WKB asymptotics for the Euler-Maxwell equations

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Abstract - The Euler-Maxwell system of equations is a complex, hydrodynamical model for the description of laser-plasma interactions. We introduce non-dimensional variables, exhibit a small parameter \( \varepsilon \) and study various WKB approximations with respect to \( \varepsilon \) for this system, under a polarization condition for the initial data. We justify an approximation by a weak Zakharov equation for times \( O(1) \) and an approximation by a Davey-Stewartson equation for times \( O(|\log \varepsilon|) \). Our key observation is that the Euler-Maxwell exhibit transparency properties, similar to the properties exhibited by Joly, Métivier and Rauch for Maxwell-Bloch systems. These properties imply in particular that in a weakly nonlinear regime, the geometric optics approximation is given by a linear equation.

1 Introduction

The Euler-Maxwell system of equations is a complex, hydrodynamical model for the description of laser-plasma interactions. We introduce non-dimensional variables, exhibit a small parameter \( \varepsilon \) and study various WKB approximations with respect to \( \varepsilon \) for this system.

We justify an approximation by a Zakharov equation for times \( O(1) \) and an approximation by a Davey-Stewartson equation for times \( O(|\log \varepsilon|) \).

The Zakharov equation describes the nonlinear interactions of the envelope of a polarized electric field with the slowly varying fluctuations of density of the ions in the plasma. It was originally derived by V. Zakharov and its collaborators in the seventies (see [30]). An ad hoc multiscale expansion of the Euler-Maxwell equations leading to the Zakharov equation is

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given in C. Sulem and P.-L. Sulem’s book [27]. To our knowledge, our results establish the first rigorous links between Euler-Maxwell and Zakharov.

The Davey-Stewartson equation was introduced in [9] in the context of water waves. Rigorous justifications of the Davey-Stewartson equation as a limit system for general nonlinear hyperbolic equations are given in [4, 7]. The Davey-Stewartson equation appears in the context of laser-plasma interactions in [24].

Our key observation is that the Euler-Maxwell equations exhibit transparency properties, in the sense of P. Donnat [10] and J.-L. Joly, G. Métivier and J. Rauch [19]. This means that interaction coefficients vanish because of the special structure of the equations. As a consequence, the weakly nonlinear geometric optics approximation (over times $O(1)$) leads to a linear limit system. To reach a nonlinear regime, it is then natural to consider solutions over longer times. In section 4, we show that the system is approximated over diffractive times $O(1/\varepsilon)$ by a Davey-Stewartson system. We show that the approximation is valid over times $O(|\log \varepsilon|)$ in the spirit of the paper of D. Lannes and J. Rauch [22] (see also [7]). In section 5, we consider generalized WKB solutions. A WKB approximation is usually defined by the annulation of the coefficients of a formal series in $\varepsilon$. We relax this condition and construct thereby WKB solutions which terms retain dependance in $\varepsilon$, a classical procedure in the domain of kinetic equations (see the paper by P. Degond and M. Lemou [12] for a quick introduction and a bibliography). In section 5, this allows to derive and justify a weak form of the Zakharov equations. This approximation is valid for times of order $O(1)$ only, because of a mathematical difficulty of the specific Zakharov equations that we derive. Finally, in section 6, we consider solutions with large amplitudes, as in [19, 3]. Infinitely accurate approximate solutions with large amplitude are constructed, but we do not justify the asymptotics.

Choosing to describe the laser-plasma interactions by the Euler-Maxwell equations, one overlooks phenomena both of collisional and non-collisional nature, such as Landau damping. However, our results are in accordance with the classical textbook descriptions [11]:

- There is a threshold frequency $\tilde{\omega}_p$ such that electromagnetic waves oscillating at frequencies $< \tilde{\omega}_p$ cannot propagate in the plasma.
- The electromagnetic waves that propagate in the plasma are polarized transversely to the direction of propagation and do not induce a perturbation of density in the plasma.
- The reaction of the plasma to the incident electromagnetic wave is
composed of a longitudinal high-frequency wave and of an acoustic wave. The polarization of the acoustic wave respects the quasineutrality principle.

1.1 Outline of the paper

In section 2.1, we introduce a non-dimensional form of the Euler-Maxwell system using orders of magnitudes of the physical parameters met in real applications, as given by [1, 25]. In section 2.2, we discuss the algebraic structure of the resulting system in the context of WKB expansions. The system takes the form of a quasilinear symmetric hyperbolic system:

$$(EM) \quad \mathcal{L}^\varepsilon(u^\varepsilon, \partial)u^\varepsilon = \frac{1}{\varepsilon} \mathcal{B}(u^\varepsilon),$$

where $\mathcal{B}$ is quadratic. In the following sections, we study specific regimes:

- Section 3 deals with WKB solutions of size $O(\varepsilon)$ over time intervals $O(1)$ (weakly nonlinear regime).
- Section 4 deals with WKB solutions of size $O(\varepsilon)$ over time intervals $O(1/\varepsilon)$ (beyond the weakly nonlinear regime).
- Section 5 deals with generalized WKB solutions of size $O(\varepsilon)$ over time intervals $O(1)$ (weakly nonlinear regime).
- Section 6 deals with WKB solutions of size $O(\sqrt{\varepsilon})$ over time intervals $O(1)$ (highly nonlinear regime).

1.2 Statement of the results

Throughout the paper, we consider waves in two space dimensions $x, z$, where $z$ is the direction of propagation and $x$ is a transverse direction. We suppose the transverse direction is one-dimensional only to simplify the formal computations.

In section 2.1, we write the physical Euler-Maxwell equations in the form $(EM)$ introduced above. After a suitable change of variable (section 2.2),
(EM) takes the form

\[
\begin{align*}
\partial_t B + \nabla \times E &= 0, \\
\partial_t E - \nabla \times B &= \frac{1}{\epsilon}(e^{n_e}v_e - e^{n_i}v_i), \\
\partial_t v_e + (v_e \cdot \nabla)v_e &= -\nabla n_e - \frac{1}{\epsilon}(E + v_e \times B), \\
\partial_t n_e + \nabla \cdot v_e + (v_e \cdot \nabla)n_e &= 0, \\
\partial_t v_i + (v_i \cdot \nabla)v_i &= -\nabla n_i + \frac{1}{\epsilon}(E + v_i \times B), \\
\partial_t n_i + \nabla \cdot v_i + (v_i \cdot \nabla)n_i &= 0,
\end{align*}
\]

where all the constants were set to 1, except the small parameter \( \epsilon \). The unknown \( u^\epsilon \) represents the non-dimensional physical variables: \( B, E \) the electromagnetic field, \( v_e, v_i \) the velocities of the electrons and of the ions, and \( n_e \) and \( n_i \) the density fluctuations from the equilibrium.

### 1.2.1 Transparency

In section 3, we describe approximate solutions to the initial value problem for (EM) by means of profiles depending on the real variables \( t, x, z, \theta = \frac{k z - \omega t}{\epsilon} \), with a periodic dependance on \( \theta \), and with amplitude \( O(\epsilon) \). This amplitude corresponds to the weakly nonlinear regime of geometrical optics. Precisely, the initial datum is \( \epsilon u^0,\epsilon \), where \( u^0,\epsilon \) is real and highly oscillating:

\[
u^0,\epsilon(x, z) = u^0(x, z)e^{ikz/\epsilon} + c.c.
\]

It has no mean mode and satisfies a polarization condition. One assumes that \( k \neq 0 \).

We show that the limit system is a linear transport equation for the leading term of the field:

\[
(\partial_t + c\partial_z)E = 0, \tag{1.1}
\]

where \( c \) is a group velocity. This is characteristic of a transparency phenomenon: that is, the solutions have the critical size in order that nonlinear effects be observed of times \( O(1) \) but the limit system is linear. We can state:

**Proposition 1.1.** The (EM) system is weakly transparent.

Proposition 1.1 is precisely stated in section 2.2 (Proposition 2.1); the notion of weak transparency is discussed in section 1.3. This results says in particular that the solutions have the critical size in order that nonlinear effects be observed of times \( O(1) \) but the limit equation (1.1) is linear.
1.2.2 Approximation by a Davey-Stewartson system

In section 4, we push the analysis over long times $O(1/\varepsilon)$ by considering profiles depending on the additional variable $\tau = \varepsilon t$ and with amplitude $O(\varepsilon)$.

The limit system is an elliptic-elliptic Davey-Stewartson system of the form

$$\begin{cases}
(i \partial_\tau + \Delta) E &= \bar{n} E, \\
H \bar{n} &= H'|E|^2,
\end{cases}$$

where $E$ is the envelope of the leading term of the WKB solution, $\bar{n}$ is a mean value of the leading mean mode of the fluctuation of density of the WKB solution, $H, H'$ are second-order differential operators in $x, z$, and $H$ is elliptic; $\Delta = \Delta_{x,z}$. The envelope $E$ is solution of the transport equation (1.1) in the variables $t, z$, and $n - \bar{n}$ satisfies a linear wave equation. In Theorem 4.2, we justify the approximation over times $O(|\log \varepsilon|)$:

**Theorem 1.2.** The electric field $E^\varepsilon$ and the (electronic or ionic) fluctuation of density $n^\varepsilon$ of the exact solution to the initial value problem for (EM) with the initial datum $\varepsilon u^{0,\varepsilon}$ satisfy the estimate

$$\frac{1}{\varepsilon} |E^\varepsilon - \varepsilon (E e^{i(kz-\omega t)/\varepsilon} + c.c.)|_{L^\infty} + \frac{1}{\varepsilon} |n^\varepsilon - \varepsilon n|_{L^\infty} = O(\varepsilon),$$

over a time interval $O(|\log \varepsilon|)$.

1.2.3 Approximation by a weak Zakharov system

In section 5, we take the same ansatz as in section 3 but we write the cascade of WKB equations in a different way, as we allow the profiles to depend on $\varepsilon$.

With the same notations as above, the leading terms of the generalized WKB solution $u^\varepsilon$ satisfy a weak Zakharov system of the form

$$\begin{cases}
i(\partial_t + \varepsilon \partial_z) E + \varepsilon \Delta E &= n E, \\
(\partial_t^2 - \Delta) n &= \varepsilon \Delta |E|^2,
\end{cases}$$

where $\Delta = \Delta_{x,z}$. We show (Proposition 5.1):

**Proposition 1.3.** The initial value problem for $(Z)_w$ is well-posed for regular solutions (that is, with sufficiently high Sobolev regularity).

Moreover, its solution $E, n$ satisfies the estimate (Theorem 5.3):

5
Theorem 1.4. The components $E^\varepsilon$ and $n^\varepsilon$ of the exact solution satisfy the estimate

$$\frac{1}{\varepsilon} |E^\varepsilon - \varepsilon (E e^{(kz-\omega t)/\varepsilon} + \text{c.c.})|_{L^\infty} + \frac{1}{\varepsilon} |n^\varepsilon - \varepsilon n|_{L^\infty} = O(\varepsilon),$$

over a time interval $O(1)$.

The velocities and the magnetic field of the approximate solution are expressed in terms of the electric field. Thus the above estimate gives a description of the solution. We cannot push the analysis to long times $O(|\log \varepsilon|)$ as in section 2.7 because we can only prove that the solutions to $(Z)_w$ have an existence time $O(1)$.

1.2.4 Formal approximations by Zakharov systems

In the last section, we sketch formal asymptotic WKB expansions to show how the fully nonlinear Zakharov equations can be derived from the Euler-Maxwell equation. With the same ansatz as in section 5, we consider solutions of amplitude $O(\sqrt{\varepsilon})$. This regime is highly nonlinear; it corresponds to the regime called strong oscillations by Cheverry, Guès and Métivier in [3]. A generalized WKB procedure leads in this setting to a limit system of the form:

$$(Z)^\varepsilon \left\{ 
\begin{array}{ll}
    i(\partial_t + c\partial_z)E + \varepsilon \Delta E &= nE, \\
    (\partial_t^2 - \Delta)n &= \Delta |E|^2.
\end{array}
\right.
$$

We can adapt Proposition 5.1 to solve the Cauchy problem for $(Z)^\varepsilon$. This gives a time existence of size $O(\sqrt{\varepsilon})$ only. Similarly, standard energy estimates for $(EM)$ yield a time existence of size only $O(\varepsilon)$ for solutions with large-amplitude $O(\sqrt{\varepsilon})$. We can construct an associated infinitely accurate approximate solution $u^\varepsilon$ (see Proposition 6.1):

Proposition 1.5. There exists a generalized WKB approximate solution $u^\varepsilon_0 \sqrt{\varepsilon}$ with initial datum $\sqrt{\varepsilon}u_0^\varepsilon$ such that its leading terms satisfy the system $(Z)^\varepsilon$; this approximate solution is infinitely accurate in the sense that

$$\mathcal{L}^\varepsilon (u^\varepsilon_0, \partial)u^\varepsilon_0 - \frac{1}{\varepsilon} \mathcal{B}(u^\varepsilon_0) = O(\varepsilon^\infty),$$

over a time interval $O(\sqrt{\varepsilon})$.

We do not justify the asymptotics, that is we do not prove a asymptotic estimate similar to (1.2) and (1.3).
Last, we suppose that the ions are cold and consider again solutions with large amplitudes \( O(\sqrt{\varepsilon}) \), with profiles depending on the variables \( t, X = x/\sqrt{\varepsilon}, z, \theta = k \cdot z - \omega t \). Then we derive the limit system:

\[
\begin{aligned}
\left( Z \right)_c \left\{ \begin{array}{c}
i(\partial_t + c\partial_z)E + \partial_X^2 E = nE, \\
(\partial_t^2 - \partial_X^2) n = \partial_X^2 |E|^2.
\end{array} \right.
\end{aligned}
\]

Proposition 5.1 does not apply to \((Z)_c\). We have the formal result:

**Proposition 1.6.** In a highly nonlinear regime, one can write a cascade of WKB equations for the Euler-Maxwell system in the cold ions regime (corresponding to three different space scales). The formal limit is the Zakharov system \((Z)_c\).

Following this result, Colin and Métévier proved that when \( c \neq 0 \), \((Z)_c\) is ill-posed in the sense of Hadamard in \( L^\infty \). We discuss the possible implications of this result in the next subsection.

### 1.3 Discussion and open problems

We need to point the limits of Theorems 1.2, 1.4 and Proposition 1.5.

The justification of the \((DS)\) system given in Theorem 1.2 is essentially weakly nonlinear. The fact that the approximation can be justified over times \( O(|\log \varepsilon|) \) does not depend on the properties of the Euler-Maxwell system; it derives from an observation of Lannes and Rauch [22]. The interest of Theorem 1.2 is to give a justification of a Davey-Stewartson system in a less transparent context than Maxwell-Bloch-type equations (see below for a discussion of transparency). Colin in [4] justified Davey-Stewartson-type approximations only in the class of Maxwell-Bloch-type systems. Colin and Lannes give in [7] another justification of Davey-Stewartson systems over logarithmic times.

