CONTINUOUS SPECTRUM ON LAMINATIONS OVER AUBRY-MATHER SETS.

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Abstract. If we perturb a completely integrable Hamiltonian system with two degrees of freedom, the perturbed flow might display, on every energy level, invariant sets that are laminations over Aubry-Mather sets of a Poincaré section of the flow. Each one of these laminations carry a unique invariant probability measure for the flow and it is interesting therefore to understand the statistical properties of these measures. From a result of Kocergin in [12], we know that mixing is *a priori* impossible. In this paper, we investigate on the possible occurrence of weak mixing.

The answer will essentially depend on the number of orbits of gaps in the Aubry-Mather set. More precisely, if the Aubry-Mather set has exactly one orbit of gaps and is hyperbolic then the special flow over it with any smooth ceiling function will be conjugate to a suspension with a constant ceiling function, failing hence to be weak mixing or even topologically weak mixing. To the contrary, if the Aubry-Mather set has more than one orbit of gaps with at least two in a *general position* then the special flow over it will in general be weak mixing.

1. Introduction.

For an integrable Hamiltonian system with N degrees of freedom, the $2N$-dimensional phase space is completely foliated with invariant N-dimensional tori on which the motion is that of a translation. The Kolmogorov-Arnold-Moser (KAM) theorem states that for a generic and sufficiently small C^∞ perturbation of such a completely integrable system, an arbitrarily large proportion (in measure) of the invariant tori will be preserved, i.e. slightly deformed into invariant tori where the dynamics is C^∞ conjugate to translation flows. These preserved tori correspond in general to frequencies that are badly approximated by rational vectors, i.e. Diophantine translation vectors. It is then a natural question to ask what type of behavior can we expect on the other invariant sets of the perturbed system, sets corresponding to the non-Diophantine frequencies.

Among such sets we find, in the closure of the KAM tori, invariant tori that carry non linearizable flows. Indeed, for the generic C^∞ perturbation of the integrable Hamiltonian system, M. Herman proved in [7] that on a residual subset among the invariant tori of the perturbed flow the dynamics is uniquely ergodic and weak mixing. Since the invariant tori he considers lie in the closure of KAM tori, to prove the result above Herman investigates on the generic behavior of a diffeomorphism that is in the closure of smoothly linearizable ones. To this end, one way is to use the successive conjugations techniques as in [1] or [3] which imply genericity of unique ergodicity and weak mixing. Another way, which can still be seen as a particular case of successive conjugations, is to consider reparametrizations of linear flows since such reparametrized flows are always in the closure of flows that are smoothly conjugate to linear ones (as it follows from the famous result of Kolmogorov on the rigidity under smooth reparametrizations of Diophantine linear flows [13]). Following this program we have studied in [5] the occurrence of weak mixing for reparametrizations of minimal translation flows on the torus $\mathbf{T}^n$, $n \geq 2$ and proved that for a non-Diophantine linear flow, the flow obtained after reparametrization with a strictly positive smooth function ϕ is in general (i.e. for ϕ in a G_δ dense subset of $C^\infty(\mathbf{T}^n, \mathbf{R}_+^*)$) weak mixing for its unique invariant measure. That is the content of the theorem we will state in §2.5.

What about the dynamics on invariant sets that are not tori? Here, we will only consider Hamiltonian systems with two degrees of freedom and a class of perturbations of completely integrable systems for which the perturbed flow displays invariant sets that are laminations over invariant Aubry-Mather sets. Indeed, consider on an energy level of the perturbed flow a two dimensional Poincaré section: by KAM theory, this section contains invariant circles and hence an invariant annulus for the Poincaré return map. In addition, the Poincaré map is under generic conditions on the perturbation a twist map for which the Aubry-Mather theory is thus valid (Cf. Section 2). Each Aubry-Mather set carries a unique invariant probability measure for the Poincaré map, and so does the laminations for the suspended flow. For the perturbations we consider the dynamics on the laminations is not weak mixing because it is a constant time suspension. Our aim in this paper is to study the display of weak mixing after time change (Cf. §2.3 and 2.4 for the definitions).

