ANALYTIC MIXING REPARAMETRIZATIONS OF IRRATIONAL FLOWS

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ABSTRACT. We give an example of a strictly positive analytic reparametrization (or time change) of an irrational flow on $\mathbf{T}^3$ that is mixing. As an immediate application we obtain perturbations of completely integrable Hamiltonian systems that display many invariant tori with mixing dynamics positively answering a problem raised by Kolmogorov.

1. DEFINITIONS AND NOTATIONS.

1.1 On the n torus $\mathbf{T}^n = \mathbf{R}^n/\mathbf{Z}^n$, a translation of vector $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbf{R}^n$ is the transformation

$$\begin{array}{ccc} \mathbf{T}^n & \to & \mathbf{T}^n, \\ (x_1, ..., x_n) & \to & (x_1 + \alpha_1, ..., x_n + \alpha_n). \end{array}$$

We denote it by R_α. The translation R_α is said to be *irrational* if the real numbers $1, \alpha_1, ..., \alpha_n$ are rationally independent, i.e. if the relation $k_1\alpha_1 + ...k_n\alpha_n = p$, where $k_1, ..., k_n$, and p are in $\mathbf{Z}$ implies $k_1 = ... = k_n = p = 0$. In this case R_α is strictly ergodic (uniquely ergodic and minimal).

1.2 The translation flow on $\mathbf{T}^n$ of vector $\alpha \in \mathbf{R}^n$ is the flow arising from the constant vector field $X(x) = \alpha$. We denote this flow by $\{R_{t\alpha}\}$. When the numbers $\alpha_1, ..., \alpha_n$ are rationally independent, i.e. none of them is a rational combination of the others, $\{R_{t\alpha}\}$ is strictly ergodic. In this case we say it is an irrational flow. Note that one of the coordinates of the corresponding vector field might be rational. More specifically, given an irrational translation R_α on $\mathbf{T}^n$, then the flow $\{R_{t(1,\alpha)}\}$ on $\mathbf{T}^{n+1}$ is irrational.

1.3 *Reparametrization of* $\{R_{t\alpha}\}$. If ϕ is a strictly positive smooth real function on $\mathbf{T}^n$, we define the reparametrization of $\{R_{t\alpha}\}$ with velocity ϕ as the flow given by the vector field $\phi(x)\alpha$, that is, by the system

$$\frac{dx}{dt} = \phi(x)\alpha.$$

The new flow has the same orbits as $\{R_{t\alpha}\}$ and preserves a measure equivalent to the Haar measure given by the density $\frac{1}{\phi}$. Moreover, if

$\{R_{t\alpha}\}$ is ergodic then so is the reparametrized flow. (For a general abstract definition of reparametrization of flows, and for the proof of measure preserving and ergodicity of the resulting flow see [12]).

1.4 *Special flows.* Given a function $f \in L^1(\mathbf{T}^n)$, $f > c > 0$, the *special flow constructed over R_α and under the function f* is the quotient flow of the action

$$\begin{array}{rcl} \mathbf{T}^n \times \mathbf{R} & \longrightarrow & \mathbf{T}^n \times \mathbf{R} \\ (x,s) & \longrightarrow & (x, s+t) \end{array}$$

by the relation $(x, s+f(x)) \sim (R_\alpha(x), s)$. This flow acts on the manifold $M_{R_\alpha,f} = \mathbf{T}^n \times \mathbf{R} / \sim$, and preserves the normalized Lebesgue measure on $M_{R_\alpha,f}$, i.e. the product of the Haar measure on the basis $\mathbf{T}^n$ with the Lebesgue measure on the fibers divided by the constant $\int_{\mathbf{T}^n} f(x)dx$. To lighten the notations, when there is no ambiguity, we will call this measure ν, and write simply M for $M_{R_\alpha,f}$.
- The function f, that measures the time needed by a point on the basis to return to it, is called the *ceiling function*. As an example, one can look at the flow $\{R_{t(\alpha,1)}\}$ as a special flow over the translation R_α and under the constant function equal to one. For the natural correspondence between reparametrization and special flows refer to section 4.
- In the case of a special flow $\{T^t\}$ over $\mathbf{T}^2$, we denote the Haar measure on the basis by $\mu = \lambda \times \lambda$, where λ designates indifferently the Lebesgue measure on the line or the Haar measure on $\mathbf{T}^1$.
- A subset R of $\mathbf{T}^2$ is called a rectangle, if it is a product of two intervals from $\mathbf{T}^1$; and for $0 < \delta < c$ we call cube on the basis, of height δ and base R, the product $R \times [0, \delta] = \bigcup_{0 \le t \le \delta} T^t(R)$.
- If I is an interval on $\mathbf{T}^1$ we agree on the notation $|I| := \lambda(I)$.
- Let $z = (x, y) \in \mathbf{T}^2$; we write, for $i \in \mathbf{N}$, $R_\alpha^i(z) = (x + i\alpha, y + i\alpha')$ the i^{th} iterate of z under R_α.
- When there is no confusion, we will denote the Birkhoff sums of the function f over the iterates of the translation R_α by:

$$f_m(z) = \sum_{i=0}^{m-1} f\left(R_\alpha^i(z)\right).$$

1.5 We also recall the definition of mixing for a measure preserving flow: a flow $\{T_t\}$ preserving a measure ν on M is said to be mixing if, for any measurable subsets A and B of M, one has

$$\lim_{t \to \infty} \nu(T^t(A) \bigcap B) = \nu(A)\nu(B).$$

1.6 We introduce finally some arithmetic notations. Let x be a real number; we denote by:

–$[x]$ the integer part of x,
–$\{x\} = x - [x]$ its fractional part,
–$\| x \| = \min(\{x\}, 1 - \{x\})$ the distance of x to the closest integer.
–When we write $\frac{p}{q} \in \mathbf{Q}$, we assume that $q \in \mathbf{N}$, $q \geq 1$, $p \in \mathbf{Z}$ and that p and q are relatively prime. And we give the following reminder on continued fractions. Let α be an irrational real number: There exists a sequence of rationals $\{\frac{p_n}{q_n}\}_{n \in \mathbf{N}}$, called the convergents of α, such that:

$$\| q_{n-1}\alpha \| < \| k\alpha \| \quad \forall k < q_n \tag{1}$$

and for any n

$$\frac{1}{q_n(q_n + q_{n+1})} \leq (-1)^n(\alpha - \frac{p_n}{q_n}) \leq \frac{1}{q_n q_{n+1}}.$$

2. Introduction.

Kocergin, in 1972 [9], gave a very simple proof of the fact that no measure-preserving flow on the torus $\mathbf{T}^2$, of class C^1 and without periodic orbits, can be mixing. He proves the more general result that *a special flow constructed over an irrational rotation of the circle with a ceiling function of bounded variation is never mixing*. Kocergin's proof is based on the Denjoy-Koksma inequality (for instance, when f is of class C^1 and $\int f = 1$, one can immediately prove that if q_n are the denominators of the converegnts of α, then $T^{q_n} \to Id$ uniformly as n goes to infinity. This *rigidity* clearly impedes mixing).

In the case where the rotation numbers are "not too well approximated" by rationals, and the ceiling function is analytic, the result of Kocergin on special flows is included in what was obtained by Kolmogorov [11] in his analysis of aperiodic measure-preserving flows on the two torus. Kolmogorov actually proves that in "general" (i.e. for a set of total measure of rotation numbers), the dynamics of the special flow under any analytic ceiling function, can be reduced by means of analytic transformation of coordinates to the dynamics of an irrational flow (the absence of mixing is a clear consequence). He also made the remark that the conjugacy to the irrational flow can be proven even when the ceiling function is less regular (obviously, for a smaller set of rotation numbers and with less regularity of the conjugating function).[1]

[1]In fact, in order to show equivalence of a special flow over a rotation of the circle R_α and under a function f with the irrational flow $R_{t(\int f\alpha, \int f)}$, it is enough to solve the cohomological equation

$$\psi \circ R_\alpha - \psi = f - \int_{\mathbf{T}} f d\lambda \qquad (*)$$

and the conjugacy will be as regular as ψ is. The original proof of this classical result now, is due to Kolmogorv [11]. A simple proof, based on the construction of a new section for which the return time is constant, can be found in [7], Proposition 2.9.5. A generalization of Kolmogorv's result to higher dimensions was given by

The case of the "bad" numbers, i.e. those that are well approximated by rationals remained open. However, for these numbers, Katok [6] used his theory of periodic approximations to show that the corresponding special flows admit fast cyclic approximations (under a regularity condition on the ceiling function), a property that rules out mixing. Combining his result with that of Kolmogorov to cover all numbers α, he has obtained (in 1970) a stro nger result than that of Kocergin (he proved the simple spectrum property along with the nonmixing), but less general, since he assumed the ceiling function of class at least C^5 (actually, to obtain cyclic approximations when the number is sufficiently well approximated, Katok does not require that much regularity on the ceiling function, but he needs it to cover the remaining numbers using Kolmogorov's method).

Maybe it is worth noticing here that according to Kolmogorov a new behavior for measure preserving flows on the torus, different from the canonical ones (discrete spectrum with a finite number of independent frequencies, and Lebesgue spectrum with countable multiplicity), was rather unexpected in the analytic case [10]. Still, Shklover in 1967 [13] (see [1] for an english version) gave examples of special flows over irrational rotations of the circle with an analytic ceiling function, that had a continuous singular spectrum (they were weak-mixing but not mixing), which contradicted Kolmogorov's intuition. In the case of measure preserving flows on $\mathbf{T}^2$ *with fixed points*, Kocergin in 1975 [8], gave explicit examples of mixing flows of class C^∞. The notion of *stretching* for the Birkhoff sums of the ceiling function, underlying the occurrence of mixing for special flows over rigid transformations, appeared in both these works (although under different aspects and because of different reasons).

From the result of Kocergin in [9] , it comes that any reparametrization of an irrational flow on $\mathbf{T}^2$ by a smooth strictly positive function fails to be mixing. Our aim in this paper is to prove that this is not anymore the case on the torus $\mathbf{T}^3$ (or on any $\mathbf{T}^n$, $n \geq 3$). We will give examples of irrational flows on $\mathbf{T}^3$ that display the mixing property when reparametrized by adequately chosen strictly positive real analytic functions. Since it is more convenient to work with special flows, we will start by giving an example of an irrational vector $(\alpha, \alpha') \in \mathbf{R}^2$ and a strictly positive real analytic function φ on $\mathbf{T}^2$, such that the

Herman [4], who also gives a more direct proof of conjugacy using the vector fields, when the conjugacy one seeks should be of class C^1. As for the cohomological equation, one sees using Fourier expansions that for a set of full measure of $\alpha \in \mathbf{R}$, if $f \in C^{1+\epsilon}(\mathbf{T})$, equation $(*)$ has a continuous solution (In the annex of [3], the cohomological equation is thoroughly studied). See also our footnote in page 34.

special flow over $R_{(\alpha,\alpha')}$ and under φ is mixing. The crucial fact is that the Denjoy-Koksma inequality for functions defined on the circle disappears in dimension higher than 1, as was proved by J-C. Yoccoz in an appendix to his thesis [14]. More precisely, he *constructs an irrational translation on $\mathbf{T}^2$, and a real analytic function ψ, having complex values and mean value zero, such that there exists a borelian subset of $\mathbf{T}^2$, Ω, with full Lebesgue measure having the following property: For any couple $(x, x') \in \Omega$, the Birkhoff sums of ψ, $\psi_m(x, x')$ corresponding to the translation $R_{\alpha,\alpha'}$, tend to infinity in modulus when m tends to infinity.* The main ingredient in Yoccoz's proof is that the denominators, $\{q_n\}_{n\in\mathbf{N}}$ and $\{q'_n\}_{n\in\mathbf{N}}$ of the convergents of α and α' are alternated, and more precisely, they are such that the sequence $...q_n, q'_n, q_{n+1}, q'_{n+1}...$ increases exponentially. We will see later, how this is used to create mixing.

