# LYAPUNOV UNSTABLE ELLIPTIC EQUILIBRIA

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ABSTRACT. We introduce a new diffusion mechanism from the neighborhood of elliptic equilibria for Hamiltonian flows in three or more degrees of freedom. Using this mechanism, we obtain the first examples of real analytic Hamiltonians that have a Lyapunov unstable non-resonant elliptic equilibrium.

## Introduction

An equilibrium $(p,q) \in \mathbb{R}^{2d}$ of an autonomous Hamiltonian flow is said to be Lyapunov stable or topologically stable if all nearby orbits remain close to 0 for all forward time.

The topological stability of the equilibria of Hamiltonian flows is one of the oldest problems in mathematical physics. The important contributions to the understanding of this problem, dating back to the 18th century, form a fundamental part of the foundation and of the evolution of the theory of dynamical systems and celestial mechanics up to our days.

The goal of this note is to give the first examples of real analytic Hamiltonians that have a Lyapunov unstable non-resonant elliptic equilibrium.

A $C^2$ function $H : (\mathbb{R}^{2d}, 0) \to \mathbb{R}$ such that $DH(0) = 0$ defines a Hamiltonian vector field $X_H(x,y) = (\partial_y H(x,y), -\partial_x H(x,y))$ whose flow $\phi_H^t$ preserves the origin.

Naturally, to study the stability of the equilibrium at the origin, one has first to investigate the stability of the linearized system at the origin. By symplectic symmetry, the eigenvalues of the linearized system come by pairs $\pm\lambda$, $\lambda \in \mathbb{C}$. It follows that if the linearized system has an eigenvalue with a non zero real part, it also has an eigenvalue with positive real part and this implies instability of the origin for the linearized system as well as for the non-linear flow.

When all the eigenvalues of the linearized system are on the imaginary axis the stability question is more intricate. In the non-degenerate case where the eigenvalues are simple, we say that the origin is an *elliptic* equilibrium. The linear system is then symplectically conjugated to a direct product of planar rotations. The arguments of the eigenvalues are called the frequencies of the equilibrium since they correspond to angles of rotation of the linearized system. The elliptic equilibrium is said to be *non-resonant* if its frequencies are rationally independent. The phenomenon of averaging out of the non-integrable part of the nonlinearity effects at a non-resonant frequency is responsible for the long time effective stability around the equilibrium : the points near the equilibrium remain in its neighborhood during a time that is greater than any negative power of their distance to the equilibrium. This can be formally studied and proved using the Birkhoff Normal Forms (BNF) at the equilibrium, that introduce action-angle coordinates in which the system is integrable up to arbitrary high degree in its Taylor series (see for example [Bi66] or [SM71]). Moreover, it was proven in [MG95, BFN15] that a typical elliptic fixed point is doubly exponentially stable in the sense that a neighboring point of the equilibrium remains close to it for an interval of time which is doubly exponentially large with respect to some power of the inverse of the distance to the equilibrium point.

In addition to the long time effective stability of non-resonant equilibria, KAM theory (after Kolmogorov Arnold and Moser), asserts that a non-resonant elliptic fixed point is in general accumulated by quasi-periodic invariant Lagrangian tori whose relative measurable density tends to one in small neighborhoods of the fixed point. This can be viewed as stability in a probabilistic sense, and is usually coined *KAM stability*. In classical KAM theory, KAM stability is established when the BNF has a non-degenerate Hessian. Further development of the theory allowed to relax the non degeneracy condition and [EFK13] proved KAM-stability of a non-resonant elliptic fixed point under the non-planarity condition of the BNF (see Section 3.c).

Despite the long time effective stability, and despite the genericity of KAM-stability, Arnold conjectured that apart from two cases, the case of a sign-definite quadratic part of the Hamiltonian, and generically for $d = 2$, an elliptic equilibrium point of a generic real analytic Hamiltonian system is Lyapunov unstable [Arn94, Section 1.8].

Although a rich literature in the direction of proving this conjecture exist in the $C^\infty$ smoothness (we mention [KMV04] below, but to give a list of contributions would exceed the scope of this introduction), the conjecture is still wide open in the real analytic category. For instance, not a single example of real analytic Hamiltonians was known that has an unstable non-resonant elliptic equilibrium.

Our goal is to give the first such examples in three or more degrees of freedom. The question in two degrees of freedom remains open.

THEOREM. – *There exists a non-resonant $\omega \in \mathbb{R}^3$ and a real entire Hamiltonian $H$ : $\mathbb{R}^6 \to \mathbb{R}$, such that the origin is a Lyapunov unstable elliptic equilibrium with frequency $\omega$ of the Hamiltonian flow $\Phi_H^t$ of $H$.*

*For any $\omega \in \mathbb{R}^d, d \geq 4$, such that $\omega_1\omega_2 < 0$, there exists a real entire Hamiltonian $H : \mathbb{R}^{2d} \to \mathbb{R}$ such that the origin is a Lyapunov unstable elliptic equilibrium with frequency $\omega$ of the Hamiltonian flow $\Phi_H^t$ of $H$.*

*Moreover, for non-resonant frequencies $\omega$, the Hamiltonians $H$ can be chosen such that the origin is KAM stable.*

Detailed statements with an explicit definition of the Hamiltonians will be given in Section 3.

In case $\omega$ is resonant, it is known that instabilities are more likely to happen. Algebraic examples were known since long time ago [LC1901, Ch26] (see [MS02, §31]). Our construction is actually based on the existence in two degrees of freedom, for resonant frequencies, of polynomial vector Hamiltonians that have orbits attracted to the origin in negative time and escaping to infinity in finite time (see Remark 1 of Section 3.a).

