WEAK MIXING FOR REPARAMETRIZED LINEAR FLOWS ON THE TORUS.

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Abstract. In this paper, we study the display of weak mixing by reparametrized linear flows on the torus $\mathbf{T}^d$, $d \geq 2$. We show that if the vector of the translation flow is Liouvillian (i.e. well approximated by rationals), then for a residual set of time change functions in the C^∞ topology, the reparametrized flow is weak mixing. If this is not the case, i.e. if the vector of the linear flow is Diophantine, it follows from a result of Kolmogorov on the two torus and its generalization to any dimension by M. Herman, that any C^∞ reparametrization of the flow is C^∞ conjugate to a linear flow. More generally, we give the optimal arithmetical condition on the vector of the translation flow that guarantees the existence of a residual set in the C^r topology of weak mixing reparametrizations. In the real analytic case, the optimal arithmetical condition for the generic display of weak mixing under time change is also given.

As a consequence of our results on reparametrizations of Liouvillian linear flows, we obtain that an aperiodic smooth flow on the two dimensional torus is in general weak mixing. We also deduce the existence on the torus of analytic diffeomorphisms that are rank one and weak mixing.

1. Introduction

1.1. We will denote by $\mathbf{T}^d$ the torus $\mathbf{R}^d/\mathbf{Z}^d$. Assume R_α is a minimal translation on $\mathbf{T}^d$ and consider on $\mathbf{T}^{d+1}$ the *irrational* translation flow:

$$\frac{dx}{dt} = (1, \alpha).$$

This flow is minimal and uniquely ergodic for the Haar measure on $\mathbf{T}^{d+1}$. Given $\phi \in C^r(\mathbf{T}^{d+1}, \mathbf{R}_+^*)$, $r \geq 1$, we define the *reparametrization*, or smooth time change, of this translation flow, with *speed* $\frac{1}{\phi}$, to be the flow given by

$$\frac{d\theta}{dt} = \frac{\alpha}{\phi(\theta, s)}, \qquad \frac{ds}{dt} = \frac{1}{\phi(\theta, s)}.$$

The reparametrized flow is strictly ergodic (minimal and uniquely ergodic) and the invariant measure is $\phi(x)dx$, where dx denotes the Haar measure on $\mathbf{T}^{d+1}$ (see [17]). Considering a Poincaré section, such a flow can be viewed as a suspension flow constructed from R_α on $\mathbf{T}^d$ and a suspension function with the same regularity C^r than ϕ. To obtain results on time change for flows, it is often more convenient to work with special flows (i.e. suspension flows) that are easier to handle, and then transfer the properties to reparametrizations. The exact definition of special flows will be given in Section 2 and the natural correspondence with reparametrization will be explained in Section 3.

Using reparametrizations of irrational flows on the two torus, Shklover, in 1967 [18], proved the existence of analytic flows that are weak mixing but not mixing. He actually shows the following result: *For any real-analytic function on the circle* $\mathbf{T}^1 = \mathbf{R}/\mathbf{Z}$, $f > 0$, *which is not a polynomial, there exists* $\alpha \in \mathbf{R} - \mathbf{Q}$ *such that the special flow constructed over the rotation* R_α *of the circle with a suspension function equal to* f *is weak mixing.* From his proof and the nature of the weak mixing property, it is easy to see that the same would hold for a dense G_δ of $\alpha \in \mathbf{R}$. By the natural correspondence with reparametrizations this yields the existence of analytic reparametrizations of irrational flows on $\mathbf{T}^2$ that are weak mixing. In the work of Shklover the suspension function is explicit, it could be any analytic function that is not a polynomial. Nevertheless, the rotation number α is obtained by going to the limit over rational numbers and the result is still a result of existence. In other words, by his method one can not obtain the explicit condition on α that implies the existence of an analytic (resp. C^∞ or C^r, $r \in \mathbf{R}_+$) reparametrization of $R_{t(1,\alpha)}$ that is weak mixing. One of our aims here is to give these arithmetic conditions. In fact, given α with some arithmetical condition (Diophantine, Liouvillian,...),we will find the optimal class of differentiability, corresponding to this condition, in which there exists a weak mixing reparametrization of $R_{t(1,\alpha)}$. We will actually prove the existence, in this class of differentiability, of a G_δ dense subset of reparametrizations of $R_{t(\alpha,1)}$ that are weak mixing. To obtain such generic dynamics we will use Baire Category techniques inspired by some works of M. Herman and A. Fathi ([10], [5]). In their turn, Fathi and Herman in [5] were inspired by constructions via periodic approximations developped in [1].

Explicit examples of weak mixing analytic special flows over irrational rotations of the circle were constructed by Katok [15]. In these examples the relation between the regularity of the ceiling function and the rate of approximation of α clearly appears. Starting from these constructions of Katok, Iwanik obtained, when $d = 1$, equivalent results to our Theorems 3 and 4 ([13] and [12]), in his study of smooth cocycles over irrational rotations of the circle.

In fact, our proofs show how to construct explicit examples (Cf Remark 2); and from the nature of weak mixing (Cf. Proposition 2) and the fact that adding a trigonometrical polynomial to the time change function does not alter weak mixing, the existence of one weak mixing time change is enough to prove the existence of a residual set of them. Still, we prefered to present a categorical argument that fits perfectly with the nature of weak mixing.

For explicit examples and a direct and geometrical approach to prove mixing features of reparametrizations, one can also refer to [6] where examples of mixing analytic reparametrizations of linear flows are produced on the torus $\mathbf{T}^3$.

Finally, in our proof, we will insist on having the results in any dimension. Indeed, in a paper on invariant tori by symplectic diffeomorphisms [9], M. Herman proves the remarkable fact that, for a generic C^∞ exact symplectic perturbation of a completely integrable symplectic diffeomorphism, the dynamics on the generic invariant tori[1] of the perturbation is very different from what one usually expects, meaning different from the C^∞ diffeomorphisms C^∞ conjugate to translations of $\mathbf{T}^n$ that we have on the tori given by KAM (Kolmogorov Arnold Moser). (See also [20] for a summary of this article). An example of such generic behavior that he finds is weak mixing, and he uses in his proof the result of Shklover for invariant tori of dimension $n = 2$, and the examples of Anosov-Katok [1] for $n \geq 3$. As remarked by Herman, rather than to use Anosov-Katok constructions it is easier to generalize Shklover's result to any dimension and obtain by the same token analytic examples, which is not yet possible by the methods in [1].