Let Y be the set of couples $(\alpha, \alpha') \in \mathbf{R}^2 - \mathbf{Q}^2$, whose sequences of best approximations q_n and q'_n satisfy the following, for any $n \geq n_0(\alpha, \alpha')$

$$\begin{aligned} q'_n &\geq e^{3q_n}, \\ q_{n+1} &\geq e^{3q'_n}. \end{aligned}$$

And let φ be the following strictly positive real analytic function on $\mathbf{T}^2$:

$$\varphi(x,y) = 1 + \mathrm{Re}\left(\sum_{k=2}^{\infty} \frac{e^{i2\pi kx}}{e^k} + \sum_{k=2}^{\infty} \frac{e^{i2\pi ky}}{e^k}\right).$$

Our main theorem is the following

Theorem 1 (Special flows). *For any $(\alpha, \alpha') \in Y$, the special flow constructed over the translation $R_{\alpha,\alpha'}$ on $\mathbf{T}^2$, with the ceiling function φ is mixing.*

Remarks. a) In his construction, J-C Yoccoz points out the elementary fact that the set Y as defined above is uncountable and dense.

b) From the proof of the main theorem, it will be easy to see that we can take for the ceiling function, instead of $\varphi(x, y)$, a function $\varphi(x, y)+P(x, y)$, where $P(x, y)$ is any trigonometric polynomial on $\mathbf{T}^2$ such that $\varphi + P > 0$. More generally, it appears that we can take any strictly positive function

$$\varphi(x,y) = 1 + \mathrm{Re}\left(\sum_{k=1}^{\infty} \frac{e^{i2\pi kx}}{a_k} + \sum_{k=1}^{\infty} \frac{e^{i2\pi ky}}{b_k}\right),$$

such that a_k and b_k satisfy for large k

$$e^{\frac{k}{2}} \leq a_k \leq e^{2k}, \qquad e^{\frac{k}{2}} \leq b_k \leq e^{2k}.$$

From Theorem 1 and the foregoing remark we will be able to derive

Corollary 1.1 (Reparametrized flows)**.** *There exists a strictly positive analytic function on $\mathbf{T}^3$, ϕ, such that, for any $(\alpha, \alpha') \in Y$, the reparametrization of the irrational flow $(\alpha, \alpha', 1)$ by $\frac{1}{\phi}$ is mixing.*

The convenience of using $\frac{1}{\phi}$ instead of ϕ in the statement of the corollary will appear clearly when we will study the natural correspondence between special flows over translations and reparametrizations of irrational flows.

Still, for a "general" vector $(\alpha, \alpha') \in \mathbf{R}^2$, it is rather rare to be able to construct mixing special flows over $R_{\alpha,\alpha'}$, even with less regular ceiling functions. More precisely we have

Theorem 2. *There exists a set $G \subset \mathbf{R}^2$ that contains a set of full measure (when intersected with any compact set) and a dense G_δ of $\mathbf{R}^2$ such that any special flow over $R_{\alpha,\alpha'}$, $(\alpha, \alpha') \in G$, with a ceiling function $\varphi \in C^4(\mathbf{T}^2)$ is not mixing.*

The proof of this theorem will be based on the fact that for a generic vector $(\alpha, \alpha') \in \mathbf{R}^2$ (in both meanings of measure and topology) the Denjoy-Koksma property still appears in dimension 2 and prevents the special flow from being mixing. Most likely, the differentiability class C^4 is not optimal in this result.

Plan of the work. What follows is an outline of the different parts of this paper. In Section 3, we are interested in special flows and we prove the main theorem, Theorem 1. The section will be divided in three parts: We start by giving a general lemma based on a Fubini integration that guarantees mixing for a special flow over $\mathbf{T}^2$. We then give the definition of *uniform stretch* for a function on an interval and we prove, using the general lemma mentioned above, that uniform stretch of the Birkhoff sums of the ceiling function implies mixing for special flows constructed over irrational translations of the torus. In the same subsection we give a simple criterion , involving only the derivatives of the Birkhoff sums, that implies uniform stretch, hence mixing. Finally, in 3.3 we check this last criterion for φ and for any $(\alpha, \alpha') \in Y$, which achieves the proof of the main theorem. In Section 4 we derive the results on reparametrizations, and we prove Corollary 1.1. The following section is devoted to the proof of Theorem 2 on absence of mixing in the general case. In Section 6 we give a generalization of our main result, relevant in particular for the higher dimension case. We finish the section with an application to the three dimensional cubic billiard. Finally in the last section we ask what might happen with less regular ceiling functions if the translation vector on the basis is Diophantine, and we give an example of a mixing special flow over a $(2 + \epsilon)-$Diophantine translation of $\mathbf{T}^2$ with a ceiling function of class C^2. We finish this introduction by giving an

Idea of the proof of the main theorem. Because of the disposition of the best approximations of α and α' the Birkhoff sums φ_m of the function φ, for any m sufficiently large, will be always stretching (i.e. have big derivatives), in one or in the other of the two directions, x or y, depending on whether m is far from q_n or far from q'_n. And this stretch will increase when m goes to infinity. So when time goes from 0 to t, t large, the image of a small typical interval I from the basis $\mathbf{T}^2$ (depending on t the intervals should be taken along the x or the y axis) will be more and more distorted and stretched in the fibers' direction. Until the image of I at time t will consist of a lot of almost vertical curves whose projection on the basis lies along a part of the trajectory under the translation $R_{\alpha,\alpha'}$. By unique ergodicity these projections become more and more uniformly distributed, and so will $T^t(I)$. For each t, and except for increasingly small subsets of it (as function of t), we will be able to cover the basis with such "typical" intervals. Besides what is true for I on the basis is true for $T^s(I)$ at any height s on the fibers. So applying Fubini in two directions, first along the other direction on the basis (for a time t all typical intervals are in the same direction), and second along the fibers, we will obtain the asymptotic uniform distribution of any measurable subset, which is, by definition, the mixing property. (See Fig. 1)

Fig. 1. An interval J and its image.

3. Mixing for special flows over translations on $\mathbf{T}^2$.

3.1. **A general lemma.** Let $\{T^t\}$ be a special flow constructed over an ergodic transformation T of $\mathbf{T}^2$ and under a ceiling function f, that we suppose bounded from below by a constant $c > 0$. Let M be the space where this flow acts (see section 1). Recall that we denote by ν the invariant probability measure $\frac{1}{\int f}\mu \times \lambda$, where $\mu = \lambda \times \lambda$ is the Haar measure on $\mathbf{T}^2$ and λ is the Lebesgue measure on the line. For an interval I on $\mathbf{T}^1$, and points $y \in \mathbf{T}^1$, $I \times \{y, 0\}$ designates the set of M situated on the basis $\mathbf{T}^2$ in the x direction, call it a horizontal interval. We use similar notations for the direction y, the vertical direction. We will denote $I \times \{y, z\} := T^z(I \times \{y, 0\})$. When $z \leq c < \inf f$, with a slight abuse, we will use the notation $\lambda(I \times \{y, z\}) = \lambda(I) = |I|$. When R is a rectangle on the basis and δ is a strictly positive number $\delta < c$, we defined a 'cube' on the basis, of base R and hight δ, to be the set $Q = R \times [0, \delta] = \bigcup_{0 \leq t \leq \delta} T^t(R)$. Obviously $\nu(Q) = \mu(R)\delta$. We have the following lemma based on Fubini's theorem.

Lemma 3.1. *If for any cube Q on the basis, of base R and height $\delta < c$, and for any $\varepsilon > 0$, there is a t_0 such that for any $t > t_0$ there is a partial partition of the circle, $\eta_t = \{C_i^{(t)}\}$ satisfying the following*

$$\eta_t \underset{t\to\infty}{\longrightarrow} \epsilon \quad (\text{ partition into points }),$$

and, depending on $t > t_0$ at least one of the two conditions is true

(i) *for all $y \in \mathbf{T}^1$, and all $C_i^{(t)}$,*

$$\lambda\left[\left(C_i^{(t)} \times \{y, 0\}\right) \bigcap T^{-t}Q\right] \geq (1-\epsilon)\delta\mu(R)\lambda(C_i^{(t)});$$

(ii) *for all $x \in \mathbf{T}^1$, and all $C_i^{(t)}$,*

$$\lambda\left[\left(\{x\} \times C_i^{(t)} \times \{0\}\right) \bigcap T^{-t}Q\right] \geq (1-\epsilon)\delta\mu(R)\lambda(C_i^{(t)});$$

then the flow $\{T^t\}$ is mixing.

A partial partition of a set X consists of disjoint subset of X and we say that a sequence of partial partition tends to the partition into points if every finite partition of X has its atoms arbitrarily well approximated by unions of sets in the successive partial partitions.

A similar lemma was used by Kocergin in [8] to this difference that we have to use twice the Fubini integration because our lemma takes into account the occurrence of mixing for intervals on the basis in one or the other direction x and y depending on t. The property implied by this lemma is stronger than that of mixing for measurable sets of the three dimensional space of the special flow. Indeed even sets of codimension two, like intervals, get uniformly distributed in all the space under the action of the flow.

Proof. To prove mixing for the flow $\{T^t\}$, it is enough to show that given two measurable subsets A and A' of M, then for any $\epsilon > 0$, when t is big enough

$$\nu\left(A' \bigcap T^{-t}A\right) > (1-\epsilon)\nu(A)\nu(A'), \tag{2}$$

indeed, if we apply this to A^c and A' we get

$$\nu\left(A \bigcap T^{-t}A'\right) < (1+\epsilon)\nu(A)\nu(A') + \epsilon\nu(A').$$

Since finite unions of cubes on the basis and of their images under the flow generate the Borel σ-algebra on M it is enough to prove (2) for any couple Q and Q' of cubes on the basis.

Also, we will place ourselves in situation (i), the other case being similar. By definition of a cube on the base, there exists an interval in $\mathbf{T}^1$, J, such that

$$Q' = \bigcup_{z=0}^{z_1} \bigcup_{y=y_1}^{y_2} J \times \{y, z\}.$$

From the hypothesis $\eta_t \to \epsilon$, there exists for t large enough, a subset S of $\mathbf{N}$ and a union of atoms from η_t, $\tilde{J} = \bigcup_{i\in S} C_i^{(t)}$, such that $\lambda\left(J\triangle\tilde{J}\right) < \epsilon\lambda(J)\nu(Q)$. Hence the set

$$\tilde{Q}' = \bigcup_{z=0}^{z_2}\bigcup_{y=y_1}^{y_2}\bigcup_{i\in S} C_i^{(t)} \times \{y,z\},$$

satisfies

$$\nu\left(Q'\triangle\tilde{Q}'\right) < \epsilon\nu(Q')\nu(Q),$$

and

$$\nu\left(\tilde{Q}'\bigcap T^{-t}Q\right) = \int\int\sum_{i\in S}\lambda\left[\left(C_i^{(t)}\times\{y,z\}\right)\bigcap T^{-t}Q\right]dydz$$

Recall that $C_i^{(t)}\times\{y,z\} = T^z(C_i^{(t)}\times\{y,0\})$, $z\le z_1$, and that we assumed $z_1 \le c$, so we can consider that (i) holds also for $\lambda\left[\left(C_i^{(t)}\times\{y,z\}\right)\bigcap T^{-t}Q\right]$, hence,

$$\begin{aligned}\nu\left(\tilde{A}\bigcap T^{-t}Q\right) &\ge \int\int\sum_{i\in S}\delta\mu(R)\lambda(C_i^{(t)})(1-\epsilon)dydz\\ &\ge \delta\mu(R)\nu(\tilde{A})(1-\epsilon).\end{aligned}$$

Finally, we have

$$\nu\left(A\bigcap T^{-t}Q\right) \ge (1-\epsilon)\nu\left(\tilde{A}\bigcap T^{-t}Q\right) \ge (1-\epsilon)^3\delta\mu(R)\nu(A).$$

The lemma is proved. □

3.2. **A mixing criterion on the ceiling-function Birkhoff sums.** Until section 4, we will be exclusively interested in special flows over ergodic translations of the two torus $\mathbf{T}^2$. First, we will use the general lemma for mixing stated above for general special flows over the torus and derive from it, in the particular case of a special flow with a translation on the basis, a criterion on the Birkhoff sums of the ceiling function that implies mixing for the flow. Then we check this criterion for the special flow $\{R_{\alpha,\alpha'},\varphi\}$, where $(\alpha,\alpha')\in Y$ and φ is the function introduced in Section 2.