Note that in the non-resonant case, we do not obtain the existence of an orbit that accumulates on the origin. Based on a different diffusion mechanism, [FMS17] gives examples of smooth symplectic diffeomorphisms of $\mathbb{R}^6$ having a non-resonant elliptic fixed point that attracts an orbit.

In [KMV04], the authors admit Mather's proof of Arnold diffusion for a cusp residual set of nearly integrable convex Hamiltonian systems in 2.5 degrees of freedom, and deduce from it that generically, a convex resonant totally elliptic point of a symplectic map in 4 dimensions is Lyapunov unstable, and in fact has orbits that converge to the fixed point.

Finally, we point to the fact that it would be interesting to decide whether the BNF at the equilibria in our examples, especially in the Diophantine case, are convergent or not. Finding unstable examples with a convergent BNF would indeed answer a question of Eliasson on the possibility of having a non locally integrable analytic Hamiltonian with a convergent BNF at a non resonant equilibrium (see [E89, EFK15]). This question does not seem to be easily tractable in our examples and we do not pursue it here.

Before we outline the idea in the diffusion mechanism that is behind the instability of the equilibria, let us make some comments about the general statement.

For $\omega \in \mathbb{R}^d$ the Hamiltonians we will be interested in are of the form

$$(*) \qquad H(x,y) = \sum_{j=1}^{d} \omega_j I_j + \mathcal{O}^3(x,y), \quad I_j = \frac{1}{2}(x_j^2 + y_j^2).$$

The Hamiltonian system associated to $H$ is given by the vector field $X_H = (\partial_y H, -\partial_x H)$, that is

$$\begin{cases} \dot{x} = \partial_y H(x,y) \\ \dot{y} = -\partial_x H(x,y). \end{cases}$$

The flow of $X_H$ has an elliptic equilibrium at the origin whose frequency is $\omega$.

Our examples in three degrees of freedom require very strong almost resonances on the vector $(\omega_1, \omega_2)$, where we assumed that $\omega_1\omega_2 < 0$.

Starting from 4 degrees of freedom and above it is possible to give examples with arbitrary frequency vectors, in particular Diophantine. Recall that $\omega$ is said to be Diophantine if there exists $\gamma, \tau > 0$ such that $|\langle k, \omega \rangle| \geq \gamma |k|^{-\tau}$, for all $k \in \mathbb{Z}^d - \{0\}$, with $\langle \cdot \rangle$ being the canonical scalar product and $|\cdot|$ its associated norm.

In his ICM talk of 1998 [He98], Herman conjectured that a real analytic elliptic equilibrium with a Diophantine frequency vector must be accumulated by a set of positive measure of KAM tori. This conjecture is still open. However, our examples can be chosen such that the Birkhoff Normal Form is non-planar, which implies KAM-stability as established in [EFK13].

In the $C^\infty$ category, examples of unstable elliptic equilibria can be obtained *via* the successive conjugation method, the Anosov-Katok method. They can be obtained in two degrees of freedom or for $\mathbb{R}^2$ symplectomorphisms, provided the frequency at the elliptic equilibrium is not Diophantine ([AK66, FS05, FS17]). In three or more degrees of freedom, smooth examples with Diophantine frequencies can be obtained through a more sophisticated version of the successive conjugation method (see [EFK15, FS17]). The Anosov-Katok examples are infinitely tangent to the rotation of frequency $\omega$ at the fixed point and as such are very different in nature from our construction. In particular, KAM stability is in general excluded in these constructions.

Again in the $C^\infty$ class but in the non-degenerate case, R. Douady gave examples in [Dou88] of Lyapunov unstable elliptic points for symplectic diffeomorphisms on $\mathbb{R}^{2n}$ for any $n \geq 2$. Douady's examples can have any chosen Birkhoff Normal Form at the origin provided its Hessian at the fixed point is non-degenerate. Douady's examples are modeled on the Arnold diffusion mechanism through chains of heteroclinic intersections between lower dimensional partially hyperbolic invariant tori that accumulate toward the origin. The construction requires a countable number of compactly supported perturbations of a completely integrable flow, and as such cannot be made real analytic.

A third diffusion mechanism, closely related to Arnold diffusion mechanism, is Herman's synchronized diffusion, and is due to Herman, Marco and Sauzin [MS02]. It is based on the following coupling of two twist maps of the annulus (the second one being integrable with linear twist): at exactly one point $p$ of a well chosen periodic orbit of period $q$ on the first twist map, the coupling consists of pushing the orbits in the second annulus up on some fixed vertical $\Delta$ by an amount that sends an invariant curve whose rotation number is a multiple of $1/q$ to another one having the same property. The dynamics of the coupled maps on the line $\{p\} \times \Delta$ will thus drift at a linear speed.

We now give a quick description of the diffusion mechanism that underlies our constructions. It is inspired by all these three mechanisms described above but is quite different from each.

DIFFUSION IN 3 DEGREES OF FREEDOM NEAR A CLOSE TO RESONANT ELLIPTIC EQULIBRIUM We start with a Hamiltonian that is a product of three rotators of frequencies $\omega_1, \omega_2, \omega_3$, where $\omega_1 \omega_2 < 0$ and the vector $\bar{\omega} = (\omega_1, \omega_2)$ is very well approached by resonant vectors. It is then possible to couple the first two degrees of freedom with the third one *via* a diffusive flow that acts in the 4-dimensional space $(x_1, y_1, x_2, y_2)$ and almost commutes with the rotator $R_{\bar{\omega}}$ (these flows have a degenerate multi-saddle and commute with any chosen resonant approximation of $\bar{\omega}$). To guarantee analyticity, the coupling parameter must be very small, making the diffusion very slow, so that a very good approximation (double exponential) of $\bar{\omega}$ by the resonant vector is required[1].