Theorem 1.4 does not yield more information on the solution than standard geometric optics, as the approximate solution satisfying the weak Zakharov system \((Z)_w\) is not closer to the exact solution than the solution \((E, n)\) given by \( n = 0 \) and the linear transport equation (1.1) (see Remark 5.4). In this sense, Theorem 1.4 does not establish much more than a formal link between Euler-Mawell and Zakharov. In strong constrast, in the context of kinetic equations, Chapman-Enskog expansions (similar to the generalized WKB expansion of section 5) effectively yield precise error estimates.

Proposition 1.5 yields very precise approximate solutions to the Euler-Maxwell equations, but they could well be unstable – see indeed the example given by Joly, Méteivier and Rauch of an approximate solution that is infinitely accurate and that is actually unstable in [19] (Theorem 11.4).
More interesting, in a sense, are the results of Proposition 2.1 and Proposition 1.6. To put these results in perspective, let us recall briefly the context of [19]. Joly, Métévier and Rauch studied in this paper large-amplitude solutions of semilinear systems with nilpotent bilinear singularities. The system (EM) does not enter this class, as the convective terms are quasilinear, and the singular bilinear terms (the above current density and Lorentz force term) do not have a triangular structure. Nevertheless, the results of [19] provide guidelines for the understanding of Propositions 2.1 and 1.6. The results of [19] can be sketched as follows: for the above mentioned class of equations (semilinear with nilpotent bilinear singularity), in a highly nonlinear regime comparable to the ansatz of section 6 (large-amplitude solutions), there exists a hierarchy of null (transparency) conditions (a), (b), (c) such that

(a) is a necessary condition for the existence of a cascade of WKB equations

(b) is a sufficient condition for the well-posedness of the system satisfied by the first profiles of the WKB expansion,

(c) is a sufficient, and almost necessary, condition for the stability of the WKB solution.

Joly, Météiver and Rauch prove that (c) \(\Rightarrow\) (b) and that (b) \(\Rightarrow\) (a). They also prove that the Maxwell-Bloch equations satisfy all three transparency properties.

Proposition 2.1 states that the Euler-Maxwell system provides another example of physical equations satisfying a condition of type (a). In other words, the Euler-Maxwell system can be used to describe large-amplitude solutions. This remark is especially important with respect to the applications, which typically concern the propagation of high energy lasers, hence large-amplitude solutions.

Proposition 1.6 together with the result of Colin and Métévier [8] hints that the Euler-Maxwell system does in general not satisfy a condition of type (b).

These remarks yield very interesting questions:

- When \(c = 0\), \((Z)_c\) reduces to the standard Zakharov system, for which existence results are known. Can then one justify the highly nonlinear formal asymptotics of Proposition 1.6 in this case? That is, in the special case when (b) holds, does (c) hold? This question is addressed

- When $c \neq 0$, that is when (b) does not hold, can one prove strong instabilities in the highly nonlinear regime of Proposition 1.6, in the spirit of [3]? This is an interesting direction for future work.

### 1.4 Notations and definitions

The parameters are introduced in (2.1). Most notations related to the WKB expansions are introduced at the beginning of section 2.2. The notion of polarization is defined in (2.5). The corresponding polarization condition for the equations of our interest is spelled out in (2.13). The corresponding compatibility condition is (2.14). The dispersion relations are given in (2.11) and (2.12). The definition of approximate solution is given in the paragraph above (2.11). The key transparency conditions are stated in Proposition 2.1. The notations pertaining to the profiles are set in the paragraph above (2.6).

### 2 The Euler-Maxwell equations

#### 2.1 A non-dimensional form of the equations

We consider a plasma as an ideal, nonhomogeneous, two-species fluid composed of electrons and ions of charge respectively $-e$ and $e$. We assume that the plasma is non relativistic, i.e. that the characteristic speeds of the particles of the plasma are small compared with the speed of light in vacuum. The laser is described by the Maxwell equations; the plasma is described by the Euler equations of conservation of momentum and mass [11, 27]:

\[
\begin{align*}
\partial_t \tilde{B} + c \nabla \times \tilde{E} &= 0, \\
\partial_t \tilde{E} - c \nabla \times \tilde{B} &= 4\pi e((n_0 + \tilde{n}^\sharp_e)v_e - (n_0 + \tilde{n}^\sharp_i)v_i), \\
m_e(n_0 + \tilde{n}^\sharp_e)(\partial_t \tilde{v}_e + (\tilde{v}_e \cdot \nabla)\tilde{v}_e) &= -\gamma_e T_e \nabla \tilde{n}^\sharp_e - e(n_0 + \tilde{n}^\sharp_e)(\tilde{E} + \frac{1}{c} \tilde{v}_e \times \tilde{B}), \\
m_i(n_0 + \tilde{n}^\sharp_i)(\partial_t \tilde{v}_i + (\tilde{v}_i \cdot \nabla)\tilde{v}_i) &= -\gamma_i T_i \nabla \tilde{n}^\sharp_i + e(n_0 + \tilde{n}^\sharp_e)(\tilde{E} + \frac{1}{c} \tilde{v}_i \times \tilde{B}), \\
\partial_t \tilde{n}^\sharp_e + \nabla \cdot ((n_0 + \tilde{n}^\sharp_e)\tilde{v}_e) &= 0, \\
\partial_t \tilde{n}^\sharp_i + \nabla \cdot ((n_0 + \tilde{n}^\sharp_i)\tilde{v}_i) &= 0,
\end{align*}
\]

together with the divergence equations

\[
\nabla \cdot \tilde{B} = 0, \quad \nabla \cdot \tilde{E} = 4\pi e(\tilde{n}^\sharp_e - \tilde{n}^\sharp_i).
\]
We are interested in the initial value problem for the Euler-Maxwell equation. We will assume that the initial data satisfy (2.1). As the conditions (2.1) are propagated by the system, the corresponding solutions of the initial value problem will then satisfy (2.1) for all times.

The space-time variable is $\tilde{t}, \tilde{y}$. In this section only, we suppose that $\tilde{y}$ is three-dimensional.

The parameters are $m_e$ and $m_i$ the masses of both species, $\gamma_e$ and $\gamma_i$ the specific heat ratios of both species, $T_i$ and $T_e$ the temperatures of both species and $n_0$ the (assumed constant and isotropic) density of the plasma at equilibrium. The variables are $\tilde{B}, \tilde{E}$ the electromagnetic field, $\tilde{v}_e, \tilde{v}_i$ the velocities of the electrons and of the ions, and $\tilde{n}_e^\sharp$ and $\tilde{n}_i^\sharp$ the density fluctuations from the equilibrium $n_0$. The nonlinear term in the Ampère equation is the current density term. The nonlinear terms in the equations of conservation of momentum are the Lorentz forces terms. Introduce $\tilde{\omega}_e$ the electronic plasma frequency:

$$\tilde{\omega}_e := \sqrt{\frac{4\pi e^2 n_0}{m_e}}.$$

For plasmas created by lasers, a typical value is $\tilde{\omega}_e = 10^{14} \text{s}^{-1}$ [1]. At first order (equivalently, for small amplitudes), one can assume that the plasma is homogeneous and neglect the nonlinear phenomena and the magnetic field. The equations become

$$\partial_t \tilde{E} = 4\pi e n_0 (\tilde{v}_e - \tilde{v}_i), \quad \partial_t \tilde{v}_e = -\frac{e}{m_e} \tilde{E}, \quad \partial_t \tilde{v}_i = \frac{e}{m_i} \tilde{E}.$$

The electric field is oscillating: $\tilde{E} = \mathcal{E}(\tilde{t}) e^{i\tilde{\omega}_e \tilde{t}}$, where $\mathcal{E}$ is the order of magnitude of the field and where $E(t)$ is a slowly varying enveloppe: $E(t) = O(1), \partial_t E \ll \tilde{\omega}$. The velocities take the form: $\tilde{v}_e = \mathcal{V}_e v_e(\tilde{t}) e^{i\tilde{\omega}_e \tilde{t}}, \tilde{v}_i = \mathcal{V}_i v_i(\tilde{t}) e^{i\tilde{\omega}_e \tilde{t}}$, where $v_e$ and $v_i$ are slowly varying, and the above equations imply

$$\tilde{\omega}^2 = 4\pi e^2 n_0 \left( \frac{1}{m_e} + \frac{1}{m_i} \right) =: \tilde{\omega}_p^2,$$

where $\tilde{\omega}_p$ is called the plasma frequency. In the limit $m_e/m_i \to 0$, one has $\tilde{\omega}_p = \tilde{\omega}_e$, that is the ionic correction term is negligible and the plasma frequency is the electronic plasma frequency.

We want to determine the orders of magnitude $\mathcal{E}, \mathcal{B}, \mathcal{V}_e, \mathcal{V}_i, \mathcal{N}_e, \mathcal{N}_i$ of the variables, the characteristic time and length $t_0$ and $l_0$ and the characteristic pulsation $\omega_0$ and wave number $k_0$ of the oscillations that we want to describe. We set $\tilde{B} = BB$, $\tilde{E} = \mathcal{E} E$, $\tilde{v}_e = \mathcal{V}_e v_e$, $\tilde{v}_i = \mathcal{V}_i v_i$, $\tilde{n}_e^\sharp = \mathcal{N}_e n_e^\sharp$, $\tilde{n}_i^\sharp = \mathcal{N}_i n_i^\sharp$, and $\tilde{t} = t_0 t, \tilde{y} = l_0 y, \tilde{\omega} = \omega_0 \omega, \tilde{k} = k_0 k$. We suppose that the fields and
the velocities are oscillating with slowly varying envelopes, that is have the form

$$\tilde{H} = \mathcal{H}(H(t), e^{i\tilde{\omega}t}), \quad \partial_t \tilde{H} \ll \tilde{\omega}. $$

First, the characteristic time and space scales are set equal to the characteristic time and space scales of the laser: $t_0 := t_l, l_0 := c t_l$, where $t_l$ is the duration of the pulse. Second, the above computation leads to the choice $\omega_0 := \tilde{\omega} \simeq \tilde{\omega}_p$, $k_0 = \frac{\omega_0}{c}$. Then $\mathcal{V}_e$ is set to be equal to the electronic thermal velocity $v_{Te}$:

$$\mathcal{V}_e = v_{Te} := \sqrt{\frac{\gamma_e T_e}{m_e}}.$$

The choice for $\mathcal{E}$ is obtained via the equation of conservation of momentum for the electrons. Neglecting the nonlinear terms and the pressure, one gets

$$\tilde{\omega}_e m_e n_0 v_{Te} = e n_0 \mathcal{E}, \quad \text{hence} \quad \mathcal{E} = \frac{m_e v_{Te} \tilde{\omega}_e}{e} = \sqrt{4\pi n_0 \gamma_e T_e}.$$

Then $\mathcal{B}$ is chosen in order that the speed of light is normalized to 1:

$$\mathcal{B} := \mathcal{E}.$$

Finally, $\mathcal{N}_e$ and $\mathcal{N}_i$ are set equal to the plasma density at equilibrium:

$$\mathcal{N}_e := n_0, \quad \mathcal{N}_i := n_0.$$

It remains to make a choice for $\mathcal{V}_i$. In the physics literature, the characteristic speed of the ions is said to be the ionic acoustic velocity

$$c_a := \sqrt{\frac{T_e}{m_i}}.$$

We discuss this choice below, in the context of our specific interest – that is, high-frequency asymptotics. Introduce the small parameter

$$\epsilon := \frac{1}{\tilde{\omega}_e t_l},$$

which depends both on the characteristics of the laser and the plasma. With this notation, the oscillations take the form

$$\exp (i(kz - \tilde{\omega}t)) = \exp \frac{i}{\epsilon} (k z - \omega t).$$

A typical value for long pulses is $\epsilon \simeq 10^{-6}$ [25]. Introduce also the parameters

$$\theta_e := \frac{\mathcal{V}_e}{c} = \frac{v_{Te}}{c}, \quad \theta_i := \frac{\mathcal{V}_i}{c}, \quad \alpha := \frac{v_{Ti}}{\mathcal{V}_i}.$$
where \( v_{T_i} \) is the ionic thermal velocity:

\[
v_{T_i} := \sqrt{\frac{\gamma_i T_i}{m_i}}.
\]

The non-relativistic assumption above implies that \( \theta_e \) and \( \theta_i \) are \( \ll 1 \). Introduce finally the small parameter

\[
\eta = \frac{m_e}{m_i},
\]

which depends only on the plasma. A typical value is \( \eta \simeq 2.5 \times 10^{-5} \) [25].

The non-dimensional system is

\[
\text{(EM)}^\sharp \left\{ \begin{align*}
\partial_t B + \nabla \times E &= 0, \\
\partial_t E - \nabla \times B &= \frac{1}{\varepsilon}(1 + n_e^\sharp) v_e - \frac{1}{\varepsilon \theta_e^\sharp} (1 + n_e^\sharp) v_i, \\
\partial_t v_e + \theta_e (v_e \cdot \nabla) v_e &= -\theta_e \frac{\nabla n_e^\sharp}{1 + n_e^\sharp} - \frac{1}{\varepsilon} (E + \theta_e v_e \times B), \\
\partial_t n_e^\sharp + \theta_e (v_e \cdot \nabla) n_e^\sharp + \theta_e (1 + n_e^\sharp) \nabla \cdot v_e &= 0, \\
\partial_t v_i + \theta_i (v_i \cdot \nabla) v_i &= -\alpha^2 \theta_i \frac{\nabla n_i^\sharp}{1 + n_i^\sharp} + \frac{1}{\varepsilon \theta_i^\sharp} (E + \theta_i v_i \times B), \\
\partial_t n_i^\sharp + \theta_i (v_i \cdot \nabla) n_i^\sharp + \theta_i (1 + n_i^\sharp) \nabla \cdot v_i &= 0.
\end{align*} \right.
\]

With \( u^\sharp := (B, E, v_e, n_e^\sharp, v_i, n_i^\sharp) \), the system has the form

\[
\text{(EM)}^\sharp \quad \mathcal{L}^\sharp(u^\sharp, \varepsilon \partial) u^\sharp = \mathcal{B}^\sharp(u^\sharp, u^\sharp),
\]

where \( \mathcal{L}^\sharp \) is a symmetrizable hyperbolic operator and where the nonlinear term \( \mathcal{B}^\sharp \) represents both the nonlinear current density and Lorentz forces terms. A symmetrizer is the diagonal matrix

\[
S^\sharp(u^\sharp) := \text{diag}(\text{Id}_{\mathbb{C}^3}, \text{Id}_{\mathbb{C}^3}, \text{Id}_{\mathbb{C}^3}, \frac{1}{(1 + n_e^\sharp)^2}, \frac{\theta_i^2}{\eta \theta_e^\sharp \text{Id}_{\mathbb{C}^3}}, \frac{\alpha^2}{(1 + n_i^\sharp)^2}).
\]

Then two natural choices appear for \( \mathcal{V}_i \). One could take \( \mathcal{V}_i \) equal to \( v_{T_i} \). Then one would have \( \alpha = 1 \), and \( \frac{\theta_i^2}{\eta \theta_e^\sharp} = \frac{T_i}{T_e} \gg 1 \). The second possibility is to choose

\[
\mathcal{V}_i := c_s.
\]

Then one has

\[
\eta = \frac{m_e}{m_i} = \frac{\theta_i^2}{\theta_e^\sharp}, \quad \alpha = \frac{T_i}{T_e} \ll 1.
\]
We choose this second possibility, because the condition \( \alpha \ll 1 \) corresponds to the condition of negligible Landau damping for acoustic waves (see [11], t.1, paragraph 1.7).