Henceforth, $\{T^t\}$ will be a special flow over an ergodic translation $R_{\alpha,\alpha'}$ of $\mathbf{T}^2$ with a smooth ceiling function f that we suppose bounded from below by $c>0$. To simplify the exposition of our proofs we will assume that $\int_{\mathbf{T}^2} f = 1$ and that

$$\frac{1}{2}\le f\le\frac{3}{2},$$

which does not cause any loss of generality in the propositions we will state. (Besides, this condition is effectively satisfied by φ if we drop the first non constant terms from its Fourier series.)

The following definition, relative to one variable real functions, will be essential in the sequel. It is similar to the definition of uniform distribution in [8].

Definition 3.1 (Uniform stretch). *Given $\varepsilon > 0$ and $K > 0$; we say that a real function g, on an interval $[a,b]$ is (ε, K)-uniformly stretching on $[a,b]$ if*

$$\sup_{[a,b]} g - \inf_{[a,b]} g \geq K,$$

and if for any u and v such that

$$\inf_{[a,b]} g \leq u \leq v \leq \sup_{[a,b]} g,$$

the set

$$I_{u,v} = \{x \in [a,b]/u \leq g(x) \leq v\},$$

has Lebesgue measure

$$(1-\varepsilon)\frac{v-u}{g(b)-g(a)}(b-a) \leq \lambda(I_{u,v}) \leq (1+\varepsilon)\frac{v-u}{g(b)-g(a)}(b-a).$$

Agree on the notation $\Delta g_{[a,b]}$ for $\sup_{x\in[a,b]} |g(x)| - \inf_{x\in[a,b]} |g(x)|$. We assume now g of class at least C^2 and we give a straightforward but useful criterion on the derivatives of g insuring its uniform stretch on the segment $[a,b]$:

Lemma 3.2 (A Criterion for uniform stretch). *If*

$$\begin{aligned} \inf_{x\in[a,b]} |g'(x)||b-a| &\geq K \\ \text{and} \quad \sup_{x\in[a,b]} |g''(x)||b-a| &\leq \varepsilon \inf_{x\in[a,b]} |g'(x)| \end{aligned}$$

then g is (ε, K)-uniformly stretching on $[a,b]$.

Proof. The first condition implies that g is monotone on $[a,b]$, we will suppose it increasing. It also implies $\Delta g_{[a,b]} = g(b) - g(a) \geq K$. To check the condition on uniformity take $u = g(c)$ and $v = g(d)$, where $a \leq c \leq d \leq b$. By Rolle's theorem there exist $\xi_1,\ \xi_2 \in [a,b]$ such that

$$\frac{v-u}{g(b)-g(a)} = \frac{g'(\xi_1)}{g'(\xi_2)}\frac{d-c}{b-a},$$

from the condition on the second derivative we have that

$$\left|\frac{g'(\xi_1)}{g'(\xi_2)} - 1\right| \leq \varepsilon,$$

hence

$$(1-\varepsilon)\frac{d-c}{b-a} \leq \frac{v-u}{g(b)-g(a)} \leq (1+\varepsilon)\frac{d-c}{b-a},$$

and the proof is over. □

Now, we will show how uniform stretch of the Birkhoff sums of f implies mixing.

Proposition 3.1 (Fundamental proposition)**.** *If there exist partial partitions of* $\mathbf{T}^1$, $\eta_t = \{C_i^{(t)}\}$, *where the* $C_i^{(t)}$ *are intervals such that*

$$\sup_{C_i^{(t)} \in \eta_t} |C_i^{(t)}| \underset{t\to\infty}{\longrightarrow} 0,$$
$$\sum_{C_i^{(t)} \in \eta_t} |C_i^{(t)}| \underset{t\to\infty}{\longrightarrow} 1,$$

and positive functions $\varepsilon(t)$ *and* $k(t)$ *such that*

$$\varepsilon(t) \underset{t\to\infty}{\longrightarrow} 0,$$
$$k(t) \underset{t\to\infty}{\longrightarrow} \infty,$$

and if the function f *is such that, for any* t *at least one of the following two conditions is true:*

(i) *for any* $m \in [\frac{t}{2}, 2t]$, *for any* $y_0 \in \mathbf{T}^1$ *and any* $C_i^{(t)}$,

$$f_m(.,y_0) \text{ is } (\varepsilon(t), k(t)) - uniformly\ stretching \text{ on } C_i^{(t)};$$

(ii) *for any* $m \in [\frac{t}{2}, 2t]$, *for any* $x_0 \in \mathbf{T}^1$ *and any* $C_i^{(t)}$,

$$f_m(x_0,.) \text{ is } (\varepsilon(t), k(t)) - uniformly\ stretching \text{ on } C_i^{(t)};$$

then the flow $\{T^t\}$ *is mixing.*

Proof of Proposition 3.1. First we prove that the conditions on η_t implies that $\eta_t \to \epsilon$ (partition into points on $\mathbf{T}^1$): As any measurable set can be approximated by intervals, it is enough to show that any interval I on the circle can be approximated for t large enough by a union of atoms of η_t. Indeed, fix $\sigma > 0$ and assume that $\sup |C_i^{(t)}| \leq \sigma$ and $\sum |C_i^{(t)}| \geq 1-\sigma$. Let $\tilde{I}$ be the union of all the atoms $C_i^{(t)}$ of η_t that intersect I, obviously $\lambda(I \triangle \tilde{I}) \leq 3\sigma$. Now, to prove the proposition, we will show that the intervals of this partition satisfy the condition of the general lemma on mixing, Lemma 3.1. In fact we will show that if we fix ε and we take t big enough; according to whether the first or second pair of conditions of Proposition 3.1 hold we will get the first or second inequality of Lemma 3.1.

Fix $\varepsilon' > 0$ and a cube Q of base R and height $\delta < \frac{1}{2}$ ($Q = \bigcup_{t=0}^{\delta} T^t(R)$ where R is a rectangle on $\mathbf{T}^2$). In the proof we will determine a t_0 that will depend only on Q and $\varepsilon' > 0$.

Let $t \geq t_0$ be fixed, and assume that for this t it is the first condition of Proposition 3.1 that holds. Take an arbitrary interval of η_t, C_i^t and an arbitrary $y_0 \in \mathbf{T}^1$ and let I be the interval $C_t^i \times \{y_0, 0\}$. In light of Lemma 3.1, we will finish if we prove

$$\lambda \left[I \bigcap T^{-t} Q \right] \geq (1-\varepsilon)\delta\mu(R)\lambda(I).$$

In the proof we will write $I := [x_1, x_2]$ while $x \in I$ will be a simplified notation of $(x, y_0, 0) \in I$. Recall from the sketch of the proof the following definitions, for any $m \in \mathbf{N}$,

$$\begin{aligned} I_m &= \{x \in I;\ 0 \le t - f_m(x, y_0) \le f(R^m_{\alpha,\alpha'}(x, y_0))\} \\ &= \{x \in I;\ N(x,t) = m\}, \end{aligned}$$

$$I_{m,\delta} = \{x \in I_m;\ 0 \le t - f_m(x, y_0) \le \delta\},$$

where $N(x,t)$ is the biggest integer m such that $t - \varphi_m(x) \ge 0$, that is the number of fibers covered by x during its motion under the action of the flow until time t. By definition

$$T^t(x) = \left(R^{N(t,x)}_{\alpha,\alpha'}(x), t - \varphi_{N(t,x)}(x) \right).$$

So $T^t(x) \in Q$ if and only if $R^{N(t,x)}_{\alpha,\alpha'}(x) \in R$ and $t - \varphi_{N(t,x)}(x) \le \delta$. The set $T^t(I_m)$ lies in M in the band over $R^m_{\alpha,\alpha'}(I)$. It will in general intersect Q when $R^m_{\alpha,\alpha'}(I) \subset R$ and the intersection would be $T^t(I_{m,\delta})$. Consequently, the set $I \bigcap T^{-t}Q$ will essentially be the union of those $I_{m,\delta}$ with m such that $R^m_{\alpha,\alpha'}(I) \subset R$. We say essentially because there are border effects: the interval $R^m_{\alpha,\alpha'}(I_m)$ might intersect R but not fall completely inside of it. But I can be thought of being so small with regard to R that whenever it hits R, it falls completely inside. This will be more precise in the following lemma.

Let R_η be what is left from the rectangle R after we have taken off from its border a narrow strip of thickness η. The rectangle R and ε' being fixed, we can choose $\eta > 0$ and a continuous function on $\mathbf{T}^2$, χ_{R_η}, such that:

- χ_{R_η} is identically zero outside R_η;
- $0 \le \chi_{R_\eta} \le 1$;
- $\int_{\mathbf{T}^2} \chi_{R_\eta} \ge (1 - \varepsilon')\mu(R)$.

In the definition of t_0 *we will ask that for any* $t \ge t_0$, $\sup_{C^i_t \in \eta_t} |C^i_t| \le \eta$. Hence, $|I| \le \eta$, and from the definition of χ_{R_η} we have

lemma 3.1.1. *Let* x_1 *be an arbitrary point in* I, *we have*

$$\lambda \left[I \bigcap T^{-t} Q \right] \ge \sum_{m \in \mathbf{N}} \chi_{R_\eta}(R^m_{\alpha,\alpha'}(x_1)) \lambda(I_{m,\delta}).$$

Proof. If m is such that

$$\chi_{R_\eta}\left(R^m_{\alpha,\alpha'}(x_1) \right) > 0,$$

because $|I| \le \eta$ we will have

$$R^m_{\alpha,\alpha'}(I) \subset R,$$

which in its turn implies

$$T^t\left(I_{m,\delta}\right) \subset Q.$$

□

Notice that only finitely many sets I_m will not be empty.

