CONTINUOUS SPECTRUM FOR MIXING SCHRÖDINGER OPERATORS

BASSAM FAYAD[1] AND YANHUI QU[2]

ABSTRACT. We give an example of a smooth volume preserving mixing flow on the three torus such that any discrete Schrödinger operator on the line with a potential generated by this flow and a Hölder sampling function has almost surely a continuous spectrum.

1. INTRODUCTION

Given a dynamical system (Ω, T, μ), a sample function $V : \Omega \to \mathbb{R}$ and $x \in \Omega$, we define the 1d Schrödinger operator generated by (Ω, T, μ), V and x as the operator on $\ell^2(\mathbb{Z})$

$$(*) \qquad (H_{T,V,x}u)_n = u_{n+1} + u_{n-1} + V(T^n x)u_n.$$

A general fact in spectral theory of 1d Schrödinger operators is that randomness of the potential is a source of localization of the spectrum. Thus an ergodic dynamical system with randomness features has in general a localized pure point spectrum. The most famous example is Anderson's model: the dynamical system is the Bernoulli shift $(X^{\mathbb{Z}}, \sigma, \mu^{\mathbb{Z}})$, where X is a subset of $\mathbb{R}$ and μ is a probability supported on X; the sample function $V : X^{\mathbb{Z}} \to \mathbb{R}$ is defined by $V(x) = x_0$, where $x = \cdots x_{-1}x_0x_1 \cdots$. It is well-known that under suitable assumption on X and μ, for $\mu^{\mathbb{Z}}$ a.e. $x \in X^{\mathbb{Z}}$, the operator $H_{\sigma,V,x}$ has pure point spectrum and the related eigenfunctions are localized, see for example [7, 6, 14]. Another kind of example is given by Bourgain and Schlag [5]. In their example, the dynamical system is $(\mathbb{T}^2, A, \text{Leb})$, where $A : \mathbb{T}^2 \to \mathbb{T}^2$ is a hyperbolic toral automorphism; the sample function $V(x) = \lambda F(x)$ with $F \in C^1(\mathbb{T}^2)$ non-constant and $\int F = 0$. For $\lambda \neq 0$ small, they established Anderson localization on the spectrum up to edges and the center. We note that $(A, \mathbb{T}^2, \text{Leb})$ is mixing by the assumption of A.

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An interesting question is to understand to which extent randomness is needed to insure localization.

For example, recently much interest in Schrödinger operators generated by the skew-shift was in part motivated by the study of the transition from complete localization to continuous spectra as randomness of the potential decreases. It is a know fact indeed that quasi-periodic potentials may display absolutely continuous spectra. The skew-shift dynamics are on one hand related to the quasi-periodic dynamics, but present on the other hand parabolic features. [1] ???

Consider the skew-shift T_α defined on $\mathbb{T}^2$ by $T_\alpha(x, y) = (x + \alpha, y + x)$, where $\alpha \in \mathbb{R}$ is irrational. For Diophantine α, Bourgain, Goldstein, and Schlag [4] proved a localization result for analytic and sufficiently large sample function and other localization results were subsequently obtained. [2] ??? We note that the system $(\mathbb{T}^2, T_\alpha, \text{Leb})$ is strictly ergodic, but not weakly mixing.

In the opposite direction, Boshernitzan and Damanik showed that for a typical skew-shift and a generic continuous sampling function, the associated Schrödinger operators have no eigenvalues for almost all base points [2]. In [3], they generalized the result to skew-shift defined in higher-dimendional torus $\mathbb{T}^k$.

The approach of [2, 3] is to prove a repetition property (MRP) (see Definition 2) for the dynamical system, that implies a Gordon property on the potential, that in turn yields absence of pure point in the spectrum. [3]??? The result in [2, 3] is in some sense on the contrary of [4].

A natural rising question is that what is the borderline of randomness for the dynamical system to produce continuous spectrum. Still in [2], they showed that almost every interval exchange transformation has (MRP) relative to Lebesgue measure. As a consequence, for generic continuous sampling function, the associated Schrödinger operators have no eigenvalues for almost all base points. It is well-known that almost every interval exchange transformation is weakly mixing [1], but it is never mixing [12]. In [10], Huang et. al. constructed a positive entropy minimal dynamical system with (TRP) (see Definition 2). As a consequence, for generic continuous sampling function, the associated Schrödinger operators have no eigenvalues for points in a residual subset.