An additional smallness in the coupling is given by the coupling parameter with the third degree of freedom that should only depend on the third action and that we can take to be exactly $I_3 = \frac{1}{2}(x_3^2 + y_3^2)$.

If only one coupling term is added in this way to the original Hamiltonian, there appears orbits that diffuse from a small (but not arbitrarily small) neighborhood of the origin towards infinity (in projection on the $(x_1, y_1, x_2, y_2)$ space, since the third action is invariant).

Of course, to get diffusion from arbitrary small neighborhoods of the origin, one has to add successive couplings that commute with increasingly better resonant approximations of $\bar{\omega}$. Each individual coupling will "act" separately and induce a segment of orbit that starts closer and closer from the origin and diffuses farther and farther towards infinity. To isolate the effect of each individual coupling from all the successive couplings is easy because these terms are extremely

[1] The requirement of double exponential approximations is not uncommon in instability results in real analytic and holomorphic dynamics as shown for example in [PM97]

small compared to it. The effect of the prior coupling terms is tamed out due to Birkhoff averaging.

DIFFUSION IN 4 DEGREES OF FREEDOM NEAR AN ARBITRARY ELLIPTIC EQULIBRIUM In the case of 4 degrees of freedom (or more) we can take the frequency vector of the equlibrium to be arbitrary, provided all the coordinates are not of the same sign. WLOG suppose that $\omega_1\omega_2 < 0$. Following an idea introduced in [EFK15] (see also [FS17]) we can use the action of the fourth degree of freedom as a parameter, in the sense that the Hamiltonians we consider will be a coupling of a Hamiltonian of the first three degrees of freedom and a rotator of frequency $\omega_4$ in the fourth degree of freedom, and with a coupling that only depends on the fourth action variable $I_4 = \frac{1}{2}(x_4^2 + y_4^2)$. Hence the action variable $I_4$ will be constant along the motion. To obtain the diffusion in the first three degrees of freedom, we first consider the three rotators of frequencies $\omega_1 + I_4, \omega_2, \omega_3$. Regardless of the values of $\omega_1$ and $\omega_2$ the vector $\omega_1 + I_4, \omega_2$ will go through rational resonances as $I_4 \to 0$, which opens the way for the application of the diffusion mechanism in 3 degrees of freedom that we discussed above, in an even simpler context in fact, since the Liouville condition on $(\omega_1, \omega_2)$ is in this case an exact resonance condition.

## 2. Notations

Let $(x, y) = (x_1, \ldots, x_d, y_1, \ldots, y_d)$ be symplectic coordinates defined on $\mathbb{R}^{2d}$. We can complexify $x$ and $y$ and introduce the complex coordinates $(\xi, \eta) \in \mathbb{C}^{2d}$

$$\xi_j = \frac{1}{\sqrt{2}}(x_j + iy_j), \quad \eta_j = \frac{1}{\sqrt{2}}(x_j - iy_j), \quad j = 1, 2, 3.$$

For a Hamiltonian $H \in C^2(\mathbb{C}^{2d}, \mathbb{C})$, the corresponding Hamiltonian flow in the $(\xi, \eta)$ coordinates is given by the following system of equations

$$\begin{cases} \dot{\xi}_j = -i\partial_{\eta_j} H(\xi, \eta) \\ \dot{\eta}_j = i\partial_{\xi_j} H(\xi, \eta) \end{cases}$$

The Hamiltonian $H$ is said to be real if $H(\xi, \bar{\xi})$ is real. We call a point $z \in \mathbb{C}^{2d}$ *real* and denote this by $z \in \mathbb{R}^{2d}$ if $\eta_j = \bar{\xi}_j$ for $j = 1, \ldots, d$. This is of course equivalent to having the coordinates $x_j$ and $y_j$ real.

In our constructions, we will only deal with real entire Hamiltonians, that are real Hamiltonians defined by a power series in $\xi_j, \eta_j$, $j = 1, \ldots, d$, that converges on all $\mathbb{C}^{2d}$.

We denote $|\cdot|$ the Euclidean norm on $\mathbb{R}^{2d}$, indifferently on the value of $d$ that will be clear from the context. We also denote indifferently $B_n$ the Euclidean ball of radius $n$ in $\mathbb{R}^{2d}$ for any value of $d$. For $k \in \mathbb{N}$, we denote by $\|H\|_{C^k(B_R)}$ the $C^k$ norm of $H$ on the ball $B_R$.

## 3. Lyapunov unstable elliptic equilibria

### 3.a. Lyapunov unstable elliptic equilibrium in three degress of freedom

For a frequency vector $\omega \in \mathbb{R}^3$ we define the Hamiltonian on $\mathbb{R}^6$

$$H_0(\xi, \eta) = \sum_{j=1}^{3} \omega_j I_j, \quad I_j = \frac{1}{2}(x_j^2 + y_j^2).$$

We suppose $\omega$ is such that there exists a sequence $\{(k_n, l_n)\} \in \mathbb{N}^* \times \mathbb{N}^*$ satisfying

$$(\mathcal{L}) \qquad 0 < |k_n\omega_1 + l_n\omega_2| < e^{-e^{n^4(k_n+l_n)}}.$$

Up to extracting we can also assume that

$$k_{n+1} \geq e^{\frac{1}{|k_n\omega_1+l_n\omega_2|}}.$$

For $n \in \mathbb{N}$ we define on $\mathbb{R}^4$ the following polynomial Hamiltonians

$$(1) \qquad F_n(x_1, x_2, y_1, y_2) = \zeta_1^{k_n}\zeta_2^{l_n} + \bar{\zeta}_1^{k_n}\bar{\zeta}_2^{l_n}, \quad \zeta_j = \frac{1}{\sqrt{2}}(x_j + iy_j).$$

We finally define a real entire Hamiltonian on $\mathbb{R}^6$

$$H(x, y) = H_0(x, y) + \sum_{n\in\mathbb{N}} e^{-n(k_n+l_n)} I_3 F_n(x_1, x_2, y_1, y_2)$$

The origin is an elliptic fixed point for the Hamiltonian flow of $H$ with frequency vector $\omega$.