Unique Ergodicity. Because $R_{\alpha,\alpha'}$ is uniquely ergodic and χ_{R_η} and f are continuous, there exist N_0 such that
$\forall N > N_0,\ \forall (x,y) \in \mathbf{T}^2,\ \ \forall m_0 \in \mathbf{N}$,

$$\frac{\sum_{j=0}^{N} \chi_{R_\eta}\left(R_{\alpha,\alpha'}^{m_0+j}(x,y)\right)}{N} \geq (1-\varepsilon')^2 \mu(R) \tag{3}$$

$$\left| \frac{\sum_{j=0}^{N} f\left(R_{\alpha,\alpha'}^{m_0+j}(x,y)\right)}{N} - 1 \right| \leq \varepsilon' \tag{4}$$

In the definition of t_0 *we ask that* $t_0 \geq 4N_0$*; and that* $t \geq t_0$ *implies on the stretch that* $k(t) \geq max\left(4N_0, \frac{1}{\varepsilon'}\right)$ *and* $\varepsilon(t) \leq \varepsilon'$. On one hand, the fact that $k(t)$ is large will allow us to make sure that I breaks down into sufficiently many intervals I_m. Consequently, with (4) we will obtain an asymptotic estimation of the number of nonempty I_m (Lemma 3.1.3) and using (3) we will estimate the proportion of the I_m that fall into R and show it is close to $\mu(R)$. On the other hand, the condition $\varepsilon(t) \leq \varepsilon'$ will allow us to give a precise estimation of $\lambda(I_{m,\delta})$ (Lemma 3.1.4). We will need the following fact:

lemma 3.1.2. *For t large enough, for any $(x,y) \in \mathbf{T}^2$,*

$$N(x,y,t) \in [\frac{t}{2}, 2t].$$

Proof. By definition of $N(x,y,t)$

$$0 \leq t - f_{N(x,y,t)}(x,y) \leq f(R_{\alpha,\alpha'}^{N(x,y,t)}(x,y)).$$

So, f being bounded by $\frac{3}{2}$,

$$0 \leq t - f_{N(x,y,t)}(x,y) \leq \frac{3}{2} \tag{5}$$

and

$$t - \frac{3}{2} \leq f_{N(x,y,t)}(x,y) \leq \frac{3}{2} N(x,y,t),$$

therefore, when $t \geq t_0 \geq 4N_0$,

$$N(x,y,t) \geq N_0.$$

By ergodicity, (4), this implies

$$\left| \frac{f_{N(x,y,t)}(x,y)}{N(x,y,t)} - 1 \right| \leq \varepsilon'.$$

This last inequality with (5) and the fact that $N(x,y,t) \geq N_0$ imply

$$\left| \frac{t}{N(x,y,t)} - 1 \right| \leq 3\varepsilon'$$

and if $\varepsilon' \leq \frac{1}{10}$ this would imply the bounds of the lemma. □

For the interval I, the above lemma gives in particular that $N_1 := N(x_1, t) \in [\frac{t}{2}, 2t]$. From the hypothesis of Proposition 3.1 and our choice of t_0 (and because we assumed that we are in situation (i) for the time t):

$$f_{N_1} \text{ is } (\varepsilon', 4N_0) - \text{uniformly stretching on } I. \tag{6}$$

Hereafter, we will suppose $f_{N_1}(x_1) \geq f_{N_1}(x_2)$, the other case being similar.

Define

$$\begin{aligned} K(I) &= \{j \in \mathbf{N} \ / \ \sup_{x \in I} f_j(R^{N_1}_{\alpha,\alpha'}(x)) \leq \Delta f_{N_1|I} - \delta\}, \\ M(I) &= \max K(I). \end{aligned}$$

To understand the meaning of these definitions and the lemmas that will just follow, imagine that the Birkhoff sums f_m are monotone decreasing on $I = [x_1, x_2]$. Consequently when we go from left to right on I the points in their trajectory under the flow up to time t will have passed through more and more fibers. That's how $T^t(I)$ splits into the $T^t(I_m)$ that we have already mentioned in the introduction when we said $T^t(I)$ breaks into many vertical strips whose projection on the basis lie on the trajectory of the translation. With this assumption on monotonicity, $f_{N_1}(x_1) - f_{N_1}(x_2) = \Delta f_{N_1|I}$ is the "delay" between x_1 and x_2: after N_1 translations, while the point x_1 reaches its last fiber, x_2 still has a distance $\Delta f_{N_1|I}$ to cover on the fibers. The number of fibers covered by x_2 by this means is $M(I)$. By ergodicity we can assume the size of a fiber to the mean value of f, that is 1. Consequently $M(I)$ is equivalent to $\Delta f_{N_1|I}$. More precisely

lemma 3.1.3. *We have*

$$\left| \frac{M(I)}{\Delta f_{N_1|I}} - 1 \right| \leq 5\varepsilon'.$$

Using the hypothesis on uniform stretch, we give in the lemma hereunder, for $j \leq M(I)$, a uniform estimation on the measure of $I_{N_1+j,\delta}$.

lemma 3.1.4. *For any j such that* $1 \leq j \leq M(I)$,

$$\lambda(I_{N_1+j,\delta}) \geq (1 - 5\varepsilon') \frac{\delta}{\Delta f_{N_1|I}} |I|.$$

Proof of the lemma 3.1.3. For any $(x, y) \in \mathbf{T}^2$,

$$f(x, y) \leq \frac{3}{2}. \tag{7}$$

From the maximality of $M(I) \in K(I)$ we have for some $x \in I$,

$$\Delta f_{N_1|I} - \delta - \frac{3}{2} \leq f_{M(I)}(R^{N_1}_{\alpha,\alpha'}(x)) \leq \Delta f_{N_1|I} - \delta \tag{8}$$

otherwise by (7) $M(I)+1$ would be in $K(I)$. But, from the stretch in (6)

$$\Delta f_{N_1|I} \geq 4N_0$$

so, using (7) again, we have for any $x \in I$, $\frac{3}{2}M(I) \geq f_{M(I)}(R^{N_1}_{\alpha,\alpha'}(x))$, we deduce

$$M(I) \geq 2N_0 \tag{9}$$

which, by unique ergodicity (4) implies for any $x \in I$

$$\left|\frac{f_{M(I)}(R^{N_1}_{\alpha,\alpha'}(x))}{M(I)} - 1\right| \leq \varepsilon'.$$

Since $N_0 \geq \frac{1}{\varepsilon'}$, (9) implies when we divide (8) by $M(I)$

$$\left|\frac{f_{M(I)}(R^{N_1}_{\alpha,\alpha'}(x))}{M(I)} - \frac{\Delta f_{N_1|I}}{M(I)}\right| \leq 2\varepsilon',$$

hence,

$$\left|\frac{\Delta f_{N_1|I}}{M(I)} - 1\right| \leq 4\varepsilon'.$$

The proof of the lemma is complete. □

Proof of the lemma 3.1.4. Whenever $j \in \mathbf{N}$ is such that

$$\inf_{x\in I}\left(t - f_{N_1+j}(x)\right) \leq 0 \leq \delta \leq \sup_{x\in I}\left(t - f_{N_1+j}(x)\right), \tag{10}$$

then by the intermediate value theorem there is an $x \in I$ such that $N(x,t) = N_1 + j$ and Lemma 3.1.2 implies $N_1 + j \in [\frac{t}{2}, 2t]$. Further, we can apply the uniform stretch hypothesis to f_{N_1+j}, we have

$$\lambda\left(I_{N_1+j,\delta}\right) \geq (1-\varepsilon(t))\frac{\delta}{\Delta f_{N_1+j|I}}|I|. \tag{11}$$

Before we derive the conclusion of the lemma from (11), we will show that whenever $1 \leq j \leq M(I)$ equation (10) is satisfied. From the definition of $N_1 = N(x_1, t)$

$$0 \leq t - f_{N_1}(x_1) \leq f\left(R^{N_1}_{\alpha,\alpha'}(x_1)\right)$$

the right hand of this equation implies for $j \geq 1$

$$t - f_{N_1+j}(x_1) \leq 0,$$

hence the left hand side of the equation (10) is valid. For the other side, because by definition $t - f_{N_1}(x_1) \geq 0$, obviously

$$\sup_{x\in I}\left(t - f_{N_1}(x)\right) \geq t - f_{N_1}(x_1) + \Delta f_{N_1|I} \geq \Delta f_{N_1|I}.$$

Since

$$t - f_{N_1+j}(x) = t - f_{N_1}(x) - f_j\left(R^{N_1}_{\alpha,\alpha'}(x)\right),$$

immediately

$$\sup_{x\in I}\left(t-f_{N_1+j}(x)\right)\geq \sup_{x\in I}\left(t-f_{N_1}(x)\right)-\sup_{x\in I} f_j\left(R_{\alpha,\alpha'}^{N_1}(x)\right),$$

but by definition of $j\in K(I)$, $\sup_{x\in I} f_j\left(R_{\alpha,\alpha'}^{N_1}(x)\right)\leq \Delta f_{N_1|I}-\delta$ so,

$$\sup_{x\in I}\left(t-f_{N_1+j}(x)\right)\geq \Delta f_{N_1|I}-\left(\Delta f_{N_1|I}-\delta\right)\geq\delta,$$

and the left hand side of (10) is also satisfied. For any $1\leq j\leq M(I)$, equation (10) is established, thus (11) holds. Yet, to have a uniform estimation we need to compare $\Delta f_{N_1+j|I}$ with $\Delta f_{N_1|I}$, when $j\leq M(I)$. For all $j\in\mathbf{N}$

$$\left|\Delta f_{j|I}\right|\leq j||f||_{C^1}|I|\leq j||f||_{C^1}\eta,$$

where η is the maximal size of an interval in η_t. If t is large enough we will have $\eta\leq\frac{\varepsilon'}{||f||_{C^1}}$. So, whenever $j\leq M(I)$, we have

$$\begin{aligned}\left|\Delta f_{j|I}\right| &\leq M(I)\varepsilon'\\ &\leq 2\varepsilon'\Delta f_{N_1|I}\end{aligned}$$

hence, for any $j\leq M(I)$

$$\left|\frac{\Delta f_{N_1+j|I}}{\Delta f_{N_1|I}}-1\right|\leq 2\varepsilon'$$

this, with (11) ends the proof of Lemma 3.1.4. □

We can conclude now using successively Lemma 3.1.1, Lemma 3.1.4, the ergodic estimation (3) and then Lemma 3.1.3:

$$\begin{aligned}\lambda\left[I\bigcap T^{-t}Q\right] &\geq \sum_{m\in\mathbf{N}}\chi_{R_\eta}\left(R_{\alpha,\alpha'}^{m}(x_1)\right)\lambda(I_{m,\delta})\\ &\geq \sum_{j=1}^{M(I)}\chi_{R_\eta}\left(R_{\alpha,\alpha'}^{N_1+j}(x_1)\right)\lambda(I_{N_1+j,\delta})\\ &\geq (1-5\varepsilon')\frac{\delta}{\Delta f_{N_1|I}}|I|\sum_{j=1}^{M(I)}\chi_{R_\eta}\left(R_{\alpha,\alpha'}^{N_1+j}(x_1)\right)\\ &\geq (1-5\varepsilon')\frac{\delta}{\Delta f_{N_1|I}}|I|(1-\varepsilon')^2M(I)\mu(R)\\ &\geq (1-5\varepsilon')^2(1-\varepsilon')^2\delta\mu(R)|I|.\end{aligned}$$

If we took ε' such that $(1-5\varepsilon')^2(1-\varepsilon')^2\geq 1-\varepsilon$ this would exactly be the inequality of lemma 3.1. Proposition 3.1 is thus proved. □

In the light of Lemma 3.2 on uniform stretch for smooth functions, we give a restatement of the fundamental proposition using the first and second derivatives of the Birkhoff sums $f_m(x,y)$. Here f is considered to be of class at least C^2.

Proposition 3.2. *Let $\eta_t = \{C_i^{(t)}\}$, $\varepsilon(t)$ and $k(t)$ be as in the statement of Proposition 3.1. If, depending on t, at least one of the following conditions holds*

(i) *for any $m \in [\frac{t}{2}, 2t]$, for any $y_0 \in \mathbf{T}^1$ and all $C_i^{(t)}$,*

. $\inf_{x\in C_i^{(t)}} \left|\frac{\partial f_m(x,y_0)}{\partial x}\right| |C_i^{(t)}| \geq k(t)$

. $\sup_{x\in C_i^{(t)}} \left|\frac{\partial^2 f_m(x,y_0)}{\partial x^2}\right| |C_i^{(t)}| \leq \varepsilon(t) \inf_{x\in C_i^{(t)}} \left|\frac{\partial f_m(x,y_0)}{\partial x}\right|$

(ii) *for any $m \in [\frac{t}{2}, 2t]$, for any $x_0 \in \mathbf{T}^1$ and all $C_i^{(t)}$,*

. $\inf_{y\in C_i^{(t)}} \left|\frac{\partial f_m(y,x_0)}{\partial y}\right| |C_i^{(t)}| \geq k(t)$

. $\sup_{y\in C_i^{(t)}} \left|\frac{\partial^2 f_m(y,x_0)}{\partial y^2}\right| |C_i^{(t)}| \leq \varepsilon(t) \inf_{y\in C_i^{(t)}} \left|\frac{\partial f_m(y,x_0)}{\partial x}\right|$

then the flow $\{T^t\}$ is mixing.