[1] improve, extend and give references...

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[3] improve, extend and give references

We are interested here in the same problem of the relation between randomness of the driving dynamical system and continuous spectrum of the associated Schrödinger operators. Using reparametrizations of linear flows, we construct examples on the three torus of mixing uniquely ergodic volume preserving flows for which the associated Schrödinger operators with a Hölder sample function have no point part in the spectrum for almost all base points. Our examples will be reparametrization of a minimal translation flow on the three torus by a smooth function Φ. We will see in Section 3 that such flows are uniquely ergodic for a measure equivalent to the Haar measure with density $\frac{1}{\phi}$. We denote by μ the Haar measure on the torus.

THEOREM 1. *There exists $(\alpha, \alpha') \in \mathbb{R}^2$ and a smooth reparametrization of the translation flow $T_{t(\alpha,\alpha',1)}$ such that the resulting flow is mixing, for its unique ergodic invariant probability measure μ, and μ a.e. $x \in \mathbb{T}^3$ is super-recurrent for its time one map T.*

As a consequence, $(\mathbb{T}^3, T, \mu)$ satisfies (MRP). Moreover for every Hölder continuous potential $V : \mathbb{T}^3 \to \mathbb{R}$, the operator $H_{T,V,x}$ has purely continuous spectrum for μ a.e. $x \in \mathbb{T}^3$.

Remark 1. (i) See Definition 1 for the definition of super-recurrence on a topological dynamical system (Ω, T, d). When T has Hölder property, it induces strongly recurrent property like (MRP).

(ii) By [2], for generic continuous function V and μ-a.e. x, the operator $H_{T,V,x}$ has continuous spectrum, however for a concrete V, it is hard to see whether it is in that residual set. On the other hand, if V is more regular, say Hölder continuous, we can directly show that $\{V(T^n x)\}$ is a Gordon potential for μ-a.e. x, as a consequence, the operator $H_{T,V,x}$ has continuous spectrum.

(iii) We note that the system $(\mathbb{T}^3, T, \mu)$ is strictly ergodic, mixing. It has zero topological entropy, nevertheless, it is topologically mixing.

To prove the super-recurrence we follow the same strategy as [2, 3] based on recurrence. Indeed, our mixing flows are reparametrizations of linear flows with a super Liouville translation vector. Naturally, the very strong periodic approximations of the flow are lost after time change, otherwise mixing would not be possible. However, one can choose the reparametrization in such a way that along a sequence of times $t_n \to \infty$, the very strong almost periodic behavior of the translation flow still appears on a set of small measure ϵ_n.

If now ϵ_n decreases, but not too rapidly, say $\epsilon_n \sim \frac{1}{n}$, then by a Borel-Cantelli argument, most of the points on the torus will be strongly recurrent along a subsequence of the sequence t_n.

The construction of the reparametrized flow follows very closely the construction in [9] of a reparametrization of a linear flow on $\mathbb{T}^3$ that is mixing but has a purely singular maximal spectral type. Indeed, the singularity of the maximal spectral type in [9] was due to the existence of very strong periodic approximations on parts of the phase space that have a slowly decaying measure.

2. SUPER-RECURRENCE, REPETITION AND GORDON POTENTIALS

Let (Ω, T) be a topological dynamical system with Ω a compact metric space and T a homeomorphism.

DEFINITION 1. Assume $x \in \Omega$. If there exist $\alpha > 1$ and an integer sequence $k_n \uparrow \infty$ such that

$$d(T^{k_n}x, x) \leq \exp(-k_n^{\alpha}),$$

then we say that x is super-recurrent with recurrent exponent α.

If μ is an invariant ergodic measure of T, we say that the system (Ω, T, μ) is super-recurrent if μ-a.e. $x \in \Omega$ is super-recurrent.

By take a further subsequence if necessary, we may and will assume $k_n \geq n$ from now on.

A bounded function $V : \mathbb{Z} \to \mathbb{R}$ is called a Gordon potential if there are positive integers $q_k \to \infty$ such that

$$\max_{1 \leq n \leq q_k} |V(n) - V(n \pm q_k)| \leq k^{-q_k}$$

for any $k \geq 1$. The Gordon condition insures that the 1d Schrödinger operator on $\ell^2(\mathbb{Z})$ with potential V has no eigenvalues [11].