1.2. We recall first the definition of weak mixing and we introduce some notations.
-A measure preserving flow $\{T^t\}$ on (L, μ) is said to be weak mixing if for all $f, g \in L^2(L, \mu)$ we have

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \left| \int_L f(T^u x) g(x) d\mu - \int_L f d\mu \int_L g d\mu \right| du = 0. \tag{1}$$

An equivalent definition is that for all measurable sets A and B

$$\mu(T^{-t}A \bigcap B) \longrightarrow \mu(A)\mu(B), \tag{2}$$

when $|t|$ goes to infinity on a set of density one over $\mathbf{R}$.
One can also prove that a flow $\{T^t\}$ is weak mixing if and only if it does not have an eigenfunction, i.e. a measurable function h, not constant, such that $h(T^t x) = e^{i\lambda t} h(x)$, for some eigenvalue $\lambda \in \mathbf{R}$.
For the equivalence between the definitions, we refer to the book of Parry [17].

-For $r \in \mathbf{R}^+ \bigcup \{+\infty\}$, we denote by $C^r(\mathbf{T}^d, \mathbf{R})$ the set of real functions on $\mathbf{R}^d$ of class C^r and $\mathbf{Z}^d$-periodic. The set $C^r(\mathbf{T}^d, \mathbf{R})$ is hence a Baire space for the C^r topology. If there is no ambiguity we might

[1]The generic invariant tori of the perturbation is not a KAM tori but is a C^0 limit of KAM tori. The restriction on the dynamics of such a tori is that they should be in the closure of the C^∞ diffeomorphisms C^∞ conjugate to translations. That is the reason why Herman investigates in [9] for generic properties in the closure of diffeomorphisms C^∞ conjugate to translations. Reparametrized irrational flows are an excellent tool to this end.

just denote these spaces by C^r and we will denote their norms by $||.||_r$. Furthermore, if we denote by $C^r(\mathbf{T}^d, \mathbf{R}^*_+)$ the set of strictly positive functions in $C^r(\mathbf{T}^d, \mathbf{R})$, it is an open set for the C^0 norm, and in particular for the C^r norm, therefore it is a Baire space.
-For $k = (k_1, ..., k_d) \in \mathbf{Z}^d$ and $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbf{R}^d$ we will use the following notations

$$< k, \alpha >= \sum_{i=1}^{d} k_i \alpha_i,$$

$$||k|| = \sup_i |k_i|,$$

$$||| < k, \alpha > ||| = \inf_{p \in \mathbf{Z}} | < k, \alpha > +p|.$$

-Given a vector $\alpha \in \mathbf{R}^d$, we say it is β *Diophantine* if there exists a constant $C > 0$ such that for every $k \in \mathbf{Z}^d - \{0\}$ we have :

$$||| < k, \alpha > ||| \geq \frac{C}{||k||^{d+\beta}}.$$

In this case we say that α is Diophantine or satisfies a Diophantine condition. If there is a constant $C > 0$ such that α verifies the above condition with $\beta = 0$ we say that α is of constant type. For commodity in stating the results we will allow the notation "0 Diophantine" for such vectors.
-We recall the arithmetic decomposition of $\mathbf{R}^d = CD \sqcup L \sqcup Q$, where $\mathbf{Q}$ designates the vectors with rationally dependent coordinates, CD the Diophantine vectors, and L the Liouvillian vectors (vectors that do not satisfy any Diophantine condition and are not in Q). Again for commodity we will use the notation α "is not $+\infty$ Diophantine" for Liouvillian vectors.

Let $d \in \mathbf{N}$, $d \geq 1$. With the above notations, we show the following:

Theorem 1. *Let $\beta \in \mathbf{R}^+ \bigcup \{+\infty\}$. Assume the irrational vector α in $\mathbf{R}^d$ is not β Diophantine. Then, for a dense G_δ of ϕ in $C^{\beta+d}(\mathbf{T}^{d+1}, \mathbf{R}^*_+)$, the reparametrization of the translation flow $R_{t(1,\alpha)}$, with speed $\frac{1}{\phi}$ is weak mixing (for the unique invariant measure).*

The result is the best one can have as shown by the following proposition due to Kolmogorov (in dimension 2, i.e. $d = 1$, [16]) and generalized to any dimension by M. Herman in [11] :

Proposition 1. *If $\alpha \in \mathbf{T}^d$ is β Diophantine and the function ϕ is C^r for some $r > \beta + d$, the reparametrization of the flow $R_{t(1,\alpha)}$, with speed $\frac{1}{\phi}$ is C^0 conjugate to a translation flow on $\mathbf{T}^{d+1}$.*

In the case of C^∞ reparametrizations ($\beta = +\infty$) the dichotomy is therefore complete between weak mixing and conjugation to translation flows and we have the following:

Corollary 1. *Let R_α be a minimal translation on $\mathbf{T}^d$. Then either one of two possibilities hold:*
(i) The vector α is Diophantine and for any $\phi \in C^\infty(\mathbf{T}^{d+1}, \mathbf{R}_+^*)$, the reparametrization of the flow $R_{t(1,\alpha)}$, with speed $\frac{1}{\phi}$ is C^∞ conjugate to a translation flow on $\mathbf{T}^{d+1}$.
(ii) The vector α is Liouvillian and for a dense G_δ of $\phi \in C^\infty(\mathbf{T}^{d+1}, \mathbf{R}_+^*)$, the reparametrization of the flow $R_{t(1,\alpha)}$, with speed $\frac{1}{\phi}$ is weak mixing (for the unique invariant measure).