As we said in the introduction, the way to obtain mixing with the mechanism introduced by J-C Yoccoz was the alternation in the occurrence of uniform stretch for the Birkhoff sums φ_m: in the x direction when t is far from q_n and in the y direction when t is far from q'_n. So the alternation between the validity of the conditions in the latter propositions is known. We state now, a sufficient criterion for mixing, that takes into account this particular game of rotation, and that involves only the Birkhoff sums of the ceiling function f (time does not appear anymore explicitly in the conditions).

Proposition 3.3 (Mixing Criterion). *Let $\{T^t\}$ be a special flow over an ergodic translation of the torus $\mathbf{T}^2$ with a ceiling function $f > 0$ of class at least C^2. If there exist sequences of real numbers τ_n, k_n, ε_n such that*

$$\begin{aligned} \tau_n &\to \infty, \\ \varepsilon_n &\to 0, \\ k_n &\to \infty, \end{aligned}$$

and a sequence of partial partitions $\eta_n = \{C_i^{(n)}\}$ where the $C_i^{(n)}$ are intervals such that

$$\begin{aligned} \sup_{C_i^{(n)}\in\eta_n} |C_i^{(n)}| &\longrightarrow 0, \\ \sum_{C_i^{(n)}\in\eta_n} |C_i^{(n)}| &\longrightarrow 1; \end{aligned}$$

satisfying

(i) *for any* $m \in [\frac{\tau_{2n}}{2}, 2\tau_{2n+1}]$*, for all* $y_0 \in \mathbf{T}^1$ *and all* $C_i^{(2n)} \in \eta_{2n}$*,*

. $\inf_{x \in C_i^{(2n)}} \left| \frac{\partial f_m(x,y_0)}{\partial x} \right| |C_i^{(2n)}| \geq k_{2n}$,

. $\sup_{x \in C_i^{(2n)}} \left| \frac{\partial^2 f_m(x,y_0)}{\partial x^2} \right| |C_i^{(2n)}| \leq \varepsilon_{2n} \inf_{x \in C_i^{(2n)}} \left| \frac{\partial f_m(x,y_0)}{\partial x} \right|$

and

(ii) *for any* $m \in [\frac{\tau_{2n+1}}{2}, 2\tau_{2n+2}]$*, for all* $x_0 \in \mathbf{T}^1$ *and all* $C_i^{(2n+1)} \in \eta_{2n+1}$*,*

. $\inf_{y \in C_i^{(2n+1)}} \left| \frac{\partial f_m(y,x_0)}{\partial y} \right| |C_i^{(2n+1)}| \geq k_{2n+1}$,

. $\sup_{y \in C_i^{(2n+1)}} \left| \frac{\partial^2 f_m(y,x_0)}{\partial y^2} \right| |C_i^{(2n+1)}| \leq \varepsilon_{2n+1} \inf_{y \in C_i^{(2n+1)}} \left| \frac{\partial f_m(y,x_0)}{\partial x} \right|$,

then $\{T^t\}$ *is mixing.*

Proof. For every $t \in [\tau_{2n}, \tau_{2n+1}]$ take for η_t the partial partition η_{2n} and for $k(t)$ and $\varepsilon(t)$ take k_{2n} and ε_{2n}.

For $t \in [\tau_{2n+1}, \tau_{2n+2}]$ take for η_t the partial partition η_{2n+1} and for $k(t)$ and $\varepsilon(t)$ take k_{2n+1} and ε_{2n+1}.

The partition η_t and the functions $k(t)$ and $\varepsilon(t)$ satisfy clearly the conditions of Proposition 3.2. □

3.3. **The Birkhoff Sums of** φ**.** In this subsection, we will consider a special flow constructed over a translation of the two torus of vector $(\alpha, \alpha') \in Y$ and under the ceiling function φ (the set Y and the ceiling function φ are those defined in the introduction). To prove Theorem 1, we will check that the Birkhoff sums of φ related to $R_{\alpha,\alpha'}$ satisfy the hypothesis of Proposition 3.3.

For any $m \in \mathbf{N}$, we have

$$\varphi_m(x,y) = m + \mathrm{Re}\left(\sum_{k=1}^{\infty} \frac{X(m,k)}{e^k} e^{i2\pi kx} + \sum_{k=1}^{\infty} \frac{Y(m,k)}{e^k} e^{i2\pi ky} \right),$$

where

$$X(m,k) = \frac{1 - e^{i2\pi mk\alpha}}{1 - e^{i2\pi k\alpha}},$$

$$Y(m,k) = \frac{1 - e^{i2\pi mk\alpha'}}{1 - e^{i2\pi k\alpha'}}.$$

We will need the following inequalities :

Lemma 3.3.

(12) *For all* $k \in \mathbf{N}^*$, $m \in \mathbf{N}$, $|X(m,k)| \leq m$;

(13) *for all* $n \in \mathbf{N}$, $k < q_n$, $m \in \mathbf{N}$, $|X(m,k)| \leq q_n$;

(14) *for all* $n \in \mathbf{N}$, $k \in]q_n, 2q_n[$, $m \in \mathbf{N}$, $|X(m,k)| \leq 2q_n$;

and for any $m \leq \frac{q_{n+1}}{2}$,

$$|X(m,q_n)| \geq \frac{2}{\pi} m; \tag{15}$$

$$|\arg(X(m,q_n))| \leq \pi \frac{(m-1)}{q_{n+1}}. \tag{16}$$

Proof. We have,

$$X(m,k) = \sum_{j=0}^{m-1} e^{i2\pi jk\alpha},$$

so the first inequality is trivial. For the other inequalities remember that by definition of the best approximations, q_n is such that

$$\| q_{n-1}\alpha \| \leq \| k\alpha \|, \quad \forall k < q_n, \tag{17}$$

and we also have

$$\frac{1}{2q_{n+1}} \leq \frac{1}{q_n + q_{n+1}} \leq ||q_n \alpha|| < \frac{1}{q_{n+1}}. \tag{18}$$

For inequality (13) write

$$|X(m,k)| \leq \frac{2}{|1 - e^{i2\pi k\alpha}|},$$

then using the inequality $\sin(\pi u) \geq 2u$, when $0 \leq u \leq \frac{1}{2}$, we have

$$\frac{2}{|1 - e^{i2\pi k\alpha}|} = \frac{1}{\sin \pi ||k\alpha||} \leq \frac{1}{2||k\alpha||},$$

since $k < q_n$, we have from (17) and the left hand side in (18), that this last term is bounded by q_n, and (13) is proved.

When $k \in]q_n, 2q_n[$, writing k under the form $q_n + q$, with $0 < q < q_n$, we obtain

$$\begin{aligned} ||k\alpha|| &\geq \frac{1}{2q_n} - \frac{1}{q_{n+1}} \\ &\geq \frac{1}{4q_n}. \end{aligned}$$

(Obviously $q_{n+1} \geq 4q_n$ since $q_{n+1} \geq e^{3e^{q_n}}$ by definition of the set Y.) With this, (14) follows in a similar way as (13).

Assume now $m \leq \frac{q_{n+1}}{2}$, and write

$$X(m, q_n) = e^{\pm i\pi(m-1)||q_n\alpha||} \frac{\sin \pi m||q_n\alpha||}{\sin \pi||q_n\alpha||},$$

from the right hand side in (18) one has $\pi m||q_n\alpha|| \leq \frac{\pi}{2}$, so

$$|X(m, q_n)| \geq \frac{2m||q_n\alpha||}{\sin(\pi||q_n\alpha||)} \geq \frac{2}{\pi}m.$$

Finally, with the same hypothesis on m, one has clearly

$$0 \leq |\arg(X(m, q_n))| = \pi(m-1)||q_n\alpha|| \leq \pi\frac{(m-1)}{q_{n+1}}.$$

□

Remark. There are of course analogous inequalities for $Y(m, k)$.

We can state now the central estimation on the φ_m that will imply uniform stretch in the x direction.

First define, for $n \in \mathbf{N}$, the set

$$I_n = \{x \in \mathbf{T}^1 \ / \ \{q_n x\} \in [\frac{1}{n}, \frac{1}{2} - \frac{1}{n}] \bigcup [\frac{1}{2} + \frac{1}{n}, 1 - \frac{1}{n}]\}.$$

Then for any integer n we have the following

Proposition 3.4. *For any $y \in \mathbf{T}^1$, for any $x \in I_n$, for any $m \in [\frac{e^{2q_n}}{2}, 2e^{2q_n'}]$, the following holds*

$$\left|\frac{\partial \varphi_m}{\partial x}(x, y)\right| \geq \frac{m}{e^{q_n}} \frac{q_n}{n}. \tag{19}$$

The proposition has clearly a counterpart in the y direction.

Proof. We have

$$\begin{aligned}
\frac{\partial \varphi_m}{\partial x}(x, y) &= \mathrm{Re}\left(\sum_{k=1}^{\infty} i2\pi k \frac{X(m, k)}{e^k} e^{i2\pi kx}\right) \\
&= \mathrm{Re}\left(i2\pi q_n \frac{|X(m, q_n)|}{e^{q_n}} e^{i2\pi q_n x}\right) + \mathrm{Re}\left(\sum_{k=1}^{q_n-1} i2\pi k \frac{X(m, k)}{e^k} e^{i2\pi kx}\right) \\
&+ \mathrm{Re}\left(\sum_{k=q_n+1}^{2q_n-1} i2\pi k \frac{X(m, k)}{e^k} e^{i2\pi kx}\right) + \mathrm{Re}\left(\sum_{k=2q_n}^{\infty} i2\pi k \frac{X(m, k)}{e^k} e^{i2\pi kx}\right) \\
&+ \mathrm{Re}\left(i2\pi q_n \frac{X(m, q_n) - |X(m, q_n)|}{e^{q_n}} e^{i2\pi q_n x}\right).
\end{aligned}$$

As we said in the introduction, it is the first term of this expression that will prevail as long as m lies between q_n and q_{n+1}, far from both of them as it is the case here. To prove the proposition we will thus bound the absolute value of the first term from below, and bound the absolute values of the others from above.

