, 2R]$, identically 0 outside $[-3R, 3R]$, and everywhere non-negative. We then define $g_R\colon \mathbb{R}^2 \to \mathbb{R}$ by the formula

$$g_R(x,y) := xy\,\eta_R(x)\,\eta_R(y). \tag{2.1}$$

The following diffeomorphisms will be of constant use in this paper:

Definition 2.1. *For $(i,j) \in \{1,2,3\}$, $z \in \mathbb{R}^2$ and $\nu > 0$, we denote by $\Phi_{i,j,z,\nu}$ the time-one map of the Hamiltonian flow generated by the function $\exp(-c\,\nu^{-\frac{2}{\alpha-1}})f_{z,\nu} \otimes_{i,j} g_R$, where $f_{z,\nu} \otimes_{i,j} g_R\colon X \to \mathbb{R}$ stands for the function*

$$s = (s_1,s_2,s_3) \mapsto f_{z,\nu}(s_i)g_R(s_j).$$

In the above definition, our convention for the Hamiltonian vector field generated by a function H is $X_H = \sum(-\frac{\partial H}{\partial y_i}\frac{\partial}{\partial x_i} + \frac{\partial H}{\partial x_i}\frac{\partial}{\partial y_i})$. Note that the Hamiltonian $\exp(-c\,\nu^{-\frac{2}{\alpha-1}})f_{z,\nu} \otimes_{i,j} g_R$ has compact support, hence it generates a complete vector field and Definition 2.1 makes sense. Actually, any $H \in G^{\alpha,L_1}(X)$ has bounded partial derivatives, hence X_H is always complete; the flow of X_H is made of Gevrey maps for which estimates are given in Appendix A.2. In the case of $\Phi_{i,j,z,\nu}$, for ν small enough we have

$$\Phi_{i,j,z,\nu} \in \mathcal{U}^{\alpha,L} \quad \text{and} \quad \|\Phi_{i,j,z,\nu} - \mathrm{Id}\|_{\alpha,L} \leqslant K\exp(-c\,\nu^{-\frac{1}{\alpha-1}}), \tag{2.2}$$

with $K := C\|g_R\|_{\alpha,L_1}$, where C is independent from i,j,z,ν and stems from (A.6). Here are the properties which make the $\Phi_{i,j,z,\nu}$'s precious. To alleviate the notations, we state them for $\Phi_{2,1,z,\nu}$ but similar properties hold for each diffeomorphism $\Phi_{i,j,z,\nu}$.

Lemma 2.2. *Let $z \in \mathbb{R}^2$ and $\nu > 0$. Then $\Phi_{2,1,z,\nu}$ satisfies:*

(a) *For every $(s_1,s_2,s_3) \in X$ such that $s_2 \in B(z,\nu)^c$,*

$$\Phi_{2,1,z,\nu}(s_1,s_2,s_3) = (s_1,s_2,s_3).$$

(b) *For every $x_1 \in \mathbb{R}$, $s_2 \in \mathbb{R}^2$ and $s_3 \in \mathbb{R}^2$,*

$$\Phi_{2,1,z,\nu}((x_1,0),s_2,s_3) = ((\widetilde{x}_1,0),s_2,s_3) \ \textit{with}\ |\widetilde{x}_1| \leqslant |x_1|.$$

(c) *For every $x_1 \in [-2R,2R]$, $s_2 \in B(z,\nu/2)$ and $s_3 \in \mathbb{R}^2$,*

$$\Phi_{2,1,z,\nu}((x_1,0),s_2,s_3) = ((\widetilde{x}_1,0),s_2,s_3) \ \textit{with}\ |\widetilde{x}_1| \leqslant \kappa\,|x_1|,$$

where $\kappa := 1 - \frac{1}{2}\exp(-c\,\nu^{-\frac{2}{\alpha-1}})$.

Proof. The dynamics of the flow generated by $f_{z,\nu} \otimes_{2,1} g_R$ can easily be understood from those of the flows generated by $f_{z,\nu}$ alone on the second factor $\mathbb{R}^2$ and by g_R alone on the first factor $\mathbb{R}^2$. Indeed

$$\Phi_{f\otimes_{2,1}g}(z_1, z_2) = (\Phi^{f(z_2)g}(z_1), \Phi^{g(z_1)f}(z_2))$$

where Φ^h denotes the time one map associated to the Hamiltonian h.

The properties (a)-(b)-(c) immediately follow from the latter expression. □

From now on, we denote simply by $|\,.\,|$ the $\|\,.\,\|_\infty$ norm in $\mathbb{R}^2$ or in $X = \mathbb{R}^6$, and by $B(s, \rho)$ the corresponding closed ball centred at s with radius ρ (the context will tell whether it is in $\mathbb{R}^2$ or $\mathbb{R}^6$).

Here is a brief outline of the strategy for proving Theorem A and obtaining, inductively, the required Ψ, z and ω:

- The diffeomorphism Ψ in Theorem A will be obtained as an infinite product (for composition) of diffeomorphisms of the form $\Phi_{i,j,z,\nu}$, with smaller and smaller values of ν so as to derive convergence in $\mathcal{U}^{\alpha,L}$ from (2.2).
- On the other hand, R will be kept fixed and the initial condition z will be obtained as the limit of a sequence contained in the ball $B(0, R) \subset X$.
- As for the non-resonant frequency vector ω in Theorem A, it will be obtained as a limit of vectors with rational coordinates with larger and larger denominators, so as to make possible a kind of "orbit synchronization" at each step of the construction.

3. The attraction mechanism

Starting from a point $z = ((x_1, 0), (x_2, 0), (x_3, 0))$, the mechanism of attraction of the point to the origin is an alternation between bringing closer to zero the x_1,x_2 or x_3 coordinates when all the coordinates of the point come back to the horizontal axes. The main ingredient is the following lemma, where we use shortcut notation $\Phi_{i,j,x,\nu}$ for $\Phi_{i,j,(x,0),\nu}$ and, for two integers Q_1, Q_2, the notation $Q_1|Q_2$ stands for "Q_1 divides Q_2".

Lemma 3.1. *Let* $\omega = (P_1/Q_1, P_2/Q_2, P_3/Q_3) \in \mathbb{Q}^3$ *with* P_i, Q_i *coprime positive integers and*

$$z = ((x_1, 0), (x_2, 0), (x_3, 0)) \in B(0, R).$$

Set

$$T_1 = \Phi_{2,1,x_2,Q_2^{-3}} \circ \Phi_{1,3,x_1,Q_1^{-3}} \circ S_\omega$$
$$T_2 = \Phi_{3,2,x_3,Q_3^{-3}} \circ \Phi_{2,1,x_2,Q_2^{-3}} \circ S_\omega$$
$$T_3 = \Phi_{1,3,x_1,Q_1^{-3}} \circ \Phi_{3,2,x_3,Q_3^{-3}} \circ S_\omega$$

Then the following properties hold.

I) *If* $Q_3|Q_1$ *and* $Q_1|Q_2$, *and*

$$x_1 \geqslant 1/Q_1, \quad x_2 \geqslant 0, \quad x_3 \geqslant 1/Q_3,$$

then there exists N *such that*

$$T_1^N(z) = ((\hat{x}_1, 0), (\hat{x}_2, 0), (\hat{x}_3, 0))$$

with

$$0 \leqslant \hat{x}_1 \leqslant x_1/2, \quad 0 \leqslant \hat{x}_2 = x_2, \quad 0 \leqslant \hat{x}_3 \leqslant x_3,$$

and $|T_1^m(z)_i| \leqslant x_i$ *for all* $m \in \{0, \dots, N\}$.