For analytic reparametrizations, we have analogous results. First define for $h > 0$ the set $C_h^w(\mathbf{T}^d, \mathbf{R}_+^*)$ of strictly positive real functions on $\mathbf{R}^d$, $\mathbf{Z}^d$-periodic and that can be extended to holomorphic functions on $B_h = \{z \in \mathbf{C}^d, \sup_i |\mathrm{Im} z_i| < h\}$. With the topology of the uniform convergence on compact sets, $C_h^w(\mathbf{T}^d, \mathbf{R}_+^*)$ is a Baire space and we have

Theorem 2. *Given an irrational vector $\alpha \in \mathbf{R}^d$, if there exist $\delta > 0$ and a sequence of $k_n \in \mathbf{Z}^d$ such that*

$$||| < k_n, \alpha > ||| \leq e^{-\delta ||k_n||},$$

then, for a dense G_δ subset of functions ϕ in the set of $C^w_{\frac{\delta}{2\pi}}$ strictly positive functions, the reparametrization of the translation flow $R_{t(1,\alpha)}$, with speed $\frac{1}{\phi}$ is weak mixing (for the unique invariant measure).

Here again the result is optimal and we have a perfect dichotomy, because if α satisfies

$$\lim_{||k|| \to \infty} \frac{-\mathrm{Log}||| < k, \alpha > |||}{||k||} = 0, \tag{3}$$

then, for any real analytic function $\phi > 0$, the reparametrization of $R_{t\alpha}$ by ϕ is analytically conjugate to a translation flow failing therefore to be weak mixing (the proof is based on the linearized equation and is analogous to the proof of Proposition 1, Cf. §2).
Remark: The condition that implies conjugacy to a translation flow is weaker than the Brjuno condition.

This paper contains four sections. Sections 2 and 3 are dedicated to the proof of the theorems 1 and 2. The essential results are obtained in Section 2 where we prove analogous theorems for special flows.

As we said before, it is possible to obtain weak mixing as defined in (2), with a direct and a geometrical method, but it appears easier to prove the equivalent spectral characterization of weak mixing, i.e. the non-existence of eigenfunctions. In Lemma 2, we state a central criterion on the ceiling-function Birkhoff-sums that guarantees the non-existence of eigenfunctions for the special flow. Then we will use Baire category arguments and the stationary phase method to study when this criterion is fulfilled in relation with the arithmetics of α and the regularity of the ceiling function φ. In Section 3 we derive the results for reparametrizations.

In the last section we show that the general aperiodic flow of class C^∞ on the two-torus is weak mixing.

Finally, we have also added an appendix due to Michael Herman on the possibility of using time t_0 maps of the flows considered above to obtain analytic diffeomorphisms on the torus, combining the weak mixing property with the rank one property.

2. Weak Mixing for Special Flows.

We recall the definition of a special flow: Given a Lebesgue space L, a measure preserving transformation T on L and an integrable strictly positive real function defined on L we define the special flow over T and under the *ceiling function* φ by inducing on $L \times \mathbf{R}/\sim$, where $\sim$ is the identification $(x, s + \varphi(x)) \sim (T(x), s)$,the action of

$$\begin{array}{ccc} L \times \mathbf{R} & \rightarrow & L \times \mathbf{R} \\ (x, s) & \rightarrow & (x, s+t). \end{array}$$

If T preserves a unique probability measure μ then the special flow will preserve a unique probability measure that is the normalized product measure of μ on the base and the Lebesgue measure on the fibers.

In this section α is an irrational vector in $\mathbf{R}^d$, $d \geq 1$, and we consider special flows constructed over the translation R_α of the torus $\mathbf{T}^d$. We will prove the following:

Theorem 3. *If the vector $\alpha \in \mathbf{R}^d$ is not β Diophantine then there exists a dense G_δ for the $C^{\beta+d}$ topology, of functions $\varphi \in C^{\beta+d}(\mathbf{T}^d, \mathbf{R}_+^*)$, such that the special flow constructed over R_α with the ceiling function φ is weak mixing.*

In the analytic case we have:

Theorem 4. *If the vector $\alpha \in \mathbf{R}^d$ is such that there exist $\delta > 0$ and a sequence of $k_n \in \mathbf{Z}^d$ satisfying*

$$||| < k_n, \alpha > ||| \leq e^{-\delta ||k_n||},$$

then, for a dense G_δ of functions φ in $C^w_{\frac{\delta}{2\pi}}(\mathbf{T}^d, \mathbf{R}_+^*)$, the special flow constructed over R_α with the ceiling function φ is weak mixing.

Optimality. Like the theorems on reparametrization, Theorems 3 and 4 are optimal, and this is easy to check. Indeed, if the cohomological equation

$$\varphi(\theta) - \int_{\mathbf{T}^d} \varphi = \chi(\theta) - \chi(\theta + \alpha), \tag{4}$$

has a continuous solution χ, the special flow $\{R_\alpha, \varphi\}$ is C^0 conjugate to a special flow over R_α with a constant ceiling function, i.e. a translation flow on $\mathbf{T}^{d+1}$ (Cf. [16] or [14] for a simple proof). Next, the study of the linearized equation (4) with Fourier techniques (see [10], Annexe §8, p. 229) confirms[2] the existence of a continuous solution χ when $\alpha \in \mathbf{T}^d$ is β Diophantine and the function φ is C^r with $r > \beta + d$. The condition on the vector α that implies *apriori* the existence of solutions in the analytic case is the one stated in (3). □

We start the proof of Theorem 3 with a very general fact about weak mixing.

Proposition 2. *Given $\alpha \in \mathbf{R}^d$, in every space $C^r(\mathbf{T}^d, \mathbf{R}_+^*)$, $C^\infty(\mathbf{T}^d, \mathbf{R}_+^*)$ or $C_h^w(\mathbf{T}^d, \mathbf{R}_+^*)$, the set of strictly positive functions φ for which the special flow $\{R_\alpha, \varphi\}$ is weak mixing is a G_δ set.*

We can find a proof of this result in the introductory book to ergodic theory of P. Halmos [7], in his proof of the second category theorem. □

Considering Proposition 2, to prove Theorem 3, it is enough to show that the set of strictly positive functions φ for which the flow is weak mixing is dense in the set $C^{\beta+d}(\mathbf{T}^d, \mathbf{R}_+^*)$.
Still, the way we prove density, we obtain that the set contains a G_δ dense subset. We then use Proposition 2 to conclude that it is a G_δ dense.