First we have

$$\left|\mathrm{Re}\left(i2\pi q_n \frac{|X(m,n)|}{e^{q_n}} e^{i2\pi q_n x}\right)\right| = \frac{|X(m,n)|}{e^{q_n}} 2\pi q_n |\sin(2\pi q_n x)|;$$

since $m \in [\frac{e^{2q_n}}{2}, 2e^{2q_n'}]$, one has $m \leq \frac{q_{n+1}}{2}$ (by definition of the set Y), so equation (15) is valid, while the fact that $x \in I_n$, implies

$$|\sin(2\pi q_n x)| \geq \frac{4}{n}.$$

Hence,

$$\left|\mathrm{Re}\left(i2\pi q_n \frac{|X(m,n)|}{e^{q_n}} e^{i2\pi q_n x}\right)\right| \geq 16 \frac{m}{e^{q_n}} \frac{q_n}{n}.$$

Next, and for any m, using (13) we have

$$\begin{aligned}\left|\mathrm{Re}\left(\sum_{k=1}^{q_n-1} i2\pi k \frac{X(m,k)}{e^k} e^{i2\pi k x}\right)\right| &\leq 2\pi \sum_{k=1}^{q_n-1} k \frac{|X(m,k)|}{e^k}, \\ &\leq 2\pi \sum_{k=1}^{q_n-1} \frac{q_n k}{e^k}, \\ &\leq 2\pi {q_n}^2.\end{aligned}$$

Similarly,

$$\left|\mathrm{Re}\left(\sum_{k=q_n+1}^{2q_n-1} i2\pi k \frac{X(m,k)}{e^k} e^{i2\pi k x}\right)\right| \leq 4\pi {q_n}^2.$$

Furthermore, from (12), one has

$$\begin{aligned}\left|Re\left(\sum_{k=2q_n}^{\infty} i2\pi k \frac{X(m,k)}{e^k} e^{i2\pi k x}\right)\right| &\leq 2\pi \sum_{k=2q_n}^{\infty} k \frac{|X(m,k)|}{e^{q_k}}, \\ &\leq 2\pi m \sum_{k=2q_n}^{\infty} \frac{k}{e^k}, \\ &\leq \frac{2\pi}{e^{\frac{3}{2}q_n}} m.\end{aligned}$$

Finally, from (16) and (12), we easily obtain

$$\begin{aligned}|X(m,q_n) - |X(m,q_n)|| &\leq 2|X(m,q_n)| \arg(X(m,q_n), \\ &\leq 2\pi \frac{(m-1)}{q_{n+1}} m;\end{aligned}$$

since $m \leq e^{2q_n'}$, and $q_{n+1} \geq e^{3q_n'}$, this implies

$$\left|Re\left(i2\pi q_n \frac{|X(m,q_n)| - X(m,q_n)}{e^{q_n}} e^{i2\pi q_n x}\right)\right| \leq 4\pi^2 e^{-q_n'} q_n \frac{m}{e^{q_n}}.$$

Combining all the bounds above, with the fact that $m \geq e^{2q_n}$, it follows that

$$\begin{aligned}\left|\frac{\partial \varphi_m}{\partial x}(x,y)\right| &\geq 16\frac{m}{e^{q_n}}\frac{q_n}{n} - 2\pi {q_n}^2 - 4\pi {q_n}^2 - 2\pi\frac{m}{e^{\frac{3}{2}q_n}} - 4\pi^2 e^{-q_n'} q_n \frac{m}{e^{q_n}},\\ &\geq 15\frac{m}{e^{q_n}}\frac{q_n}{n}.\end{aligned}$$

□

For the second derivatives a straightforward bound is enough here:

Proposition 3.5. *There is a constant C such that for any integer m and any $(x,y) \in \mathbf{T}^2$, one has*

$$\left|\frac{\partial^2 \varphi_m}{\partial x^2}(x,y)\right| \leq Cm.$$

The proof is direct because φ is C^2. □

Proof of Theorem 1. We are going to prove for φ the validity of the mixing criterion given by Proposition 3.3. Take for $\{\tau_n\}$ the sequence

$$\begin{aligned}\tau_{2n} &= e^{2q_n},\\ \tau_{2n+1} &= e^{2q_n'}.\end{aligned}$$

For η_{2n} take a partition of the set I_n defined above with intervals $C_i^{(2n)}$, of lengths bounded between e^{-q_n} and $2e^{-q_n}$. Because $\lambda(I_n) \geq 1 - \frac{4}{n}$, η_{2n} satisfies the conditions of Proposition 3.3.

Moreover, from Propsitions 3.4 and 3.5, for any $m \in [\frac{\tau_{2n}}{2}, 2\tau_{2n+1}]$, for all $C_i^{(2n)} \in \eta_{2n}$, and all $y_0 \in \mathbf{T}^1$, one has

$$\inf_{x \in C_i^{(2n)}} \left|\frac{\partial \varphi_m(x,y_0)}{\partial x}\right| |C_i^{(2n)}| \geq \frac{1}{2}\frac{q_n}{n},$$

and

$$\begin{aligned}\sup_{\{x \in C_i^{(2n)}\}} \left|\frac{\partial^2 \varphi_m(x,y_0)}{\partial x^2}\right| |C_i^{(2n)}| &\leq 2Cme^{-q_n},\\ &\leq 2C\frac{n}{q_n} \inf_{x \in C_i^{(2n)}} \left|\frac{\partial f_m(x,y_0)}{\partial x}\right|.\end{aligned}$$

Hence, condition (i) of Proposition 3.3 is satisfied once we take $k_{2n} = \frac{1}{2}\frac{q_n}{n}$ and $\varepsilon_{2n} = 2C\frac{n}{q_n}$. These sequences obviously go to infinity and to zero respectively when n goes to infinity as required in the criterion. The checking of the criterion in the y direction (condition (ii)), when $m \in [\frac{\tau_{2n+1}}{2}, 2\tau_{2n+2}]$, is exactly similar.

The proof of Theorem 1 is over. □

4. Mixing reparametrizations of irrational flows on $\mathbf{T}^3$.

In this section we consider irrational flows on $\mathbf{T}^3$ of vector $(\alpha, \alpha', 1)$ in $\mathbf{R}^3$, where $(\alpha, \alpha') \in \mathbf{R}^2$ is such that $R_{\alpha,\alpha'}$ is a minimal translation on $\mathbf{T}^2$:

$$\begin{cases} \frac{dx}{dt} = \alpha, \\ \frac{dy}{dt} = \alpha', \\ \frac{dz}{dt} = 1. \end{cases}$$

It is not hard to see that the reparametrization of such a flow by a strictly positive smooth function $\frac{1}{\phi}$ can be interpreted as a special flow over $R_{\alpha,\alpha'}$ on $\mathbf{T}^2$ with the ceiling function defined by

$$\varphi_{\alpha,\alpha'}(x, y) = \int_0^1 \phi(x + s\alpha, y + s\alpha', s)ds.$$

And conversely, given a special flow $\{R_{\alpha,\alpha'}, \varphi_{\alpha,\alpha'}\}$, if there is a smooth and strictly positive function ϕ on $\mathbf{T}^3$ satisfying the linear equation above, the flow can be viewed as a reparametrization of $\{T^t_{(\alpha,\alpha',1)}\}$ with velocity $\frac{1}{\phi}$.

Before we prove Corollary 1.1 we will derive from Theorem 1 the following weaker result:

Proposition 4.1. *For any irrational vector $(\alpha, \alpha') \in \mathbf{R}^2/\mathbf{Q}^2$, there exists a strictly positive real analytic function on $\mathbf{T}^3$, $\phi_{\alpha,\alpha'}$ such that*

$$\varphi(x, y) = \int_0^1 \phi_{\alpha,\alpha'}(x + s\alpha, y + s\alpha', s)ds,$$

where φ is the real analytic function we used in the main proposition. In particular, if $(\alpha, \alpha') \in Y$, the reparametrization of the irrational flow $R_{t(\alpha,\alpha',1)}$ by the function $\frac{1}{\phi_{\alpha,\alpha'}}$ is mixing.

Remark. Eventually, to make sure that the solution $\phi_{\alpha,\alpha'}$ is positive, we will need to take away from φ the first terms of its Fourier series.

Proof. The easiest way to find a solution $\phi_{\alpha,\alpha'}$ for the above equation is to look for it under the special form

$$\phi_{\alpha,\alpha'}(x, y, z) = 1 + \mathrm{Re}\left(\sum_{k=1}^{\infty} d_{k,l_k} e^{i2\pi kx} e^{i2\pi l_k z}\right) + \mathrm{Re}\left(\sum_{k=1}^{\infty} b_{k,l'_k} e^{i2\pi ky} e^{i2\pi l'_k z}\right),$$

where l_k and l'_k will be chosen later in $\mathbf{Z}$, following the lines of [13]. We have,

$$\int_0^1 \phi_{\alpha,\alpha'}(x + s\alpha, y + s\alpha', s)ds = 1 \quad + \quad \mathrm{Re}\left(\sum_{k>1} d_{k,l_k} \frac{e^{i2\pi(k\alpha+l_k)} - 1}{i2\pi(k\alpha + l_k)} e^{i2\pi kx}\right)$$

$$+ \quad \mathrm{Re}\left(\sum_{k>1} b_{k,l'_k} \frac{e^{i2\pi(k\alpha'+l'_k)} - 1}{i2\pi(k\alpha' + l'_k)} e^{i2\pi ky}\right),$$

and $\phi_{\alpha,\alpha'}$ is formally a solution of the linear equation if we take

$$d_{k,l_k} = \frac{i2\pi(k\alpha + l_k)}{e^{i2\pi(k\alpha+l_k)} - 1} e^{-k},$$
$$b_{k,l'_k} = \frac{i2\pi(k\alpha + l'_k)}{e^{i2\pi(k\alpha'+l'_k)} - 1} e^{-k},$$

which is possible to do because α and α' are irrational. In this case,

$$|d_{k,l_k}| = \frac{\pi|k\alpha + l_k|}{|\sin(\pi(k\alpha + l_k))|} e^{-k},$$
$$|b_{k,l'_k}| = \frac{\pi|k\alpha' + l'_k|}{|\sin(\pi(k\alpha' + l'_k))|} e^{-k}.$$

So, if we choose l_k to be the closest relative integer to $-k\alpha$, we will have

$$|k\alpha + l_k| \leq \frac{1}{2},$$

hence,

$$\frac{|k\alpha + l_k|}{|\sin(\pi(k\alpha + l_k))|} \leq \frac{1}{2},$$

moreover, with this choice of l_k, $|l_k| \leq |\alpha|k + 1$, therefore

$$\begin{aligned} |d_{k,l_k}| &\leq \frac{\pi}{2} e^{-k} \\ &\leq \frac{\pi}{2} e^{-\frac{1}{|\alpha|+2}(k+|l_k|)}. \end{aligned}$$

We make a similar choice of l'_k, and the function $\tilde{\phi}_\alpha$ we obtain will thus be analytic and strictly positive. The proposition is proved. □

Proof of Corollary 1.1. Let $(\alpha, \alpha') \in Y$ be fixed for the moment. Consider the special flow over $R_{\alpha,\alpha'}$ with a ceiling function

$$\varphi_{\alpha,\alpha'}(x, y) = 1 + \mathrm{Re}\left(\sum_{k=1}^{\infty} d_k e^{i2\pi kx} + \sum_{k=1}^{\infty} b_k e^{i2\pi ky} \right).$$

We see from the proof of the main theorem that a sufficient condition for the flow to be mixing is that the coefficients d_k and b_k satisfy the following

(c1) For any $k \in \mathbf{N}$ large enough,

$$|d_k| \leq e^{-k}, \qquad |b_k| \leq e^{-k},$$

(c2) For any $n \in \mathbf{N}$ large enough,

$$|d_{q_n}| \geq e^{-2q_n}, \qquad |b_{q'_n}| \geq e^{-2q'_n}.$$

Define now on $\mathbf{T}^3$

$$\phi(x, y, z) := 1 + \mathrm{Re}\left(\sum_{k=1}^{\infty} \sum_{|l|\leq k} \frac{e^{-k}}{k} \left(e^{i2\pi kx} e^{i2\pi lz} + e^{i2\pi ky} e^{i2\pi lz} \right) \right). \tag{20}$$

Corollary 1.1 will then follow from the above remark and

Proposition 4.2. *For any vector* $(\alpha,\alpha')\in Y\bigcap[-1,1]^2$ *it is true that the analytic function*

$$\varphi_{\alpha,\alpha'}(x,y)=\int_0^1\phi(x+s\alpha,y+s\alpha',s)ds,$$

satisfies both (c1) and (c2).

Proof. As in the proof of Proposition 4.1, a direct computation gives

$$\varphi_{\alpha,\alpha'}(x,y)=1+\mathrm{Re}\left(\sum_{k=1}^{\infty}d_ke^{i2\pi kx}+\sum_{k=1}^{\infty}b_ke^{i2\pi ky}\right)$$

with

$$a_k \quad = \quad \frac{e^{-k}}{k}\sum_{|l|\leq k}\frac{\sin(\pi(k\alpha+l_k))}{\pi(k\alpha+l)}$$

and a similar expression for b_k. Clearly $|d_k|\leq e^{-k}$ and (c1) is satisfied.