II) *If* $Q_1|Q_2$ *and* $Q_2|Q_3$, *and*

$$x_1 \geqslant 0, \quad x_2 \geqslant 1/Q_2, \quad x_3 \geqslant 1/Q_3,$$

then there exists N *such that*

$$T_2^N(z) = ((\hat{x}_1, 0), (\hat{x}_2, 0), (\hat{x}_3, 0))$$

with

$$0 \leqslant \hat{x}_1 \leqslant x_1, \quad 0 \leqslant \hat{x}_2 \leqslant x_2/2, \quad 0 \leqslant \hat{x}_3 = x_3,$$

and $|T_2^m(z)_i| \leqslant x_i$ *for all* $m \in \{0, \dots, N\}$.

III) *If* $Q_2|Q_3$ *and* $Q_3|Q_1$, *and*

$$x_1 \geqslant 1/Q_1, \quad x_2 \geqslant 0, \quad x_3 \geqslant 1/Q_3,$$

then there exists N *such that*

$$T_3^N(z) = ((\hat{x}_1, 0), (\hat{x}_2, 0), (\hat{x}_3, 0))$$

with

$$0 \leqslant \hat{x}_1 = x_1, \quad 0 \leqslant \hat{x}_2 \leqslant x_2, \quad 0 \leqslant \hat{x}_3 \leqslant x_3/2,$$

and $|T_3^m(z)_i| \leqslant x_i$ *for all* $m \in \{0, \dots, N\}$.

Proof. We will prove the Lemma for T_2 since it will be the first map that we will use in the sequel. The proof for the maps T_1 and T_3 follows exactly the same lines.

The hypothesis $x_2 \geqslant 1/Q_2$ implies that the orbit of $z_2 = (x_2, 0)$ under the rotation R_{ω_2} enters the Q_2^{-3} neighborhood of z_2 only at times that are multiples of Q_2. Moreover $R_{\omega_2}^{\ell Q_2}(z_2) = z_2$. A similar remark holds for z_3.

Since $Q_3 > Q_2$, we consider the action of $\mathcal{T} := \Phi_{2,1,x_2,Q_2^{-3}} \circ S_\omega$ first. Since $Q_1 \mid Q_2$, if $s = (s_1, s_2, s_3)$ with $s_1 = (u_1, 0)$ and $s_2 = (u_2, 0)$, by Lemma 2.2:

$$\mathcal{T}^m(s) = (s_{1,m}, R^m_{\omega_2}(s_2), R^m_{\omega_3}(s_3)) \quad \text{for all } m \in \mathbb{N},$$

with

$$|s_{1,m}| \leqslant |s_1| .$$

Consider now the orbit of z under the full diffeomorphism T_2. Since $Q_2 \mid Q_3$, the previous remark shows that one has to take the effect of $\Phi_{3,2,x_3,Q_3^{-3}}$ into account only for the iterates of order $m = \ell Q_3$. One therefore gets

$$T_2^m(z) = (z_{1,m}, z_{2,m}, R^m_{\omega_3}(z_3)), \quad \text{for all } m \in \mathbb{N},$$

where in particular $z_{2,\ell Q_3} = (x_{2,\ell Q_3}, 0)$ with

$$0 < x_{2,(\ell+1)Q_3} \leqslant (1 - \tfrac{1}{2}\exp(-cQ_3^{\frac{6}{\alpha-1}}))x_{2,\ell Q_3},$$

and where

$$z_{2,\ell Q_3+\ell'} = R^{\ell'}_{\omega_2}(z_{2,\ell Q_3}), \quad 1 \leqslant \ell' \leqslant Q_3 - 1,$$
$$|z_{1,m}| \leqslant x_1, \quad \text{for all } m \in \mathbb{N}.$$

We let L be the smallest integer such that $0 < x_{2,LQ_3} \leqslant x_2/2$ and get the conclusion with $N = LQ_3$. □

4. Proof of Theorem A

The proof is based on an iterative process (Proposition 4.2) which is itself based on the following preliminary result. For positive integers q_1, q_2, q_3, the notation $q_3|q_1|q_2$ means "q_3 divides q_1 and q_1 divides q_2".

Proposition 4.1. *Let $\omega = (p_1/q_1, p_2/q_2, p_3/q_3) \in \mathbb{Q}^3_+$ with $q_3|q_1|q_2$ and $z = ((x_1, 0), (x_2, 0), (x_3, 0)) \in B(0, R)$ with $x_1, x_2, x_3 > 0$ and $x_2 \geqslant 1/q_2$. Then, for any $\eta > 0$, there exist*

(a) *$\overline{\omega} = (\overline{p}_1/\overline{q}_1, \overline{p}_2/\overline{q}_2, \overline{p}_3/\overline{q}_3)$ such that $\overline{q}_3|\overline{q}_1|\overline{q}_2$, the orbits of the translation of vector $\overline{\omega}$ on $\mathbb{T}^3$ are η-dense and $|\overline{\omega} - \omega| \leqslant \eta$;*
(b) *$\bar{z} = ((\overline{x}_1, 0), (\overline{x}_2, 0), (\overline{x}_3, 0))$ such that $0 < \overline{x}_i \leqslant x_i/2$ for every $i \in \{1, 2, 3\}$ and $\overline{x}_2 \geqslant 1/\overline{q}_2$;*
(c) *$z' \in X$, $\widehat{x}_1 \in (\overline{x}_1 + \frac{1}{\overline{q}_1^3}, x_1)$ and $N \in \mathbb{N}$, such that $|z' - z| \leqslant \eta$ and the diffeomorphism*

$$\mathcal{T} = \Phi_{2,1,\overline{x}_2,\overline{q}_2^{-3}} \circ \Phi_{1,3,\widehat{x}_1,\overline{q}_1^{-3}} \circ \Phi_{3,2,x_3,\overline{q}_3^{-3}} \circ \Phi_{2,1,x_2,q_2^{-3}} \circ S_{\overline{\omega}}$$

satisfies

$$\mathcal{T}^N(z') = \bar{z}$$

and $|\mathcal{T}^m(z')_i| \leqslant (1+\eta)x_i$ *for* $m \in \{0,\dots,N\}$.

Moreover, $\overline{q}_1$, $\overline{q}_2$ *and* $\overline{q}_3$ *can be taken arbitrarily large.*

Proof of Proposition 4.1. We divide the proof into three steps.

1. First choose coprime integers $\widehat{p}_3$ and $\widehat{q}_3$ with $\widehat{q}_3$ large multiple of q_2, so that

$$q_1|q_2|\widehat{q}_3, \qquad x_2 \geqslant \frac{1}{q_2}, \qquad x_3 \geqslant \frac{1}{\widehat{q}_3}, \qquad \frac{1}{\widehat{q}_3} < \eta \tag{4.1}$$

and the new rotation vector

$$\widehat{\omega} = (p_1/q_1, p_2/q_2, \widehat{p}_3/\widehat{q}_3)$$

satisfies $|\widehat{\omega} - \omega| < \eta$. Set

$$\widehat{T}_2 = \Phi_{3,2,x_3,\widehat{q}_3^{-3}} \circ \Phi_{2,1,x_2,q_2^{-3}} \circ S_{\widehat{\omega}}.$$

By Lemma 3.1 **II)**, there exist $\widehat{N} \in \mathbb{N}$ and $\widehat{z} = ((\widehat{x}_1,0),(\widehat{x}_2,0),(\widehat{x}_3,0))$ such that $\widehat{T}_2^{\widehat{N}}(z) = \widehat{z}$, with

$$\widehat{x}_1 \leqslant x_1, \quad \widehat{x}_2 \leqslant x_2/2, \quad \widehat{x}_3 = x_3,$$

and $\left|\widehat{T}_2^m(z)_i\right| \leqslant x_i$ for all $m \in \{0,\dots,\widehat{N}\}$.