Depending on whether the Aubry-Mather set (we will call Aubry-Mather sets the Cantor minimal invariant sets exclusively) has one or many *holes* (Cf. Definition 1 in §2.6) the results will be very contrasting. More precisely, given an Aubry-Mather set $\mathcal{A}_\alpha$ of the Poincaré map, we denote by $\mathcal{S}_\alpha$ the corresponding invariant laminations by the flow and prove the following results: if the Aubry-Mather set $\mathcal{A}_\alpha$ has one hole and is hyperbolic, then any smooth reparametrization of the flow has its restriction to $\mathcal{S}_\alpha$ conjugate to a suspension flow over $\mathcal{A}_\alpha$ with a constant ceiling function, failing thus to be weak mixing (Cf. Theorem 3.2 in §3). To the contrary, if $\mathcal{A}_\alpha$ has more than one hole with at least two in a *general position* (Cf. Definition 2 in §2.6), then arbitrarily close to 1 (in the C^∞ topology), there exists a function ϕ such that the reparametrization of the perturbed flow by ϕ is weak mixing on $\mathcal{S}_\alpha$ (Theorem 3.3).

REMARK 1. We do not know how often two holes of an Aubry-Mather set are in a general position. The arithmetic condition between the point projections of two gaps that defines them as being in general position (Cf. Definition 2) is generic but we do not know how to lift the condition into a generic property on the Aubry-Mather sets. However, we will see that for any irrational α, the numbers 0 and 1/2 are in a general position with respect to α (Cf. the proof of Proposition 4.6). Hence, considering two fold coverings, an Aubry-Mather set with one hole becomes an Aubry-Mather with two holes (one in 0, and one in 1/2) in a general position for which Theorem 3.3 applies (Cf. Corollary 3.4).

REMARK 2. It is worth noticing here that an Aubry-Mather set is typically hyperbolic (as proved by Patrice Le Calvez [14], Cf. §2.7.1 below) and has one hole (the single gap Theorem by Veerman [20]). Hence the absence of weak mixing prevails on the laminations. Furthermore, Theorem 3.2 can be viewed as a statement of rigidity under time change in the case of non-Diophantine frequencies when the invariant tori degenerate into Aubry-Mather sets, similar to the rigidity result of Kolmogorov in the case of invariant tori carrying Diophantine linear flows [13], and surprisingly contrasting with the instability results for invariant tori with Liouvillian frequencies [7], [5] (See §2.5).

2. THE SETTING.

2.1. Let (θ, r, s, u) be a system of coordinates on $\mathbf{T}\times\mathbf{R}\times\mathbf{T}\times\mathbf{R}$. Given a real function $H(\theta, r, s)$ of class C^∞, and a real number ε, we consider the smooth Hamiltonian system given by

$$\hat{H}(\theta, r, s, u) := u + \frac{r^2}{2} + \varepsilon H(\theta, r, s). \tag{1}$$

We consider the standard symplectic structure $\omega_0 = d\theta\wedge dr + ds\wedge du$. We denote by $X^t_{\hat{H}}$ the Hamiltonian flow corresponding to $\hat{H}$, that is the flow given by the vector field $X_{\hat{H}}$ satisfying the formula $\omega_0(X_{\hat{H}}, .) = d\hat{H}(.)$. Since $ds/dt = \partial\hat{H}/\partial u = 1$, the coordinate s can be assimilated to time. On an energy surface $\mathcal{E}_{\hat{H}=\hat{H}_0}$, consider the Poincaré section $s = 0$ parametrized by $(\theta, r)\in\mathbf{T}\times\mathbf{R}$

$$u := \hat{H}_0 - \frac{r^2}{2} - \varepsilon H(\theta, r, 0).$$

It was proved by Robinson [17] that for the generic $H \in C^\infty(\mathbf{T}\times\mathbf{R}\times\mathbf{T}, \mathbf{R})$, the flow $X^t_{\hat{H}}$ satisfies the Kupka Smale condition. In this case, for ε small enough, there exists $R > 0$ such that the projection on (θ, r), $|r| \le R$, of the Poincaré section map is a monotone twist map, denote it f. From KAM theory, we know that there exists an invariant annulus for this map inside $\mathbf{T} \times [-\frac{R}{2}, \frac{R}{2}]$; denote this annulus by A_0.