Now, since $|\alpha|\leq 1$, if we consider the closest integer l_{q_n} to $-q_n\alpha$, it satisfies $|l_{q_n}|\leq q_n$. Next, notice that

$$\frac{|\sin(\pi(q_n\alpha+l_{q_n}))|}{\pi|q_n\alpha+l_{q_n}|}\geq\frac{2}{\pi},$$

while for any $l\in\mathbf{Z}-\{l_{q_n}\}$

$$\frac{|\sin(\pi(q_n\alpha+l))|}{\pi|q_n\alpha+l|}\leq\frac{2}{\pi}|\sin(\pi q_n\alpha)|=O(\frac{1}{q_{n+1}}),$$

and (c2) now follows. □

5. Absence of mixing in the "general" case.

This section is devoted to the

Proof of Theorem 2. We will designate by G the set of vectors $(\alpha,\alpha')\in\mathbf{R}^2$ for which there is a sequence of integers r_n such that for any $\varphi\in C^4(\mathbf{T}^2)$ with $\int_{\mathbf{T}^2}\varphi d\mu=1$, the following holds

(G1) $\varphi_{r_n}-r_n$ converges uniformly to 0 when n tends to infinity
(G2) $R^{r_n}(\alpha,\alpha')\underset{n\to\infty}{\longrightarrow}Id_{\mathbf{T}^2}$.

This fact obviously implies that $T^{r_n}\to Id_{M_\varphi}$ when $n\to\infty$, where $\{T^t\}$ is the special flow over $R_{\alpha,\alpha'}$ and under φ. In this case, we say the flow is *rigid*, and we call the r_n rigidity times. Clearly, rigidity impedes mixing.

First we remark that the set G contains the set of vectors such that given any function $\varphi\in C^4(\mathbf{T}^2)$ of integral 1, the linearized equation $\varphi-1=\psi\circ R_{\alpha,\alpha'}-\psi$, has a continuous solution (measurable would be enough). But we can prove using Fourier analysis that the latter set, consisting of somehow "badly approximated" vectors is of total Lebesgue measure ([3], in the annex).

Next, we will prove that G contains a dense G_δ. For this we will prove the following lemmas

Lemma 5.1. *There exists a G_δ dense set $\hat{G} \in \mathbf{R}^2$ such that for any $(\alpha, \alpha') \in \hat{G}$ there exist infinitly many triplets $(p, p', q) \in \mathbf{Z} \times \mathbf{Z} \times \mathbf{N}$ such that*

(a) $|\alpha - \frac{p}{q}| < \frac{1}{e^q}, \quad p \wedge q = 1;$

(a') $|\alpha' - \frac{p'}{q+1}| < \frac{1}{e^q}, \quad p' \wedge (q+1) = 1.$

Here $p \wedge q = 1$ stands for p and q relatively prime.

Lemma 5.2. *The set $\hat{G}$ is included in G.*

Clearly, with these lemmas and the definition of the set G the proof of the proof of the theorem will follow.

Proof of lemma 5.1. Denote by $\mathcal{O}_q$ the set of vectors of $\mathbf{R}^2$ for which (a) and (a') are realizable with q. It is an open set and for any integer n, $\bigcup_{q \geq n} \mathcal{O}_q$ is open and dense because it contains the points $(\frac{p}{q}, \frac{p'}{q+1})$, $q \geq n$. Hence, $\hat{G} = \bigcap_{n \in \mathbf{N}} \bigcup_{q \geq n} \mathcal{O}_q$ is a residual subset (dense G_δ), and the lemma is proved. □

Proof of lemma 5.2. Take $(\alpha, \alpha') \in \hat{G}$. Choose a sequence (p_n, p'_n, q_n) satisfying (a) and (a'). Clearly (G2) is satisfied for the sequence $r_n = q_n(q_n + 1)$. On the other hand we can write the Fourier series of φ

$$\varphi(x, y) = 1 + \sum_{(k,l) \in \mathbf{Z}^2 - \{0,0\}} d_{k,l} e^{i2\pi kx} e^{i2\pi ly},$$

since φ is of class C^4 we have

$$|d_{k,l}| = o\left(\frac{1}{|k|^4 + |l|^4}\right).$$

For any $m \in \mathbf{N}$, write

$$\varphi_m(x, y) = m + \sum_{(k,l) \in \mathbf{N}^2 - \{0,0\}} d_{k,l} \frac{1 - e^{i2\pi m(k\alpha + l\alpha')}}{1 - e^{i2\pi(k\alpha + l\alpha')}} e^{i2\pi kx} e^{i2\pi lz}.$$

We will check that $\varphi_{r_n} - r_n$ converges uniformly to zero, with $r_n = q_n(q_n + 1)$. For any $k, l,\ |k|, |l| < q_n$ we have

$$k\alpha + l\alpha' = k\frac{p_n}{q_n} + l\frac{p'_n}{q_n + 1} + O(\frac{q_n}{e^{q_n}}),$$

since $q_n \wedge (q_n + 1)p_n = 1$, this implies

$$|||k\alpha + l\alpha'||| \geq \frac{1}{2q_n(q_n + 1)},$$

where $|||.|||$ is the distance to the closest integer. Meanwhile, (a) and (a') imply

$$|||r_n(k\alpha + l\alpha')||| = O(\frac{q_n^2}{e^{q_n}}).$$

Hence, for $k, l,\ |k|, |l| < q_n$, one has

$$\left|\frac{1-e^{i2\pi r_n(k\alpha+l\alpha')}}{1-e^{i2\pi(k\alpha+l\alpha')}}\right| = O\left(\frac{q_n^4}{e^{q_n}}\right)$$

and

$$\sum_{|k|,|l|<q_n, kl\neq 0} |d_{k,l}| \left|\frac{1-e^{i2\pi r_n(k\alpha+l\alpha')}}{1-e^{i2\pi(k\alpha+l\alpha')}}\right| = o(1).$$

On the other hand

$$\begin{aligned}\sum_{|k|+|l|\geq q_n} |d_{k,l}| \left|\frac{1-e^{i2\pi r_n(k\alpha+l\alpha')}}{1-e^{i2\pi(k\alpha+l\alpha')}}\right| &\leq r_n \sum_{|k|+|l|\geq q_n} |d_{k,l}|, \\ &= r_n \sum_{|k|+|l|\geq q_n} o\left(\frac{1}{|k|^4+|l|^4}\right), \\ &= o(1).\end{aligned}$$

We showed that (G2) holds for the sequence r_n and the lemma is proved. □

6. Generalization and application.

Let $\{T, M, \rho\}$ be a dynamical system and consider the product transformation $T \times R_{\alpha,\alpha'}$ on $\mathbf{T}^2 \times M$, with $(\alpha, \alpha') \in Y$. This transformation preserves the product measure of ρ with the Haar measure on $\mathbf{T}^2$, μ. Now consider special flows over $T \times R_{\alpha,\alpha'}$ with a ceiling function that depends only on the variables of $\mathbf{T}^2$ and that we take equal to φ of the main theorem. We will denote this special flow by $\{T \times R_{\alpha,\alpha'}, \varphi\}$. Reasoning exactly as in the proof of the main theorem we obtain

Theorem 3. *If $T \times R_{\alpha,\alpha'}$ is uniquely ergodic then $\{T \times R_{\alpha,\alpha'}, \varphi\}$ is mixing (for the invariant measure $\rho \times \mu$).*

This theorem permits to generalize Theorem 1 to higher dimensional tori

Corollary 3.1 (Higher dimension)**.** *If $\beta \in \mathbf{R}^k$ is such that the translation $R_{\alpha,\alpha',\beta}$ on $\mathbf{T}^{k+2}$ is irrational, then $\{R_{(\alpha,\alpha',\beta)}, \varphi\}$ is mixing. This special flow can also be viewed as a mixing analytic reparametrization of the irrational flow $\{T^t_{(\alpha,\alpha',\beta,1)}\}$ on $\mathbf{T}^{k+3}$.*

Corollary 1.1 on mixing reparametrizations can be stated in a slightly more general way

Theorem 4. *There is a dense and uncountable subset $\mathcal{B} \subset \mathbf{R}^3$ and a dense subset $F \subset C^w_{\frac{1}{2\pi}}(\mathbf{T}^3, \mathbf{R}^{+,*})$, such that, any reparametrization of an irrational translation on $\mathbf{T}^3$ with vector $\beta \in \mathcal{B}$ by $\frac{1}{\phi}$, $\phi \in F$, is mixing.*

We will apply this corollary to the cubic billiard. A cubic billiard is the dynamical system obtained when we consider the uniform motion of a point mass inside a cube C in the Euclidian space $\mathbf{R}^3$, with the usual laws of reflection when the point hits a side. For the moment, velocity is taken to be constant equal to unity. The phase space of this dynamical system, that we call M is obtained from the direct product $C \times \mathbf{S}^2$, where $\mathbf{S}^2$ is the sphere of unit velocities, by identifying pairs of the form (ξ, v), (ξ, v') for $\xi \in \partial C$, v and $v' \in \mathbf{S}^2$ and $v - v' = 2(n(\xi), v)n(\xi)$; $n(\xi)$ being the unit exterior normal to ∂C at the point ξ. We denote by $\{T^t\}$ the corresponding phase flow. It preserves the measure induced on M by the volume measure on $C \times \mathbf{S}^2$, $d\xi dv$. Actually, in this description we neglected the elements $(\xi, v) \in M$ that might ever meet the vertices of the cube, this exceptional set being of volume zero. Given a direction $v = (\alpha, \beta, \gamma)$, $\alpha^2 + \beta^2 + \gamma^2 = 1$, $\alpha\beta\gamma \neq 0$, where the coordinate axes are given by the vertices of C, we denote by $C(v)$ the "cube" $C \times v$ in the phase space. The trajectory of a point $(c, v) \in M$ will be a straight line in $C(v)$ until it hits one side of the cube. After reflection the point goes into $C(v')$ where v' is obtained from v by just changing the sign of one of the coordinates α, β, γ of v, depending on the side the point hits. So, the union of those eight cubes, that we call M_v is an invariant subset of the phase space under the flow action. To visualise the motion on M_v we put the cubes $C(v)$ and $C(v')$ side by side in $\mathbf{R}^3$ (along the incidence side) and we can view the trajectory of the point before and after the reflection not as a broken line but rather as a straight one that crosses from $C(v)$ to $C(v')$. By gluing in this fashion the eight components of M_v we obtain a cube in $\mathbf{R}^3$ on which the flow is the translation with direction v. So as for the general planar polygon billiard with rational angles (see [15]), we have density of the trajectories in the configuration space for almost every initial direction.

Assume now as above that the motion in the cube is rectilinear with elastic reflection on the boundary, but suppose that the modulus of the velocity depends on the position ξ in an analytic way, the point going faster or slower than unity speed but never stopping or changing direction unless it hits a side of the cube. If the velocity distribution is given by a function $\frac{1}{\phi}$ where ϕ is in the dense space $F \subset C^w_{\frac{1}{2\pi}}(\mathbf{T}^3, \mathbf{R}^{+,*})$ of Theorem 4, then there is an uncountable dense set of initial directions ($v \in \mathcal{B}$) for which the corresponding flow on the configuration space is mixing. This turns out to be impossible to do in the case of a polygonal billiard with rationally related angles since by a result of Katok [5], any special flow over an interval exchange transformation built under a function of bounded variation is not mixing.

7. A mixing special flow over a Diophantine irrational translation on $\mathbf{T}^2$.

We say that $\hat{\alpha} \in \mathbf{R}^n$ satisfies a *diophantine condition with exponent* β, if there exist a positive constant C such that for any vector $k \in \mathbf{Z}^n - \{0\}$ we have :

$$|||<k,\hat{\alpha}>||| \geq \frac{C}{|k|^{n+\beta}}; \tag{21}$$

where $<.,.>$ is the canonical scalar product on $\mathbf{R}^n$ and $|.|$ its associated norm, while $|||.|||$ is the distance to the closest integer.