2. Next, consider a vector of the form

$$\widetilde{\omega} = (\widetilde{p}_1/\widetilde{q}_1, p_2/q_2, \widehat{p}_3/\widehat{q}_3)$$

with coprime $\widetilde{p}_1$ and $\widetilde{q}_1$, and

$$\widehat{q}_3|\widetilde{q}_1, \qquad \widehat{x}_1 > \frac{1}{\widetilde{q}_1}, \qquad \frac{\widehat{q}_3}{\widetilde{q}_1} < \eta, \tag{4.2}$$

so that in particular

$$\frac{\widehat{x}_1}{2} > \frac{1}{\widetilde{q}_1^3}. \tag{4.3}$$

Set

$$\widetilde{T}_3 = \Phi_{1,3,\widehat{x}_1,\widetilde{q}_1^{-3}} \circ \Phi_{3,2,x_3,\widehat{q}_3^{-3}} \circ S_{\widetilde{\omega}}.$$

By Lemma 3.1 **III)**, there exist $\widetilde{N} \in \mathbb{N}$ and $\widetilde{z} = ((\widetilde{x}_1,0),(\widetilde{x}_2,0),(\widetilde{x}_3,0))$ such that $\widetilde{T}_3^{\widetilde{N}}(\widehat{z}) = \widetilde{z}$ with

$$\widetilde{x}_1 = \widehat{x}_1, \quad \widetilde{x}_2 \leqslant \widehat{x}_2 \leqslant x_2/2, \quad \widetilde{x}_3 \leqslant \widehat{x}_3/2 = x_3/2, \tag{4.4}$$

and $\left|\widetilde{T}_3^m(\widehat{z})_i\right| \leqslant \widehat{x}_i$ for all $m \in \{0,\dots,\widetilde{N}\}$.

Define now

$$\mathbf{T} = \Phi_{1,3,\widehat{x}_1,\widetilde{q}_1^{-3}} \circ \Phi_{3,2,x_3,\widehat{q}_3^{-3}} \circ \Phi_{2,1,x_2,q_2^{-3}} \circ S_{\widetilde{\omega}}.$$

Choosing $\widetilde{q}_1$ in (4.2) large enough and $\widetilde{p}_1$ properly, one can assume that $\widetilde{\omega}$ is arbitrarily close to $\widehat{\omega}$, so that $S_{\widetilde{\omega}}$ is arbitrarily C^0-close to $S_{\widehat{\omega}}$ on the ball $B = B(0, |z| + 1)$, and moreover that $\Phi_{1,3,\widehat{x}_1,\widetilde{q}_1^{-3}}$ is arbitrarily C^0-close to Id on B. As a consequence, one can assume that $\mathbf{T}$ is arbitrarily C^0-close to $\widehat{T}_2$ on B. Hence one can choose $\widetilde{\omega}$ with $|\widetilde{\omega} - \omega| < \eta$ such that there exists $\mathbf{z}$ with $|\mathbf{z} - z| < \eta$ which satisfies

$$\mathbf{T}^{\widehat{N}}(\mathbf{z}) = \widehat{z}, \qquad |\mathbf{T}^m(\mathbf{z})_i| \leqslant (1+\eta)x_i \quad \text{for all } m \in \{0, \dots, \widehat{N}\}.$$

Moreover, using Lemma 2.2, one proves by induction that:

$$\mathbf{T}^m(\widehat{z})_2 \in B(x_2, \widehat{q_2}^{-3})^c, \qquad \mathbf{T}^m(\widehat{z}) = \widetilde{T}_3^{\;m}(\widehat{z}) \quad \text{for all } m \in \{0, \dots, \widetilde{N}\}.$$

As a consequence

$$\mathbf{T}^{\widehat{N}+\widetilde{N}}(\mathbf{z}) = \widetilde{T}_3^{\widetilde{N}}(\widehat{z}) = \widetilde{z}$$

and $|\mathbf{T}^m(\mathbf{z})_i| < (1+\eta)x_i$ for all $m \in \{0, \dots, \widehat{N} + \widetilde{N}\}$.

3. It remains now to perturb $\mathbf{T}$ in the same way as above to bring the first component of $\widetilde{z}$ closer to the origin. Consider coprime integers $\overline{p}_2$ and $\overline{q}_2$ such that

$$\widetilde{q}_1 | \overline{q}_2, \qquad x_2 \geqslant 1/\overline{q}_2, \qquad \widetilde{x}_2 \geqslant 1/\overline{q}_2, \qquad \frac{\widetilde{q}_1}{\overline{q}_2} < \eta, \tag{4.5}$$

and such that the vector

$$\overline{\omega} = (\widetilde{p}_1/\widetilde{q}_1, \overline{p}_2/\overline{q}_2, \widehat{p}_3/\widehat{q}_3) \tag{4.6}$$

satisfies $|\overline{\omega} - \omega| < \eta$. Set now

$$\mathcal{T} = \Phi_{2,1,\widetilde{x}_2,\overline{q}_2^{-3}} \circ \Phi_{1,3,\widehat{x}_1,\widetilde{q}_1^{-3}} \circ \Phi_{3,2,x_3,\widehat{q}_3^{-3}} \circ \Phi_{2,1,x_2,q_2^{-3}} \circ S_{\overline{\omega}}.$$

As above, a proper choice of $\overline{p}_2$ and $\overline{q}_2$ satisfying (4.5) makes $\mathcal{T}$ arbitrarily C^0 close to $\mathbf{T}$ and yields the existence of a $z' \in X$ such that $|z' - z| < \eta$, satisfying

$$\mathcal{T}^{\widehat{N}+\widetilde{N}}(z') = \widetilde{z}, \qquad |\mathcal{T}^m(z')_i| < (1+\eta)x_i \quad \text{for all } m \in \{0, \dots, \widehat{N} + \widetilde{N}\}.$$

Set

$$\overline{T}_1 = \Phi_{2,1,\widetilde{x}_2,\overline{q}_2^{-3}} \circ \Phi_{1,3,\widehat{x}_1,\overline{q}_1^{-3}} \circ S_{\overline{\omega}}.$$

Using Lemma 2.2 and Lemma 3.1 **I)**, one proves by induction that now for $m \geqslant 0$:

$$\mathcal{T}^m(\widetilde{z})_2 \in B(x_2, \overline{q}_2^{-3})^c, \qquad \mathcal{T}^m(\widetilde{z})_3 \in B(x_3, \overline{q}_3^{-3})^c, \qquad \mathcal{T}^m(\widetilde{z}) = \overline{T}_1^m(\widetilde{z}).$$

By Lemma 3.1 **I)** there exists $\overline{N}$ such that

$$\overline{T}_1^{\overline{N}}(\widetilde{z}) = \bar{z} = ((\overline{x}_1, 0), (\overline{x}_2, 0), (\overline{x}_3, 0))$$

with

$$\overline{x}_1 \leqslant \widetilde{x}_1/2 \leqslant x_1/2, \quad \overline{x}_2 = \widetilde{x}_2 \leqslant x_2/2, \quad \overline{x}_3 \leqslant \widetilde{x}_3 \leqslant x_3/2,$$

and $\left|(\overline{T}_1{}^m(\widetilde{z})_i\right| \leqslant \widetilde{x}_i \leqslant x_i$ for all $m \in \{0,\dots,\overline{N}\}$. As a consequence, setting $N = \widehat{N} + \widetilde{N} + \overline{N}$:

$$\mathcal{T}^N(z') = \bar{z}, \qquad |\mathcal{T}^m(z')_i| \leqslant (1+\eta)x_i \quad \text{for all } m \in \{0,\dots,N\}.$$