2.2. A compact set $\mathcal{A}$ in the annulus $A_0 \subset \mathbf{T} \times \mathbf{R}$ is said to be f-ordered if it is invariant by f, if it projects injectively on $\mathbf{T}$ and if the restriction of f to $\mathcal{A}$ preserves the natural order given by the projection. From the Aubry-Mather theory it follows from the fact that f is a monotone twist map that for every $\alpha \in [\alpha_1, \alpha_2]$, where α_i, $i = 1, 2$ are the rotation numbers on the boundaries of A_0, there exists a minimal f-ordered set $\mathcal{A}_\alpha \subset A_0$. If α is irrational the set $\mathcal{A}_\alpha$ is either a Cantor set or a continuous graph $(\theta, \psi(\theta))$. In both cases order preserving implies that the restriction f_α of f to $\mathcal{A}_\alpha$ is semi-conjugate to the irrational rotation R_α just like orientation preserving homeomorphisms of the circle with irrational rotation number are. The case of a Cantor set corresponding to a homeomorphism called a *Denjoy counter-example* that displays a wandering interval, i.e., an interval disjoint from all its iterates.

We will be chiefly interested in this paper with the case where $\mathcal{A}_\alpha$ is a Cantor set. The semi-conjugacy between f_α and the rotation is obtained in the following way: we choose a point u on the Cantor set and we project it to a point on the circle, say 0. Then to each iterate of u by f_α we associate the iterate of 0 by R_α. Then we extend by continuity using minimality and bridging between any two points of the Cantor set separated by a gap by projecting them on the same point. Doing so we get a map h that is continuous from $\mathcal{A}_\alpha$ onto $\mathbf{T}$ and satisfies $h \circ f_\alpha = R_\alpha \circ h$. Besides the map h is injective except on endpoints of gaps, hence we can define an inverse h^{-1} except at the points of countably many orbits of the rotation corresponding to different orbits of gaps on $\mathcal{A}_\alpha$; we will call each such orbit a *hole*. At the points where h is not invertible, we take h^{-1} to be the right extremity of the corresponding gap. The map h^{-1} thus defined will be right semi-continuous.

We recall that $\mathcal{A}_\alpha$ lies on a Lipschitz graph and that the projection on $\mathbf{T}$ is bi-Lipschitz. This will be of importance in §4.3 as we will reduce the study of special flows over Aubry-Mather sets to the study of special flows over rotations on the circle.

Finally, to any Aubry-Mather set $\mathcal{A}_\alpha$ of the Poincaré section map defined above corresponds for the flow $X^t_{\hat{H}}$ a minimal invariant set $\mathcal{S}_\alpha$ on which the dynamics is a suspension over f_α with a constant ceiling function equal to one (since $ds/dt = 1$). The only invariant probability measure on $\mathcal{S}_\alpha$ is the normalized product of the invariant measure of f_α on $\mathcal{A}_\alpha$ with the Lebesgue measure on the fibers. Hence the spectrum of the flow $X^t_{\hat{H}}$ on $\mathcal{S}_\alpha$ is not continuous since the function $e^{i2\pi s}$ is an eigenfunction of the flow. The eigenfunction being continuous the flow is not topologically weak mixing (Cf. §2.4).

2.3. *Reparametrizations.* We address the following question: *How sensitive is the dynamics on $\mathcal{S}_\alpha$ under time change of the flow, i.e., if all the orbits of $X^t_{\hat{H}}$ are kept intact but the velocity along them is modified?*

A smooth reparametrization of a flow given by a smooth vector field is done by multiplication of the original vector field by some smooth and strictly positive real function ϕ. In the case of a Hamiltonian system, if we restrict our study to an energy surface $\mathcal{E}_{\hat{H}=\hat{H}_0}$ of the flow $X^t_{\hat{H}}$, a time change can be performed using the Hamiltonian

$$\hat{H}_\phi := \phi(\theta, r, s, u)(\hat{H}(\theta, r, s, u) - \hat{H}_0), \tag{2}$$

for some smooth real function ϕ, $\phi > 0$. The energy surface $\mathcal{E}_{\hat{H}_0}$ is indeed invariant by the Hamiltonian flow corresponding to $\hat{H}_\phi$ denoted by $X^t_{\hat{H}_\phi}$; and if we write the vector field of $X^t_{\hat{H}_\phi}$ on $\mathcal{E}_{\hat{H}_0}$, we notice that it is exactly the vector field of $X^t_{\hat{H}}$ multiplied by ϕ.

It is a general fact that the time change of a uniquely ergodic system is uniquely ergodic (Cf. for example [16]). Here the unique invariant probability measure of the reparametrized flow is absolutely continuous with respect to the original measure and has density $\frac{1}{\phi}$.