**Validity of the Euler-Maxwell model.** The hydrodynamic model is realistic for intensities \( \sim 10^{16} - 10^{17} \text{ W.cm}^{-2} \) and relativistic effects are negligible for intensities \( \leq 10^{18} \text{ W.cm}^{-2} \) [1]. Introduce \( \lambda_D \) the Debye length

\[
\lambda_D := \sqrt{\frac{\gamma_e T_e}{m_e \bar{\omega}_e^2}}.
\]

Landau damping of electromagnetic waves is negligible for oscillations with a wave number \( \tilde{k} \) such that \( \tilde{k} \lambda_D \leq 0.3 \) ([11], t.2, paragraph 10.3). For plasmas created by lasers, a typical value is \( \lambda_D \approx 10^{-6} \text{ m} \). The condition is thus \( \tilde{k} \leq 0.3 \times 10^6 \text{ m}^{-1} \). Then the choice \( k_0 = \tilde{\omega}_e / c \approx 10^6 \text{ m}^{-1} \) seems reasonable. Landau damping of acoustic waves is negligible for small \( \alpha \) [11], that is when the ions are much colder than the electrons.

### 2.2 Preliminary notations and results

The main notations are introduced in this section. We also state and verify in this section the important transparency relations (Proposition 2.1).

We start with (EM), where \( \eta \) and \( \alpha \) are given by (2.2). The change of variables for small amplitudes

\[
1 + n_e^\# = e^{n_e}, 
1 + n_i^\# = e^{n_i},
\]

applied to (EM) leads to the system

\[
\left\{
\begin{align*}
\begin{aligned}
\partial_t B + \nabla \times E &= 0, \\
\partial_t E - \nabla \times B &= \frac{1}{\varepsilon}((1 + n_e + f(n_e))v_e \\
&- \frac{1}{\varepsilon} \theta_e (1 + n_i + f(n_i))v_i, \\
\partial_t v_e + \theta_e (v_e \cdot \nabla) v_e &= -\theta_e \nabla n_e - \frac{1}{\varepsilon} (E + \theta_e v_e \times B), \\
\partial_t n_e + \theta_e \nabla \cdot v_e + \theta_e (v_e \cdot \nabla) n_e &= 0 \\
\partial_t v_i + \theta_i (v_i \cdot \nabla) v_i &= -\alpha^2 \theta_i \nabla n_i + \frac{1}{\varepsilon} \theta_i (E + \theta_i v_i \times B), \\
\partial_t n_i + \theta_i \nabla \cdot v_i + \theta_i (v_i \cdot \nabla) n_i &= 0,
\end{aligned}
\end{align*}
\right.
\]

(EM)
where \( f(x) := e^x - 1 - x \). The fields \( B \) and \( E \) and the velocities \( v_e, v_i \) are vectors in \( \mathbb{C}^3 \), and the densities \( n_e, n_i \) are scalar. The variable in (EM) is

\[ u^e = (B, E, v_e, n_e, v_i, n_i) \in \mathbb{C}^{14}. \]

The space variable is

\[ y = (z, x) \in \mathbb{R}^2, \]

where \( z \) is the direction of propagation and \( x \) is a scalar transverse direction. We assume that \( x \) is scalar only to make the formal computations manageable. Our results apply to the 3-dimensional equations. One needs to distinguish between a transverse component \( u_\perp \) of \( u \) and a longitudinal component \( u_\parallel \) of \( u \), defined as

\[ u_\perp = (B_x, B_y, E_y, v_{ex}, v_{ey}, v_{ix}, v_{iy}) \in \mathbb{C}^8, \]

and

\[ u_\parallel = (B_z, E_z, v_{ez}, n_e, v_{iz}, n_i) \in \mathbb{C}^6. \]

Note that the densities are entries of the longitudinal variable. In the variable \((u_\perp, u_\parallel)\), the equations take the form

\[
\begin{aligned}
L(\partial_t, \partial_z)u_\perp + C \partial_x u_\parallel + L(u, (\partial_x, 0, \partial_z))u_\perp &= \frac{1}{\varepsilon}B_\perp(u_\parallel, u_\perp) \\
&\quad + \frac{1}{\varepsilon}(B_{Lo\perp}(u_\parallel, u_\perp) + R_\perp(u_\parallel, u_\perp)), \\
M(\partial_t, \partial_z)u_\parallel + C^* \partial_x u_\perp + M(u, (\partial_x, 0, \partial_z))u_\parallel &= \frac{1}{\varepsilon}B_\parallel(u_\parallel, u_\parallel) \\
&\quad + \frac{1}{\varepsilon}(B_{Lo\parallel}(u_\perp, u_\perp) + R_\parallel((u_\parallel, u_\parallel)).
\end{aligned}
\]

The variables \( u_\perp \) and \( u_\parallel \) thus obey evolution equations governed by distinct linear symmetric hyperbolic operators \( L \) and \( M \) with constant coefficients. How these variables are coupled will prove to be crucial in the following. The coupling terms are

- by the linear transverse derivative terms \( C \partial_x \) and \( C^* \partial_x \),
- by the quasilinear convective terms \( L \) and \( M \),
- by the bilinear current density term \( B \), by the bilinear Lorentz force term \( B_{Lo} \) and by the semilinear term \( R \) that comes from the change of variable (2.3).
We now precisely define the symbols used in (EM). For $\beta = (\omega, k) \in \mathbb{R} \times \mathbb{R}$, one sets

$$L(\beta \partial \theta) := \begin{pmatrix} -\omega \partial \theta & 0 & 0 & -k \partial \theta & 0 & 0 & 0 & 0 \\ 0 & -\omega \partial \theta & 0 & 0 & 0 & 0 \\ 0 & k \partial \theta & -\omega \partial \theta & 0 & -1 & 0 & \frac{\theta_k}{\theta_c} & 0 \\ -k \partial \theta & 0 & 0 & -\omega \partial \theta & 0 & -1 & 0 & \frac{\theta_k}{\theta_c} \\ 0 & 0 & 1 & 0 & -\omega \partial \theta & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -\omega \partial \theta & 0 & 0 \\ 0 & 0 & -\frac{\theta_k}{\theta_c} & 0 & 0 & 0 & -\omega \partial \theta & 0 \\ 0 & 0 & 0 & -\frac{\theta_k}{\theta_c} & 0 & 0 & 0 & -\omega \partial \theta \end{pmatrix},$$

and

$$M(\beta \partial \theta) := \begin{pmatrix} -\omega \partial \theta & 0 & 0 & 0 & 0 & 0 \\ 0 & -\omega \partial \theta & -1 & 0 & \frac{\theta_k}{\theta_c} & 0 \\ 0 & 1 & -\omega \partial \theta & \theta_e k \partial \theta & 0 & 0 \\ 0 & 0 & \theta_e k \partial \theta & -\omega \partial \theta & 0 & 0 \\ 0 & 0 & -\frac{\theta_k}{\theta_c} & 0 & 0 & -\omega \partial \theta & \alpha^2 \theta_i k \partial \theta \\ 0 & 0 & 0 & 0 & \theta_e k \partial \theta & -\omega \partial \theta \end{pmatrix}.$$
The linear coupling term with a transverse derivative are

\[ C := \begin{pmatrix} C_M & 0 \\ 0 & C_E(\alpha) \end{pmatrix}, \quad C_1 := \begin{pmatrix} C_M & 0 \\ 0 & 0 \end{pmatrix}, \quad C^* := \begin{pmatrix} C_M^* & 0 \\ 0 & C_E(1)^* \end{pmatrix}, \]

where \( C_M^* \) and \( C_E^* \) are the adjoints of \( C_M \) and \( C_E \) respectively. Note that \( C^* \) is not quite the adjoint of \( C \) when \( \alpha \neq 1 \). One has \( C, C_1 \in \mathcal{L}(\mathbb{C}^6, \mathbb{C}^8), C^* \in \mathcal{L}(\mathbb{C}^8, \mathbb{C}^6) \).

Let \( a := (a_x, 0, a_z) \in \mathbb{R}^3 \). The convective terms are

\[ L(u, a)u'_{\perp} := \theta_e(v_e \cdot a)(0_{\mathbb{C}4}, v'_{ex}, v'_{ey}, 0_{\mathbb{C}2}) + \theta_i(v_i \cdot a)(0_{\mathbb{C}4}, 0, v'_{ix}, v'_{iy}), \]

and

\[ M(u, a)u'_{\parallel} := \theta_e(v_e \cdot a)(0_{\mathbb{C}2}, v'_{ex}, n'_e, 0, 0) + \theta_i(v_i \cdot a)(0_{\mathbb{C}2}, 0, 0, v'_{ix}, n'_i). \]

When \( a = (0, 0, a_z) \), one writes \( L(u_{\parallel}, a_z) \) instead of \( L(u, a) \), and similarly when \( a = (a_x, 0, 0) \), one writes \( L(u_{\perp}, a_x) \) instead of \( L(u, a) \). Similar notations are used for \( M \).

The current density terms are the bilinear maps \( B_{\perp} : \mathbb{C}^6 \times \mathbb{C}^8 \to \mathbb{C}^8 \) and \( B_{\parallel} : \mathbb{C}^6 \times \mathbb{C}^6 \to \mathbb{C}^6 \), respectively defined by

\[ B_{\perp}(u_{\parallel}, u_{\perp}) := (0_{\mathbb{C}4}, n_e v_{ex} - \frac{\theta_i}{\theta_e} n_i v_{ix}, n_e v_{ey} - \frac{\theta_i}{\theta_e} n_i v_{iy}, 0_{\mathbb{C}4}), \]

and

\[ B_{\parallel}(u_{\parallel}, u'_{\perp}) := (0, v_{ex}' - \frac{\theta_i}{\theta_e} n_i v_{ix}', 0_{\mathbb{C}4}). \]

The higher order semilinear terms are \( \mathcal{R}_{\perp} : \mathbb{C}^6 \times \mathbb{C}^8 \to \mathbb{C}^8 \) and \( \mathcal{R}_{\parallel} : \mathbb{C}^6 \times \mathbb{C}^6 \to \mathbb{C}^6 \), defined by

\[ \mathcal{R}_{\perp}(u_{\parallel}, u_{\perp}) := (0_{\mathbb{C}2}, f(n_e) v_{ex} - \frac{\theta_i}{\theta_e} f(n_i) v_{ix}, f(n_e) v_{ey} - \frac{\theta_i}{\theta_e} f(n_i) v_{iy}, 0_{\mathbb{C}4}), \]

and

\[ \mathcal{R}_{\parallel}(u_{\parallel}, u'_{\perp}) := (0, f(n_e) v_{ex}' - \frac{\theta_i}{\theta_e} f(n_i) v_{ix}' - f(n_i) v_{iy}' - 0_{\mathbb{C}4}). \]

The nonlinear Lorentz force terms are \( B_{L_o\perp} : \mathbb{C}^6 \times \mathbb{C}^8 \to \mathbb{C}^8 \) and \( B_{L_o\parallel} : \mathbb{C}^8 \times \mathbb{C}^8 \to \mathbb{C}^6 : \)

\[ B_{L_o} := B'_{L_o\perp} + B''_{L_o\parallel}, \]

where

\[ B'_{L_o\perp}(u_{\parallel}, u_{\perp}) := (0_{\mathbb{C}4}, -\theta_e v_{eg} B_z, \theta_e v_{ex} B_z, \frac{\partial^2}{\partial e} v_{iy} B_z, -\frac{\partial^2}{\partial e} v_{ex} B_z), \]

16
\[ B''_{L, \perp}(u, u) := (0_C, \theta e_v e z B_y, -\theta e_v e z B_x, \frac{\theta^2}{\theta e} v_{i x} B_y, \frac{\theta^2}{\theta e} v_{i x} B_x), \]

\[ B_{L, 0} := B'_{L, \perp} + B''_{L, \perp}, \]

and

\[ B_{L, \parallel}(u, u') := (0_C, -\theta(e v e x B'_y - v e y B'_x), 0, \frac{\theta^2}{\theta e} (v_{i x} B'_y - v_{i y} B'_x), 0). \]

**Initial datum.** In the following, \( u^{0, \varepsilon} \) denotes a real highly oscillating initial datum

\[ u^{0, \varepsilon} = u_{-1}^0 e^{-ikz/\varepsilon} + u_1^0 e^{ikz/\varepsilon}, \tag{2.4} \]

where \( u_{-1}^0 \) is the complex conjugate of \( u_1^0 \). One supposes that the initial datum is regular, that is \( u_{-1}^0, u_1^0 \in H^\infty(\mathbb{R}^2) \). It would of course be possible to deal with initial data with a finite (but sufficiently large) regularity. We suppose in addition that \( u^{0, \varepsilon} \) is polarized with respect to a transverse characteristic phase \( \beta = (\omega, k) \), with \( k \neq 0 \). This means that we choose \( \beta = (\omega, k) \) with \( k \neq 0 \) and such that \( \det L(\beta) = 0 \) and \( u_p^0 \) such that

\[ \pi_L(p\beta)u_p^0 = u_p^0, \tag{2.5} \]

for \( p = \pm 1 \). We represent \( u^{0, \varepsilon} \) by the profile \( u^0 \):

\[ u^{0, \varepsilon}(x, z) = [u^0(x, z, \theta)]_{\theta = kz/\varepsilon} = [u_{-1}^0 e^{-i\theta} + u_1^0 e^{i\theta}]_{\theta = kz/\varepsilon}. \]

Any vector \( u \in \mathbb{C}^8 \) is decomposed into ker\((1 - \pi(p\beta)) \oplus \ker(p\beta) \); the components of \( u \) in this decomposition are referred to as polarized and non-polarized component of \( u \). The initial condition must also satisfy (2.1). We take for instance

\[ u^0 = (B^0, E^0, v e^0, 0, v i^0, 0), \]

with

\[ \nabla \cdot B^0 = 0, \quad \nabla \cdot E^0 = 0. \]

The polarization (2.5) is spelled out in (2.13).