Assume $V : \Omega \to \mathbb{R}$ is continuous. Define

$$V_x(n) = V(T^n x), \;\; x \in \Omega, n \in \mathbb{Z}.$$

We now assume that M is a smooth compact manifold and T is a C^1 diffeomorphism of M. Let d be the Riemann metric on M. Then we have the following simple consequence of super-recurrence.

PROPOSITION 1. *If $x \in M$ is super-recurrent and $V : M \to \mathbb{R}$ is Hölder, then V_x is a Gordon potential.*

Proof. Since T is C^1, T is Lipschitz. Let $L > 1$ be the Lipschitz constant. Assume V is β-Hölder with Hölder constant C_1. Let $\alpha > 1$ be the recurrent exponent of x. Let $\{k_n : n \geq 0\}$ be the sequence related to x. Since $k_n \geq n$, for $1 \leq l \leq k_n$ we have

$$\begin{aligned} |V_x(l) - V_x(l \pm k_n)| &= |V(T^l x) - V(T^{l \pm k_n} x)| \\ &\leq C_1 d(T^l x, T^{l \pm k_n} x)^\beta \\ &\leq C_1 [L^l d(x, T^{\pm k_n} x)]^\beta \\ &\leq C_1 [L^{2k_n} e^{-k_n^\alpha}]^\beta \\ &= C_1 \exp\left(-\beta(k_n^\alpha - 2k_n \ln L)\right) \\ &\leq n^{-k_n} \end{aligned}$$

as soon as n is big enough. By the definition, V_x is a Gordon potential. □

The following definition was introduced in [2]. Denote the forward orbit of x by $\mathcal{O}^+(x) := \{T^n x : n \geq 0\}$.

DEFINITION 2. A sequence $\{\omega_k : k \geq 0\} \subset \Omega$ has the repetition property if for any $\epsilon > 0$ and $r \in \mathbb{N}$, there exists $q \in \mathbb{N}$ such that $d(\omega_k, \omega_{k+q}) \leq \epsilon$ for $k = 0, 1, \cdots, rq$. Let

$$PRP(\Omega, T) := \{x \in \Omega : \mathcal{O}^+(x) \text{ has the repetition property }\}.$$

We say that (Ω, T) satisfies the *topological repetition property (TRP)* if $PRP(\Omega, T) \neq \emptyset$. Fix a T-ergodic measure μ. We say that (Ω, T, μ) satisfies the *metric repetition property (MRP)* if $\mu(PRP(\Omega, T)) > 0$.

PROPOSITION 2. *If $x \in M$ is super recurrent, then $x \in PRP(M, T)$. Consequently, if (M, T, μ) is super-recurrent, then it satisfies (MRP).*

Proof. Let $\alpha > 1$ be the recurrent exponent of x. Since T is at least C^1, T is Lipschitz. Let $L > 1$ be the Lipschitz constant. Write $q = k_n$, for any $s \in \mathbb{N}$ and $0 \leq t < q$, by the Lipschitz property we get

$$d(T^{(s+1)q+t}x, T^{sq+t}x) \leq L^{sq+t} d(T^q x, x) \leq L^{sq+t} e^{-q^\alpha} \leq L^{(s+1)q} e^{-q^\alpha}.$$

Now fix $\epsilon > 0$ and $r \in \mathbb{N}$. Since k_n can be sufficient large, we can choose some $q = k_n$ such that

$$d(T^{(s+1)q+t}x, T^{sq+t}x) \leq \epsilon$$

for any $0 \leq s \leq r$. By the definition, $\mathcal{O}^+(x)$ has the repetition property and consequently $x \in PRP(M, T)$.

The last result follows from the definition. □

The second part of Theorem 1 follows from the first part of Theorem 1, Proposition 1, Proposition 2 and the above mentioned spectral consequence of the Gordon potentials.

We now proceed to the construction of the reparametrized flow.

3. SUPER-RECURRENT MIXING FLOWS

We start with some notations and reminders on reparametrizations and special flows.