We finally change the notation of (4.6) and write

$$\overline{\omega} = (\overline{\omega}_1, \overline{\omega}_2, \overline{\omega}_3) = (\overline{p}_1/\overline{q}_1, \overline{p}_2/\overline{q}_2, \overline{p}_3/\overline{q}_3),$$

so that in particular $\widetilde{q}_1 = \overline{q}_1$, $\widehat{q}_3 = \overline{q}_3$ and

$$\overline{q}_3 \mid \overline{q}_1, \qquad \overline{q}_1 \mid \overline{q}_2.$$

Hence the orbits of $S_{\overline{\omega}}$ are $\overline{q}_2$-periodic. Moreover, from (4.3) and the equality $\widehat{x}_1 = \widetilde{x}_1$, one deduces

$$\widehat{x}_1 - \overline{x}_1 > \frac{1}{\overline{q}_1^3}.$$

Note finally that the last conditions in (4.1), (4.2) and (4.5) now read

$$\frac{1}{\overline{q}_3} < \eta, \qquad \frac{\overline{q}_3}{\overline{q}_1} < \eta, \qquad \frac{\overline{q}_1}{\overline{q}_2} < \eta.$$

Fix $(\theta_1,\theta_2,\theta_3) \in \mathbb{T}^3$ and recall that $\overline{q}_3 \mid \overline{q}_1$ and $\overline{q}_1 \mid \overline{q}_2$. By the first inequality one can first find $\ell_3 \in \mathbb{N}$ such that $R^{\ell_3}_{\overline{\omega}_3}(0)$ is η-close to θ_3. Then, by the second inequality there is an $\ell_1 \in \mathbb{N}$ such that $R^{\ell_1\overline{q}_3+\ell_3}_{\overline{\omega}_1}(0)$ is η-close to θ_1. Finally, by the last inequality there is an $\ell_2 \in \mathbb{N}$ such that $R^{\ell_2\overline{q}_1+\ell_1\overline{q}_3+\ell_3}_{\overline{\omega}_2}(0)$ is η-close to θ_2. This proves that $S^{\ell_2\overline{q}_1+\ell_1\overline{q}_3+\ell_3}_{\overline{\omega}}(0,0,0)$ is η-close to $(\theta_1,\theta_2,\theta_3)$, so that the orbits of $S_{\overline{\omega}}$ are η-dense on $\mathbb{T}^3$. This concludes the proof. □

Definition 4.1. *Given $z = (z_1,z_2,z_3) \in X$, we say that a diffeomorphism Φ of X is z-admissible if $\Phi \equiv \mathrm{Id}$ on*

$$\{s \in X : |s_i| \leqslant \tfrac{11}{10}|z_i|, i = 1,2,3\}.$$

Proposition 4.2. *Let $\omega = (p_1/q_1, p_2/q_2, p_3/q_3) \in \mathbb{Q}^3_+$ with $q_3|q_1|q_2$ and $z = ((x_1,0),(x_2,0),(x_3,0)) \in B(0,R)$ with $x_1,x_2,x_3 > 0$ and $x_2 \geqslant 1/q_2$. Suppose $\Phi \in \mathcal{U}^{\alpha,L}$ is z-admissible and $\|\Phi_{2,1,x_2,q_2^{-3}} \circ \Phi - \mathrm{Id}\|_{\alpha,L} < \epsilon$, where ϵ is defined by Lemma A.2, and let*

$$T := \Phi_{2,1,x_2,q_2^{-3}} \circ \Phi \circ S_\omega.$$

Assume that $z_0 \in X$ and $M \geqslant 1$ are such that $T^M(z_0) = z$. Then, for any $\eta > 0$, there exist

(a) $\overline{\omega} = (\overline{p}_1/\overline{q}_1, \overline{p}_2/\overline{q}_2, \overline{p}_3/\overline{q}_3)$ *such that* $\overline{q}_3|\overline{q}_1|\overline{q}_2$*, the orbits of the translation of vector* $\overline{\omega}$ *on* $\mathbb{T}^3$ *are* η*-dense and* $|\overline{\omega}-\omega| \leqslant \eta$*;*

(b) $\bar{z} = ((\overline{x}_1, 0), (\overline{x}_2, 0), (\overline{x}_3, 0))$ *such that* $0 < \overline{x}_i \leqslant x_i/2$ *for every* $i \in \{1, 2, 3\}$ *and* $\overline{x}_2 \geqslant 1/\overline{q}_2$*;*

(c) $\bar{z}_0 \in X$ *such that* $|\bar{z}_0 - z_0| \leqslant \eta$*, and* $\overline{M} \geqslant M$*, and* $\bar{\Phi} \in \mathcal{U}^{\alpha,L}$ $\bar{z}$*-admissible, so that the diffeomorphism*

$$\overline{T} := \Phi_{2,1,\overline{x}_2,\overline{q}_2^{-3}} \circ \bar{\Phi} \circ S_{\overline{\omega}}$$

satisfies $\overline{T}^{\overline{M}}(\bar{z}_0) = \bar{z}$ *and* $\left|\overline{T}^m(\bar{z}_0)_i\right| \leqslant (1+\eta)x_i$ *for all* $m \in \left\{M, \dots, \overline{M}\right\}$*.*

(d) *Moreover,* $\|\Phi_{2,1,\overline{x}_2,\overline{q}_2^{-3}} \circ \bar{\Phi} - \Phi_{2,1,x_2,q_2^{-3}} \circ \Phi\|_{\alpha,L} \leqslant \eta$*.*

Proof of Proposition 4.2. Take $\overline{\omega}, \bar{z}, N, z', \widehat{x}_1$ as in Proposition 4.1 and let

$$\mathcal{T} = \Phi_{2,1,\overline{x}_2,\overline{q}_2^{-3}} \circ \Phi_{1,3,\widehat{x}_1,\overline{q}_1^{-3}} \circ \Phi_{3,2,x_3,\overline{q}_3^{-3}} \circ \Phi_{2,1,x_2,q_2^{-3}} \circ S_{\overline{\omega}}$$

so that $\mathcal{T}^N(z') = \bar{z}$ and $|\mathcal{T}^m(z')_i| \leqslant (1+\eta)x_i$ for all $m \in \{0, \dots, N\}$. If we define

$$\overline{T} = \Phi_{2,1,\overline{x}_2,\overline{q}_2^{-3}} \circ \Phi_{1,3,\widehat{x}_1,\overline{q}_1^{-3}} \circ \Phi_{3,2,x_3,\overline{q}_3^{-3}} \circ \Phi_{2,1,x_2,q_2^{-3}} \circ \Phi \circ S_{\overline{\omega}}$$

then, since Φ is z-admissible and $|z - z'| < \eta$, we get $\mathcal{T}^m(z') = \overline{T}^m(z')$ for all $m \in \{0, \dots, N\}$, hence $\overline{T}^N(z') = \bar{z}$ and $\left|\overline{T}^m(z')_i\right| \leqslant (1+\eta)x_i$ for all $m \in \{0, \dots, N\}$.