In the sequel, we will discuss whether a reparametrization given by (2) can yield a continuous spectrum on the laminations $\mathcal{S}_\alpha \subset \mathcal{E}_{\hat{H}_0}$. Since we are considering perturbations of completely integrable Hamiltonians we will pay a special attention to reparametrizations that are small perturbations, that is, in time change functions ϕ that are close to one.

2.4. *Continuous spectrum.* We say that a measure preserving flow (T^t, X, μ) has a *continuous spectrum* if and only if it does not have an eigenfunction, i.e., a measurable non constant complex function h such that $h(T^t x) = e^{i\lambda t} h(x)$, at almost every $x \in X$ for some $\lambda \in \mathbf{R}$.

An equivalent property is *weak mixing*: We say that a measure preserving flow (T^t, X, μ) has the weak mixing property if and only if for all measurable sets A and B

$$\mu(T^{-t}A \bigcap B) \longrightarrow \mu(A)\mu(B),$$

when $|t|$ goes to infinity on a set of density one over $\mathbf{R}$.

For the equivalence between the definitions see for example the book by Parry [16].

A flow is said to be *topologically weak mixing* if it does not have a continuous non constant eigenfunction.

2.5. *The case of invariant tori.* In the case where the invariant set $\mathcal{A}_\alpha$ of the Poincaré map is a smooth circle and f_α is C^∞ conjugate to the translation R_α (KAM invariant tori), the question of possibly having a continuous spectrum after time change was asked by Kolmogorov and treated by himself [13], Shklover [18], Herman [9] and others. In [5] the following dichotomy was proved:

THEOREM. *Let R_α be a minimal translation on $\mathbf{T}^d$. Then either one of two possibilities hold:*
(i) The vector α is Diophantine, and for any $\phi \in C^\infty(\mathbf{T}^{d+1}, \mathbf{R}_+^*)$, the reparametrization of the flow $R_{t(1,\alpha)}$, with speed ϕ is C^∞ conjugate to a translation flow on $\mathbf{T}^{d+1}$.
(ii) The vector α is Liouville (i.e. not Diophantine), and for a dense G_δ of $\phi \in C^\infty(\mathbf{T}^{d+1}, \mathbf{R}_+^*)$, the reparametrization of the flow $R_{t(1,\alpha)}$, with speed ϕ is weak mixing (for its unique invariant measure).

Alternative (i) was proven by Kolmogorov for $d = 1$ and generalized to any dimension by Herman.

2.6. *The case of laminations.*

DEFINITION. 1. *If the Aubry-Mather set has exactly k orbits of wandering intervals, we say it is an Aubry-Mather set with k holes.*

Assume I and J are two gaps whose endpoints lie on distinct orbits of f_α on $|\mathcal{A}_\alpha$ and let β_I and β_J be their point projections on the circle by some semi-conjugacy of f_α to R_α. It is easy to see that the difference $\beta_I - \beta_J$ does not depend on the choice of the semi-conjugacy.

DEFINITION. 2. *Holes in a general position. We say that the holes of $\mathcal{A}_\alpha$, corresponding to I and J, are in a general position if $\beta = \beta_I - \beta_J$ satisfies*

$$||q_n \beta|| \underset{n\to\infty}{\not\longrightarrow} 0, \tag{3}$$

where q_n, $n \in \mathbf{N}$ are the denominators in the successive convergents of α, and $||.||$ is the distance to the closest integer.

We remind that it is possible to define the sequence q_n in the following unique way: $q_{-1} = q_0 = 1$, and for every $n \in \mathbf{N}$

$$\| q_n \alpha \| < \| k\alpha \|, \qquad \forall k < q_{n+1}. \tag{4}$$

The sequence q_n can equally be defined as satisfying the recurrence relation

$$q_n = a_n q_{n-1} + q_{n-2} \ \text{ for } \ n \geq 1, \quad q_{-1} = q_0 = 1, \tag{5}$$

where the a_n are the partial quotients in the continued fraction expansion of α (Cf. for example [8], Chapter V).

REMARK. Given an irrational number α, the set of numbers β, satisfying (3) is a dense G_δ in $\mathbf{R}$ of full measure on any compact interval.