3.1. The translation flow on $\mathbb{T}^n$ of vector $\alpha \in \mathbb{R}^n$ is the flow arising from the constant vector field $X(x) = \alpha$. We denote this flow by $\{R_{t\alpha}\}$. When the numbers $\alpha_1, ..., \alpha_n$ are rationally independent, i.e. none of them is a rational combination of the others, $\{R_{t\alpha}\}$ is uniquely ergodic for the Haar measure μ on the torus. In this case we say it is an irrational flow.

3.2. *Reparametrized flows.* If ϕ is a strictly positive smooth real function on $\mathbb{T}^n$, we define the reparametrization of $\{R_{t\alpha}\}$ with velocity ϕ as the flow given by the vector field $\phi(x)\alpha$, that is, by the system

$$\frac{dx}{dt} = \phi(x)\alpha.$$

The new flow has the same orbits as $\{R_{t\alpha}\}$ and preserves a measure equivalent to the Haar measure given by the density $\frac{1}{\phi}$. Moreover, if $\{R_{t\alpha}\}$ is uniquely ergodic then so is the reparametrized flow (see [**?**]).

3.3. *Special flows.* The reparametrizations of linear flows can be viewed as special flows above toral translations. We give the formal definition.

DEFINITION 3. Given a Lebesgue space L, a measure preserving transformation T on L and an integrable strictly positive real function defined on L we define the special flow over T and under the *ceiling function* φ by inducing on $L \times \mathbb{R}/\sim$, where $\sim$ is the identification $(x, s + \varphi(x)) \sim (T(x), s)$,the action of

$$\begin{array}{rcl} L \times \mathbb{R} & \to & L \times \mathbb{R} \\ (x,s) & \to & (x, s+t). \end{array}$$

If T preserves a unique probability measure λ then the special flow will preserve a unique probability measure that is the normalized product measure of λ on the base and the Lebesgue measure on the fibers.

We will be interested in special flows above minimal translations $R_{\alpha,\alpha'}$ of the two torus and under smooth functions $\varphi(x,y) \in C^\infty(\mathbb{T}^2, \mathbb{R}_+^*)$ that we will denote by $T^t_{\alpha,\alpha',\varphi}$. We denote $M_\varphi = \{(z,s) : z \in \mathbb{T}^2, s \in$

$[0, \varphi(z)\}$. We will still denote by μ the product of the Haar measure of $\mathbb{T}^2$ with the normalized Lebesgue measure on the line.

In all the sequel we will use the following notation, for $m \in \mathbb{N}$,

$$S_m\varphi(z) = \sum_{l=0}^{m-1} \varphi(z + l(\alpha, \alpha'))$$

With this notation, given $t \in \mathbb{R}_+$ we have for $\xi \in M_\varphi$, $\xi = (z, s)$

$$T^t\xi = \left(R_{\alpha,\alpha'}^{N(t,s,z)}(z), t + s - \varphi_{N(t,s,z)}(z)\right) \tag{3.1}$$

where $N(t, s, z)$ is the largest integer m such that $t + s - \varphi_m(x) \geq 0$, that is the number of fibers covered by (z, s) during its motion under the action of the flow until time t.

3.4. Mixing. We also recall the definition of mixing for a measure preserving flow: a flow $\{T_t\}$ preserving a measure ν on M is said to be mixing if, for any measurable subsets A and B of M, one has

$$\lim_{t\to\infty} \nu(T^t A \bigcap B) = \nu(A)\nu(B).$$

By standard equivalence between special flows and reparametrizations (see for example [8]), Theorem 1 follows form

THEOREM 2. *There exists a vector* $(\alpha, \alpha') \in \mathbb{R}^2$ *and a smooth strictly positive function* φ *defined over* $\mathbb{T}^2$ *such that the special flow* $T^t_{\alpha,\alpha',\varphi}$ *is mixing and* μ*-a.e.* $\xi \in M_\varphi$ *is super-recurrent for* $T^1_{\alpha,\alpha',\varphi}$.

We will now undertake the construction of the special flow $T^t_{\alpha,\alpha',\varphi}$, following the same steps as [9]. We will first choose a special translation vector on $\mathbb{T}^2$, then we will give two criteria on the Birkhoff sums of the special function φ above $R_{\alpha,\alpha'}$ that will guarantee mixing and super-recurrence respectively. Finally we build a smooth function φ satisfying these criteria.