Let

$$\bar{\Phi} := \Phi_{1,3,\widehat{x}_1,\overline{q}_1^{-3}} \circ \Phi_{3,2,x_3,\overline{q}_3^{-3}} \circ \Phi_{2,1,x_2,q_2^{-3}} \circ \Phi, \tag{4.7}$$

so that, indeed, $\overline{T} = \Phi_{2,1,\overline{x}_2,\overline{q}_2^{-3}} \circ \bar{\Phi} \circ S_{\overline{\omega}}$. Notice that we can write $\Phi_{2,1,\overline{x}_2,\overline{q}_2^{-3}} \circ \bar{\Phi} = \Phi^{u_3} \circ \Phi^{u_2} \circ \Phi^{u_1} \circ \Psi$ (notation of Lemma A.2), where $\Psi = \Phi_{2,1,x_2,q_2^{-3}} \circ \Phi$ and the Gevrey-(α, L_1) norms of u_1, u_2, u_3 are controlled by Lemma 2.1; we thus get (d) by applying (A.8), choosing $\overline{q}_1, \overline{q}_2, \overline{q}_3$ sufficiently large.

Comparing $\overline{T}$ and T in C^0-norm in the ball $B(0, |z_0| + 1)$, since we can take $\overline{\omega}$ arbitrarily close to ω and the $\overline{q}_i$'s arbitrarily large, we can find $\bar{z}_0 \in X$ such that $|\bar{z}_0 - z_0| \leqslant \eta$ and $\overline{T}^M(\bar{z}_0) = z'$. We thus take $\overline{M} = M + N$, so that $\overline{T}^{\overline{M}}(\bar{z}_0) = \bar{z}$ and $\left|\overline{T}^m(\bar{z}_0)_i\right| \leqslant (1+\eta)x_i$ for all $m \in \left\{M, \dots, \overline{M}\right\}$.

To finish the proof of (c), just observe that $\bar{\Phi} \in \mathcal{U}^{\alpha,L}$ and $\bar{\Phi}$ is $\bar{z}$-admissible since $\overline{x}_i \leqslant x_i/2$ and $\overline{q}_1^{-3} \leqslant \widehat{x}_1/10$, $\overline{q}_3^{-3} \leqslant x_3/10$ (possibly increasing $\overline{q}_1$ and $\overline{q}_3$ if necessary). $\square$

Clearly, Proposition 4.2 is tailored so that it can be applied inductively. The gain obtained when going from T to $\overline{T}$ is twofold : on the one hand the orbit of the new initial point $\bar{z}_0$ is pushed further close to the origin, and on the other hand the rotation vector at the origin is changed to behave increasingly like an non-resonant vector.

Proof of Theorem A. Let $\gamma > 0$. We pick

$$\omega^{(0)} = (p_1^{(0)}/q_1^{(0)}, p_2^{(0)}/q_2^{(0)}, p_3^{(0)}/q_3^{(0)}) \in \mathbb{Q}_+^3$$

with $q_3^{(0)}|q_1^{(0)}|q_2^{(0)}$, and $x_1^{(0)}, x_2^{(0)}, x_3^{(0)} > 0$ so that $x_2^{(0)} \geqslant 1/q_2^{(0)}$ and

$$z_0^{(0)} := ((x_1^{(0)}, 0), (x_2^{(0)}, 0), (x_3^{(0)}, 0)) \in B(0, R/2).$$

Let $\Phi^{(0)} := \mathrm{Id}$ and $M^{(0)} := 0$. Define

$$T^{(0)} := \Psi^{(0)} \circ S_{\omega^{(0)}} \quad \text{with} \quad \Psi^{(0)} := \Phi_{2,1,x_2^{(0)},1/(q_2^{(0)})^3} \circ \Phi^{(0)}.$$

Choosing $q_2^{(0)}$ sufficiently large, we have $\|\Psi^{(0)} - \mathrm{Id}\|_{\alpha,L} \leqslant \min\{\epsilon/2, \gamma/2\}$ by (2.2). The hypotheses of Proposition 4.2 hold for $z^{(0)} = z_0^{(0)}$.

We apply Proposition 4.2 inductively by choosing inductively a sequence $(\eta^{(n)})_{n\geqslant 1}$ such that

$$\eta^{(n)} \leqslant \min\Big\{\frac{\epsilon}{2^{n+1}}, \frac{\gamma}{2^{n+1}}, 1/10\Big\}, \qquad \sum_{k=n+1}^{\infty} \eta^{(k)} \leqslant \frac{\eta^{(n)}}{\overline{q}_2^{(n)}}$$

(where $\overline{q}_2^{(n)}$ is determined at the nth step of the induction). We get sequences $(\omega^{(n)})_{n\geqslant 0}$, $(z_0^{(n)})_{n\geqslant 0}$, $(z^{(n)})_{n\geqslant 0}$, $(T^{(n)})_{n\geqslant 0}$, $(M^{(n)})_{n\geqslant 0}$, with

$$z^{(n)} = ((x_1^{(n)}, 0), (x_2^{(n)}, 0), (x_3^{(n)}, 0)), \quad 0 < x_i^{(n+1)} \leqslant x_i^{(n)}/2$$

and $T^{(n)} = \Psi^{(n)} \circ S_{\omega^{(n)}}$ with $\Psi^{(n)} = \Phi_{2,1,x_2^{(n)},1/(q_2^{(n)})^3} \circ \Phi^{(n)} \in \mathcal{U}^{\alpha,L}$, so that

$$\text{(4.8)} \quad \left|\omega^{(n+1)} - \omega^{(n)}\right| \leqslant \eta^{(n+1)}, \quad \left|z_0^{(n+1)} - z_0^{(n)}\right| \leqslant \eta^{(n+1)}, \quad \|\Psi^{(n+1)} - \Psi^{(n)}\|_{\alpha,L} \leqslant \eta^{(n+1)}.$$

We also have

$$\text{(4.9)} \quad \left|(T^{(n+1)^m}(z_0^{(n+1)}))_i\right| \leqslant 1.01 x_i^{(j)} \quad \text{for all } m \in \{M^{(j)}, \ldots, M^{(j+1)}\} \text{ with } j \leqslant n.$$

In view of (4.8), the sequences $(z_0^{(n)})$, $(\omega^{(n)})$ and $(\Psi^{(n)})$ are Cauchy. We denote their limits by z_0^{∞}, ω^{∞} and Ψ^{∞}. Notice that $\|\Psi^{\infty} - \mathrm{Id}\|_{\alpha,L} \leqslant \gamma$.

We obtain that $\mathbf{T} := \Psi^{\infty} \circ S_{\omega^{\infty}}$ satisfies $|\mathbf{T}^m(z_0^{\infty})| \underset{m\to+\infty}{\longrightarrow} 0$, because the ball $B(0,R)$ is a compact subset of X which contains all the points

$T^{(n+1)^m}(z_0^{(n+1)})$ and on which $T^{(n)} \underset{n\to+\infty}{\longrightarrow} \mathbf{T}$ in the C^0 topology, hence $T^{(n+1)^m}(z_0^{(n+1)}) \underset{n\to+\infty}{\longrightarrow} \mathbf{T}^m(z_0^\infty)$ for each m and, in (4.9), we can first let n tend to ∞ and then use the fact that $x_i^{(j)} \downarrow 0$ and $M^{(j)} \uparrow \infty$ as j tends to ∞.