2.7. *Hyperbolic Aubry-Mather sets.* Our first result, related to Aubry-Mather sets with only one hole, will be stated just for *hyperbolic* Aubry-Mather sets (i.e. admitting stable and unstable bundles) and will be based on the following facts:

2.7.1. The assumption of hyperbolicity is relevant since P. Le Calvez has shown that under the same generic condition that $X^t_{\hat{H}}$ should be Kupka Smale, it is true for an open and dense set of $\alpha \in \mathbf{R}$ that the Aubry-Mather set with rotation number α is hyperbolic [14].

2.7.2. When this is the case, A. Fathi shows that the Hausdorff dimension of the Aubry-Mather set is equal to zero [4] (he shows even more, that the union of the hyperbolic Aubry-Mather sets has Hausdorff dimension zero).

2.7.3. The sizes of $f^i_\alpha(I)$ decrease to zero since they are disjoint intervals on a Lipschitz graph. The hyperbolicity assumption on $\mathcal{A}_\alpha$ forces these sizes to decrease in fact geometrically.

3. The results.

3.1. The notations are as in the precedent section: the flow corresponding to the Hamiltonian given in (1) is denoted by $X^t_{\hat{H}}$. The flow arising from the reparametrization (2) of $X^t_{\hat{H}}$, on the energy level $\mathcal{E}_{\hat{H}=\hat{H}_0}$ and with speed ϕ, is denoted by $X^t_{\hat{H}_\phi}$. On the energy surface $\mathcal{E}_{\hat{H}=\hat{H}_0}$ we consider the Poincaré section $s = 0$ and an annulus A_0 on this section invariant by the Poincaré return map (§2.1). An Aubry-Mather set corresponding to a frequency α is denoted by $\mathcal{A}_\alpha$, and $\mathcal{S}_\alpha$ denotes the lamination over $\mathcal{A}_\alpha$ invariant by the flow $X^t_{\hat{H}}$ as well as by the flow $X^t_{\hat{H}_\phi}$. Finally, the time change function ϕ, defined on $\mathbf{T}\times\mathbf{R}\times\mathbf{T}\times\mathbf{R}$ is assumed to be of class at least C^1 and strictly positive. Naturally, the reparametrized flow is as close to the initial flow as ϕ is close to 1.

3.2. Recall Definition 1 of (2.6) on the number of holes of an Aubry-Mather set.

Theorem. *If $\mathcal{A}_\alpha$ is a hyperbolic Aubry-Mather set and has one hole, then, for any time change function ϕ of class C^1 with mean value 1, the restriction of the flow $X^t_{\hat{H}_\phi}$ to $\mathcal{S}_\alpha$ is C^0 conjugate to the initial flow, i.e. to a suspension flow above $(\mathcal{A}_\alpha, f_\alpha)$ with a constant suspension function equal to one.*

It immediately follows that, under these hypothesis on $\mathcal{A}_\alpha$, the restriction of the reparametrized flow to $\mathcal{S}_\alpha$ is never topologically weak mixing.

3.3. In the case $\mathcal{A}_\alpha$ is an Aubry-Mather sets with more than one hole ($\mathcal{A}_\alpha$ hyperbolic or not), we have with the notion of general position given in Definition 2 of (2.6)

Theorem. *If $\mathcal{A}_\alpha$ is an Aubry-Mather set with at least two holes in a general position, then arbitrarily close (in the C^∞ topology) to the constant function equal to one, there exists a function $\phi \in C^\infty((\mathbf{T}\times\mathbf{R})^2, \mathbf{R}^*_+)$*

such that the restriction of the reparametrized flow $X^t_{\hat{H}_\phi}$ to $\mathcal{S}_\alpha$ is weak mixing for the unique invariant measure.

3.4. Assume now that $\mathcal{A}_\alpha$ is any given Aubry-Mather set on the Poincaré section. We can always assume that the semi-conjugacy to R_α in §2.2 projects an orbit of a hole of $\mathcal{A}_\alpha$ by f_α to the orbit of 0 by R_α. If we consider a two fold covering of the Poincaré section, we obtain an Aubry-Mather set $\overline{\mathcal{A}}_\alpha$ for the Poincaré map of the Hamiltonian flow given by $\overline{H}(\theta, r, s, u) := \hat{H}(2\theta, r, s, u)$. The set $\overline{\mathcal{A}}_\alpha$ now has at least two holes with one projecting in 0 and one in $\frac{1}{2}$. From the last theorem we will be able to deduce the following