3.5. Choice of the translation on $\mathbb{T}^2$. *Reminder on continued fractions.* Let α be an irrational real number: There exists a sequence of rationals $\{\frac{p_n}{q_n}\}_{n\in\mathbb{N}}$, called the convergents of α, such that:

$$\| q_{n-1}\alpha \| < \| k\alpha \| \quad \forall k < q_n \tag{3.2}$$

and for any n

$$\frac{1}{q_n(q_n + q_{n+1})} \leq (-1)^n(\alpha - \frac{p_n}{q_n}) \leq \frac{1}{q_n q_{n+1}}. \tag{3.3}$$

We recall also that any irrational number $\alpha \in \mathbb{R} - \mathbb{Q}$ has a writing in continued fraction

$$\alpha = [a_0, a_1, a_2, ...] = a_0 + 1/(a_1 + 1/(a_2 + ...)),$$

where $\{a_i\}_{i\geq 1}$ is a sequence of integers ≥ 1, $a_0 = [\alpha]$. Conversely any sequence $\{a_i\}_{i\in\mathbb{N}}$ corresponds to a unique number α. The convergents of α are given by the a_i in the following way:

$$\begin{aligned} p_n &= a_n p_{n-1} + p_{n-2} \quad \text{for} \quad n \geq 2, p_0 = a_0, p_1 = a_0 a_1 + 1, \\ q_n &= a_n q_{n-1} + q_{n-2} \quad \text{for} \quad n \geq 2, q_0 = 1, q_1 = a_1. \end{aligned}$$

Following [15] and as in [8], we take α and α' satisfying

$$q'_n \geq e^{(q_n)^5}, \tag{3.4}$$

$$q_{n+1} \geq e^{(q'_n)^5}. \tag{3.5}$$

Vectors $(\alpha, \alpha') \in \mathbb{R}^2$ satisfying (3.4) and (3.5) are obtained by an adequate choice of the sequences $a_n(\alpha)$ and $a_n(\alpha')$. Moreover, it is easy to see that the set of vectors satisfying (3.4) and (3.5) is a continuum (Cf. [15], Appendix 1).

3.6. Mixing criterion. We will use the criterion on mixing for a special flow $T^t_{\alpha,\alpha',\varphi}$ studied in [8]. It is based on the uniform stretch of the Birkhoff sums $S_m\varphi$ of the ceiling function above the x or the y direction alternatively depending on whether m is far from the q_n or from q'_n. From [8], Propositions 3.3, 3.4 and 3.5 we have the following sufficient mixing criterion. We denote by $\{x\} \in [0, 1)$ the fractional part of a real number x.

PROPOSITION 3 (Mixing Criterion). *Let* (α, α') *be as in* (3.5) *and* $\varphi \in C^2(\mathbb{T}^2, \mathbb{R}^*_+)$. *If for every* $n \in \mathbb{N}$ *sufficiently large, we have a set* I_n *equal to* $[0, 1]$ *minus two intervals whose lengths converge to zero such that:*

- $m \in \left[\frac{e^{2(q_n)^4}}{2}, 2e^{2(q'_n)^4}\right] \implies |D_x S_m \varphi(x, y)| \geq \frac{m}{e^{(q_n)^4}} \frac{q_n}{n}$, *for any* $y \in \mathbb{T}$ *and* $\{q_n x\} \in I_n$;
- $m \in \left[\frac{e^{2(q'_n)^4}}{2}, 2e^{2(q_{n+1})^4}\right] \implies \left|D_y S_m \varphi(x, y)\right| \geq \frac{m}{e^{(q'_n)^4}} \frac{q'_n}{n}$, *for any* $x \in \mathbb{T}$ *and* $\{q'_n y\} \in I_n$;

Then the special flow $T^t_{\alpha,\alpha',\varphi}$ *is mixing.*

3.7. Super-recurrence Criterion. We give now a condition on the Birkhoff sums of φ above $R_{\alpha,\alpha'}$ that is sufficient to insure super-recurrence for $T^1_{\alpha,\alpha',\varphi}$.