The orbits of the translation of vector $\omega^{(n)}$ on $\mathbb{T}^3$ being $\eta^{(n)}$-dense and $\overline{q}_2^{(n)}$-periodic, we see that ω^∞ defines a minimal translation on $\mathbb{T}^3$. Indeed, given $\theta \in \mathbb{T}^3$ and $\epsilon > 0$, we can choose $n, m \in \mathbb{N}$ so that $\eta^{(n)} \leqslant \epsilon/2$, $\operatorname{dist}(m\omega^{(n)} - \theta, \mathbb{Z}^3) \leqslant \eta^{(n)}$ and $m < \overline{q}_2^{(n)}$. Then,

$$\operatorname{dist}(m\omega^\infty - \theta, \mathbb{Z}^3) \leqslant \eta^{(n)} + m\left|\omega^\infty - \omega^{(n)}\right| \leqslant \eta^{(n)} + \overline{q}_2^{(n)} \sum_{k=n+1}^{\infty} \eta^{(k)} \leqslant 2\eta^{(n)}$$

which is $\leqslant \epsilon$. Hence the orbit of 0 under the translation of vector ω^∞ is ϵ-dense for every ϵ, which entails that ω^∞ is non-resonant.

The proof of Theorem A is thus complete. □

Appendix A. Gevrey functions, maps and flows

A.1. **Gevrey functions and Gevrey maps.** We follow Section 1.1.2 and Appendix B of [LMS18], with some simplifications stemming from the fact that here we only need to consider functions satisfying uniform estimates on the whole of a Euclidean space.

The Banach algebra of uniformly Gevrey-(α, L) functions. Let $N \geqslant 1$ be integer and $\alpha \geqslant 1$ and $L > 0$ be real. We define

$$G^{\alpha,L}(\mathbb{R}^N) := \{f \in C^\infty(\mathbb{R}^N) \mid \|f\|_{\alpha,L} < \infty\},$$

$$\|f\|_{\alpha,L} := \sum_{\ell\in\mathbb{N}^N} \frac{L^{|\ell|\alpha}}{\ell!^\alpha} \|\partial^\ell f\|_{C^0(\mathbb{R}^N)}.$$

We have used the standard notations $|\ell| = \ell_1 + \cdots + \ell_N$, $\ell! = \ell_1! \ldots \ell_N!$, $\partial^\ell = \partial_{x_1}^{\ell_1} \ldots \partial_{x_N}^{\ell_N}$, and

$$\mathbb{N} := \{0, 1, 2, \ldots\}.$$

The space $G^{\alpha,L}(\mathbb{R}^N)$ turns out to be a Banach algebra, with

$$\|fg\|_{\alpha,L} \leqslant \|f\|_{\alpha,L}\|g\|_{\alpha,L} \tag{A.1}$$

for all $f, g \in G^{\alpha,L}(\mathbb{R}^N)$, and there are "Cauchy-Gevrey inequalities": if $0 < L' < L$, then all the partial derivatives of f belong to $G^{\alpha,L'}(\mathbb{R}^N)$ and, for each $p \in \mathbb{N}$,

$$\sum_{m\in\mathbb{N}^N;\ |m|=p} \|\partial^m f\|_{\alpha,L'} \leqslant \frac{p!^\alpha}{(L-L')^{p\alpha}} \|f\|_{\alpha,L} \tag{A.2}$$

(see [MS03]).

The Banach space of uniformly Gevrey-(α, L) maps. Let $N, M \geqslant 1$ be integer and $\alpha \geqslant 1$ and $L > 0$ be real. We define

$$G^{\alpha,L}(\mathbb{R}^N, \mathbb{R}^M) := \{F \in C^\infty(\mathbb{R}^N, \mathbb{R}^M) \mid \|F\|_{\alpha,L} < \infty\},$$
$$\|F\|_{\alpha,L} := \|F_{[1]}\|_{\alpha,L} + \cdots + \|F_{[M]}\|_{\alpha,L}.$$

This is a Banach space.

When $N = M = 2n$, we denote by $\mathrm{Id} + G^{\alpha,L}(\mathbb{R}^{2n}, \mathbb{R}^{2n})$ the set of all maps of the form $\Psi = \mathrm{Id} + F$ with $F \in G^{\alpha,L}(\mathbb{R}^{2n}, \mathbb{R}^{2n})$. This is a complete metric space for the distance $\mathrm{dist}(\mathrm{Id} + F_1, \mathrm{Id} + F_2) = \|F_2 - F_1\|_{\alpha,L}$. We use the notation

$$\mathrm{dist}(\Psi_1, \Psi_2) = \|\Psi_2 - \Psi_1\|_{\alpha,L}$$

as well. We then define

$$\mathcal{U}^{\alpha,L} \subset \mathrm{Id} + G^{\alpha,L}(\mathbb{R}^{2n}, \mathbb{R}^{2n})$$

as the subset consisting of all Gevrey-(α, L) symplectic diffeomorphisms of $\mathbb{R}^{2n}$ which fix the origin and are C^∞-tangent to Id at the origin. This is a closed subset of the complete metric space $\mathrm{Id} + G^{\alpha,L}(\mathbb{R}^{2n}, \mathbb{R}^{2n})$.

Composition with close-to-identity Gevrey-(α, L) maps. Let $N \geqslant 1$ be integer and $\alpha \geqslant 1$ and $L > 0$ be real. We use the notation $(\mathbb{N}^N)^* := \mathbb{N}^N \setminus \{0\}$ and define

$$\mathcal{N}^*_{\alpha,L}(f) := \sum_{\ell \in (\mathbb{N}^N)^*} \frac{L^{|\ell|\alpha}}{\ell!^\alpha} \|\partial^\ell f\|_{C^0(\mathbb{R}^N)},$$

so that $\|f\|_{\alpha,L} = \|f\|_{C^0(\mathbb{R}^N)} + \mathcal{N}^*_{\alpha,L}(f)$.

Lemma A.0. *Let $L_1 > L$. There exists $\epsilon_{\mathrm{c}} = \epsilon_{\mathrm{c}}(N, \alpha, L, L_1)$ such that, for any $f \in G^{\alpha,L_1}(\mathbb{R}^N)$ and $F = (F_{[1]}, \ldots, F_{[N]}) \in G^{\alpha,L}(\mathbb{R}^N, \mathbb{R}^N)$, if*

$$\mathcal{N}^*_{\alpha,L}(F_{[1]}), \ldots, \mathcal{N}^*_{\alpha,L}(F_{[N]}) \leqslant \epsilon_{\mathrm{c}},$$

then $f \circ (\mathrm{Id} + F) \in G^{\alpha,L}(\mathbb{R}^N)$ and $\|f \circ (\mathrm{Id} + F)\|_{\alpha,L} \leqslant \|f\|_{\alpha,L_1}$.

Proof. Since $L < L_1$, we can pick $\mu > 1$ such that $\mu L^\alpha < L_1^\alpha$; we then choose $a > 0$ such that $(1 + a)^{\alpha-1} \leqslant \mu$ and set $\lambda := \big(N(1 + 1/a)\big)^{\alpha-1}$. We will prove the lemma with $\epsilon_{\mathrm{c}} := (L_1^\alpha - \mu L^\alpha)/\lambda$.