First we state a classical general lemma on weak mixing for ergodic special flows, the proof of which can be found in [2] for example. In this lemma $\{T^t\}$ will be the special flow constructed from an ergodic automorphisim T of a Lebesgue space L and from a function $f > 0$.

Lemma 1. *The flow $\{T^t\}$ is weak mixing if and only if, for any λ in $\mathbf{R}^*$ the equation*

$$h(T(x)) = e^{i\lambda f(x)} h(x) \tag{5}$$

does not have a non zero measurable solution h.

Now we return to the special flow constructed over the rotation automorphism R_α of the torus $\mathbf{T}^d$ with the ceiling function φ and from the Lemma we just stated we obtain a criterion, on the Birkhoff sums of φ, which guarantees weak mixing for the special flow. We will always denote the Birkhoff sums by

[2]The exact result as stated in [10] is the following: Assume α is β Diophantine and let $\theta = d + \beta$ (when $\beta = 0$ take $\theta = d + \epsilon$). Then if $r > \theta$, and $\varphi \in C^r(\mathbf{T}^d)$, there exists $\psi \in C^{r-\theta}(\mathbf{T}^d)$ satisfying the equation (4) if $r - \theta$ is not integer; if $r - \theta$ is integer then the solution ψ is $C^{r-\theta-1}$ and "smooth in the sense of Zygmund". In both cases it is continuous.

$$\varphi_m(\theta) = \sum_{k=0}^{m-1} \varphi(\theta + k\alpha).$$

Lemma 2 (Criterion for weak mixing)**.** *If for every g in $L^2(\mathbf{T}^d, \mathbf{C})$, and for every λ in $\mathbf{R}^*$, we have*

$$\inf_{m\in\mathbf{N}} \left| \int_{\mathbf{T}^d} e^{i\lambda\varphi_m(\theta)} g(\theta) d\theta \right| = 0,$$

then the flow over R_α with the ceiling function φ is weak mixing.

Proof of Lemma 2. Since R_α is ergodic any measurable solution of (5) has constant modulus and is therefore in L^2. Suppose h is a solution of (5), we have

$$h(\theta + m\alpha) = e^{i\lambda\varphi_m(\theta)} h(\theta)$$

and for any $k \in \mathbf{Z}^d$

$$\begin{aligned} e^{-i2\pi m<k,\alpha>} \int_{\mathbf{T}^d} h(\theta) e^{i2\pi<k,\theta>} d\theta &= \int_{\mathbf{T}^d} h(\theta + m\alpha) e^{i2\pi<k,\theta>} d\theta, \\ &= \int_{\mathbf{T}^d} e^{i\lambda\varphi_m(\theta)} h(\theta) e^{i2\pi<k,\theta>} d\theta. \end{aligned}$$

Should the condition of the Lemma be satisfied we would have

$$\inf_{m\in\mathbf{N}} \left| \int_{\mathbf{T}^d} e^{i\lambda\varphi_m(\theta)} h(\theta) e^{i2\pi<k,\theta>} d\theta \right| = 0.$$

Which implies that $\int_{\mathbf{T}^d} h(\theta) e^{i2\pi<k,\theta>} d\theta = 0$ for any $k \in \mathbf{Z}^d$, hence h is zero. □

Remark: It is enough to check the condition of the lemma for the characters $\chi_j(\theta) = e^{i2\pi<j,\theta>}$ that form a basis of L^2.

Given the irrational vector $\alpha \in \mathbf{T}^d$ and a positive number β, that can be zero or infinite define

$$\mathcal{A}(\alpha) = \left\{\varphi \in C^{\beta+d}(\mathbf{T}^d, \mathbf{R}) | \forall\lambda \in \mathbf{R}^*, \forall j \in \mathbf{N}, \inf_{m\in\mathbf{N}} \left| \int_{\mathbf{T}^d} e^{i\lambda\varphi_m(\theta)} \chi_j(\theta) d\theta \right| = 0 \right\}.$$

In light of Lemma 2 density in Theorem 3 will follow from

Proposition 3. *If α is not β Diophantine then $\mathcal{A}(\alpha)$ is a G_δ dense subset of $C^{\beta+d}(\mathbf{T}^d, \mathbf{R})$.*

Proof of Proposition 3. For $j, p, k \in \mathbf{N}^3$, define

$$\mathcal{A}_{(j,p,k)}(\alpha) = \{\varphi \in C^{\beta+d}(\mathbf{T}^d, \mathbf{R}) | \forall\lambda \in [\frac{1}{p}, p], \inf_{m\in\mathbf{N}} \left| \int_{\mathbf{T}^d} e^{i\lambda\varphi_m(\theta)} \chi_j(\theta) d\theta \right| < \frac{1}{k}\}.$$

We have

$$\mathcal{A}(\alpha) = \bigcap_{j\in\mathbf{N}} \bigcap_{p\in\mathbf{N}} \bigcap_{k\in\mathbf{N}} \mathcal{A}_{(j,p,k)}(\alpha).$$

The set $\mathcal{A}_{(j,p,k)}(\alpha)$ is obviously open (we took $\lambda \in [\frac{1}{p}, p]$ for this purpose), and we just have to prove that it is dense, which will be the concern of the rest of the section.

First we write the fact that α is not β Diophantine:
There exist a sequence $k_n \in \mathbf{Z}^d$ with $\lim_{n\to\infty} ||k_n|| = +\infty$ and a sequence d_n of positive real numbers with $\lim_{n\to\infty} d_n = +\infty$, such that

$$|||<k_n,\alpha>||| < \frac{1}{d_n}\frac{1}{||k_n||^{d+\beta}}. \tag{6}$$

(If $\beta = +\infty$,i.e. α is a Liouvillian vector, we put $||k_n||^n$ in (6) instead of $||k_n||^{d+\beta}$.)