PROPOSITION 4 (Super-recurrecne criterion). *If for n sufficiently large, we have for any x such that* $1/n^2 \leq \{q_n x\} \leq 1/n - 1/n^2$ *and for any* $y \in \mathbb{T}$

$$\left| S_{q_n q'_n} \varphi(x,y) - q_n q'_n \right| \leq \frac{1}{e^{(q_n q'_n)^2}}, \tag{3.6}$$

then μ-almost every $z = (x,y,s) \in M_\varphi$ *is super recurrent for the special flow* $T^t_{\alpha,\alpha',\varphi}$ *as in Definition 1.*

Proof. Denote by T^t the flow $T^t_{\alpha,\alpha',\varphi}$ and let $t_n = q_n q'_n$. From (3.1) we have that

$$T^{t_n}(x,y,s) = (x + t_n\alpha, y + t_n\alpha', s + t_n - S_{t_n}\varphi(x,y)).$$

From (3.3), (3.4) and (3.5) we get that $\|t_n \alpha\| \leq \frac{1}{e^{t_n^3}}$. Now, for x such that $1/n^2 \leq \{q_n x\} \leq 1/n - 1/n^2$, for any $y \in \mathbb{T}$ and for $s \in [0, \varphi(x,y))$ we have from (3.6) that $z = (x,y,s)$ satisfies (for the Euclidean distance) $d(z, T^{t_n} z) \leq \frac{2}{e^{t_n^2}}$.

Now, the set $\mathcal{C}_n = \{x \in \mathbb{T} : 1/n^2 \leq \{q_n x\} \leq 1/n - 1/n^2\}$ has Lebesgue measure larger than $1/n$. As q_n increases very fast, we have that the sets $\mathcal{C}_n$ are almost independent, from which it follows by Borel Cantelli type lemmas that Lebsgue a.e. $x \in \mathbb{T}$ belongs to infinitely many of the $\mathcal{C}_n$. Thus almost every $z \in M_\varphi$ is super-recurrent. □

3.8. Choice of the ceiling function φ. Let (α, α') be as above and define

$$f(x,y) = 1 + \sum_{n \geq 2} X_n(x) + Y_n(y)$$

where

$$X_n(x) = \frac{1}{e^{(q_n)^4}} \cos(2\pi q_n x) \tag{3.7}$$

$$Y_n(y) = \frac{1}{e^{(q'_n)^4}} \cos(2\pi q'_n y). \tag{3.8}$$

Using Criterion 3 we proved in [8] that the flow $T^t_{\alpha,\alpha',f}$ is mixing. In order to keep this criterion valid but have in addition the conditions of Criterion 4 satisfied we modify the ceiling function in the following way:

• We keep $Y_n(y)$ unchanged.

• We replace $X_n(x)$ by a trigonometric polynomial $\tilde{X}_n$ with integral zero, that is essentially equal to 0 for $\{q_n x\} < 1/n$ and whose derivative has its absolute value bounded from below by $q_n/e^{(q_n)^4}$ for $\{q_n x\} \in [2/n, 1/2 - 1/n] \cup [1/2 + 2/n, 1 - 1/n]$. The first two properties of $\tilde{X}_n$ will yield Criterion 4 while the lower bound on the absolute value of its derivative will insure Criterion 3.

More precisely, the following Proposition enumerates some properties that we will require on $\tilde{X}_n$ and its Birkhoff sums, and that will be sufficient for our purposes

PROPOSITION 5. *Let (α, α') be as in Section (3.5). There exists a sequence of trigonometric polynomials $\tilde{X}_n(x)$ satisfying*

(1) $\int_{\mathbb{T}} \tilde{X}_n(x)dx = 0$;

(2) For any $r \in \mathbb{N}$, *for every* $n \geq N(r)$, $\| \tilde{X}_n \|_{C^r} \leq \frac{1}{e^{\frac{(q_n)^4}{2}}}$;

(3) For $\{q_n x\} \leq \frac{1}{n}$, $|\tilde{X}_n(x)| \leq \frac{1}{e^{(q_n q_n')^4}}$;

(4) For $\frac{2}{n} \leq \{q_n x\} \leq \frac{1}{2} - \frac{1}{n}$, *it holds* $\tilde{X}_n'(x) \geq \frac{q_n}{e^{(q_n)^4}}$, *as well as for* $\frac{1}{2} + \frac{2}{n} \leq \{q_n x\} \leq 1 - \frac{1}{n}$, *it holds* $\tilde{X}_n'(x) \leq -\frac{q_n}{e^{(q_n)^4}}$;

(5) $\| S_{q_n} \sum_{l \leq n-1} \tilde{X}_l \| \leq \frac{1}{e^{(q_n q_n')^4}}$;

(6) For any $m \in \mathbb{N}$, $\| S_m \sum_{l \leq n-1} \tilde{X}_l' \| \leq q_n$.