In sections 3, 4 and 5, we consider solutions of amplitude \( O(\varepsilon) \), that is with the initial datum \( \varepsilon u^{0, \varepsilon} \). In section 6, we consider solutions of amplitude \( O(\sqrt{\varepsilon}) \), that is with the initial datum \( \sqrt{\varepsilon} u^{0, \varepsilon} \).

**Ansätze.** One looks for profiles \( u \) depending on a certain set of variables, among which the angular variable \( \theta \in \mathbb{T} \). The profiles are periodic in \( \theta \). The Fourier series decomposition is \( u = \sum_{p \in \mathbb{Z}} e^{ip\theta} u_p \). The notation \( (u)_p = u_p \) is also used for the \( p \)th Fourier coefficient of \( u \). A comma then makes the
distinction between an index coming from a WKB expansion and the index of a Fourier mode: \((u_m)_p = u_{m,p}\).

In section 3, we describe approximate solutions in the form

\[
u^\varepsilon(t, x, z) = \varepsilon \left[ \sum_{m=0}^{m_0} \varepsilon^m u_m(t, x, z, \theta) \right]_{\theta = (kz - \omega t)/\varepsilon}, \tag{2.6}\]

over time intervals \(O(1)\) in \(t\) (weakly nonlinear WKB approximation in the geometric optics regime).

In section 4, we describe approximate solutions in the form

\[
u^\varepsilon(t, x, z) = \varepsilon \left[ \sum_{m=0}^{m_0} \varepsilon^m u_m(\tau, t, x, z, \theta) \right]_{\tau = \varepsilon t, \theta = (kz - \omega t)/\varepsilon}, \tag{2.7}\]

over time intervals \(O(1)\) in \(\tau\) (WKB approximation for solutions with large amplitudes in the diffractive optics regime). In order that this ansatz be coherent over times \(O(1/\varepsilon)\) in \(t\), one looks for correctors \(u_m, m \geq 1\), satisfying the following sublinear growth condition: for all \(\tau \in [0, \tau_0]\),

\[
\lim_{t \to \infty} \frac{1}{t} \| u_m(\tau, t) \|_{\dot{H}^s(\mathbb{R}^2 \times T)} = 0, \tag{2.8}\]

where the Sobolev index \(s\) is real and large. The profiles will be seen to be continuous in \(\tau, t\) with values in \(H^s\), so that the above estimate will make sense.

In section 5, we describe approximate solutions in the form

\[
u^\varepsilon(t, x, z) = \varepsilon \left[ \sum_{m=0}^{m_0} \varepsilon^m u_m(\tau, t, x, z, \theta) \right]_{\tau = \varepsilon t, \theta = (kz - \omega t)/\varepsilon}, \tag{2.9}\]

over time intervals \(O(1)\) in \(t\). The difference with (2.6) is that in (2.9) the profiles are allowed to depend on \(\varepsilon\). Such an asymptotic expansion is called of Chapman-Enskog type in the context of kinetic equations (see [12] for a quick introduction and references). We call them generalized WKB expansions. To ensure that (2.9) makes sense as an asymptotic expansion, one assumes that for each \(m\), \(\varepsilon u_{m+1}\) is negligible to \(u_m\) in \(H^s\), for a large \(s\).

In section 6, we consider the specific regime \(\theta_i = O(\sqrt{\varepsilon})\) and we describe approximate solutions in the form

\[
u^\varepsilon(t, x, z) = \varepsilon^{1/2} \left[ \sum_{m=0}^{m_0} \varepsilon^m u_m(t, X, z, \theta) \right]_{X = x/\sqrt{\varepsilon}, \theta = (kz - \omega t)/\varepsilon}, \tag{2.10}\]

18
over time intervals $O(1)$ in $t$.

In all cases, one denotes by $u^\varepsilon$ the profile that represents the solution:

$$u^\varepsilon = \varepsilon^s [u^\varepsilon],$$

in the sense of (2.6), (2.7), (2.9) or (2.10). The parameter $s$ equals 1 in (2.6) and (2.9), corresponding to a standard amplitude $O(\varepsilon)$; it equals 1 in (2.7), corresponding to a large amplitude $O(\varepsilon)$ – because the solution is considered over diffractive time scales in this case; finally $s = 1/2$ in (2.10), corresponding to a large amplitude $O(\sqrt{\varepsilon})$.

If $u^\varepsilon$ of the form (2.6), (2.7), (2.9) or (2.10) is solution of (EM), then $u^\varepsilon$ is solution of

$$\begin{cases}
L(\beta \partial_\theta) u^\varepsilon_\perp + L(\varepsilon \partial_0, \varepsilon \partial_z) u^\varepsilon_\perp + C \varepsilon \partial_\perp u^\varepsilon_\parallel + \varepsilon^s L(u^\varepsilon_\parallel, (\varepsilon \partial_x, 0, k \partial_\theta + \varepsilon \partial_z)) u^\varepsilon_\perp \\
= \varepsilon^s \left( B_\perp(u^\varepsilon_\parallel, u^\varepsilon_\perp) + B_{Lo\perp}(u^\varepsilon_\parallel, u^\varepsilon_\perp) + R_\perp(\varepsilon^s u^\varepsilon_\parallel, u^\varepsilon_\perp) \right),
\end{cases}$$

$$\begin{cases}
M(\beta \partial_\theta) u^\varepsilon_\parallel + M(\varepsilon \partial_0, \varepsilon \partial_z) u^\varepsilon_\parallel + C^* \varepsilon \partial_\parallel u^\varepsilon_\perp + \varepsilon^s M(u^\varepsilon_\parallel, (\varepsilon \partial_x, 0, k \partial_\theta + \varepsilon \partial_z)) u^\varepsilon_\perp \\
= \varepsilon^s \left( B_\parallel(u^\varepsilon_\parallel, u^\varepsilon_\parallel) + B_{Lo\parallel}(u^\varepsilon_\parallel, u^\varepsilon_\parallel) + R_\parallel(\varepsilon^s u^\varepsilon_\parallel, u^\varepsilon_\parallel) \right).
\end{cases}$$

In the geometric optics approximations (that is for (2.6), (2.9) and (2.10)), $\partial_0 = \partial_t$. In the diffractive optics approximation (2.7), $\partial_0 = \varepsilon \partial_x + \partial_t$. For the ansatz (2.10), one has to change $\varepsilon \partial_x$ into $\sqrt{\varepsilon} \partial_X$ in (EM).

We now describe standard, and less standard, WKB procedures for the (EM) equations. The idea is to plug in (EM) the above ansätze, write the resulting system as a formal series in $\varepsilon$ and equal the first terms in the series to 0 to construct an approximate solution.

In this view, we need to introduce the following notations. Plugging any ansatz in the nonlinear terms and ordering the terms according to the powers of $\varepsilon$, one sets

$$L(u^\varepsilon_\parallel, k \partial_\theta) u^\varepsilon_\parallel := \sum_{m \geq 0} \varepsilon^{ms} L_m, \quad B_\perp(u^\varepsilon_\parallel, u^\varepsilon_\perp) := \sum_{m \geq 0} \varepsilon^{ms} B_{\perp m},$$

$$B_{Lo\perp}(u^\varepsilon_\parallel, u^\varepsilon_\perp) := \sum_{m \geq 0} \varepsilon^{ms} B_{Lo\perp m}, \quad R_\perp(\varepsilon^s u^\varepsilon_\parallel, u^\varepsilon_\perp) := \sum_{m \geq 0} \varepsilon^{ms} R_{\perp m},$$

For (2.6), (2.7) and (2.9), one sets

$$L(u^\varepsilon_\parallel, (\partial_x, 0, \partial_z)) u^\varepsilon_\parallel := \sum_{m \geq 0} \varepsilon^m L'_m.$$
For (2.10), one sets
\[ L(u_\perp, \partial_X)u_\perp^\varepsilon := \sum_{m \geq 0} \varepsilon^{m/2} L_m', \quad L(u_\parallel, \partial_z)u_\perp^\varepsilon := \sum_{m \geq 0} \varepsilon^{m/2} L_m'. \]

Similar notations are used for the longitudinal terms.

Given an asymptotic expansion of the type (2.6), one sets for all 0 \leq m \leq m_0:
\[ R_m := (R_{\perp m}, R_{\parallel m}), \]
where
\[ R_{\perp m} := L(\beta \partial_\theta)u_{\perp m} + L(\partial_t, \partial_z)u_{\perp m-1} + C\partial_x u_{\parallel m-1} + L_{m-1} + L'_{m-2} - B_{\perp m-1} - B_{L_0 \perp m-1} - R_{\perp m-2}, \]
and
\[ R_{\parallel m} := M(\beta \partial_\theta)u_{\parallel m} + M(\partial_t, \partial_z)u_{\parallel m-1} + C^*\partial_x u_{\perp m-1} + M_{m-1} + M'_{m-2} - B_{\parallel m-1} - B_{L_0 \parallel m-1} - R_{\parallel m-2}, \]
and one defines the residual of \( u^\varepsilon \) as
\[ R^\varepsilon := \sum_{m=0}^{m_0} \varepsilon^m R_m. \]

Given an asymptotics expansion of the type (2.7), one adds the term \( \partial_t u_{m-2} \) to \( R_m \) to obtain the \( m \)th term of the residual.

Given an asymptotic expansion \( u^\varepsilon \) of the type (2.9), one sets \( r_0 := R_0 \), and for all 0 \leq m \leq m_0:
\[ r_{\perp m}^\varepsilon := L(\beta \partial_\theta)u_{\perp m} + \tilde{\Pi}_L(\beta)L(\partial_t, \partial_z)\tilde{\Pi}_L(\beta)u_{\perp m-1} + (1 - \tilde{\Pi}_L(\beta))L(\partial_t, \partial_z)u_{\perp m-1} + (1 - \tilde{\Pi}_L(\beta))C_1\partial_x u_{\parallel m-1} + C_2\partial_x u_{\parallel m-1} + L_{m-1} + L'_{m-2} - (1 - \tilde{\Pi}_L(\beta))B_{\perp m-1} - \delta(m, 1)\tilde{\Pi}_L(\beta)B_{\perp 0} - B_{L_0 \perp m-1} - R_{\perp m-2} + \varepsilon\tilde{\Pi}_L(\beta)L(\partial_t, \partial_z)(1 - \tilde{\Pi}_L(\beta))u_{\perp m} + \varepsilon\tilde{\Pi}_L(\beta)C_1\partial_x u_{\parallel m} - \varepsilon\tilde{\Pi}_L(\beta)B_{\perp m}, \]
where \( \delta \) is the Kronecker symbol, and for 0 \leq m \leq m_0:
\[ r_{\parallel m}^\varepsilon := R_{\parallel m}. \]

The residual is \( r^\varepsilon := \sum_{m=0}^{m_0} \varepsilon^m r_m^\varepsilon. \) The difference between \( R^\varepsilon \) and \( r^\varepsilon \) is the presence of terms with a prefactor \( \varepsilon \) in \( r_m^\varepsilon. \) These terms are:
• the linear terms in \( \varepsilon C \partial_x \) and \( \varepsilon \tilde{\Pi}_L \) — to include these terms in the equation for the terms in \( O(\varepsilon^m) \) rather than in the equation for the terms in \( O(\varepsilon^{m+1}) \) allows to derive a Schrödinger equation in a geometric optics regime, as in [13] and [2], and

• the nonlinear term \( \varepsilon \tilde{\Pi}_L(\beta)B \). The presence of this term at this level allows to couple the equation for \( \tilde{u}_{1,0} \) with the equation for \( u_{1,0} \), and thus to derive a weak form of the Zakharov equations in the weakly nonlinear regime in spite of the transparency properties stated in Proposition 2.1. The coupling term is of size \( O(\varepsilon) \) and this is why we say that the coupling is weak.

Given an asymptotic expansion \( u^\varepsilon \) of the type (2.10), one sets

\[
\bar{R}_m := \bar{R}_m(\varepsilon),
\]

where

\[
\bar{R}_m := L(\beta \partial \theta)u_{m-2} + C \partial_x u_{m-1} + L_{m-1} + L_{m-2} + L_{m-3} - B_{1,m-1} - B_{L0,m-1} - R_{1,m-2},
\]

and

\[
\bar{R}_m := M(\beta \partial \theta)u_{m-2} + C^* \partial_x u_{m-1} + M_{m-1} + M_{m-2} + M_{m-3} - B_{1,m-1} - B_{L0,m-1} - R_{m-2},
\]

and the residual is \( \tilde{R}^\varepsilon := \sum_{m=0}^{m_0} \varepsilon^m \bar{R}_m \).

One calls (local) approximate solution at a certain order \( l \) any asymptotic expansion of the above types such that the first \( l - 1 \) terms of its residual vanish identically over a time interval \([0, t(\varepsilon)]\), or \([0, \tau(\varepsilon)] \times \mathbb{R}_t\), depending on the time regime under consideration.

**Characteristic variety.** The dispersion relation is

\[
\det L(\beta) \det M(\beta) = 0.
\]

One computes

\[
\begin{align*}
\det L(\beta) &= (-i\omega)^4(\omega^2 - k^2 - 1 - \left(\frac{\theta_i}{\theta_e}\right)^2)^2, \\
\det M(\beta) &= (-i\omega)^2((\omega^2 - \alpha k^2 \theta_e^2)(\omega^2 - 1 - k^2 \theta_e^2) \\
&\quad - (\omega^2 - k^2 \theta_e^2)(\frac{\theta_i}{\theta_e})^2).
\end{align*}
\]
The plasma frequency is
\[ \omega_p := \sqrt{1 + \left( \frac{\theta_i}{\theta_e} \right)^2}. \]

With this notation, one has \( \tilde{\omega}_e \omega_p = \tilde{\omega}_p \), where \( \tilde{\omega}_p \) is the (physical) plasma frequency introduced in section 2.1. The characteristic variety is the union of branches \((t), (l), (s)\), for which the expansions in the limit \( \theta_i \to 0 \) are
\[
(t) \quad \omega^2_{(t)} = \omega^2_p + k^2,
\]
\[
(l) \quad \omega^2_{(l)} = \omega^2_p + k^2 \theta^2_e + \left( \frac{1}{1 + k^2 \theta^2_e} - 1 \right) + O(\theta^4_i),
\]
\[
(s) \quad \omega^2_{(s)} = k^2 \theta^2_i \left( \frac{\alpha^2}{1 + k^2 \theta^2_e} \right) + O(\theta^4_i),
\]
locally uniformly in \( k \). The behaviour for large \( k \) is investigated by noticing that \( |\omega| \leq C|k| \), and setting \( \omega(k) = k\omega'(1/k) \). The equation satisfied by \( \omega' \) is
\[
(\omega'^2 - \alpha^2 \theta^2_i)(\omega'^2 - \theta^2_e(1 + \frac{1}{k^2 \theta^2_e})) - k^2 (\omega'^2 - \theta^2_e) \left( \frac{\theta_i}{\theta_e} \right)^2 = 0,
\]
which for small \( k \) gives \( \omega'^2_{(l)} = \theta^2_e + O(k^2) \) and \( \omega'^2_{(s)} = \alpha^2 \theta^2_i + O(k^2) \) (\( \omega' \) is analytic for small \( k \), see [17] or [28]). This gives for large \( k \) :
\[
(l) \quad \omega_{(l)} = \pm \theta_e k + O\left( \frac{1}{k} \right),
\]
\[
(s) \quad \omega_{(s)} = \pm \alpha \theta_i k + O\left( \frac{1}{k} \right).
\]
The branch \((t)\) corresponds to transverse characteristic phases (i.e. \( \beta \) such that \( \det L(\beta) = 0 \)) and the branches \((l)\) and \((s)\) correspond to longitudinal characteristic phases (i.e. \( \beta' \) such that \( \det M(\beta') = 0 \)).