We introduce now a sequence of real functions on $\mathbf{T}^d$

$$\psi^{(n)}(\theta) = \frac{\cos(2\pi <k_n,\theta>)}{\sqrt{d_n}||k_n||^{d+\beta}}.$$

(If $\beta = +\infty$, we take n instead of $d+\beta$.)
Finally let m_n be the integer part of $d_n^{\frac{3}{4}}||k_n||^{d+\beta}$ (in the sequel, the essential fact about m_n will be that $\sqrt{d_n}||k_n||^{d+\beta} \ll m_n \ll |||<k_n,\alpha>|||^{-1}$).

We will need the following lemmas (the first one is direct from the definition of $\psi^{(n)}$)

Lemma 3. *If $||.||_{\beta+d}$ denotes the $C^{\beta+d}$ norm, then we have*

$$||\psi^{(n)}||_{d+\beta} \underset{n\to\infty}{\longrightarrow} 0.$$

(If $\beta = +\infty$, we take n instead of $d+\beta$.)

From our choice of m_n,

Lemma 4. *We have*

$$\psi^{(n)}_{m_n}(\theta) = X_n \cos\left(2\pi <k_n,\theta> + \phi_n\right),$$

where X_n is a real positive number satisfying $X_n \geq \frac{1}{\pi} d_n^{\frac{1}{4}}$.

Proof of Lemma 4. We have

$$\begin{aligned}
\psi^{(n)}_{m_n}(\theta) &= \mathrm{Re}\left(\frac{1-e^{i2\pi m_n<k_n,\alpha>}}{1-e^{i2\pi<k_n,\alpha>}}\frac{e^{i2\pi<k_n,\theta>}}{\sqrt{d_n}||k_n||^{d+\beta}}\right), \\
&= \mathrm{Re}\left(e^{i\pi(m_n-1)<k_n,\alpha>}\frac{\sin\left(\pi m_n <k_n,\alpha>\right)}{\sin\left(\pi <k_n,\alpha>\right)}\frac{e^{i2\pi<k_n,\theta>}}{\sqrt{d_n}||k_n||^{d+\beta}}\right), \\
&= X_n \cos\left(2\pi <k_n,\theta> + \phi_n\right),
\end{aligned}$$

if we let

$$X_n = \frac{\sin(\pi m_n < k_n, \alpha >)}{\sin(\pi < k_n, \alpha >)} \frac{1}{\sqrt{d_n}||k_n||^{d+\beta}}.$$

Since $m_n ||| < k_n, \alpha > ||| \leq d_n{}^{-\frac{1}{4}}$, we have

$$\frac{\sin(\pi m_n < k_n, \alpha >)}{\sin(\pi < k_n, \alpha >)} \geq \frac{2}{\pi} m_n,$$

hence

$$X_n \geq \frac{2}{\pi} \frac{m_n}{\sqrt{d_n}||k_n||^{d+\beta}} \geq \frac{1}{\pi} d_n{}^{\frac{1}{4}}.$$

□

Comment: The functions $\psi^{(n)}$ we introduced are "α" periodic (in the sense that $\psi^{(n)}(x + j\alpha) \sim \psi^{(n)}(x)$) as long as the number of iteration of R_α, is such that $j ||| < k_n, \alpha > ||| = o(1)$. Consequently, the Birkhoff sums of $\psi^{(n)}$ up to m_n will look like $m_n \psi^{(n)}$. If m_n is moreover such that $\frac{m_n}{D_n} \to \infty$, where D_n is the denominator in the expression of $\psi^{(n)}$, then $\psi^{(n)}_{m_n}$ will have great oscillations. This essential phenomenon will allow us to check the criterion of Lemma 2 by means of the stationnary phase method. The idea next, is to perturb a given function by adding to it a function $\psi^{(n)}$ that will "produce mixing" at time m_n. Here we see that the less α is Diophantine, the larger we can take m_n and D_n, and the smaller (i.e. more differentiable) will be the perturbation $\psi^{(n)}$ that we have to add to the given ceiling function to produce some mixing.

From the estimation in Lemma 4 and a basic fact on stationnary phase, we have the following fundamental result :

Lemma 5. *For every $g \in C^\infty(\mathbf{T}, \mathbf{R})$, and for every $\lambda > 0$, we have as n goes to infinity*

$$\int_{\mathbf{T}^d} e^{i\lambda \psi^{(n)}_{m_n}(\theta)} g(\theta) d\theta = O(\frac{1}{\sqrt{\lambda X_n}}).$$

Proof of Lemma 5. From a lemma on stationnary phase[3] that can

[3]The exact statement of Dieudonné is the following: Assume g and h are two functions in $C^\infty([a,b], \mathbf{R})$, such that h' has only one zero $c \in]a,b[$ with $g(c) \neq 0$ and $h"(c) \neq 0$). Then for any t large enough, if $h"(c) > 0$

$$\int_a^b g(x) e^{ith(x)} dx = \left(\frac{\pi}{2th"(c)} \right)^{\frac{1}{2}} g(c) e^{ith(c) + \frac{i\pi}{4}} + O(\frac{1}{t}).$$

If $h"(c) < 0$ the same is true if we change $\frac{i\pi}{4}$ to $-\frac{i\pi}{4}$ and $h"(c)$ to $-h"(c)$ in the above expression. If $g(c) = 0$ the same equality holds.

be found in [4] (Chapter IV, paragraph (4.8)) we know that for every g in $C^\infty(\mathbf{T},\mathbf{R})$ we have for large $|X_n|$

$$\int_{\mathbf{T}^1} e^{i\lambda X_n \cos 2\pi x} g(x)dx \leq \frac{C(g)}{\sqrt{\lambda X_n}}$$

where the constant $C(g)$ only depends (as n goes to infinity) on the C^0 norm of g, $||g||_0$. Now k_n is a vector different from zero in $\mathbf{Z}^d$, and we can suppose the first coordinate k_n^1 different from zero. From the inequality above we have when n is large enough

$$\int_{\mathbf{T}^1} e^{i\lambda X_n \cos(2\pi k_n^1\theta_1 + u_n)} g(\theta_1,\theta)d\theta_1 \leq \frac{C(g)}{\sqrt{\lambda X_n}},$$

where $\theta = (\theta_2, ..., \theta_d)$ and $u_n = 2\pi(< k_n, \theta > -k_n^1\theta_1) + \phi_n$. Integrating along the other variables we obtain the result. □

Remark: If g is a C^∞ function, and g_n is a sequence converging to g in norm L^1, then we have from the above lemma

$$\lim_{n\to\infty} \int_{\mathbf{T}^d} e^{i\lambda\psi_{m_n}^{(n)}(\theta)} g_n(\theta)d\theta \underset{n\to\infty}{\longrightarrow} 0.$$

Now, we are ready to prove that $\mathcal{A}_{(j,p,k)}(\alpha)$ is dense in $C^{\beta+d}(\mathbf{T}^d,\mathbf{R})$.