Before we prove this Proposition let us show how it allows to produce the example of Theorem 2. Define for some $n_0 \in \mathbb{N}$

$$\varphi(x, y) = 1 + \sum_{n=3}^{\infty} \tilde{X}_n(x) + Y_n(y) \tag{3.9}$$

that is of class C^∞ from Property (2) of $\tilde{X}_n$ and from the definition of Y_n in (3.8). From (2) again, we can choose n_0 sufficiently large so that φ is strictly positive. Furthermore, we have

THEOREM 3. *Let $(\alpha, \alpha') \in \mathbb{R}^2$ be as in Section 3.5 and φ be given by (3.9). Then the special flow $T^t_{\alpha,\alpha',\varphi}$ satisfies the conditions of Propositions 3 and 4 and hence the conclusion of Theorem 2*

Proof. The second part of Proposition 3 is valid exactly as in [8] since Y_n has not been modified.

Let $m \in [e^{2(q_n)^4}/2, 2e^{2(q'_n)^4}]$ and x be such that $3/n \leq \{q_n x\} \leq 1/2 - 2/n$. From (3.5) we get for any $l \leq m$ that $2/n \leq \{q_n(x + m\alpha)\} \leq 1/2 - 1/n$ hence by Property (4) of $\tilde{X}_n$

$$S_m \tilde{X}'_n(x) \geq \frac{mq_n}{e^{(q_n)^4}}.$$

On the other hand, Properties (2) and (6) imply that

$$\begin{aligned} \| S_m\varphi' - S_m\tilde{X}'_n \| &\leq q_n + m \sum_{l \geq n+1} \frac{1}{e^{\frac{(q_l)^4}{2}}} \\ &\leq q_n + \frac{2m}{e^{\frac{(q_{n+1})^4}{2}}} \\ &= o(\frac{mq_n}{e^{(q_n)^4}}) \end{aligned}$$

for the current range of m. With a similar computation for $1/2 + 2/n \leq \{q_n(x + m\alpha)\} \leq 1 - 1/n$, the criterion of Proposition 3 thus holds true.

Let now x be as in Proposition 4, that is $1/n^2 \leq \{q_n x\} \leq 1/n - 1/n^2$. From (3.5) we have for any $l \leq q_n q'_n$ that $0 \leq \{q_n(x + l\alpha)\} \leq 1/n$, hence Property (3) implies

$$|S_{q_n q'_n} \tilde{X}_n(x)| \leq \frac{q_n q'_n}{e^{(q_n q'_n)^4}} \leq \frac{1}{e^{(q_n q'_n)^3}}. \tag{3.10}$$

From Properties (2) and (5) we get for n sufficently large

$$\begin{aligned} \| S_{q_n q'_n} \sum_{l \neq n} \tilde{X}_l \| &\leq \frac{q_n q'_n}{e^{(q_n q'_n)^4}} + q_n q'_n \sum_{l \geq n+1} \frac{1}{e^{\frac{(q_l)^4}{2}}} \\ &\leq \frac{1}{e^{(q_n q'_n)^3}}. \end{aligned} \tag{3.11}$$

On the other hand, it follows from thew definition of convergents in Section 3.5 and (3.3) that for any $y \in \mathbb{T}$, for any $|j| < q'_n$, we have

$$\begin{aligned} |S_{q'_n} e^{i2\pi jy}| &= \left| \frac{\sin(\pi j q'_n \alpha')}{\sin(\pi j \alpha')} \right| \\ &\leq \frac{\pi j q'_n}{q'_{n+1}}, \end{aligned} \tag{3.12}$$

which, using (3.4) and (3.5), yields for Y_l as in (3.8)

$$\|S_{q'_n} \sum_{l<n} Y_l\| \leq \frac{1}{e^{(q_n q'_n)^3}} \tag{3.13}$$

while clearly

$$\|S_{q'_n} \sum_{l>n} Y_l\| \leq e^{-\frac{(q'_{n+1})^4}{2}} \leq \frac{1}{e^{(q_n q'_n)^3}} \tag{3.14}$$

and

$$\|S_{q'_n} Y_n\| \leq \frac{q'_n}{e^{(q'_n)^4}} \leq \frac{1}{e^{(q_n q'_n)^3}} \tag{3.15}$$

Putting together (3.13)–(3.15) yields

$$\|S_{q_n q'_n} \sum_{l=3}^{\infty} Y_l\| \leq \frac{1}{2e^{(q_n q'_n)^2}} \tag{3.16}$$

In conclusion, (3.6) follows from (3.10), (3.11), and (3.16). □

It remains to construct $\tilde{X}_n$ satisfying (1)-(6).