Let f and F be as in the statement, and $g := f \circ (\mathrm{Id} + F)$. Computing the Taylor expansion of $g(x + h) = f(x + h + F(x + h))$ at $h = 0$, we get, for each $k \in \mathbb{N}^N$, $\dfrac{1}{k!}\partial^k g =$

$$\sum_{\substack{\ell,m,n\in\mathbb{N}^N\\ m+n=k}} \frac{(\partial^{\ell+n} f)\circ(\mathrm{Id}+F)}{\ell!\, n!} \sum_{\substack{k^1,\dots,k^{|\ell|}\in(\mathbb{N}^N)^*\\ k^1+\cdots+k^{|\ell|}=m}} \frac{\prod_{i=1}^{N} \prod_{\ell_1+\cdots+\ell_{i-1}<p\leqslant \ell_1+\cdots+\ell_i} \partial^{k^p} F_{[i]}}{k^1!\cdots k^{|\ell|}!}$$

with the convention that an empty sum is 0 and an empty product is 1. Note that if $\ell=0$, then necessarily $m=0$ and the corresponding contribution to the sum is $\frac{1}{k!}(\partial^k f)\circ(\mathrm{Id}+F)$, whereas $\ell\neq 0$ implies $m\neq 0$ and $k\neq 0$.

We have $\|g\|_{C^0(\mathbb{R}^N)} \leqslant \|f\|_{C^0(\mathbb{R}^N)}$ and, for each $k\in(\mathbb{N}^N)^*$,

$$\frac{1}{k!}\,\|\partial^k g\|_{C^0} \leqslant \frac{1}{k!}\,\|\partial^k f\|_{C^0} + \sum_{\substack{\ell,m,n\in\mathbb{N}^N\\ \ell\neq 0,\ m+n=k}} \frac{\|\partial^{\ell+n} f\|_{C^0}}{\ell!\, n!} \sum_{\substack{k^1,\dots,k^{|\ell|}\in(\mathbb{N}^N)^*\\ k^1+\cdots+k^{|\ell|}=m}} \frac{P}{k^1!\cdots k^{|\ell|}!}$$

with $P := \prod_{i=1}^{N} \prod_{\ell_1+\cdots+\ell_{i-1}<p\leqslant\ell_1+\cdots+\ell_i} \|\partial^{k^p}F_{[i]}\|_{C^0}$. Multiplying by $L^{|k|\alpha}/k!^{\alpha-1}$ and taking the sum over k, we get

$$\|g\|_{\alpha,L} \leqslant \sum_{k\in\mathbb{N}^N} \frac{L^{|k|\alpha}}{k!^{\alpha}}\|\partial^k f\|_{C^0} + S \tag{A.3}$$

with

$$S := \sum_{\ell\in(\mathbb{N}^N)^*,\ m,n\in\mathbb{N}^N} \frac{L^{|m+n|\alpha}\|\partial^{\ell+n} f\|_{C^0}}{\ell!n!(m+n)!^{\alpha-1}} \sum_{\substack{k^1,\dots,k^{|\ell|}\in(\mathbb{N}^N)^*\\ k^1+\cdots+k^{|\ell|}=m}} \frac{P}{k^1!\cdots k^{|\ell|}!} \tag{A.4}$$

with the same P as above.

Inequality (A.7) from [MS03] says that, if $s\geqslant 1$ and $k^1,\dots,k^s\in(\mathbb{N}^N)^*$ with $k^1+\cdots+k^s=m$, then $k^1!\cdots k^s!\leqslant N^s m!/s!$. Hence, in each term of the sum S, we can compare $D := \ell!n!(m+n)!^{\alpha-1}k^1!\cdots k^{|\ell|}!$ and $\tilde D := \ell!n!(\ell+n)!^{\alpha-1}k^1!^{\alpha}\cdots k^{|\ell|}!^{\alpha}$: we have

$$\frac{\tilde D}{D} = \Big(\frac{k^1!\cdots k^{|\ell|}!(\ell+n)!}{(m+n)!}\Big)^{\alpha-1} \leqslant \Big(\frac{N^{|\ell|}m!(\ell+n)!}{|\ell|!(m+n)!}\Big)^{\alpha-1}$$
$$\leqslant \Big(\frac{N^{|\ell|}(\ell+n)!}{\ell!\, n!}\Big)^{\alpha-1} \leqslant \lambda^{|\ell|}\mu^{|n|},$$

where the last inequality stems from our choice of λ and μ, using $\frac{(\ell+n)!}{\ell!\,n!} \leqslant (1+1/a)^{|\ell|}(1+a)^{|n|}$. Inserting $\dfrac{1}{D} \leqslant \dfrac{\lambda^{|\ell|}\mu^{|n|}}{\tilde D}$ in (A.4), we obtain

$$S \leqslant \sum_{\ell\in(\mathbb{N}^N)^*,\ n\in\mathbb{N}^N} \frac{L^{|n|\alpha}\lambda^{|\ell|}\mu^{|n|}\|\partial^{\ell+n}f\|_{C^0}}{\ell!n!(\ell+n)!^{\alpha-1}} \sum_{k^1,\dots,k^{|\ell|}\in(\mathbb{N}^N)^*} \frac{L^{|k^1+\cdots+k^{|\ell|}|\alpha}P}{k^1!^\alpha\cdots k^{|\ell|}!^\alpha}.$$

The inner sum over $k^1,\dots,k^{|\ell|}\in(\mathbb{N}^N)^*$ coincides with the product $\mathcal{N}^*_{\alpha,L}(F_{[1]})^{\ell_1}\cdots\mathcal{N}^*_{\alpha,L}(F_{[N]})^{\ell_N}$, which is $\leqslant \epsilon_c^{|\ell|}$ by assumption. Hence, coming back to (A.3), we get

$$\|g\|_{\alpha,L} \leqslant \sum_{\ell,n\in\mathbb{N}^N} \frac{(\mu L^\alpha)^{|n|}(\lambda\epsilon_c)^{|\ell|}\|\partial^{\ell+n}f\|_{C^0}}{\ell!n!(\ell+n)!^{\alpha-1}} = \sum_{k\in\mathbb{N}^N} \frac{(\mu L^\alpha+\lambda\epsilon_c)^{|k|}\|\partial^k f\|_{C^0}}{k!^\alpha}$$

(we have used $\mu\geqslant 1$ to absorb the first term of the right-hand side of (A.3) in the contribution of $\ell=0$). The conclusion follows from our choice of ϵ_c. □

A.2. **Estimates for Gevrey flows.** We need some improvements with respect to [MS03] and [LMS18] for the estimates of the flow of a small Gevrey vector field.

Lemma A.1. *Suppose $\alpha\geqslant 1$ and $0<L<L_1$.*

(i) For every integer $N\geqslant 1$, there exists $\epsilon_f=\epsilon_f(N,\alpha,L,L_1)$ such that, for every vector field $X\in G^{\alpha,L_1}(\mathbb{R}^N,\mathbb{R}^N)$, if $\|X\|_{\alpha,L_1}\leqslant\epsilon_f$, then the time-1 map Φ of the flow generated by X belongs to $\mathrm{Id}+G^{\alpha,L}(\mathbb{R}^N,\mathbb{R}^N)$ and

$$\|\Phi-\mathrm{Id}\|_{\alpha,L}\leqslant\|X\|_{\alpha,L_1}. \tag{A.5}$$

(ii) For every integer $n\geqslant 1$, there exists $\epsilon_H=\epsilon_H(n,\alpha,L,L_1)$ such that, for every $u\in G^{\alpha,L_1}(\mathbb{R}^{2n})$, if $\|u\|_{\alpha,L_1}\leqslant\epsilon_H$, then the time-1 map Φ^u of the Hamiltonian flow generated by u belongs to $\mathrm{Id}+G^{\alpha,L}(\mathbb{R}^{2n},\mathbb{R}^{2n})$ and

$$\|\Phi^u-\mathrm{Id}\|_{\alpha,L}\leqslant 2^\alpha(L_1-L)^{-\alpha}\|u\|_{\alpha,L_1}. \tag{A.6}$$