THEOREM 1. – *The origin is a Lyapunov unstable equilibrium of the Hamiltonian flow $\Phi_H^t$ of $H$. More precisely, for every $n \geq 1$, there exists $\zeta_n \in \mathbb{R}^8$, such that $|\zeta_n| \leq \frac{1}{n}$, and $\tau_n \geq 0$ such that $|\Phi_H^{\tau_n}(\zeta_n)| \geq n$.*

**Remark 1** (Resonant frequencies in two degrees of freedom). *If $\omega_1$ and $\omega_2$ are such that $k\omega_1 + l\omega_2 = 0$ for some $k, l \geq 1$ and $k + l > 2$, then the flow of $H(x_1, x_2, y_1, y_2) = \omega_1 I_1 + \omega_2 I_2 + \zeta_1^k\zeta_2^l + \bar{\zeta}_1^k\bar{\zeta}_2^l$ has an elliptic fixed point with frequency $(\omega_1, \omega_2)$ that is Lyapunov unstable. This is because the Hamiltonian flow of $\zeta_1^k\zeta_2^l + \bar{\zeta}_1^k\bar{\zeta}_2^l$ commutes with that of $\omega_1 I_1 + \omega_2 I_2$ and has invariant lines that go through the origin such that any point on such a line converges to the origin for negative times and goes to infinity in finite time in the future (see Section 4).*

### 3.b. Lyapunov unstable elliptic equilibrium in four degrees of freedom

In 4 degrees of freedom (or more), our method yields unstable elliptic equilibria for any frequency vector, provided its coordinates are not all of the same sign. Suppose for instance that $\omega = (\omega_1, \dots, \omega_4)$ is such that $\omega_1\omega_2 < 0$.

We assume $(\omega_1, \omega_2)$ non-resonant (the resonant case follows from Remark 1). By Dirichlet principle, there exists a sequence $(k_n, l_n) \in \mathbb{N}^* \times \mathbb{N}^*$ such that

$$|k_n\omega_1 + l_n\omega_2| < \frac{1}{k_n^2}.$$

WLOG, we assume that $k_n\omega_1 + l_n\omega_2 < 0$. Then, for $I_{4,n} = -(k_n\omega_1 + l_n\omega_2)/k_n \in (0, \frac{1}{k_n^3})$, it holds

$$(\mathcal{R}) \qquad k_n(\omega_1 + I_{4,n}) + l_n\omega_2 = 0.$$

Since $(\omega_1, \omega_2)$ is non-resonant, and since the smallness of $I_{4,n}$ is decided by $k_n$ and $l_n$, we can, up to extracting, additionally ask that for all $j \leq n-1$

$$(\mathcal{NR}) \qquad k_j(\omega_1 + I_{4,n}) + l_j\omega_2 \neq 0, \text{and } k_n \geq e^{\frac{1}{|k_j(\omega_1+I_{4,n})+l_j\omega_2|}}.$$

We define the Hamiltonian on $\mathbb{R}^8$

$$H_0(x,y) = (\omega_1 + I_4)I_1 + \sum_{j=2}^{4} \omega_j I_j, \quad I_j = \frac{1}{2}(x_j^2 + y_j^2).$$

For $n \in \mathbb{N}$ we recall the definition on $\mathbb{R}^4$ of the polynomial Hamiltonians

$$F_n(x_1, x_2, y_1, y_2) = \xi_1^{k_n}\xi_2^{l_n} + \bar{\xi}_1^{k_n}\bar{\xi}_2^{l_n}, \quad \xi_j = \frac{1}{\sqrt{2}}(x_j + iy_j).$$

We finally define the following real entire Hamiltonian on $\mathbb{R}^8$

$$H(x,y) = H_0(x,y) + \sum_{n\in\mathbb{N}} e^{-n(k_n+l_n)} I_3 F_n(x_1, x_2, y_1, y_2)$$

The origin is an elliptic fixed point for the Hamiltonian flow of $H$ with frequency vector $\omega$.

THEOREM 2. – *The origin is a Lyapunov unstable equilibrium for the Hamiltonian flow of $H$. More precisely, for every $n \geq 1$, there exists $\zeta_n \in \mathbb{R}^8$, such that $|\zeta_n| \leq \frac{1}{n}$, and $\tau_n \geq 0$ such that $|\Phi_H^{\tau_n}(\zeta_n)| \geq n$.*

### 3.c Non degenerate Birkhoff Normal Forms.

For $\omega \in \mathbb{R}^d$ non-resonant and a Hamiltonian $H$ as in $(*)$, we let $N_H$ be the *Birkhoff Normal Form* of $H$ at the origin. The BNF is the unique formal power series $N_H$ in the action variables $I_j$ such that the Hamiltonian $H$ is conjugated to $N_H$ *via* a formal exact symplectic mapping of the form $\Phi(\xi,\eta) = (\xi,\eta) + \phi(\xi,\eta)$, where $\phi(\xi,\eta)$ is a formal power series in $\xi,\eta$ such that $\phi = \mathcal{O}^2(\xi,\eta)$. For the Birkhoff Normal Form at a Diophantine, and more generally at any non-resonant elliptic equilibrium, one can consult for example [SM71].