COROLLARY. *Arbitrarily close (in the C^∞ topology) to the constant function equal to one, there exists a C^∞ function ϕ such that the restriction of the flow $X^t_{\overline{H}_\phi}$ to the lamination $\overline{\mathcal{S}}_\alpha$ over $\overline{\mathcal{A}}_\alpha$ is weak mixing for the unique invariant measure.*

4. Proofs.

4.1. *Reduction to special flows.* On the energy surface $\mathcal{E}_{\hat{H}=\hat{H}_0}$, the flow $X^t_{\hat{H}}$ is a constant time suspension over the Poincaré map f of the Poincaré section $s = 0$. We restrict f to the invariant annulus A_0 on which we introduce a pair of coordinates $(\tilde{r}, \tilde{\theta})$. The reparametrized flow $X^t_{\hat{H}_\phi}$ can be viewed as a special flow over (f, A_0) with a ceiling function ψ given by

$$\psi(\tilde{r}, \tilde{\theta}) := \int_0^1 \frac{1}{\phi(X^s_{\hat{H}}(\tilde{r}, \tilde{\theta}, s))} ds. \tag{6}$$

In the proof of the Theorems, it will be more convenient to work with special flows rather than with reparametrizations. For Theorem 3.3, we will construct a special flow above (f_α, A_α) that is weak mixing and show that the special function we will use can be obtained from a smooth time change function ϕ via the formula (6).

4.2. *Proof of Theorem 3.2.* By the formula (6), if ϕ is of class C^1 then so will be the function ψ. Hence, the proof of the theorem will be accomplished if we show that the special flow over f_α and under any function that is the restriction over $\mathcal{A}_\alpha$ of a function $\psi \in C^1(A_0, \mathbf{R}_+)$ is C^0 conjugate to a suspension with a constant ceiling function. This in its turn will follow if we prove the existence of a continuous solution on $\mathcal{A}_\alpha$ to the equation

$$\xi - \xi \circ f_\alpha = \psi - \int_{\mathcal{A}_\alpha} \psi(\tilde{r}, \tilde{\theta}) d\mu_\alpha, \tag{7}$$

where μ_α is the unique invariant measure on $\mathcal{A}_\alpha$ by f_α (Cf. for example [11] Chapter 4).

By the Theorem of Gottschalk and Hedlund ([6] or [8] Chapter IV), equation (7) will have a continuous solution if we prove that $|\psi_m(x) - \psi_m(x')|$ is uniformly bounded for $m \in \mathbf{N}$, $x, x' \in \mathcal{A}_\alpha$ (here $\psi_m(x)$ denotes the Birkhoff sums of ψ relative to f_α, i.e. $\psi_m(x) = \psi(x) + \psi(f_\alpha x) + ... + \psi(f_\alpha^{m-1}x)$.

4.3. *Reducing to special flows over rotations of the circle.* Assume now that $\mathcal{A}_\alpha$ is a hyperbolic Aubry-Mather set with only one hole. To the special flow over $(\mathcal{A}_\alpha, f_\alpha)$ and under the function ψ we associate a special flow over $(\mathbf{T}^1, R_\alpha)$ and under a function φ and showing that the sums $S_m\psi(x) - S_m\psi(x')$ above f_α are uniformly bounded is equivalent to showing that $S_m\varphi(\theta) - S_m\varphi(\theta')$ above R_α are uniformly bounded. The function φ has the following properties

i) The function φ is strictly positive and for any $0 \leq a < b < 1$

$$\varphi(b) - \varphi(a) = \sum_{R_\alpha^k(0) \in]a,b]} \Delta_k, \tag{8}$$

where the number Δ_k is the difference between the values of ψ at the right and left endpoints of the k^{th} gap. The reason for (8) is that we assumed that $(f_\alpha, \mathcal{A}_\alpha)$ is hyperbolic: which implies (Cf. §2.6.2) that the union of the gaps on the Lipschitz graph where $\mathcal{A}_\alpha$ lies has full Lebesgue measure. Since ψ is of class C^1 we have that the variations of ψ are concentrated on the gaps and since the projection on $\mathbf{T}$ is bi-Lipschitz we get that the corresponding variations of φ are concentrated on the orbit of 0 as stated in (8).