Figure 1: The characteristic variety for the Euler-Maxwell system.

The characteristic variety is pictured on Figure 1. It matches the classical physics textbooks descriptions (see for instance [11], t.2, paragraph 10.3). In this figure, \( \theta_e, \theta_i \) and \( \alpha \) have numerical values much greater than their values in realistic physical situations.

In the following, we fix a phase \( \beta = (\omega, k) \) satisfying \( \det L(\beta) = 0 \) and \( k \neq 0 \). Then \( \det M(\beta) \neq 0 \). That is, the initial polarization of the electromagnetic field is transverse.
For any characteristic phase, only the harmonics $-1, 0, 1$ are characteristic (see Figure 1). Thus we say that a transverse profile $u_\perp$ is polarized with respect to $\beta$ when

$$u_\perp = u_{\perp, -1}e^{-i\theta} + u_{\perp, 0} + u_{\perp, 1}e^{i\theta}, \quad \text{and} \quad \Pi_L(\beta)u_\perp = u_\perp.$$  

The space $\text{Ker } L(\beta)$ has dimension 2. A basis is composed of the vectors

$$(0, \frac{k}{\omega} E_x, E_x, 0, \frac{1}{i\omega} E_x, 0, \frac{-1}{i\omega} \theta_e E_x, 0), \quad E_x \neq 0,$$

and

$$(-\frac{k}{\omega} E_y, 0, 0, E_y, 0, \frac{1}{i\omega} E_y, 0, \frac{-1}{i\omega} \theta_e E_y), \quad E_y \neq 0.$$  

The transverse polarization condition $\pi_L(\beta)u_\perp = u_\perp$ reads

$$\begin{cases} B_x = -\frac{k}{\omega} E_y, & B_y = \frac{k}{\omega} E_x, \\ v_{ex} = \frac{1}{i\omega} E_x, & v_{ey} = \frac{-1}{i\omega} \theta_e E_x, \end{cases} \quad \text{(2.13)}$$

The transverse compatibility condition $\pi_L(\beta)u_\perp = 0$ reads

$$\begin{cases} \frac{k}{\omega} B_y + E_x + \frac{1}{i\omega} (v_{ex} - \frac{\theta_i}{\theta_e} v_{ix}) = 0, \\ -\frac{k}{\omega} B_x + E_y + \frac{1}{i\omega} (v_{ey} - \frac{\theta_i}{\theta_e} v_{iy}) = 0 \end{cases} \quad \text{(2.14)}$$

The polarization condition

$$(L(0)u_\perp, M(0)u_\parallel) = 0 \quad \text{amounts to} \quad E = 0, \quad v_e = \frac{\theta_i}{\theta_e} v_i.$$  

The equation $M(0)u_\parallel = (b_0, b_1, b_2, b_3, b_4, b_5)$ implies (elliptic inversion)

$$B_z = -\frac{1}{i\omega} b_0,$$  

and

$$E_z = \frac{-\omega^2}{\det M(\beta)} \left( \frac{-1}{i\omega} (\omega^2 - k^2 \theta_e^2) (\omega^2 - k^2 \alpha^2 \theta_i^2) b_1 \\ - (\omega^2 - k^2 \alpha^2 \theta_i^2) (b_2 + \frac{k \theta_e}{\omega} b_3) \\ + \frac{\theta_i}{\theta_e} (\omega^2 - k^2 \theta_e^2) (b_4 + \frac{k \theta_e}{\omega} b_5) \right), \quad \text{(2.16)}$$

23
The equation \( M(\beta)u_\parallel = C^*_1(b_0, b_1, b_2, b_3, b_4, b_5) \) implies
\[
n_e = \frac{k\theta_e}{\omega} v_{ez} \text{ and } B_z = 0. \tag{2.17}
\]
The transparency properties of (EM) are stated in the following proposition.

**Proposition 2.1.** The nonlinear terms in (EM) satisfy:

(t1) Let \( u \) be such that \( u_\perp \) is polarized with respect to \( \beta \). Then for \( p \in \{-1, 0, 1\} \),
\[
\left( L(u_\parallel, k\partial_\theta)u_\perp - B'_L\omega_L(u_\parallel, u_\perp) \right)_p = -B'_L\omega_L(u_\parallel, u_\perp, k_\theta, u_\perp, 0).
\]

(t2) Let \( u_\perp \) and \( u'_\perp \) be profiles that are polarized with respect to \( \beta \). Then
\[
\pi_M(0)\left( B_{L\parallel}(u_\perp, u'_\perp) + B_{L\parallel}(u'_\perp, u_\perp) \right)_0 = 0.
\]

(t3) Let \( u_\parallel \) and \( u'_\parallel \) be two longitudinal profiles such that there exists two profiles \( u_\perp \) and \( u'_\perp \), which are both polarized with respect to \( \beta \) and such that \( u_\parallel = M(\beta)^{-1}C^*u_\perp \), \( u'_\parallel = M(\beta)^{-1}C^*u'_\perp \). Then
\[
\left( M(u_\parallel, k\partial_\theta)u'_\parallel + M(u'_\parallel, k\partial_\theta)u_\parallel \right)_0 = 0.
\]

(t4) The nonlinear transverse current density term is not transparent in the sense that for any profile \( u_\perp \neq 0 \) polarized with respect to \( \beta \), for all \( u_\parallel \in \mathbb{C}^6 \), for all \( p \in \{-1, 1\} \):
\[
\pi_L(p\beta)(B_\perp(\pi_L(p\beta)u_\perp, u_\parallel))_p = 0 \iff n_e = 0.
\]

**Proof.** (t1) The electronic term in the \( x \) direction is
\[
(v_{ez,k\partial_\theta}v_x - v_{ez}B_y)_p = \sum_{p_1 + p_2 = p, p_2 \neq 0} v_{ez,p_1}(ip_2kv_{ex,p_2} - B_{y,p_2}) + v_{ez,p}B_{y,0} = 0.
\]

The electronic term in the \( y \) direction is
\[
(v_{ez,k\partial_\theta}v_y + v_{ez}B_x)_p = \sum_{p_1 + p_2 = p, p_2 \neq 0} v_{ez,p_1}(ip_2kv_{ey,p_2} + B_{x,p_2}) + v_{ez,p}B_{x,0} = 0.
\]
The ionic terms are similar.

(t2) One computes

\[
\left( v_{ez}B'_y - v_{ey}B'_x + v'_{ez}B_y - v'_{ey}B_x \right)_0
\]

\[
= \frac{k}{\omega} \left( \frac{1}{ip_1\omega} + \frac{1}{ip_2\omega} \right) E_{x,p_1} E'_{x,p_2}
\]

\[
+ v_{ex,0}B'_{y,0} + v'_{ex,0}B_{y,0} - (v_{ey,0}B'_{x,0} + v'_{ey,0}B_{x,0}).
\]

The terms within the sum all vanish (and the same is true for the ionic terms). It follows that the only possibly nonzero entry of the left hand side (t2) is

\[
(v_{ex,0} - \frac{\theta_i}{\theta_e} v_{ix,0}) B'_{y,0} + (v'_{ex,0} - \frac{\theta_i}{\theta_e} v'_{ix,0}) B_{y,0}
\]

\[
- (v_{ey,0} - \frac{\theta_i}{\theta_e} v_{iy,0}) B'_{x,0} - (v'_{ey,0} - \frac{\theta_i}{\theta_e} v'_{iy,0}) B_{x,0},
\]

and with the polarization condition for \( u_0 \) and \( u'_0 \), these four terms all vanish.

(t3) For all indices \( p_1, p_2 \in \mathbb{Z} \) such that \( p_1 + p_2 = 0 \) and \( (p_1, p_2) \neq (0, 0) \), one has

\[
v_{ez,p_1}(ip_2k)v'_{ez,p_2} + v'_{ez,p_2}(ip_1k)v_{ez,p_1} = (ip_2k)(v_{ez,p_1}v'_{ez,p_2} - v'_{ez,p_2}v_{ez,p_1})
\]

\[
= 0.
\]

The hypothesis implies that

\[
n_{ep} = \frac{k\theta_e}{\omega} v_{ezp}, \text{ and } n'_{ep} = \frac{k\theta_e}{\omega} v'_{ezp}, \text{ for all } p \neq 0.
\]

Hence, for all \( p_1, p_2 \) such that \( p_1 + p_2 = 0 \) and \( (p_1, p_2) \neq (0, 0) \), one has

\[
v_{ez,p_1}(ip_2k)n'_{ez,p_2} + v'_{ez,p_2}(ip_1k)n_{ez,p_1} = ip_2k\frac{k\theta}{\omega} (v_{ez,p_1}v'_{ez,p_2} - v'_{ez,p_2}v_{ez,p_1})
\]

\[
= 0.
\]

The same computations show that the ionic terms vanish as well, and this proves the third transparency equality.

(t4) One computes for \( p \in \{-1, 1\} \) :

\[
(B_{\perp}(\pi_L(p\beta)u_{\perp}, u_{\parallel}))_p = \frac{n_c}{ip\omega}(0_2, E_{xp}, E_{yp}, 0_4).
\]

and the equivalence follows. \( \square \)
3 Approximation by a linear transport equation

We use in this section the transparency equalities stated in Proposition 2.1 to show that the WKB asymptotics of (EM) is given by a linear transport equation in the geometrical optics limit.

Let $u^\varepsilon$ be an approximate solution of the form (2.6) at a certain order $\geq 1$, with the initial datum $\varepsilon u^{0,\varepsilon}$. $R_0 = 0$ gives the polarization equalities:

\[
\begin{aligned}
L(\beta \partial_\theta) u_{\perp 0} &= 0, \\
M(\beta \partial_\theta) u_{\parallel 0} &= 0.
\end{aligned}
\]

Then

\[
\Pi_L(\beta) u_{\perp 0} = u_{\perp 0}, \quad (3.1)
\]

and, as $\beta \in \text{Char } L$:

\[
\tilde{u}_{\parallel 0} = 0. \quad (3.2)
\]

The leading mean modes are polarized:

\[
\pi_L(0) u_{\perp 0,0} = u_{\perp 0,0}, \quad \pi_M(0) u_{\parallel 0,0} = u_{\parallel 0,0},
\]

which gives explicitly:

\[
E_{0,0} = 0, \quad v_{e0,0} = \frac{\theta_i}{\theta_e} v_{i0,0}. \quad (3.3)
\]

$R_{1,0} = 0$ implies $\partial_t B_{0,0} + \nabla \times E_{0,0} = 0$. The initial value for $B_{0,0}$ is zero, this implies $B_{0,0} = 0$ at all times. Next we examine the nonlinear terms in the equation for $(\pi_L(0) u_{\perp 0,0}, \pi_M(0) u_{\parallel 0,0})$:

- the current density terms vanish: $\pi_L(0) B_{\perp 0} = 0$, and $\pi_M(0) B_{\parallel 0} = 0$.
- the convective terms are

\[
L_{0,0} = (L(u_{\parallel 0,0}) k \partial_\theta) u_{\perp 0,0}, \quad M_{0,0} = (M(u_{\parallel 0,0}) k \partial_\theta) u_{\parallel 0,0}.
\]

They vanish since $\tilde{u}_{\parallel 0} = 0$.
- the nonlinear Lorentz force terms vanish as well: $\tilde{u}_{\parallel 0} = 0$ and $B_{0,0} = 0$ imply that $B_{L0,0} = 0$. Besides, the transparency equality (t2) from Proposition 2.1 implies that $\pi_M(0) B_{L0,0} = 0$. 

26
The equation \( R_{1,0} = 0 \) is thus linear. It follows that the equation for \( n_{i0,0} \) and \( v_{i0,0} \) is a linear wave equation (the usual geometric optics approximation of the Euler equations) with no source term. The initial value of the mean mode being zero, it follows that the mean mode of \( n_{i0} \) and \( v_{i0} \) vanishes at all times: \( n_{i0,0} = 0, v_{i0,0} = 0 \). With (3.3), this implies \( n_{e0,0} = 0 \) and \( v_{e0,0} = 0 \). One has finally

\[
u_{0,0} = 0, \quad u_{\parallel 0} = 0,
\]

The leading oscillating mode satisfies the transport equation

\[
T_{\beta}(\partial)\pi_L(p\beta)u_{\perp 0,p} = 0,
\]

for \( p \in \{-1, 1\} \), where \( T_{\beta}(\partial) \) is the transport operator \( \partial_t + \frac{k}{\omega} \partial_z \). (3.5) is explicitly

\[
(\partial_t + \frac{k}{\omega} \partial_z) \begin{pmatrix} E_{0x,p} \\ E_{0y,p} \end{pmatrix} = 0.
\]

The nonpolarized part of the first corrector satisfies for \( p \neq 0 \):

\[
(1 - \pi_L(p\beta))u_{\perp 1,p} = -L(ip\beta)^{-1}L(\partial)\pi_L(p\beta)u_{\perp 0,p}, \quad \text{(3.6)}
\]

\[
u_{\parallel 1,p} = -M(ip\beta)^{-1}C^* \pi_L(p\beta)\partial_x u_{\perp 0,p}. \quad \text{(3.7)}
\]

Note that \( L(ip\beta)^{-1}L(\partial)\pi_L(p\beta) \) is a differential operator in \( z \) only. One has in particular \( u_{\perp 1,0} = \pi_L(0)u_{\perp 1,0} \) and \( u_{\parallel 1,0} = \pi_M(0)u_{\parallel 1,0} \).