We say that $N_H$ is *non-planar* or *non-degenerate* if there does not exist any vector $\gamma$ such that for every $I$ in some neighborhood of 0

$$\langle \nabla N_H(I), \gamma \rangle = 0.$$

In [EFK15] the following was proven

THEOREM 3 ([EFK15]). – *Let $H : (\mathbb{R}^{2d}, 0) \to \mathbb{R}$ be a real analytic function of the form $(*)$ and assume that $\omega$ is non-resonant. If $N_H$ is non-planar, then in any neighborhood of $0 \in \mathbb{R}^{2d}$ the set of real analytic KAM-tori for $X_H$ is of positive Lebesgue measure and density one at 0.*

Real analytic KAM-tori are invariant Lagrangian tori on which the flow generated by $H$ is real analytically conjugated to a minimal translation flow on the torus $\mathbb{R}^d/\mathbb{Z}^d$.

THREE DEGREES OF FREEDOM. With the same hypothesis as in Section 3.a, we modify the definitions of the Hamiltonians $H_0$ and $H$ on $\mathbb{R}^6$ as follows

$$\widetilde{H}_0(x,y) = (\omega_1 + I_3^3)I_1 + (\omega_2 + I_3^4)I_2 + \omega_3 I_3,$$
$$\widetilde{H}(x,y) = \widetilde{H}_0(x,y) + \sum_{n\in\mathbb{N}} e^{-n(k_n+l_n)} I_3 F_n(x_1, x_2, y_1, y_2).$$

We can assume that $k_0 + l_0 > 10$, hence $\widetilde{H}_0$ gives the BNF of $\widetilde{H}$ at the origin up to order 5 in the action variables. But $\nabla \widetilde{H}_0(I) = (\omega_1 + I_3^3, \omega_2 + I_3^4, \omega_3 + 3I_3^2 I_1 + 4I_3^3 I_2)$

is clearly non-planar, and this implies that $\nabla \widetilde{H}$ is non-planar. We then have the following

THEOREM 4. – *The origin is a Lyapunov unstable equilibrium of the Hamiltonian flow $\Phi^t_{\widetilde{H}}$ of $\widetilde{H}$. Moreover, the Birkhoff normal form of $H$ at the origin is non-planar, hence the equilibrium is KAM-stable.*

FOUR DEGREES OF FREEDOM. We define the Hamiltonian on $\mathbb{R}^8$

$$\widetilde{H}_0(\xi,\eta) = (\omega_1 + I_4)I_1 + (\omega_2 + I_4^2)I_2 + (\omega_3 + I_4^3)I_3 + \omega_4 I_4.$$

Similarly to Section 3.b, we take $I_{4,n} \in (0, \frac{1}{k_n^3})$ such that

$$(\widetilde{\mathcal{R}}) \qquad k_n(\omega_1 + I_{4,n}) + l_n(\omega_2 + I_{4,n}^2) = 0.$$

Up to extracting we can additionally ask that for all $j \leq n-1$

$$(\widetilde{\mathcal{N}R}) \quad k_j(\omega_1 + I_{4,n}) + l_j(\omega_2 + I_{4,n}^2) \neq 0, \quad k_n \geq e^{\frac{1}{|k_j(\omega_1+I_{4,n})+l_j(\omega_2+I_{4,n}^2)|}}.$$

We then define the real entire Hamiltonian on $\mathbb{R}^8$

$$\widetilde{H}(x,y) = \widetilde{H}_0(x,y) + \sum_{n\in\mathbb{N}} e^{-n(k_n+l_n)} I_3 F_n(x_1,x_2,y_1,y_2).$$

Here also, it is clear that $\nabla \widetilde{H}(I)$ is non-planar. We have the following.

THEOREM 5. – *The origin is a Lyapunov unstable equilibrium of the Hamiltonian flow $\Phi^t_{\widetilde{H}}$ of $\widetilde{H}$. Moreover, the Birkhoff normal form of $H$ at the origin is non-degenrate, hence the equilibrium is KAM-stable.*

## 3. A Gronwall inequality

LEMMA 1. – *Suppose $F, G \in C^2(\mathbb{R}^{2d}, \mathbb{R})$, $\omega \in \mathbb{R}^d$, $A, r, R, a, T > 0$ such that $2a^2ATe^{aAT} \leq \sqrt{a} \ll 1$ and*

- $H(x,y) = \sum_{j=1}^{d} \omega_j I_j + aF(x,y)$
- $\bar{H}(x,y) = \sum_{j=1}^{d} \omega_j I_j + aF(x,y) + a^2 G(x,y)$
- $\|F\|_{C^2(B_{R+1})} \leq A, \quad \|G\|_{C^2(B_{R+1})} \leq A$
- *For all $s \in [0,T]$ : $\Phi^s_H(B_r) \subset B_R$*

*Then, for all $s \in [0,T]$ and for all $z \in B_r$ :*

$$\left|\Phi^s_H(z) - \Phi^s_{\bar{H}}(z)\right| \leq 2a^2ATe^{aAT}$$

*Proof.* We use the complex coordinates $\xi(s)$ and $\eta(s)$ for the solution $\Phi^s_H(z)$, and $\bar{\xi}(s)$ and $\bar{\eta}(s)$ for the solution $\Phi^s_{\bar{H}}(z)$. Define the matrix $U = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}$, and introduce $\zeta(s) = e^{isU}\begin{pmatrix}\xi(s) \\ \eta(s)\end{pmatrix}$, and $\bar{\zeta}(s) = e^{isU}\begin{pmatrix}\bar{\xi}(s) \\ \bar{\eta}(s)\end{pmatrix}$. Since $\|e^{isU}\| = 1$, it suffices to control $\|X(s)\|$, for $X(s) = \zeta(s) - \bar{\zeta}(s)$. The Hamiltonian equations and the bounds on $F$ and $G$ then yield