Fix a function $\varphi \in C^{\beta+d}(\mathbf{T}^d,\mathbf{R})$ and take $\epsilon > 0$ arbitrarily small. We can assume the integral of φ is zero and arbitrarily close to it find a C^∞ coboundary, i.e. a function of the type $\widehat{\varphi} = \eta \circ R_\alpha - \eta$, where η is of class C^∞ such that

$$||\varphi - \widehat{\varphi}||_{d+\beta} < \epsilon.$$

Indeed, any trigonometric polynomial with zero integral is a coboundary. Then for $n \in \mathbf{N}$ we define

$$\widehat{\varphi}^{(n)} = \widehat{\varphi} + \psi^{(n)}.$$

Now, if we take n large enough, we will have, from Lemma 3:

$$||\varphi - \widehat{\varphi}^{(n)}||_{d+\beta} < 2\epsilon. \tag{7}$$

On the other hand, considering a subsequence if necessary, we can assume that $\widehat{\varphi}_{m_n}$ converges to a fixed cocycle $\eta \circ R_\beta - \eta$, then, applying Lemma 5 and the remark that follows we have

$$\int_{\mathbf{T}^d} e^{i\lambda\widehat{\varphi}_{m_n}^{(n)}(\theta)}\chi_j(\theta)d\theta = \int_{\mathbf{T}^d} e^{i\lambda\psi_{m_n}^{(n)}(\theta)} e^{i\lambda\widehat{\varphi}_{m_n}(\theta)}\chi_j(\theta)d\theta \underset{n\to\infty}{\longrightarrow} 0.$$

(We used Lemma 5 for $g(\theta) = e^{i\lambda(\eta\circ R_\beta(\theta)-\eta(\theta))}\chi_j(\theta)$, and the fact that $e^{i\lambda\widehat{\varphi}_{m_n}(\theta)}\chi_j(\theta) \to e^{i\lambda(\eta\circ R_\beta(\theta)-\eta(\theta))}\chi_j(\theta)$ uniformly when n goes to infinity.)

So when n is large enough $\widehat{\varphi}^{(n)}$ is in $\mathcal{A}_{(j,p,k)}(\alpha)$. The real number ϵ being arbitrarily small this completes the proof of density of $\mathcal{A}_{(j,p,k)}(\alpha)$.

Remark 1. For the C^∞ case, i.e. β is infinite, we take $\varphi \in C^\infty$ and for an arbitrary $l \in \mathbf{N}$, with the same arguments used above, we find a function C^l close to φ and in $\mathcal{A}_{(j,p,k)}(\alpha)$.

Proposition 3 is proved. □

Proof of Theorem 3. In light of Lemma Proposition implies that the set of strictly positive functions φ for which the flow is weak mixing contains a G_δ dense subset. Together with Proposition 2, this proves Theorem 3. □

Remark 2. In Lemma 5, we could establish a more precise and uniform bound on the integral that only depends on the C^1 norm of g. This would enable us to check the validity of Lemma 2 to the function $\psi = \sum \psi^{(n)}$ and obtain thus an explicit ceiling function under which the special flow is weak mixing. Moreover, if we add to this function polynomials, the criterion in Lemma 2 remains valid and density hence follows .

Before we close this section, we want to show how the proof applies to the analytic case.

Proof of Theorem 4. Take for $\psi^{(n)}$ the functions

$$\psi^{(n)}(\theta) = Re\left(\frac{e^{i2\pi<k_n,\theta>}}{n^{-2}e^{\delta||k_n||}}\right).$$

We have

$$||\psi^{(n)}|| \underset{n\to\infty}{\longrightarrow} 0,$$

for the norm on $C^w_{\frac{\delta}{2\pi}}$ of uniform convergence on compact sets in the open band $B_{\frac{\delta}{2\pi}}$.

Now, for m_n we take the integer part of $n^{-1}e^{\delta||k_n||}$, which obviously satisfies

$$m_n||| < k_n, \alpha > ||| \underset{n\to\infty}{\rightarrow} 0,$$

while

$$\frac{m_n}{n^{-2}e^{\delta||k_n||}} \underset{n\to\infty}{\longrightarrow} \infty.$$

This leads to the proof, as in Theorem 3. □

3. Reparametrization of Translation Flows.

In this section we consider reparametrizations of irrational flows, with speed $\frac{1}{\phi}$:

$$\frac{d\theta}{dt} = \frac{\alpha}{\phi(\theta,s)}, \qquad \frac{ds}{dt} = \frac{1}{\phi(\theta,s)},$$

where $\phi > 0$ is of class at least C^1, R_α is a minimal translation on $\mathbf{T}^d$; $\theta \in \mathbf{T}^d$ and $s \in \mathbf{R}$.

Considering a Poincaré section, this flow can be viewed as a special flow constructed from R_α on $\mathbf{T}^d$ and from the ceiling function given by

$$R(\phi)(\theta) = \varphi(\theta) = \int_0^1 \phi(\theta + \xi\alpha, \xi) d\xi. \tag{8}$$

Remark: If we consider the section $(\theta, 0)$, this formula just translates the fact that the return time to that section is the integral of one over the speed along the fibres.