It is possible to construct approximate solutions at all orders and to prove their stability over times \( O(1) \) \([18]\), and even \( O(|\log\epsilon|) \) \([22]\). Such a result is contained in the result of the following section.

The amplitude of the solution was chosen in order that nonlinear effects be apparent for times \( O(1) \), but because of transparency properties of (EM), the limit system is linear. It is thus natural to push the analysis over diffractive times \( O(1/\epsilon) \).

### 4 Approximation by a Davey-Stewartson equation

In this section, we rigorously derive an approximation by a Davey-Stewartson system over times \( O(|\log\epsilon|) \). We use the technique of separation of time scales introduced by Lannes in \([20]\) and generalize to quasilinear equations the result of Lannes and Rauch \([22]\).

Let \( u^\epsilon \) be an approximate solution of the form (2.7) at a certain order \( \geq 2 \), with the initial datum \( \epsilon u^{0,\epsilon} \). The leading terms of \( u^\epsilon \) satisfy the polarization
equations (3.1), (3.2), (3.3), and the equations (3.4), (3.5), (3.6), (3.7). Apply the projector \( \Pi_L(\beta) \) on the left of the equation \( R_2 = 0 \) to find

\[
(\partial_t + iR(\partial, \partial))\Pi_L(\beta)u_{\perp 0} + T_\beta(\partial)\Pi_L(\beta)u_{\perp 1} = \Pi_L(\beta)(B''_{\perp 0,1} + B_{\perp 1} - L_0'),
\]

where \( R \) is the second-order spatial differential operator

\[
R(\partial, \partial) := \Pi_L(\beta)(L(\partial)L^{-1}(\partial) + CM(\beta)^{-1}C^* \partial_z^2) \Pi_L(\beta).
\]

The nonlinear terms are bilinear coupling terms with \( u_{1,0} \). The nonlinear term \( \Pi_L(\beta)(B''_{\perp 0,1} - L_1) \) vanishes with the transparency equality (t1) from Proposition 2.1. The equation \( R_{2,0} = 0 \) implies

\[
\begin{align*}
\pi_L(0)L(\partial)\pi_L(0)u_{\perp 1,0} &+ \pi_L(0)C\partial_z\pi_M(0)u_{\parallel 1,0} = \pi_L(0)(-L'_{0,0} - L_{1,0} + B_{\perp 0,1,0}), \\
\pi_M(0)\pi_M(0)u_{\parallel 1,0} &+ \pi_M(0)C\partial_z\pi_M(0)u_{\perp 1,0} = \pi_M(0)B_{\perp 0,1,0}.
\end{align*}
\]

\( \pi_L(0)L(\partial)\pi_L(0) \) is the scalar operator \( \partial_t \pi_L(0) \) and \( \pi_M(0)M(\partial)\pi_M(0) \) is a linear wave operator in \( t, z \) with velocities \( \pm\theta_\lambda \sqrt{\alpha^2 + 1} \) (check it by a direct computation or see Proposition 2 of [20] or Proposition 2.8 of [28] and Figure 1). The nonlinear terms are quasilinear coupling terms:

- First,

\[
B_{\perp 0,1,0} - L_{1,0} = (B_{\perp 0,1}(u_{\parallel 1}, u_{\perp 0}) - L(u_{\parallel 1}, k\partial_\theta)u_{\perp 0})_0 = -B''_{\perp 0,1}(u_{\parallel 1}, u_{\perp 0})_0,
\]

which with (3.7) is seen to contribute to a term in \( \partial_x |E_{0y,p}|^2 \) in the equation for \( v_{1x,0} \).

- Second, \( \pi_L(0)L'_{0,0} = \pi_L(0)(L(u_{\perp 0}, \partial_x)u_{\perp 0})_0 \), which contributes to a term in \( \partial_x |E_{0x,p}|^2 \) in the equation for \( v_{1x,0} \),

- and with the second transparency equality (t2) from Proposition 2.1 and (3.6):

\[
\begin{align*}
\pi_M(0)B_{\perp 0,1,0} &+ \pi_M(0)(B'_{\perp 0})(1 - \Pi_L(\beta))u_{\perp 1,0} + \pi_M(0)(B'_{\perp 0})(1 - \Pi_L(\beta))u_{\perp 1,0})_0 \\
&= -\pi_M(0)(B'_{\perp 0})(L^{-1}(i\beta)L(\partial)u_{\perp 0,0}u_{\perp 0})_0 + \pi_M(0)(B'_{\perp 0})(L^{-1}(i\beta)L(\partial)u_{\perp 0,0}u_{\perp 0})_0,
\end{align*}
\]
which contributes to a term in $\partial_z(|E_{0x,p}|^2 + |E_{0y,p}|^2)$ in the equation for $v_{i1z,0}$.

Introduce the family of linear operators $G_{T_\beta}^h, h > 0$, defined over the space $C^0([0, \tau_0] \times \mathbb{R}_t, H^s(\mathbb{R}^2 \times \mathbb{T}))$ by

$$G_{T_\beta}^h(v)(\tau, t, x, z, \theta) = \frac{1}{h} \int_0^h v(\tau, t + t', x, z + \frac{k}{\omega} t', \theta) \, dt'.$$

Set $G_{T_\beta} = \lim_{h \to \infty} G_{T_\beta}^h v$ when this limit exists in $C^0([0, \tau_0] \times \mathbb{R}_t, H^s(\mathbb{R}^2 \times \mathbb{T}))$. $G_{T_\beta}$ is the averaging operator associated with the transport $T_\beta$ [20]. It satisfies the following properties:

**Proposition 4.1 (Lannes [20]).**

a) If $v$ satisfies $T_\beta(\partial) v = 0$, then $G_{T_\beta} v$ is well defined and satisfies $G_{T_\beta} v = v$.

b) $v$ satisfies the sublinear growth condition (2.8) if and only if $G_{T_\beta} T_\beta v = 0$.

c) Let $v_1$ and $v_2$ satisfy $(\partial_t + c_1 \partial_z) v_i = 0$. Then if $c_1 = c_2 = k/\omega$, then $G_{T_\beta}(v_1 v_2) = v_1 v_2$, otherwise $G_{T_\beta}(v_1 v_2) = 0$.

Applying the averaging projector $G_{T_\beta}$ to (4.1), one finds for $p \in \{-1, 1\}$ :

$$(\partial_t + i R_p(\partial, \theta)) \pi_L(p \beta) u_{\perp 0, p} = \pi_L(p \beta)(B'' L_{0, \perp 1} + B_{\perp 1})(G_{T_\beta} u_{\parallel 1, 0}, \pi_L(p \beta) u_{\perp 0, p}),$$

(4.4)

because the corrector $u_1$ is assumed to have a sublinear growth. The nonlinear Schrödinger equation (4.4) is explicitly

$$(i\partial_t + \frac{\omega^2_p}{(p \omega)^3} \partial_z^2 + \left( \frac{\gamma(p \beta)}{\omega} 0 \right) \partial_x^2) \begin{pmatrix} E_{0x,p} \\ E_{0y,p} \end{pmatrix} = \frac{\omega^2_p}{p \omega} G_{T_\beta}(\frac{k \theta_i}{\omega} v_{i1z,0} + n_{i1,0}) \begin{pmatrix} E_{0x,p} \\ E_{0y,p} \end{pmatrix},$$

(4.5)

where the dispersion coefficients are computed via (2.15) and (2.16) respectively. One has

$$\gamma(p \beta) = -\frac{k \omega}{\det M(\beta)} \left( \frac{-pk}{\omega^2} (\omega^2 - k^2 \theta_e^2) (\omega^2 - \alpha^2 k^2 \theta^2) + \frac{p k \theta^2}{\omega^2} (\omega^2 - \alpha^2 k^2 \theta_e^2) + \frac{1}{p \omega} \frac{k \theta_i^2 \theta^2_e}{\omega^2} (\omega^2 - k^2 \theta_e^2) \right).$$

29
One checks that for $\theta_i$ small enough $\gamma(p\beta)$ has the sign of $p$, so that $R_p$ is elliptic.

Applying $G_{T\beta}$ to (4.2)-(4.3), one finds

$$H_1(\partial_x, \partial_z)G_{T\beta}u_{1,0} = H_2(\partial_x, \partial_z)(u_{\perp 0,1}, u_{\perp 0,-1}),$$

where $H_1$ is given by

$$H_1(\partial) := \begin{pmatrix} -\frac{k}{\omega^2} \omega^2_0 \partial_z & 0 & \theta_i(\alpha^2 + 1) \partial_x \\ 0 & -\frac{k}{\omega^2} \omega^2_0 \partial_z & \theta_i(\alpha^2 + 1) \partial_z \\ \theta_i \partial_x & \theta_i \partial_z & 0 \end{pmatrix},$$

in a basis of $\text{Ran} \pi_L(0) \oplus \text{Ran} \pi_M(0)$, and

$$H_2(\partial)(u, u') := -\frac{\theta_i}{\omega^2} \begin{pmatrix} \omega^2_0 \partial_x \\ 0 \\ (1 + (\frac{\partial_i}{\partial_z})^3) \partial_z \end{pmatrix} uu'.$$

One checks that the symbol of $H_1$ is invertible over the sphere $S^2$. The system (4.4)-(4.6) is the announced elliptic-elliptic Davey-Stewartson system (DS).

The equation for $\pi_L(p\beta)u_{\perp 1,p}$ is

$$L(\partial)\pi_L(p\beta)(B_{L0\perp 1} + B_{\perp 1}((1 - G_{T\beta})u_{1,0} = 0.$$

This equation is supplemented by the initial datum

$$(1 - G_{T\beta})u_{1,0}(\tau, 0, x, z) = -H_1(\partial)^{-1}H_2(\partial)(u^0_{\perp 1}, u^0_{\perp -1})(x, z).$$

The equation for $\pi_L(p\beta)u_{\perp 1,p}, p \neq 0$, is

$$T_{\beta}(\partial)\pi_L(p\beta)u_{\perp 1,p} = \pi_L(p\beta)(B_{L0\perp 1} + B_{\perp 1}((1 - G_{T\beta})u_{1,0, p}, \pi_L(p\beta)u_{0, p}),$$

together with a null initial datum. This ensures that $\pi_L(p\beta)u_{\perp 1,p}$ has sub-linear growth. The equations for the second corrector are

$$(1 - \Pi_L(\beta))u_{\perp 2} = L(i\beta)^{-1}(-L(\partial)u_{\perp 1} - C\partial_x u_{\| 1} - L_{\perp} + B_{\perp 1} + B_{L0\perp 1}),$$
$$(1 - \Pi_M(\beta))u_{\| 2} = M(i\beta)^{-1}(-M(\partial)u_{\| 1} - C^*\partial_x u_{\perp 1} + B_{L0\| 1}).$$

The approximate solution is now defined as

$$u^\varepsilon = u_0 + \varepsilon u_1 + \varepsilon^2((1 - \Pi_L(\beta))u_{\perp 2}, (1 - \Pi_M(\beta))u_{\| 2}).$$
It satisfies the estimates
\[ \|u_0(t)\|_{L^\infty([0,\tau_0],H^s)} \leq C_0, \quad \|u_1, u_2(t)\|_{L^\infty([0,\tau_0],H^s)} \leq C_1 t, \quad (4.7) \]
where \(\tau_0\) is an existence time for the (DS) system and where \(C_0\) and \(C_1\) depend on \(\tau_0\) and \(s\) but not on \(\varepsilon\) nor on \(t\). As \(u^0 \in H^\infty\), \(s\) can be taken arbitrarily large. One could possibly obtain a smaller constant \(C_1\) with additional assumptions on the initial data and the secular growth results of [21]. In the residual, the higher-order semilinear terms are estimated like
\[ \|f(n_{e1})\|_{H^s} \leq \exp(c_s\|n_{e1}\|_{H^s}), \]
where the constant \(c_s\) depends only on \(s\). This gives
\[ \|R^\varepsilon(t)\|_{L^\infty([0,\tau_0],H^s)} \leq C_\varepsilon^3 e^{c_s C_1 t} t, \quad (4.8) \]
where the constant \(C\) does not depend on \(\varepsilon\) nor on \(t\). In (4.8), the Sobolev index \(s\) needs to be chosen slightly smaller than in (4.7).

**Theorem 4.2.** Let \(s > 5/2\), such that (4.7) and (4.8) hold. Given a real initial datum \(\varepsilon u^{0,\varepsilon} \in H^s(\mathbb{R}^2)\) of the form (2.4) and satisfying the polarization condition (2.5); given a perturbative initial profile \(w^0 \in H^s(\mathbb{R}^2 \times \mathbb{T})\):

- there exists a unique approximate solution \(u^\varepsilon\) at order 2 of the form (2.7); \(u^\varepsilon\) is regular and defined over a time interval \([0, \tau^*/\varepsilon]\), with \(\tau^* > 0\). The leading terms of \(u^\varepsilon\) satisfy the Davey-Stewartson system (4.4)-(4.6) of the form (DS);

- there exists a unique regular local solution \(v^\varepsilon\) to (EM) with the initial datum
  \[ \varepsilon(u^{0,\varepsilon} + \varepsilon^\gamma w^{0,\varepsilon}), \quad \gamma > 2, \]
  where \(w^{0,\varepsilon}\) is represented by the profile \(w^0\).