$$\dot{X} = MX + B(X), \quad X(0) = 0$$

with $\|M\| \leq aA$ and $\|B\| \leq Aa^2 + Aa\|X\|^2$. As long as $\|X(s)\| \leq \sqrt{a}$ for $s \leq t$, Gronwall's inequality implies that

$$\|X(t)\| \leq 2Aa^2te^{\|M\|t} \leq Aa^2te^{aAt}$$

which allows to conclude due to the condition $2a^2ATe^{aAT} \leq \sqrt{a} \ll 1$. Note that the bound on $\|X(t)\|$ also allows to make sure that $\Phi^s_{\bar{H}}(B_r) \subset B_{R+1}$ for $t \leq T$ which insures the validity of the bound on $M$ and $B$ that are used in the control estimates. □

## 4. An unstable flow on $\mathbb{R}^4$

Recall the definition of the following real Hamiltonians

$$F_n(x_1, x_2, y_1, y_2) = \xi_1^{k_n}\xi_2^{l_n} + \bar{\xi}_1^{k_n}\bar{\xi}_2^{l_n}.$$

PROPOSITION 1. *There exist $t_n < e^{n(k_n+l_n)}$ and $z_n \in \mathbb{R}^4$ such that $|z_n| = \frac{1}{n}$ and $|\Phi^{t_n}_{F_n}(z_n)| = n+2$ and $\Phi^t_{F_n}(B_{\frac{1}{n}}) \subset B_{n+2}$ for every $t \leq t_n$.*

*Proof.* It suffices to show that there exists $t'_n < e^{n(k_n+l_n)}$ and $|z'_n| = \frac{1}{n}$ such that $\Phi^{t'_n}_{F_n}(z'_n) \geq n+2$. Indeed, we then take $t_n$ to be the first $t$ such that $\Phi^t_{F_n}(B_{\frac{1}{n}})$ intersects the circle around 0 of radius $n+2$ and $z_n$ such that $|z_n| = \frac{1}{n}$ be a point such that $\Phi^{t_n}_{F_n}(z_n) = n+1$. Clearly $t_n \leq t'_n$ and the pair $(z_n, t_n)$ satisfies the conditions of the proposition.

In the rest of this proof, we use the notation $k = k_n, l = l_n$ and $u = \sqrt{l/k}$, $\alpha = k+l-1$. We assume $\alpha \geq 2$. WLOG, we suppose that $u \leq 1$.

Pick and fix $\nu, \nu' \in (0,1)$ such that

$$\frac{1}{4} + k\nu + l\nu' = 1.$$

Define a subset of $\mathbb{R}^4$,

$$\Delta := \left\{(x_1, x_2, y_1, y_2) \in \mathbb{R}^4 : (\xi_1, \xi_2) = (\bar{\eta}_1, \bar{\eta}_2) = \left(re^{i2\pi\nu}, ure^{i2\pi\nu'}\right), r \in \mathbb{R}\right\}.$$

It is straightforward from the Hamiltonian equations of $F_n$ that $\Delta$ is invariant by the flow $\Phi^t_{F_n}$. Moreover the restriction of the vector field on $\Delta$ is given by

$$\dot{r} = ku^l r^\alpha.$$

Hence if we start with $r_0 = \frac{1}{n}$ we see that

$$r(t)^{\alpha-1} = \frac{1}{n^{\alpha-1} - (\alpha-1)ku^l t}.$$

Define then $t'_n$ such that $r(t'_n) = n+2$. Note that $t_n \leq T_n := (n+2)^{\alpha-1}/(ku^l(\alpha-1)) < e^{n(k_n+l_n)}$ since $T_n$ is an explosion time of $r(t)$ with the initial condition $r_0 = \frac{1}{n}$. □

COROLLARY 1. – *Let* $a \in (e^{-2e^{n^3(k_n+l_n)}}, e^{-e^{n^3(k_n+l_n)}})$. *Let*

$$H(\xi_1,\xi_2,\eta_1,\eta_2) = \omega_1\xi_1\eta_1 + \omega_2\xi_2\eta_2 + aF_n + a^2G_n$$

*with* $\|G_n\|_{C^2(B_{2n})} \leq e^{4n(k_n+l_n)}$.

*If* $(\mathcal{L})$ *holds, there exist* $t_n < e^{n(k_n+l_n)}$ *and* $z_n \in \mathbb{R}^4$ *such that* $|z_n| = \frac{1}{n}$ *and* $|\Phi_H^{\frac{t_n}{a}}(z_n)| \geq n+1$.

*Proof.* From $(\mathcal{L})$, there exists $\omega'_1$ such that $|\omega'_1 - \omega_1| < e^{-e^{n^4(k_n+l_n)}}$ and $|k_n\omega'_1 + l_n\omega_2| = 0$. Then, $\{\omega'_1\xi_1\eta_1 + \omega_2\xi_2\eta_2, F_n\} = 0$. Hence if we define

$$H'(\xi_1,\xi_2,\eta_1,\eta_2) = \omega'_1\xi_1\eta_1 + \omega_2\xi_2\eta_2 + aF_n,$$

we get that

$$|\Phi_{H'}^{\frac{t}{a}}(z)| = \left|\Phi^{\frac{t}{a}}_{\omega'_1\xi_1\eta_1+\omega_2\xi_2\eta_2}\left(\Phi^{\frac{t}{a}}_{aF_n}(z)\right)\right| = |\Phi^{\frac{t}{a}}_{aF_n}(z)| = |\Phi^{t}_{F_n}(z)|.$$

Hence, for $t_n < e^{n(k_n+l_n)}$ given by Proposition 1, $|\Phi_{H'}^{\frac{t_n}{a}}(z_n)| = n+2$ and $\Phi_{H'}^{\frac{t}{a}}(B_{\frac{1}{n}}) \subset B_{n+2}$ for every $t \leq t_n$.