Since we assumed that $(f_\alpha, \mathcal{A}_\alpha)$ is hyperbolic we also have (Cf. §2.6.3.) that the size of the k^{th} gap decreases geometrically as $k \to \pm\infty$. The function ψ being Lipschitz we obtain the following

ii) The sequence $\{|\Delta_k|\}_{k \in \mathbf{Z}}$ decrease geometrically when $|k| \to \infty$, e.g. there exist $C > 0$ and $0 < \Delta < 1$ such that for any $k \in \mathbf{Z}$

$$|\Delta_k| \leq C\Delta^{|k|}. \tag{9}$$

We will actually need the following weaker property on the Δ_k:

$$\sum_{k \in \mathbf{Z}} |k\Delta_k| < +\infty. \tag{10}$$

iii) Moreover, we clearly have

$$\sum_{i=-\infty}^{+\infty} \Delta_i = 0. \tag{11}$$

In conclusion, to obtain the theorem we just need to prove the following

PROPOSITION. *Given any rotation on the circle R_α of the circle and a real function φ satisfying i) — iii), we have*

$$\sup_{m\in\mathbf{N}^*}\ \sup_{(\theta,\theta')\in\mathbf{T}} \Big|\sum_{i=0}^{m-1}\varphi(\theta'+i\alpha)-\sum_{i=0}^{m-1}\varphi(\theta+i\alpha)\Big| < +\infty.$$

Proof. For the proof of the proposition we can assume that the integral of φ vanishes. From (8) we have that the derivative of φ in the sense of the distributions is

$$D\varphi = \sum_{k\in\mathbf{Z}} \Delta_k \delta_{k\alpha},$$

where δ_z denotes the Dirac measure concentrated on z.

Define for every $k \in \mathbf{Z}$

$$\sigma_k \ := \sum_{j=-\infty}^{k} \Delta_j = -\sum_{j=k+1}^{+\infty} \Delta_j$$

and denote by e_k the function on the circle, of zero integral satisfying

$$De_k = \delta_{k\alpha} - \delta_{(k+1)\alpha}.$$

Since $\Delta_k = \sigma_k - \sigma_{k-1}$, we have

$$\varphi = \sum_{k\in\mathbf{Z}} \sigma_k e_k.$$

Now, if we use the usual notation $S_m e_k$ for the Birkhoff sums of e_k above R_α we notice that for any $m > 0$ the following is true

$$DS_m e_k = \delta_{k-(m-1)\alpha} - \delta_{(k+1)\alpha}$$

hence

$$||S_m e_k||_{L^\infty} \leq 1$$

which implies

$$\begin{aligned} ||S_m\varphi||_{L^\infty} &\leq \sum_{k\in\mathbf{Z}} |\sigma_k| \\ &\leq \sum_{i\in\mathbf{Z}} (1+|i|)|\Delta_i|. \end{aligned}$$

Condition (10) concludes the proof of the proposition and hence of Theorem 3.2. □

4.4. *Proof of Theorem 3.3.* The proof of weak mixing we will give in this section is similar to the one produced by Katok and Stepin in [10] for interval exchange transformations.

4.5. *Reduction to special flows.* Assume $\mathcal{A}_\alpha$ is an Aubry-Mather set having two orbits of wandering intervals one in 0 and the other in β, and that β satisfies condition (3) of Definition 2. Define the set

$$\mathfrak{L}_\alpha = \{\epsilon \in]0,1[, \quad \frac{1}{\epsilon} \notin \mathbf{Q} + \alpha\, \mathbf{Q}\}.$$

Theorem 3.3 will follow if we prove the following

PROPOSITION *For any $\epsilon \in \mathfrak{L}_\alpha$, the special flow over R_α with the ceiling function*

$$\chi^\epsilon = (1-\epsilon)\chi_{[0,\beta[} + (1+\epsilon)\chi_{[\beta,1[} \tag{12}$$

is weak mixing (for the unique invariant measure).

Indeed, the Aubry-Mather set $\mathcal{A}_\alpha$ has a gap in 0 and a gap in β that separate it in two parts. Hence we can find two disjoint open sets of $(\mathbf{T}\times\mathbf{R})^2$ each one containing a part of the lamination S_α. We then choose the time change function ϕ^ϵ in (2) of class C^∞ and constant on each one of the two open sets with values $(1-\epsilon)^{-1}$ on one, and $(1+\epsilon)^{-1}$ on the other. By the formula (6), the restriction of the reparametrized flow to S_α can be viewed as the special flow over f_α with the ceiling function (12). By semi-conjugacy, the weak mixing of this special flow will follow from Proposition 4.5. Furthermore, it is clearly possible to construct the ϕ^ϵ as close to 1 in the C^∞ topology as ϵ goes to zero in (12).