Moreover, there exists \(\tilde{C} > 0\), depending on \(\|u^0\|_{H^s}\) and on \(\gamma\), and \(\varepsilon_0 > 0\), depending on \(\tilde{C}\), such that for all \(0 < \varepsilon < \varepsilon_0\), \(v^\varepsilon\) is defined over \([0, \tilde{C} |\log \varepsilon|]\) and satisfies the estimate
\[ \sup_{0 \leq t \leq \tilde{C} |\log \varepsilon|} \frac{1}{\varepsilon} \| (v^\varepsilon - u^\varepsilon)(t) \|_{L^\infty(\mathbb{R}^2)} = o(\varepsilon). \quad (4.9) \]

The estimate in Theorem 1.2 follows from (4.9).
Proof. As $s > 5/2$, the injection $H^s(\mathbb{R}^2 \times \mathbb{T}) \hookrightarrow W^{1,\infty}(\mathbb{R}^2 \times \mathbb{T})$ holds with a constant $c_s$ (which is slightly bigger than the constant $c_s$ from (4.8)). Set

$$\bar{u}^\varepsilon(t, x, \theta) := u^\varepsilon(\varepsilon t, t, x, \theta).$$

We look for the exact solution $v^\varepsilon$ in the form of a profile $v^\varepsilon$ of the form (2.6): $v^\varepsilon = \varepsilon [v^\varepsilon]$, where $v^\varepsilon$ is a perturbation of $u^\varepsilon$:

$$v^\varepsilon = \bar{u}^\varepsilon + w^\varepsilon.$$

The initial datum for $w^\varepsilon$ is $\varepsilon^3 w^0$. Introduce the notations:

$$\mathcal{L}(\partial) := \left( \begin{array}{c} L(\partial) & C \partial_x \\ C^* \partial_x & M(\partial) \end{array} \right), \quad \mathcal{L}(u, \nabla) := (\mathcal{L}(u, \nabla), M(u, \nabla)),$$

$$\mathcal{B}(u, u) := ((B_\perp + B_{L_\perp})(u_\parallel, u_\perp), B_\parallel(u_\parallel, u_\parallel) + B_{L_\parallel}(u_\perp, u_\perp)), \quad \mathcal{C}(u, u) := (\mathcal{R}_\perp(u_\parallel, u_\perp), \mathcal{R}_\parallel(u_\parallel, u_\parallel)).$$

The equation satisfied by $u^\varepsilon$ is

$$\mathcal{L}(\partial_t - \frac{\omega}{\varepsilon} \partial_\theta, \partial_x, \partial_z) u^\varepsilon + \mathcal{L}(\varepsilon u^\varepsilon, (\partial_x, 0, \partial_z + \frac{k}{\varepsilon} \partial_\theta)) u^\varepsilon = \mathcal{B}(u^\varepsilon, u^\varepsilon) + \frac{1}{\varepsilon} \mathcal{C}(\varepsilon u^\varepsilon, u^\varepsilon) + \mathcal{R}^\varepsilon.$$

The equation to be satisfied by $w^\varepsilon$ in order that $v^\varepsilon$ be an exact solution is

$$\mathcal{L} w^\varepsilon + \mathcal{L}(\varepsilon (u^\varepsilon + w^\varepsilon)) w^\varepsilon + \mathcal{L}(\varepsilon w^\varepsilon) u^\varepsilon = B_s(u^\varepsilon) w^\varepsilon + B(w^\varepsilon, w^\varepsilon) + \frac{1}{\varepsilon} \left( \mathcal{C}(\varepsilon(u^\varepsilon + w^\varepsilon), u^\varepsilon + w^\varepsilon) - \mathcal{C}(\varepsilon u^\varepsilon, u^\varepsilon) \right) + \mathcal{R}^\varepsilon, \quad (4.10)$$

where $B_s(u) w := B(u, w) + B(w, u)$. Classical results [23] provide the existence and uniqueness of $w^\varepsilon$ over a time interval $[0, t_0]$, where $t_0 > 0$ is independent of $\varepsilon$. We want to prove that $w^\varepsilon$ exists and is small on longer times $O(\|\log \varepsilon\|)$, as in [22] and [7]. In this view, we review the classical proof of the $H^s$ estimate for the linear operator

$$\mathcal{L}_1(u^\varepsilon + w^\varepsilon) := \mathcal{L} + \mathcal{L}(u^\varepsilon + w^\varepsilon) := \mathcal{L} + \mathcal{L}(\varepsilon u^\varepsilon + w^\varepsilon), \partial_x, 0, \partial_z + \frac{k}{\varepsilon} \partial_\theta),$$

and follow carefully the dependence of the constants on $u^\varepsilon$ and $w^\varepsilon$. We work on a time interval $[0, t(\varepsilon)]$. The first step is the proof of an $L^2$ estimate. The symmetry of $\mathcal{L}$ implies that

$$2 \Re(\mathcal{L}_1(u^\varepsilon + w^\varepsilon)) v, v)_{L^2} = \partial_t \|v\|_{L^2}^2 + 2 \Re(\mathcal{L}(\varepsilon(u^\varepsilon + w^\varepsilon))) v, v)_{L^2},$$

32
for all \( v \in C^0([0, t(\varepsilon)], L^2(\mathbb{R}^2 \times T)) \), where \( (\cdot, \cdot)_{L^2} \) is the scalar product on \( L^2(\mathbb{R}^2 \times T) \). The symmetry of \( \mathcal{L} \) gives

\[
|(\mathcal{L}(\varepsilon(u^\varepsilon + w^\varepsilon))v, v)_{L^2}| \leq C_0(t)\|v\|_{L^2}^2,
\]

where

\[
C_0(t) := C(\|u^\varepsilon(t)\|_{W^{1,\infty}(\mathbb{R}^2 \times T)} + \|w^\varepsilon(t)\|_{W^{1,\infty}(\mathbb{R}^2 \times T)}),
\]

with a constant \( C \) that depends only on \( \mathcal{L} \). Hence one has

\[
\|v(t)\|_{L^2}^2 \leq \|v(0)\|_{L^2}^2 + \int_0^t (C_0(t'))\|v(t')\|_{L^2}^2 + 2\|\mathcal{L}(u^\varepsilon + w^\varepsilon)v(t')\|_{L^2}\|v(t')\|_{L^2} dt',
\]

and Gronwall’s lemma gives

\[
\|v(t)\|_{L^2} \leq e^{C_0(t)}(\|v(0)\|_{L^2} + 2 \int_0^t e^{-C_0(t')}\|\mathcal{L}(u^\varepsilon + w^\varepsilon)v(t')\|_{L^2} dt'),
\]

with \( C_0(t) := \int_0^t C_0(t') dt' \). The second step is an \( L^2 \) estimate for the commutator:

\[
[\mathcal{L}(u^\varepsilon + w^\varepsilon), \partial^\alpha]w^\varepsilon = \mathcal{L}(u^\varepsilon + w^\varepsilon)\partial^\alpha w^\varepsilon - \partial^\alpha(\mathcal{L}(u^\varepsilon + w^\varepsilon)w^\varepsilon) = \sum_{\beta+\gamma=\alpha, 0<|\beta|} \mathcal{L}(\partial^\beta(u^\varepsilon + w^\varepsilon))\partial^\gamma w^\varepsilon,
\]

where \( 0 < |\alpha| \leq s \) and \( \partial = \varepsilon \partial_x, \varepsilon \partial_z \) or \( k \partial_\theta \). For the first term a rough estimate suffices:

\[
\|\mathcal{L}(\partial^\beta u^\varepsilon)\partial^\gamma w^\varepsilon(t)\|_{L^2} \leq C\|u^\varepsilon(t)\|_{H^s}\|w^\varepsilon(t)\|_{H^s}.
\]

Now if \( |\alpha| = 1 \), one has

\[
\|\mathcal{L}(\partial^\alpha w^\varepsilon)w^\varepsilon(t)\|_{L^2} \leq C\|w^\varepsilon(t)\|_{W^{1,\infty}}\|w^\varepsilon(t)\|_{H^1}.
\]

If \( |\alpha| > 1 \): if \( |\beta| = 1 \), then

\[
\|\mathcal{L}(\partial^\beta w^\varepsilon)\partial^\gamma w^\varepsilon(t)\|_{L^2} \leq C\|w^\varepsilon(t)\|_{W^{1,\infty}}\|w^\varepsilon(t)\|_{H^s}.
\]

In the last three estimates, \( C \) depends only on \( \mathcal{L} \). If \( |\beta| > 1 \), we use an H"older estimate and two interpolation Gagliardo-Nirenberg estimates:

\[
\begin{align*}
\|\mathcal{L}(\partial^\beta w^\varepsilon)\partial^\gamma w^\varepsilon\|_{L^2} & \leq C\|\partial^\beta w^\varepsilon\|_{L^2}^{\frac{|\beta|-1}{|\beta|-1}}\|\partial^\gamma w^\varepsilon\|_{L^2}^{\frac{|\gamma|-1}{|\beta|-1}} \\
& \leq C\|w^\varepsilon\|_{W^{1,\infty}}^{\frac{1-|\beta|-1}{|\beta|-1}}\|\partial|\beta| w^\varepsilon\|_{L^2}^{\frac{|\beta|-1}{|\beta|-1}}\|w^\varepsilon\|_{W^{1,\infty}}\|\partial|\alpha| w^\varepsilon\|_{L^2}^{\frac{|\beta|-1}{|\gamma|-1}} \\
& \leq C\|w^\varepsilon(t)\|_{W^{1,\infty}}\|w^\varepsilon\|_{H^s}.
\end{align*}
\]
One retrieves finally, for $0 < |\alpha| \leq s$, the classical commutator estimate:

$$
\|L_1(\varepsilon u^\varepsilon + \varepsilon w^\varepsilon) - \varepsilon^3 w^\varepsilon\|_{L^2} \leq C_s(t)\|w^\varepsilon(t)\|_{H^s},
$$

with $C_s(t) := C(\|u^\varepsilon(t)\|_{H^s} + \|w^\varepsilon(t)\|_{W^{1,\infty}})$, where $C$ depends only on $C$ and $s$. One has therefore

$$
\|L_1(\varepsilon u^\varepsilon + \varepsilon w^\varepsilon) - \varepsilon^3 w^\varepsilon\|_{L^2} \leq \|L_1(\varepsilon u^\varepsilon + \varepsilon w^\varepsilon)\|_{H^s} + C_s(t)\|w^\varepsilon(t)\|_{H^s},
$$

hence the $H^s$ estimate

$$
\|w^\varepsilon(t)\|_{H^s} \leq e^{C_0(t)}(\|w^0\|_{H^s} + \int_0^t e^{C_0(t')}\left(\|L_1(\varepsilon u^\varepsilon + \varepsilon w^\varepsilon)\|_{H^s} + C_s(t')\|w^\varepsilon(t')\|_{H^s} dt'\right)\). \tag{4.11}
$$

The definition of $\mathcal{C}$ and standard Moser’s inequalities imply

$$
\frac{1}{\varepsilon}\|\mathcal{C}(\varepsilon(u^\varepsilon + w^\varepsilon), u^\varepsilon + w^\varepsilon)(t)\|_{H^s}
\leq \varepsilon M(t)(1 + \|u^\varepsilon(t)\|_{H^s} + \|w^\varepsilon(t)\|_{H^s})\|w^\varepsilon(t)\|_{H^s},
$$

where $M$ depends only on $\mathcal{C}$ and on the $L^\infty$ norms of $u^\varepsilon(t)$ and $w^\varepsilon(t)$. With (4.10), one has thus

$$
\|L_1(u^\varepsilon + w^\varepsilon)w^\varepsilon\|_{H^s} \leq \left(C\|u^\varepsilon\|_{H^s} + \varepsilon M(t)(1 + \|u^\varepsilon\|_{H^s})\|w^\varepsilon\|_{H^s}\right)\|w^\varepsilon(t)\|_{H^s}
+ (C + \varepsilon M(t))\|w^\varepsilon(t)\|_{H^s}^2 + \|R^\varepsilon(t)\|_{H^s}. \tag{4.12}
$$

The estimate (4.7) implies the existence of $\mathcal{C}$ (depending on $\|u^0\|_{H^s}$) such that for all $\mathcal{C} > 0$, there exists $\varepsilon_1 > 0$ such that for all $0 < \varepsilon < \varepsilon_1$,

$$
\sup_{0 \leq t \leq \mathcal{C}|\log \varepsilon|} \|u^\varepsilon\|_{H^s} \leq \mathcal{C}. \tag{4.13}
$$

The end of the proof consists in a classical continuation argument. Let $\mathcal{C} > 0$ and $\varepsilon_1 > 0$ such that (4.13) holds. Then with (4.8),

$$
\sup_{0 \leq t \leq \mathcal{C}|\log \varepsilon|} \|R^\varepsilon(t')dt'\|_{H^s} \leq C\varepsilon^{-3-C_0C_1} \log \varepsilon. \tag{4.14}
$$

As $w^\varepsilon$ is continuous in $t$, it makes sense to define

$$
t^\varepsilon(\varepsilon) := \sup \{0 < t \leq \mathcal{C}|\log \varepsilon|, \sup_{0 \leq t' \leq t} \|w^\varepsilon(t')\|_{H^s} = o(\varepsilon^2)\}.
$$
Let then \( t \leq t^* (\varepsilon) \). One has

\[
|C_0(t), C_s(t)| \leq CC(c_s + \varepsilon_0) \leq C'.
\]

The a priori estimate (4.11) and the commutator estimate (4.12) imply

\[
\|w^\varepsilon(t)\|_{H^s} \leq C\|w^0\|_{H^s} + 2 \int_0^t e^{-C_0(t)}(C'' \|w^\varepsilon(t')\|_{H^s} + \|R^\varepsilon(t')\|_{H^s}) \, dt',
\]

where \( C'' \) is defined via \( C, C' \) and \( M \). Gronwall’s lemma gives finally

\[
e^{-C_0(t)}\|w^\varepsilon(t)\|_{H^s} \leq e^{2C''t}(\|w^0\|_{H^s} + \sup_{0 \leq t \leq \tilde{C}|\log \varepsilon|} \|R^\varepsilon\|_{H^s}).
\]

With (4.14) and the bound \( C_0(t) \leq C't \), this gives finally

\[
\sup_{0 \leq t \leq \tilde{C}|\log \varepsilon|} \|w^\varepsilon(t)\|_{H^s} \leq C_2(\varepsilon^{\gamma - \tilde{C} (2C' + C'')} + \|\log \varepsilon\|^2 \varepsilon^{3 - \tilde{C}c_sC_1}).
\]

Recalling that \( 2 < \gamma \), choosing \( \tilde{C} \) (hence \( \varepsilon_0 \)) small enough, one obtains \( t^* (\varepsilon) = \tilde{C}|\log \varepsilon| \). With the standard blowup criterion for solutions of quasi-linear symmetric hyperbolic operators, this implies that the time existence of \( w^\varepsilon \) is \( \geq \tilde{C}|\log \varepsilon| \). Finally, the estimate \( \|w^\varepsilon(t)\|_{H^s} = o(\varepsilon^2) \) for \( t \leq \tilde{C}|\log \varepsilon| \) is the desired estimate (4.9).