Now since $|\omega'_1 - \omega_1| < e^{-e^{n^4(k_n+l_n)}} \leq e^{-n(k_n+l_n)}a^2$, we have that

$$H(\xi_1,\xi_2,\eta_1,\eta_2) = \omega'_1\xi_1\eta_1 + \omega_2\xi_2\eta_2 + aF_n + a^2G'_n$$

with $\|G'_n\|_{C^2(B_{2n})} \leq e^{4n(k_n+l_n)} + 1$. Note also that $\|F_n\|_{C^2(B_{2n})} \leq e^{n(k_n+l_n)}$.

Let $A = e^{4n(k_n+l_n)} + 1$. Observe that for $T = \frac{t_n}{a}$ we have $2a^2ATe^{aAT} = 2aAt_ne^{At_n} \leq \sqrt{a} \ll 1$.

We can thus apply Lemma 1, with $r = \frac{1}{n}$, $R = n+2$, and deduce that for all $s \in [0,t_n]$ and for all $z \in B_{\frac{1}{n}}$ :

$$\left|\Phi_H^{\frac{s}{a}}(z) - \Phi_{H'}^{\frac{s}{a}}(z)\right| \leq 2aAt_ne^{At_n} \leq 1$$

and the conclusion of the corollary thus holds. □

## 5. Proofs of Theorems 1 and 4

*Proof of Theorem 1.* We fix $n \in \mathbb{N}$ and want to show that there exists $\zeta_n \in \mathbb{R}^6$, such that $|\zeta_n| \leq \frac{2}{n}$, and $\tau_n \geq 0$ such that $|\Phi_H^{\tau_n}(\zeta_n)| \geq n$.

Note that for any value $\mathbf{I} \in \mathbb{R}_+$, the set $\{\eta_3 = \bar{\xi}_3; \quad I_3 = \xi_3\bar{\xi}_3 = \mathbf{I}\}$ is invariant under all the flows we consider in this construction.

We fix from here on $\mathbf{I} := e^{-e^{n^3(k_n+l_n)}}$. Next, we are interested in the restriction of the flow of $H$ to the $(x_1,x_2,y_1,y_2)$ space, for $(x_3,y_3)$ such that $x_3^2+y_3^2 = \mathbf{I}$. We will finish if we show that there exists $z_n \in \mathbb{R}^4$, such that $|z_n| \leq \frac{1}{n}$, and $\tau_n \geq 0$ such that $|\Phi_H^{\tau_n}(z_n)| \geq n$. Indeed, for $\zeta_n$ we then pick any $\xi_3$ such that $\xi_3\bar{\xi}_3 = \mathbf{I}$ and let $\zeta_n = (z_n,\xi_3,\bar{\xi}_3)$.

For $j \leq n-1$, let $b_j = \frac{e^{-j(k_j+l_j)}}{k_j\omega_1+l_j\omega_2}$ and define the real Hamiltonians

$$\chi_j(\xi_1,\xi_2,\xi_3,\eta_1,\eta_2,\eta_3) = -ib_j\xi_3\eta_3(\xi_1^{k_j}\xi_2^{l_j} - \eta_1^{k_j}\eta_2^{l_j}).$$

that satisfy

$$\{H_0,\chi_j\} = -e^{-j(k_j+l_j)}\xi_3\eta_3 F_j(\xi_1,\xi_2,\eta_1,\eta_2).$$

Let $\bar{\chi}_{n-1} = \sum_{j\leq n-1}\chi_j$. Since $k_n \geq e^{\frac{1}{|k_j\omega_1+l_j\omega_2|}}$ for any $j \leq n-1$, we have for any $z \in \mathbb{R}^4$ such that $|z| \leq 2n$ and provided $(x_3,y_3)$ are such that $\xi_3\bar{\xi}_3 = \mathbf{I}$

$$\text{(2)} \qquad \left|\Phi^1_{\bar{\chi}_{n-1}}(z) - z\right| \leq e^{-n(k_n+l_n)}$$

and since $H \circ \Phi^1_{\bar{\chi}_{n-1}} = H + \{H,\bar{\chi}_{n-1}\} + \{\{H,\bar{\chi}_{n-1}\},\bar{\chi}_{n-1}\} + \ldots$

$$H \circ \Phi^1_{\bar{\chi}_{n-1}} = H_0 + \sum_{j\geq n} e^{-j(k_j+l_j)}\mathbf{I}F_j(\xi_1,\xi_2,\eta_1,\eta_2) + \mathbf{I}^2F(x,y)$$

with $\|F\|_{C^2(B_{2n})} \leq \frac{1}{|k_{n-1}\omega_1+l_{n-1}\omega_2|^2}$.