Proof of Proposition 4.5. We will use a classical general lemma on weak mixing for special flows, the proof of which can be found for example in [2]. In the lemma $\{T^t\}$ can be any special flow constructed from an ergodic automorphism (T, L, μ) of a Lebesgue space M and under a summable function $f > 0$. For the special flow we will consider the normalized measure $\frac{1}{\int f d\mu} d\mu ds$ where ds denotes Lebesgue measure on the fibers.

LEMMA. *The flow $\{T^t\}$ has a continuous spectrum if and only if for any λ in $\mathbf{R}^*$ the equation*

$$h(T(x)) = e^{i\lambda f(x)} h(x), \tag{13}$$

does not have a non zero measurable solution h.

In our case (13) becomes

$$h(x+\alpha) = e^{i\lambda\chi^\epsilon(x)} h(x), \tag{14}$$

where

$$\begin{aligned} e^{i\lambda\chi^\epsilon(x)} &= e^{i\lambda(1-\epsilon)} \quad if x \in [0,\beta[, \\ e^{i\lambda\chi^\epsilon(x)} &= e^{i\lambda(1+\epsilon)} \quad if x \in [\beta,1[. \end{aligned}$$

We will use the following

LEMMA. *Let α be an irrational number. If the number $\beta \in]0,1[$ satisfies* (3), *and $w(x)$ is a complex function defined on $\mathbf{T}^1$ such that*

$$\begin{aligned} w(x) &= z_1, \quad if x \in [0,\beta[, \\ w(x) &= z_2, \quad if x \in [\beta,1[, \end{aligned}$$

with $|z_1| = |z_2| = 1$ and $z_1 \neq z_2$; then the equation

$$h(x+\alpha) = w(x)h(x), \tag{15}$$

does not admit any measurable solution h.

This Lemma was stated and proved by Katok and Stepin in [10] in the case where α is not of constant type (the sequence a_n in 5 is unbounded). Their proof was based on fast cyclic approximations for irrational rotations of non-constant type. The proof of the absence of solutions in the case where α is of constant type is due to Veech [19] (a complete discussion with proofs can be found in [15]).

The last lemma implies that a necessary condition on an eventual eigenvalue λ for the special flow of Proposition 4.5 is that

$$\lambda\epsilon = l\pi, \text{ for some } l \in \mathbf{Z}.$$

But then a corresponding solution of (14) would satisfy

$$h(x+\alpha) = e^{il\pi} e^{i\frac{l\pi}{\epsilon}} h(x),$$

hence $e^{il\pi} e^{i\frac{l\pi}{\epsilon}}$ should be an eigenvalue of R_α, that is

$$l + \frac{l}{\epsilon} = k\alpha + 2p,$$

for some integers k and p, which contradicts the fact that $\epsilon \in \mathcal{L}_\alpha$.

Therefore equation (14) does not have non trivial solutions and the special flow of Proposition 4.5 is weak mixing. Theorem 3.3 is thus proved. □

4.6. *Proof of the Corollary 3.4.* The corollary of Theorem 3.3 will follow if we prove the following

PROPOSITION. *For any $\alpha \in \mathbf{R} - \mathbf{Q}$, and any $\epsilon \in \mathcal{L}_\alpha$, the special flow over R_α with the ceiling function $\chi^\epsilon = (1-\epsilon)\chi_{[0,\frac{1}{2}[} + (1+\epsilon)\chi_{[\frac{1}{2},1[}$ is weak mixing for its unique invariant measure.*

Proof. From Proposition 4.5, we just have to prove that 0 and $\frac{1}{2}$ are independent with respect to $\alpha \in \mathbf{R} - \mathbf{Q}$.

Since two consecutive denominators of the convergents of α are relatively prime (this can be easily obtained by recurrence from (5)), one at least among two consecutive q_n and q_{n+1} is odd. We extract by this means a subsequence of the best approximations of α for which (3) of Definition 2 holds for $\frac{1}{2}$. Proposition 4.6 is thus proved. □

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