From this correspondence between reparametrizations and special flow, we can give a proof to our main theorems using the results of Section 2 .

Proof of Theorem 1. Fix the vector α not β-Diophantine and consider the irrational flow $R_{t(1,\alpha)}$. It is a classical fact, as in Proposition 2 of Section 2, that the set of $\phi > 0$ for which the reparametrization of $R_{t(1,\alpha)}$ with speed $\frac{1}{\phi}$ is weak mixing is a G_δ, in particular for the topology $C^{\beta+d}$. We want to prove density. We introduce

$$\mathcal{B}(\alpha) = \{\phi \in C^{\beta+d}(\mathbf{T}^{d+1}, \mathbf{R}_+^*) \mid R(\phi) \in \mathcal{A}(\alpha)\},$$

and

$$\mathcal{B}_{(j,p,k)}(\alpha) = \{\phi \in C^{\beta+d}(\mathbf{T}^{d+1}, \mathbf{R}_+^*) \mid R(\phi) \in \mathcal{A}_{(j,p,k)}(\alpha)\}.$$

The function $R(\phi)$ being the ceiling function obtained from ϕ by (8).

As in the proof of Theorem 3, we need only to proof that $\mathcal{B}_{(j,p,k)}(\alpha)$ is $C^{\beta+d}$ dense in $C^{\beta+d}(\mathbf{T}^{d+1}, \mathbf{R}_+^*)$. We fix ϕ and we take $\widehat{\phi}$ a trigonometrical polynomial close to ϕ. Obviously, $\widehat{\varphi} = R(\widehat{\phi})$ (given by (8)) will be a trigonometrical polynomial and, by the proof of Theorem 2, for n sufficiently large, $\widehat{\varphi} + \psi^{(n)} \in \mathcal{A}_{(j,p,k)}(\alpha)$
Remember that

$$\psi^{(n)}(\theta) = \mathrm{Re}\left(\frac{e^{i2\pi<k_n,\theta>}}{\sqrt{d_n}||k_n||^{d+\beta}}\right).$$

And define

$$\Psi^{(n)}(\theta, s) = \frac{i2\pi\left(<k_n,\alpha> + l_n\right)}{e^{i2\pi(<k_n,\alpha>+l_n)} - 1}\mathrm{Re}\left(\frac{e^{i2\pi<k_n,\theta>}e^{i2\pi l_n s}}{\sqrt{d_n}||k_n||^{d+\beta}}\right),$$

where l_n is chosen to be the closest integer to $-<k_n, \alpha>$. A straightforward computation implies

$$R\left(\Psi^{(n)}\right) = \psi^{(n)}.$$

We still have to check one lemma:

Lemma 6. *If* $||.||_{d+\beta}$ *designates the* $C^{\beta+d}$ *norm, one has*

$$||\Psi^{(n)}||_{d+\beta} \underset{n\to\infty}{\longrightarrow} 0.$$

Proof of Lemma 6. The choice of l_n such that $| < k_n, \alpha > +l_n| < \frac{1}{2}$ implies

$$\left| \frac{i2\pi\left(< k_n, \alpha > +l_n\right)}{e^{i2\pi(<k_n,\alpha>+l_n)} - 1} \right| < \frac{\pi}{2}.$$

Since $|l_n| \leq ||k_n||(\sum |\alpha_j|) + 1$, $\Psi^{(n)}$, just like $\psi^{(n)}$, goes to 0 in the $C^{d+\beta}$ topology when n goes to infinity. □

Now, because the correspondence in (8), $\phi \to R(\phi)$, is linear we will have $R(\widehat{\phi} + \Psi^{(n)}) = \widehat{\varphi} + \psi^{(n)} \in \mathcal{A}_{(j,p,k)}(\alpha)$, or equivalently $\widehat{\phi} + \Psi^{(n)} \in \mathcal{B}_{(j,p,k)}(\alpha)$. By Lemma 6 and the choice of $\widehat{\phi}$ this last function is close to ϕ which proves density of $\mathcal{B}_{(j,p,k)}(\alpha)$. Theorem 1 on reparametrization is thus proved (β finite). For the C^∞ and analytic case we make the same modifications as in the precedent section in our choice of the $\Psi^{(n)}$'s. □

4. Aperiodic flows on the two torus

In this section we consider smooth flows on the two torus given by the system of differential equations

$$\frac{du}{dt} = A(u,v), \qquad \frac{dv}{dt} = B(u,v), \tag{9}$$

where A and B are of class C^∞.

When the flow in (9) has no singularities and no closed orbits we will say it is *aperiodic* (although this is not the standard definition of apriodic flows). We can define the corresponding set of smooth vector fields:

$$\mathcal{E} = \{(A,B) \in C^\infty(\mathbf{T}^2,\mathbf{R})\times C^\infty(\mathbf{T}^2,\mathbf{R}) \text{ such that the flow in (9) is aperiodic.}\}$$

It follows from standard arguments that the set $\mathcal{E}$ is a G_δ in $C^\infty(\mathbf{T}^2,\mathbf{R})\times C^\infty(\mathbf{T}^2,\mathbf{R})$, therefore it is a Baire space (for the C^∞ topology, and it makes sense to state a "general" or more precisely generic property on it with regard to the Baire category. The theorem we want to prove is the following:

Theorem 5. *For a dense* G_δ *of* (A,B) *in* $\mathcal{E}$*, the flow given by* (9) *is weak mixing for its unique invariant measure.*

Proof of Theorem 5. *Unique ergodicity:* For aperiodic flows, a result of Kneser and Siegel (see the paper of Siegel [19]) states that there exists on the torus, a closed non-self-intersecting curve Γ of class C^∞ transversal to the flow such that any point of Γ comes back to Γ in a finite time and all trajectories of the flow intersect Γ. Therefore the flow can be viewed as a special flow over Γ. It follows from smoothness of Γ and of the flow that the return map T on the basis and the return time function f (the ceiling function) associated to this special flow are of class C^∞. We call α the rotation number of T on Γ. By our assumption on the flow, α has to be irrational, therefore T is uniquely ergodic and so is the flow.