3.9. Proof of Proposition 5. Consider on $\mathbb{R}$ a C^∞ function, $0 \leq \theta \leq 1$ such that

$$\begin{aligned} \theta(x) &= 0 \text{ for } x \in (-\infty, 2] \\ \theta(x) &= 1 \text{ for } x \in [3, +\infty). \end{aligned}$$

Then we define on the circle the function

$$\theta^n(x) := \left(\theta(nx) - \theta(n(x - 1 + \frac{3}{n}))\right),$$

and let

$$\begin{aligned} U_n(x) &:= \left(\theta^n(4q_n x) - \theta^n(4q_n(x - \frac{1}{4q_n})\right) \\ V_n(x) &:= \int_{-\infty}^{x} U_n(s)ds \\ W_n(x) &:= V_n(x) - V_n(x - \frac{1}{2q_n}) \end{aligned}$$

and finally

$$\tilde{X}_n(x) := \frac{1}{e^{(q_n)^4}} \sum_{k=0}^{q_n-1} W_n(x + \frac{k}{q_n})$$

that we view as a function on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. It is easy to check (1),(2),(3) and (4) of Proposition 5 for $\hat{X}_n$.

Now we consider the Fourier series of $\hat{X}_n(x) = \sum_{k\in\mathbb{Z}} \hat{X}_{n,k}e^{i2\pi kx}$ and let

$$\tilde{X}_n := \sum_{k=-q_{n+1}+1}^{q_{n+1}-1} \hat{X}_{n,k}e^{i2\pi kx}.$$

From the order of the truncation and the C^r norms of $\hat{X}_n$ it is easy to deduce that for any $r \in \mathbb{N}$

$$\|\tilde{X}_n - \hat{X}_n\|_{C^r} \leq \frac{1}{e^{(q_n q_n')^5}}$$

which allows to check (1), (2), (3) and (4) for $\tilde{X}_n$.

Proof of Property (5). As for (3.12), using the definition of convergents in Section 3.5, we obtain for any $x \in \mathbb{T}$, and for any $|k| < q_n$

$$|S_{q_n}e^{i2\pi kx}| \leq \frac{\pi k q_n}{q_{n+1}},$$

hence for $\tilde{X}_l := \sum_{k=-q_{l+1}+1}^{q_{l+1}-1} \hat{X}_{l,k}e^{i2\pi kx}$ and $l \leq n-1$ we have

$$\begin{aligned}\|S_{q_n}\tilde{X}_l\| &\leq \frac{\pi q_n^2}{q_{n+1}} \sum_{k=-q_{l+1}+1}^{q_{l+1}-1} |\hat{X}_{l,k}| \\ &\leq \frac{2\pi q_n^3}{q_{n+1}}\|\hat{X}_l\|\end{aligned}$$

from which Property (5) follows.

Proof of Property (6). For any $|k| < q_n$ we have

$$\begin{aligned}|S_m e^{i2\pi kx}| &= \left|\frac{\sin(\pi mk\alpha)}{\sin(\pi k\alpha)}\right| \\ &\leq \frac{1}{|\sin(\pi k\alpha)} \\ &\leq q_n.\end{aligned}$$

Thus, for $l \leq n-1$, we use that $|\hat{X}_{l,k}| \leq \frac{1}{(2\pi|k|)^3}\|D_x^3\hat{X}_l\|$ and get that

$$\begin{aligned}\|S_m\tilde{X}_l'\| &\leq \sum_{k=-q_{l+1}+1}^{q_{l+1}-1} \frac{1}{(2\pi k)^2} q_{l+1}\|D_x^3\hat{X}_l\| \\ &\leq \frac{1}{12} q_{l+1}\|D_x^3\hat{X}_l\|\end{aligned}$$

from which Property (6) follows. □

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