Introduce $a := \mathbf{I}e^{-n(k_n+l_n)}$. We then have

$$H \circ \Phi^1_{\bar{\chi}_{n-1}} = H_0 + aF_n(\xi_1,\xi_2,\eta_1,\eta_2) + a^2G(x,y)$$

with $\|G\|_{C^2(B_{2n})} \leq e^{n(k_n+l_n)}$. We can thus apply Corollary 1 and get that there exist $t_n < e^{n(k_n+l_n)}$ and $z_n \in \mathbb{R}^4$ such that $|z_n| = \frac{1}{n}$ and $|\Phi^{\frac{t_n}{a}}_{H\circ\Phi^1_{\bar{\chi}_{n-1}}}(z_n)| \geq n+1$. Finally, (2) then implies that $|\Phi^{\frac{t_n}{a}}_H(z_n)| \geq n$. □

*Proof of Theorem 4.* We keep the same definitions of $\chi_j$ and $\bar{\chi}_j$ as in the above proof and observe that, since since $H - \widetilde{H} = I_3^3(I_1 + I_3I_2)$, it still holds that

$$\widetilde{H} \circ \Phi^1_{\bar{\chi}_{n-1}} = H_0 + \sum_{j\geq n} e^{-j(k_j+l_j)}\mathbf{I}F_j(\xi_1,\xi_2,\eta_1,\eta_2) + \mathbf{I}^2\widetilde{F}(x,y)$$

with $\|\widetilde{F}\|_{C^2(B_{2n})} \leq \frac{1}{|k_{n-1}\omega_1+l_{n-1}\omega_2|^2}$. The rest of the proof is the same as that of Theorem 1. □

## 6. Proofs of Theorems 2 and 5

*Proof of Theorem 2.* We fix $n \in \mathbb{N}$ and want to show that there exists $\zeta_n \in \mathbb{R}^8$, such that $|\zeta_n| \leq \frac{2}{n}$, and $\tau_n \geq 0$ such that $|\Phi^{\tau_n}_H(\zeta_n)| \geq n$.

Note that for any value $\mathbf{I} \in \mathbb{R}_+$, the set $\{\eta_3 = \bar{\xi}_3; \quad I_3 = \xi_3\bar{\xi}_3 = \mathbf{I}\}$ is invariant under all the flows we consider in this construction. The same is true for any value $\mathbf{J} \in \mathbb{R}_+$ and for the set $\{\eta_4 = \bar{\xi}_4; \quad I_4 = \xi_4\bar{\xi}_4 = \mathbf{J}\}$.

We fix from here on $I_4 = \mathbf{J} := I_{4,n}$ and $I_3 = \mathbf{I} := e^{-e^{n^3(k_n+l_n)}}$. Having fixed $I_3$ and $I_4$, we are interested in the restriction of the flow of $H$ to the $(x_1,x_2,y_1,y_2)$ space. We will finish if we show that there exists $z_n \in \mathbb{R}^4$, such that $|z_n| \leq \frac{1}{n}$, and $\tau_n \geq 0$ such that $|\Phi^{\tau_n}_H(z_n)| \geq n$. Indeed, for $\zeta_n$ we then pick any $(\xi_3,\eta_3)$ real such that $\xi_3\eta_3 = \mathbf{I}$, any $(\xi_4,\eta_4)$ real such that $\xi_4\eta_4 = \mathbf{J}$ and let $\zeta_n = (z_n,\xi_3,\eta_3,\xi_4,\eta_4)$.

Let $b_j = \frac{e^{-j(k_j+l_j)}}{k_j(\omega_1+\mathbf{J})+l_j\omega_2}$ and define the real Hamiltonians

$$\chi_j(\xi_1,\xi_2,\xi_3,\eta_1,\eta_2,\eta_3) = -ib_j\xi_3\eta_3(\xi_1^{k_j}\xi_2^{l_j} - \eta_1^{k_j}\eta_2^{l_j}).$$

that satisfy

$$\{H_0,\chi_j\} = -e^{-j(k_j+l_j)}\xi_3\eta_3 F_j(\xi_1,\xi_2,\eta_1,\eta_2).$$

Let $\bar{\chi}_{n-1} = \sum_{j\le n-1}\chi_j$. Since $k_n \ge e^{\frac{1}{|k_j(\omega_1+\mathbf{J})+l_j\omega_2|}}$ for any $j \le n-1$, we have for any $z \in \mathbb{R}^4$ such that $|z| \le 2n$ and provided that $(x_3,y_3,x_4,y_4)$ are such that $\xi_3\bar{\xi}_3 = \mathbf{I}$ and $\xi_4\bar{\xi}_4 = \mathbf{J}$

$$\left|\Phi^1_{\bar{\chi}_{n-1}}(z) - z\right| \le e^{-n(k_n+l_n)} \tag{3}$$

and

$$H\circ\Phi^1_{\bar{\chi}_{n-1}} = H_0 + \sum_{j\ge n} e^{-j(k_j+l_j)}\mathbf{I}F_j(\xi_1,\xi_2,\eta_1,\eta_2) + \mathbf{I}^2F(\xi,\eta)$$

with

$$\|F\|_{C^2(B_{2n})} \le \max_{j\le n-1}\frac{1}{|k_j(\omega_1+\mathbf{J})+l_j\omega_2|^2}.$$

Let $a := \mathbf{I}e^{-n(k_n+l_n)}$. We then have due to ($\mathcal{N}R$)

$$H\circ\Phi^1_{\bar{\chi}_{n-1}} = H_0 + aF_n(\xi_1,\xi_2,\eta_1,\eta_2) + a^2G(\xi,\eta)$$

with $\|G\|_{C^2(B_{2n})} \le e^{4n(k_n+l_n)}$. Due to ($\mathcal{R}$), we can apply Corollary 1 and get that there exist $t_n < e^{n(k_n+l_n)}$ and $z_n \in \mathbb{R}^4$ such that $|z_n| = \frac{1}{n}$ and $|\Phi^{\frac{t_n}{a}}_{H\circ\Phi^1_{\bar{\chi}_{n-1}}}(z_n)| \ge n+1$. Finally, (3) then implies that $|\Phi^{\frac{t_n}{a}}_H(z_n)| \ge n$.

□

*Proof of Theorem 5.* We just replace $\omega_2$ by $\omega_2 + \mathbf{J}^2$ everywhere in the proof of Theorem 2. □

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