Building upon the previous result, we get

Lemma A.2. *Suppose $\alpha\geqslant 1$ and $0<L<L_1$. Then there exist $C=C(n,\alpha,L,L_1)$ and $\epsilon=\epsilon(n,\alpha,L,L_1)$ such that, if $r\geqslant 1$, $u_1,\dots,u_r\in G^{\alpha,L_1}(\mathbb{R}^{2n})$, $\Psi\in\mathrm{Id}+G^{\alpha,L}(\mathbb{R}^{2n},\mathbb{R}^{2n})$ and*

$$\|\Psi-\mathrm{Id}\|_{\alpha,L}+C\big(\|u_1\|_{\alpha,L_1}+\cdots+\|u_r\|_{\alpha,L_1}\big)\leqslant\epsilon, \tag{A.7}$$

then

$$\|\Phi^{u_r}\circ\cdots\circ\Phi^{u_1}\circ\Psi-\Psi\|_{\alpha,L}\leqslant C\big(\|u_1\|_{\alpha,L_1}+\cdots+\|u_r\|_{\alpha,L_1}\big) \tag{A.8}$$

(with the same notation as in Lemma A.1(ii) for the Φ^{u_i}'s).

Proof of Lemma A.1. (i) Let us pick $L' \in (L, L_1)$. We will prove the statement with $\epsilon_{\mathrm{f}} := \epsilon_{\mathrm{c}}(N, \alpha, L, L')$ (notation from Lemma A.0).

Let X be as in the statement. We write the restriction of its flow to the time-interval $[0,1]$ in the form $\Phi(t) = \mathrm{Id} + \xi(t)$, with $t \in [0,1] \mapsto \xi(t) \in C^\infty(\mathbb{R}^N, \mathbb{R}^N)$ characterised by

$$\xi(t) = \int_0^t X \circ \big(\mathrm{Id} + \xi(\tau)\big)\, \mathrm{d}\tau \quad \text{for all } t \in [0,1].$$

We will show that ξ belongs to $\mathcal{B} := \{\, \psi \in C^0\big([0,1], G^{\alpha,L}(\mathbb{R}^N, \mathbb{R}^N)\big) \mid \|\psi\| \leqslant \|X\|_{\alpha,L_1} \,\}$, which is a closed ball in a Banach space.

Lemma A.0 shows that the formula $\mathcal{F}(\psi)(t) := \int_0^t X \circ \big(\mathrm{Id} + \psi(\tau)\big)\, \mathrm{d}\tau$ defines a map from $\mathcal{B}$ to $\mathcal{B}$. Moreover, if $\psi, \psi^* \in \mathcal{B}$ satisfy

$$\|\psi^*(t) - \psi(t)\|_{\alpha,L} \leqslant A(t) \ \text{ for all } t \in [0,1],$$

where $t \in [0,1] \mapsto A(t)$ is continuous, then for each t and i,

$$\mathcal{F}(\psi^*)(t)_{[i]} - \mathcal{F}(\psi)(t)_{[i]} = \int_0^t \mathrm{d}\tau \sum_{j=1}^N \int_0^1 \mathrm{d}\theta \\ \partial_{x_j} X_{[i]} \circ \big(\mathrm{Id} + (1-\theta)\psi(\tau) + \theta\psi^*(\tau)\big)\big(\psi^*(\tau)_{[j]} - \psi(\tau)_{[j]}\big),$$

whence

$$\|\mathcal{F}(\psi^*)(t) - \mathcal{F}(\psi)(t)\|_{\alpha,L} \leqslant K \int_0^t A(\tau)\, \mathrm{d}\tau \ \text{ with } K := \max_{i,j} \|\partial_{x_j} X_{[i]}\|_{\alpha,L'}$$

(we have $K < \infty$ by (A.2) and we have used Lemma A.0 and (A.1)). Iterating this, we get

$$\|\mathcal{F}^p(\psi^*) - \mathcal{F}^p(\psi)\| \leqslant \frac{K^p}{p!} \|\psi^* - \psi\| \ \text{ for all } p \in \mathbb{N},$$

which shows that $\mathcal{F}^p$ is a contraction for p large enough. The map $\mathcal{F}$ thus has a unique fixed point in $\mathcal{B}$, and this fixed point is ξ.

(ii) Let $L' := (L + L_1)/2$. For any $u \in G^{\alpha,L_1}(\mathbb{R}^{2n})$, inequality (A.2) with $p = 1$ reads

$$\sum_{m \in \mathbb{N}^{2n};\ |m|=1} \|\partial^m u\|_{\alpha,L'} \leqslant (L_1 - L')^{-\alpha} \|u\|_{\alpha,L_1}.$$

The left-hand side is precisely the (α, L')-Gevrey norm of the Hamiltonian vector field generated by u. Therefore, point (i) shows that the conclusion holds with $\epsilon_{\mathrm{H}} = (L_1 - L')^\alpha \epsilon_{\mathrm{f}}(2n, \alpha, L, L')$.

Proof of Lemma A.2. Let us pick $L' \in (L, L_1)$. We will show the statement with

$$C := 2^{\alpha}(L_1 - L')^{-\alpha}, \quad \epsilon := \min\big\{\epsilon_{\mathrm{c}}(2n, \alpha, L, L'), C\epsilon_{\mathrm{H}}(n, \alpha, L', L_1)\big\}$$

by induction on r.

The induction is tautologically initialized for $r = 0$. Let us take $r \geqslant 1$ and assume that the statement holds at rank $r-1$. Given $u_1, \dots, u_r \in G^{\alpha,L_1}(\mathbb{R}^{2n})$ and $\Psi \in \mathrm{Id} + G^{\alpha,L}(\mathbb{R}^{2n}, \mathbb{R}^{2n})$ satisfying (A.7), we set $\chi := \Phi^{u_{r-1}} \circ \cdots \circ \Phi^{u_1} \circ \Psi$, which satisfies

$$\|\chi - \Psi\|_{\alpha,L} \leqslant C\big(\|u_1\|_{\alpha,L_1} + \cdots + \|u_{r-1}\|_{\alpha,L_1}\big)$$

by the induction hypothesis, and observe that we also have

$$\|\Phi^{u_r} - \mathrm{Id}\|_{\alpha,L'} \leqslant C\|u_r\|_{\alpha,L_1}$$

since $\|u_r\|_{\alpha,L_1} \leqslant \epsilon_{\mathrm{H}}(n, \alpha, L', L_1)$. Now

$$\begin{aligned}\|\Phi^{u_r} \circ \cdots \circ \Phi^{u_1} \circ \Psi - \Psi\|_{\alpha,L} &\leqslant \|(\Phi^{u_r} - \mathrm{Id}) \circ \chi\|_{\alpha,L} + \|\chi - \Psi\|_{\alpha,L} \\ &\leqslant \|\Phi^{u_r} - \mathrm{Id}\|_{\alpha,L'} + \|\chi - \Psi\|_{\alpha,L}\end{aligned}$$

since $\|\chi - \mathrm{Id}\|_{\alpha,L} \leqslant \|\Psi - \mathrm{Id}\|_{\alpha,L} + \|\chi - \Psi\|_{\alpha,L} \leqslant \|\Psi - \mathrm{Id}\|_{\alpha,L} + C\big(\|u_1\|_{\alpha,L_1} + \cdots + \|u_{r-1}\|_{\alpha,L_1}\big) \leqslant \epsilon_{\mathrm{c}}(2n, \alpha, L, L')$ and we are done.

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