In the previous section we constructed a mixing special flow built over an irrational translation of the two torus and under an analytic ceiling function. But the vector of the translation we used had perforce very good rational approximations, i.e. infinitely many vectors $k \in \mathbf{Z}^n - \{0\}$ such that

$$|||<k,\hat{\alpha}>||| \leq \frac{1}{e^{|k|}}.$$

Indeed, if the vector of the translation on the basis is Diophantine, and the ceiling function is regular enough the flow will be conjugate to an irrational flow.[2] So, one might look, for a Diophantine vector, what happens with less regular ceiling functions.

Here we give an example of a mixing special flow over an irrational translation of $\mathbf{T}^2$ *with a* C^2 *ceiling function; the vector of the translation,* $\hat{\alpha}$, *being* $(2+\epsilon)$*-diophantine for any* $\epsilon > 0$.

From what what was said about the cohomological equation in the footnotes, note that under this Diophantine condition on $\hat{\alpha}$, any special flow over $R_{\hat{\alpha}}$ with a ceiling function of class C^r, $r > 4$, is conjugate to an irrational flow on $\mathbf{T}^3$. However, for a β-Diophantine translation vector on the torus $\mathbf{T}^n$ (and for this one in particular), we do not know if it is possible to realize the best one can expect, that is a $C^{n+\beta-\epsilon}$ ceiling function for which the flow is mixing. (In the case of weak mixing the question is fully answered in [2]).

We will first choose properly the vector $\hat{\alpha} \in \mathbf{R}^2$, then we will give the expression of the ceiling function φ under which the special flow over $R_{\hat{\alpha}}$ is mixing. We prove mixing using the same criterion given by the fundamental proposition in the analytic case.

7.1. Construction of $\hat{\alpha}$.

[2]See our footnote 1. The precise result in [3], is the following: Assume α is β diophantine and let $\theta = d + \beta$ (when $\beta = 0$ take $\theta = d + \epsilon$). Then if $r > \theta$, and $\varphi \in C^r(\mathbf{T}^d)$, there exists $\psi \in C^{r-\theta}(\mathbf{T}^d)$ satisfying the cohomological equation (*) if $r - \theta$ is not integer; if $r - \theta$ is integer then the solution ψ is $C^{r-\theta-1}$ and "smooth in the sense of Zygmund".

First recall that any irrational number $\alpha \in \mathbf{R}-\mathbf{Q}$ has a writing in continued fraction

$$\alpha = [a_0, a_1, a_2, ...] = a_0 + 1/(a_1 + 1/(a_2 + ...)),$$

where $\{a_i\}_{i\geq 1}$ is a sequence of integers ≥ 1, $a_0 = [\alpha]$. Conversely any sequence $\{a_i\}_{i\in\mathbf{N}}$ corresponds to a unique number α. The convergents of α are given by the a_i in the following way:

$$p_n = a_n p_{n-1} + p_{n-2} \quad \text{for} \quad n \geq 2, p_0 = a_0, p_1 = a_0 a_1 + 1,$$
$$q_n = a_n q_{n-1} + q_{n-2} \quad \text{for} \quad n \geq 2, q_0 = 1, q_1 = a_1.$$

In our construction of $\hat{\alpha} = (\alpha, \alpha')$, we will keep in mind the mechanism that allowed us to obtain mixing in the analytic case, namely the alternation between the denominators of the convergents of α and those of α'. Although here we can not space them too much because if we want the vector (α, α') to be Diophantine, α and α' should at least be so individually. To avoid good approximations also, we will ask that the convergents of α and α' should be relatively prime. The arithmetic properties we need are summarized and proven to be realizable in the following proposition:

Proposition 7.1. *One can choose two real numbers α and α' rationally independent in $[0,1]$ such that for any n:*

$$n^4 {q'}^2_{n-1} \leq q_n \leq 2n^4 {q'}^2_{n-1}, \tag{22}$$
$$n^4 q_n^2 \leq q'_n \leq 2n^4 q_n^2 \tag{23}$$

and

$$q_n \wedge q'_{n-1} = 1, \tag{24}$$
$$q'_n \wedge q_n = 1. \tag{25}$$

Here, $q_n \wedge q'_{n-1} = 1$ stands for q_n and q'_{n-1} are relatively prime.

Proof. We will construct by induction the sequences $\{a_n\}_{n\geq 1}$ and $\{a'_n\}_{n\geq 1}$ corresponding to α and α'. Assume the a_i and a'_i, $i \leq n-1$, are chosen such that (22)-(25) hold until step $n-1$. Because q_{n-1} and q'_{n-1} are relatively prime there exists an integer $\tau_n < q'_{n-1}$ such that

$$\tau_n q_{n-1} \equiv -q_{n-2}[q'_{n-1}],$$

meaning that q'_{n-1} divides $\tau_n q_{n-1} + q_{n-2}$. Now, choose $a_n = \tau_n + \rho_n$ where ρ_n is such that

$$\rho_n \wedge q'_{n-1} = 1 \quad \text{and} \quad n^4 {q'}^2_{n-1} \leq \rho_n q_{n-1} \leq \frac{3}{2} n^4 {q'}^2_{n-1}.$$

In this case

$$q_n = a_n q_{n-1} + q_{n-2} = \rho_n q_{n-1} + \tau_n q_{n-1} + q_{n-2},$$

since $\tau_n \leq q'_{n-1}$ we have clearly $n^4 {q'}^2_{n-1} \leq q_n \leq 2n^4 {q'}^2_{n-1}$, and (22) is verified at step n. On the other hand, from the recurrence hypothesis $q_{n-1} \wedge q'_{n-1} = 1$ and from our choice of ρ_n, it follows that $\rho_n q_{n-1} \wedge q'_{n-1} =$

1, while by definition $\tau_n q_{n-1} + q_{n-2}$ is a multiple of q'_{n-1}. Consequently $q_n \wedge q'_{n-1} = 1$ and (24) is true at step n.
We follow the same lines in the construction of a'_n in order to implement (23) and (25): Take $a'_n = \tau'_n + \rho'_n$ such that q_n divides $\tau'_n q'_{n-1} + q'_{n-2}$, while $\rho'_n \wedge q_n = 1$ and (23) holds. Then we use (24) that we implemented above to prove (25) at step n. □

This proposition will allow us to prove the following on $\hat{\alpha} = (\alpha, \alpha')$:

Proposition 7.2. *The vector $\hat{\alpha}$ is $2+\epsilon$-diophantine for any $\epsilon > 0$*

Proof. Fix $\epsilon > 0$. We have to show that for any $(k,l) \in \mathbf{Z}^2$, $(k,l) \neq (0,0)$, $|k|+|l|$ sufficiently large

$$|||k\alpha + l\alpha'||| \geq \frac{1}{(|k|+|l|)^{4+\epsilon}}. \tag{26}$$

We will first treat of the case when $kl = 0$. Assume $l = 0$, and take n such that $q_{n-1} \leq k < q_n$. First note that (22) and (23) imply

$$q'_n \leq 8n^{12} {q'}^4_{n-1}, \tag{27}$$

$$q_n \leq 8n^{12} q^4_{n-1}, \tag{28}$$

since $\frac{p_n}{q_n}$, $n \in \mathbf{N}$ are the best approximations of α, one has for $k < q_n$

$$\begin{aligned} |||k\alpha||| &\geq \frac{1}{2q_n}, \\ &\geq \frac{1}{16n^{12}q^4_{n-1}}, \end{aligned}$$

since $k \geq q_{n-1}$ when n is sufficiently large this leads to

$$|||k\alpha||| \geq \frac{1}{|k|^{4+\epsilon}}.$$

Using (28) we can obtain the same result when it is k that vanishes.

Now suppose that both k and l are unequal to 0. There exists $n \in \mathbf{N}$ such that

$$q'_{n-1} \leq |k|+|l| < q'_n.$$

We will separate two cases:

(i) $q'_{n-1} \leq |k|+|l| < q_n$,
(ii) $q_n \leq |k|+|l| < q'_n$.

Case (i). Recall that

$$\alpha = \frac{p_n}{q_n} - \frac{1}{q_n q_{n+1}} + h.o.t.$$

$$\alpha' = \frac{p'_{n-1}}{q'_{n-1}} - \frac{1}{q'_{n-1}q'_n} + h.o.t.$$

where in $h.o.t.$ we have rests that are less than $\frac{1}{q'_{n+2}}$. As $|k| < q_n$, and $|l| < q_n$ one has using (22) and (23)

$$\begin{aligned} |k\alpha + l\alpha' - k\frac{p_n}{q_n} - l\frac{p'_{n-1}}{q'_{n-1}}| &\leq \frac{1}{q_{n+1}} + \frac{q_n}{q'_{n-1}q'_n} + h.o.t., \\ &\leq \frac{1}{{q_n}^4} + \frac{1}{n^4 q_n q'_{n-1}} + h.o.t., \\ &= o(\frac{1}{q'_{n-1}q_n}). \end{aligned}$$

On the other hand, because $q_n \wedge q'_{n-1} = 1$ and $q_n \wedge p_n = 1$, $k < q_n$ imply

$$||k\frac{p_n}{q_n} - l\frac{p'_{n-1}}{q'_{n-1}}|| \geq \frac{1}{q'_{n-1}q_n}.$$

With the estimation above this implies

$$||k\alpha + l\alpha'|| \geq \frac{1}{2q'_{n-1}q_n},$$

and if we use (22) one more time

$$\begin{aligned} ||k\alpha + l\alpha'|| &\geq \frac{1}{2n^4{q'_{n-1}}^3}, \\ &\geq \frac{1}{{q'_{n-1}}^4}, \\ &\geq \frac{1}{(|k|+|l|)^4}. \end{aligned}$$

Case (ii). This time, we write

$$\begin{aligned} \alpha &= \frac{p_n}{q_n} - \frac{1}{q_n q_{n+1}} + h.o.t. \\ \alpha' &= \frac{p'_n}{q'_n} - \frac{1}{q'_n q'_{n+1}} + h.o.t. \end{aligned}$$

and the same computations give

$$\begin{aligned} ||k\alpha + l\alpha'|| &\geq \frac{1}{2q'_n q_n}, \\ &\geq \frac{1}{2n^4{q_n}^3}, \\ &\geq \frac{1}{{q_n}^4}, \\ &\geq \frac{1}{(|k|+|l|)^4}. \end{aligned}$$

□

Define now

$$\varphi(x,y) = 1 + \mathrm{Re}\left(\sum_{k=1}^{\infty} \frac{e^{i2\pi q_k x}}{k^2 {q_k}^2} + \sum_{k=1}^{\infty} \frac{e^{i2\pi q_k' y}}{k^2 {q_k'}^2}\right);$$

then, clearly φ is C^2, and we have

Theorem 5. *The special flow over $R_{\hat{\alpha}}$ with the ceiling function φ is mixing.*

The proof of this theorem is similar to the one given in the analytic case. Indeed, we can easily check the criterion of Proposition 3.3 in Section 3.2:
-when t is in $[n^3 q_n^2, n^3 {q_n'}^2]$, uniform stretch is proved in the x direction,
-when t is in $[n^3 {q'}_n^2, n^3 q_{n+1}^2]$ uniform stretch is proved in the y direction.

□

ACKNOWLEDGMENTS. I wish to thank Patrice Le Calvez for his continuous interest in this work, ever since its first version of January 1997. His advice and comments were very helpful, particularly in the statement of the fundamental proposition on mixing, Proposition 3.1. I also wish to thank the referee for the ameliorations he suggested.
The final version of this work was carried on during my stay at IMPA (Rio De Janeiro, Brazil) in the spring of 1999. I wish to thank the members of this institution for their warm hospitality and for many thought-provoking conversations.
I am especially indebted to Michael Herman for much advice, encouragement and inspiration.

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