Weak mixing: Reasoning as in the previous sections we only need to prove that the couples (A, B) for which the flow (9) is weak mixing are dense in $\mathcal{E}$; the G_δ property being obtained from standard remarks on weak mixing. Pick $(A_0, B_0) \in \mathcal{E}$, and let Γ_0 be a Siegel curve for the corresponding flow that we view as a special flow over Γ_0, denoted by $\{T_0, f_0\}$. To this end, we will need to consider special flows above Γ_0 built with a diffeomorphism T and a ceiling function f that are C^∞ perturbations of T_0 and f_0. These special flows are clearly flows on the torus that are C^∞ close to the special flow $\{T_0, f_0\}$, i.e. to the flow given by (A_0, B_0). Therefore the flows $\{T, f\}$ correspond to vector fields (A, B) that are C^∞ perturbations of (A_0, B_0).

We are ready now to prove density of the weak mixing flows. First of all we perturb T_0 in the C^∞ topology into a diffeomorphism T on Γ_0 that has a Diophantine rotation number α. By Herman's theorem [10], the map T_0 is C^∞-conjugated to R_α. After conjugacy, we obtain a special flow over R_α with a smooth ceiling function. We can then perturb R_α into a Liouvillian rotation, and finally use Theorem 3 to obtain a weak mixing flow after smooth perturbation of the ceiling function. □

Appendix: A remark on the existence of rank one and weak mixing analytic diffeomorphisms of the torus.

Let (X, T, μ) be a dynamical system. We say that T is of *rank one* if there is a sequence of Rokhlin towers whose levels generate the σ-algebra of all measurable sets.

For instance, any minimal translation on the torus is of rank one [3]. Later on, we will be using this fact.

It is known from [1] that weak mixing rank one transformations are abundant in the neighbourhood of translations in the C^∞ topology. What about the analytic case ?

We state here the following fact communicated to us by M. Herman [8]:

Theorem 6. *Assume φ is a given real analytic strictly positive function on the circle, that is not a polynomial. Fix $t_0 \in \mathbf{R}$ such that $t_0 / \int_{\mathbf{T}} \varphi(x)dx \in \mathbf{R} - \mathbf{Q}$. Then, there exists a G_δ dense susbet of $\mathbf{R}$, $\mathcal{W}_\varphi$, such that: for any $\alpha \in \mathcal{W}_\varphi$, the time map T^{t_0} of the special flow constructed over R_α with the ceiling function φ, is of rank one and weak mixing.*

Proof of Theorem 6. The function φ is fixed. Let $W_\varphi \subset \mathbf{R}$ be the set of numbers α for which the time T^{t_0} of the special flow constructed over R_α and under the ceiling function φ is weak mixing. We claim that W_φ is a dense G_δ of $\mathbf{R}$. Like in the previous sections, we only need to prove that W_φ is dense. By the result of Shklover [18], we know that sinec φ is not a polynomial, as close as we want from any α there is a real number α' such that the special flow constructed over $R_{\alpha'}$ with the ceiling function φ is weak mixing. But then, any time t_0 map of this flow is weak mixing (we use for this the spectral characterization of weak mixing and the fact that the maximal spectral measure of a time t_0 map of a flow is continuous if and only if the maximal spectral measure of the flow is continuous). Density of W_φ is thus proved.

On the other hand we denote by V_φ the set of numbers α for which the time T^{t_0} of the special flow constructed over R_α and under the ceiling function φ is of rank one. We claim that V_φ is a dense G_δ of $\mathbf{R}$. Here again, it is easy to see that V_φ is a G_δ in $\mathbf{R}$. For this, one uses the definition of the rank one property for a dynamical system (X, T, μ): Let $\{\mathcal{P}_i\}_{i\in\mathbf{N}}$ be a generating sequence of finite measurable partitions. The transformation T is of rank one if and only if for any $i \in \mathbf{N}$, and for every $n \in \mathbf{N}$, there exists a Rokhlin tower for T that refines $\mathcal{P}_i$ up to $\frac{1}{n}$, that is, if any atom of $\mathcal{P}_i$ is up to a precision $\frac{1}{n}$ a union of levels of the tower (when we say up to $\frac{1}{n}$ we mean that the measure of the symetrical difference is smaller than $\frac{1}{n}$). For a fixed partition $\mathcal{P}_i$ and a fixed precision $\frac{1}{n}$, the property of $\frac{1}{n}-$refining $\mathcal{P}_i$ is celarly an open condition on T. So if we denote by $V_{\varphi,i,n}$ the set of α such that the time T^{t_0} of the special flow constructed over R_α and under the ceiling function φ satisfies the above condition for the integers i and n; $V_{\varphi,i,n}$ will be open in $\mathbf{R}$. Hence, $V_\varphi = \bigcap_n \bigcap_i V_{\varphi,i,n}$ is a G_δ. We still need to show it is dense. But by the result of Kolmogorov, we know in particular that if α is Diophantine, then the special flow over R_α with the ceiling function φ is analytically conjugate to the minimal translation flow on the torus of vector $\left(\alpha / \int_{\mathbf{T}}^1 \varphi(x)dx, 1/\int_{\mathbf{T}}^1 \varphi(x)dx\right)$. From now on we assume α is Diophantine so that this conjugacy holds. Because the rank one property is invariant by conjugacy, the time t_0 map of the special flow will be of rank one if the time t_0 map of the translation flow, $R_{t_0(\alpha/\int\varphi, 1/\int\varphi)}$, is of rank one. As we mentionned above, this will be true when the latter translation is minimal. With the assumption on t_0 made in the hypothesis of the theorem, we will just have to discard the countable set of α consisting of $\frac{\int_{\mathbf{T}}\varphi(x)dx}{t_0}\mathbf{Q} + \mathbf{Q}$. Of course, we will still remain with a dense set of Diophantine numbers α. Hence, we proved that V_φ is a G_δ dense in $\mathbf{R}$.

The subset of $\mathbf{R}$, $\mathcal{W}_\varphi$, intersection of V_φ and W_φ, both G_δ dense, is a G_δ dense. $\square$

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