5 Approximation by a weak Zakharov equation

In this section, we consider an approximate solution \( u^\varepsilon \) of the form (2.9) at order \( l \), with \( l \) arbitrary large and with the initial datum \( \varepsilon u^{0, \varepsilon} \). The equation \( r_1 = 0 \) gives again the polarization equations (3.1), (3.2) and (3.3). One checks again that the equation \( r_{1,0} = 0 \) is trivial and that (3.4) holds. \( r_1^\varepsilon = 0 \) implies again (3.6) and (3.7) for the nonpolarized part of the first corrector, and

\[
\Pi_L(\beta)L(\partial)\Pi_L(\beta)u_{\perp 1} + \varepsilon \Pi_L(\beta)(L(\partial)(1 - \Pi_L(\beta)))u_{\perp 1} + C_1 \partial_{x^1} u_{|| 1} = \varepsilon \Pi_L(\beta)B_{\perp 1}.
\]

With (3.6) and (3.7), this is a nonlinear Schrödinger equation. The Schrödinger operator is \( T_\beta(\partial) + \varepsilon iR(\partial, \partial) \), with the notation of the previous section. The nonlinear term is a weak coupling term with \( u_{1,0} \). The equation is explicitly

\[
(T_\beta(\partial) + \varepsilon iR(\partial, \partial)) \left( \begin{array}{c} E_{0x,p} \\ E_{0y,p} \end{array} \right) = \varepsilon \frac{\omega_p^2 n_{e1,0}}{ip\omega} \left( \begin{array}{c} E_{0x,p} \\ E_{0y,p} \end{array} \right), \quad (5.1)
\]
\[ r_{2,0}^c = 0 \] implies again (4.2) and (4.3), which by direct computations explicitly gives:

\[
\begin{cases}
\omega_p^2 \partial_t v_{i1x,0} + \theta_i (\alpha + 1) \partial_x n_{i1,0} = - \frac{\theta_i}{\omega^2} \omega^2 \partial_x (|E_{0x,p}|^2 + |E_{0y,p}|^2), \\
\omega_p^2 \partial_t v_{i1z,0} + \theta_i (\alpha + 1) \partial_z n_{i1,0} = - \frac{\theta_i}{\omega^2} (1 + \frac{\theta_i}{\theta_c}) \partial_z (|E_{0x,p}|^2 + |E_{0y,p}|^2), \\
\partial_t n_{i1,0} + \theta_i (\partial_x v_{i1x,0} + \partial_z v_{i1z,0}) = 0.
\end{cases}
\]  

(5.2)

Changing \( n_{i1,0} \) into \( \varepsilon n_{i1,0} \), one sees that the system (5.1)-(5.2) is the announced weak Zakharov system \((Z)_w\). One checks that the component in the \( y \) direction of

\[ \pi_L(0)(L_{1,0}^r + L_{1,0} - B_{L_o \perp 1,0}) \]

vanishes identically. Hence \( \partial_t v_{i1y,0} = 0 \).

**Higher-order terms.** One determines the equations for the higher-order terms by induction. In the following, \( m > 1 \) and one assumes that the profiles

\[
\begin{cases}
u_k, \ k < m, \\
(1 - \Pi_L(\beta))u_{\perp m}, \\
u_{\parallel m}, \\
u_{m,0,0},
\end{cases}
\]

have been uniquely determined. Any operator, or vector, that can be expressed in terms of these profiles is denoted by \( F_m \).

In the equation \( r_{m+1} = 0 \), one checks that \( u_{\parallel 0} = 0 \) implies that \( L_m, M_m, L_{m-1}', M_{m-1}', B_{\perp m}, B_{\parallel m}, B_{L_o \perp m}, R_{\perp m}, R_{\parallel m} \) are the above type. One also has \( B_{L_o m} = F_m \) as a consequence of (t2). It follows that

\[
(1 - \Pi_L(\beta))u_{\perp m+1} = - \mathbb{L}(i\beta)^{-1} (1 - \Pi_L(\beta))L(\partial_t, \partial_z)u_{\perp m} + F_m,
\]

(5.3)

\[
\tilde{u}_{\parallel m+1} = - \mathbb{M}(i\beta)(-1)C^* \partial_x u_{\perp m} + F_m,
\]

(5.4)

\[
(1 - \pi_M(0))u_{\parallel m+1,0} = F_m.
\]

(5.5)

The equation for \( \tilde{\Pi}_L(\beta)u_{\perp m} \) is

\[
(T_{\beta}(\beta) + \varepsilon iR(\partial, \partial))\tilde{\Pi}_L(\beta)u_{\perp m} = \varepsilon \tilde{\Pi}_L(\beta)(B_{\perp}(u_{\parallel 1}, \tilde{\Pi}_L(\beta)u_{\perp m}) + B_{\perp}(u_{\parallel m+1,0}, u_{\perp 0})) + F_m,
\]

(5.6)
The last step of the induction is the determination of the evolution equation for \( \pi_L(0)u_{\perp m+1,0} \) and \( \pi_M(0)u_{\| m+1,0} \). The condition \( r_{m+1,0} = 0 \) is

\[
\begin{aligned}
\pi_L(0)L(\partial)\pi_L(0)u_{\perp m+1,0} + \pi_L(0)(C \partial_x u_{\| m+1,0} + L_{m+1,0} - B_{L,\perp m+1,0} + L'_{m,0}) &= F_{m,0}, \\
\pi_M(0)M(\partial)\pi_M(0)u_{\| m+1,0} + \pi_M(0)C^* \partial_x u_{\perp m+1,0} &= \pi_M(0)B_{L,\| m+1,0} + F_{m,0}.
\end{aligned}
\]

The nonlinear terms are:

- \( \pi_L(0)(L_{m+1,0} - B_{L,\perp m+1,0}) \), which gives a term in \( E_{0y,p} \partial_x E_{my,-p} + E_{my,p} \partial_x E_{0y,-p} + c.c., \)
- \( \pi_L(0)L'_{m,0} \), which gives a term in \( E_{0x,p} \partial_x E_{mx,-p} + E_{mx,p} \partial_x E_{0x,-p} + c.c., \)
- \( \pi_M(0)B_{L,\| m+1,0} \), which gives a term in \( E_{0x,p} \partial_x E_{mx,-p} + E_{mx,p} \partial_x E_{0x,-p} + c.c \) and a term in \( E_{0y,p} \partial_x E_{my,-p} + E_{my,p} \partial_x E_{0y,-p} + c.c., \)

where \( \text{c.c.} \) means complex conjugate. The equations are explicitly,

\[
\begin{aligned}
\omega_p^2 \partial_t v_{im+1x,0} + \theta_i(\alpha^2 + 1) \partial_x n_{im+1,0} &= -\frac{\theta_i \omega_p^2}{\omega^2} \left( (E_{0x,p} \partial_x E_{mx,-p} + E_{mx,p} \partial_x E_{0x,-p} + c.c.) \\
+ (E_{0y,p} \partial_x E_{my,-p} + E_{my,p} \partial_x E_{0y,-p} + c.c.) \right) + F_{m,0}, \\
\omega_p^2 \partial_t v_{im+1z,0} + \theta_i(\alpha^2 + 1) \partial_z n_{im+1,0} &= -\frac{\theta_i \omega_p^2}{\omega^2} \left( 1 + \left( \frac{\theta_i}{\theta^e} \right)^3 \left( (E_{0x,p} \partial_x E_{mx,-p} + E_{mx,p} \partial_x E_{0x,-p} + c.c.) \\
+ (E_{0y,p} \partial_x E_{my,-p} + E_{my,p} \partial_x E_{0y,-p} + c.c.) \right) \right) + F_{m,0}, \\
\partial_t n_{im+1,0} + \theta_i(\partial_x v_{im+1x,0} + \partial_z v_{im+1z,0}) &= F_{m,0},
\end{aligned}
\]

(5.7)

for \( p \in \{ -1, 1 \} \). The system satisfied by \( (\Pi_L(\beta)u_{\perp m, u_{m+1,0}}) \) is (5.6)-(5.7); it corresponds to (5.1)-(5.2) linearized around the leading terms \( (\Pi_L(\beta)u_{\perp 0, u_{1,0}}) \).

**The approximate solution.** We first give an existence, uniqueness and regularity result for \( (Z)_w \), derived from T. Ozawa and Y. Tsutsumi [26].

**Proposition 5.1.** For all \( s > 1 \), for all \( E^0 \in H^{s+2}(\mathbb{R}^2) \), there exists \( T > 0 \) independent of \( \varepsilon \) and a unique solution \( E, n \) to the initial value problem for
the system

\[(Z)_w \begin{cases}
    i(\partial_t + \partial_z)E + \varepsilon(\partial_x^2 + \partial_z^2)E = nE, \\
    (\partial_t^2 - (\partial_x^2 + \partial_z^2))n = \varepsilon(\partial_x^2 + \partial_z^2)|E|^2,
\end{cases}\]

with \(E(t = 0) = E^0, n(t = 0) = 0, \partial_t n(t = 0) = 0\). The solution \((E, n)\) belongs \(C^1([0, t], H^s)^2\); it is uniformly bounded with respect to \(\varepsilon\) in \(W^{1, \infty}([0, T], H^s) \times L^\infty([0, T], H^{s+1})\).

Moreover, \(\varepsilon E\) is uniformly bounded with respect to \(\varepsilon\) in \(L^\infty(0, T], H^{s+2})\).

Proof. As in [26], introduce the unknown \(F := \partial_t E\). The equation satisfied by \(F\) is

\[i(\partial_t + \partial_z)F + \varepsilon(\partial_x^2 + \partial_z^2)F = (\partial_t n)(E_0 + \int_0^t F(t') \, dt') + nF.\]  

(5.8)

To handle the \(\partial_z\) term in the Schrödinger equation, introduce the unknown \(G := \partial_z E\). The equation satisfied by \(G\) is

\[i(\partial_t + \partial_z)G + \varepsilon(\partial_x^2 + \partial_z^2)G = (\partial_z n)(E_0 + \int_0^s F(t') \, dt') + nG.\]  

(5.9)

\(E, F\) and \(G\) are linked by

\[i(F + G) + \varepsilon(\partial_x^2 + \partial_z^2)E = n(E_0 + \int_0^s F(t') \, dt'),\]

or equivalently

\[i(F + G) - (n - 1)(E_0 + \int_0^t F(t') \, dt') = (1 - \varepsilon(\partial_x^2 + \partial_z^2))E.\]  

(5.10)

The key is that \((1 - \varepsilon(\partial_x^2 + \partial_z^2))\) is elliptic. The norm of its inverse (as an operator \(H^s \to H^{s+2}\)) is \(O(1/\varepsilon)\), but this factor is annihilated by the \(\varepsilon\) in front of the Laplace operator in the wave equation:

\[(\partial_t^2 - (\partial_x^2 + \partial_z^2))n = \varepsilon(\partial_x^2 + \partial_z^2)|E|^2.\]  

(5.11)

The initial data are \(F^0 = i(\varepsilon(\partial_x^2 + \partial_z^2)E^0) - \partial_z E^0 \in H^s\), and \(G^0 = \partial_z E^0 \in H^s\). The proof consists in a fixed point theorem for the integral versions of (5.8), (5.9) and (5.11).

Introduce the group of unitary operators acting on \(H^s(\mathbb{R}^2)\), for all \(\sigma : S^\sigma(t) := \exp(-it(\varepsilon(\partial_x^2 + \partial_z^2) + i\partial_z)).\)
Introduce $T > 0$ and the vector space
\[ X(T) := L^\infty([0, T], H^s) \times L^\infty([0, T], H^s) \times (L^\infty([0, T], H^{s+1}) \cap W^{1, \infty}([0, T], H^s)), \]
and the functions $N_1, N_2, N_3$ defined on $X(T)$ by
\[
N_1(t) := S^\varepsilon(t)F^0 - i \int_0^t S^\varepsilon(t - t')(\partial_t n)(E_0 + \int_0^{t'} F(t'') \, dt'') + nF) \, dt',
\]
\[
N_2(t) := S^\varepsilon(t)G^0 - i \int_0^t S^\varepsilon(t - t')(\partial_z n)(E_0 + \int_0^{t'} F(t'') \, dt'') + nG) \, dt',
\]
\[
N_3(t) := \varepsilon \int_0^t \sin((-\partial_x^2 + \partial_z^2)^{1/2}(t - t'))(-\partial_x^2 + \partial_z^2)^{1/2}|E|^2(t') \, dt',
\]
for $0 \leq t \leq T$, where $F^0 := \partial_t E|_{t=0}$, and $G^0 := \partial_z E^0$. The condition $s > 1$ ensures that $H^s(\mathbb{R}^2)$ is an algebra. The following estimates hold
\[
\|N_1(F, G, n)(t)\|_{H^s} \leq \|F^0\|_{H^s}(1 + T\|\partial_t n\|_{L^\infty([0, T], H^s)}) + T\|n\|_{L^\infty([0, T], H^s)}\|F\|_{L^\infty([0, T], H^s)} + T^2\|\partial_t n\|_{L^\infty([0, T], H^s)}\|F\|_{L^\infty([0, T], H^s)},
\]
\[
\|N_2(F, G, n)(t)\|_{H^s} \leq \|G^0\|_{H^s} + T\|F^0\|_{H^s}\|\partial_z n\|_{L^\infty([0, T], H^s)} + T\|n\|_{L^\infty([0, T], H^s)}\|G\|_{L^\infty([0, T], H^s)} + T^2\|\partial_z n\|_{L^\infty([0, T], H^s)}\|F\|_{L^\infty([0, T], H^s)},
\]
\[
\|N_3(F, G, n)(t)\|_{H^{s+1}} \leq \varepsilon T\||E|^2\|_{L^\infty([0, T], H^{s+2})} \leq \varepsilon CT\|E\|_{L^\infty([0, T], H^{s+2})}^2 \quad (5.12)
\]
\[
\|\partial_t N_3(F, G, n)(t)\|_{H^s} \leq \varepsilon T\||E|^2\|_{L^\infty([0, T], H^{s+2})} \leq \varepsilon CT\|E\|_{L^\infty([0, T], H^{s+2})}^2. \quad (5.13)
\]
Now (5.10) gives
\[
\|E\|_{L^\infty([0, T], H^{s+2})} \leq \|i(F + G) - (n - 1)(E^0 + \int_0^t F(t') \, dt')\|_{L^\infty([0, T], H^s)} \leq (1 + (1 + T)\|n\|_{L^\infty([0, T], H^s)})\|F\|_{L^\infty([0, T], H^s)} + \|G\|_{L^\infty([0, T], H^s)} + (1 + \|n\|_{L^\infty([0, T], H^s)})\|E^0\|_{L^\infty([0, T], H^s)}. \quad (5.14)
\]
Remark that \( \|F_0, G_0\|_{H^s} \leq C\|E_0\|_{H^{s+2}} \). Then (5.14) plugged in (5.12) and (5.13) allows to choose \( R > 0 \) (depending on \( \|E_0\|_{H^s} \)) and \( T > 0 \) (depending on \( R \)) such that \((N_1, N_2, N_3)\) is an application from the ball of center 0 and radius \( R \) in \( X(T) \) to itself. The key is that \( \partial_z n \) is as regular as \( \partial_t n \) in the wave equation (5.11). The same techniques show that \((N_1, N_2, N_3)\) is also a contraction. A classical fixed point argument concludes the proof. \( \square \)

**Remark 5.2.** A similar result holds when the initial datum \((E_0, n_0, n_1)\) is in \( H^{s+2} \times H^{s+1} \times H^s \). The condition \( s > 1 \) is far from optimal. The proof of Ginibre, Tsutsumi and Velo \cite{GTV2} could indeed be adapted to \((Z)\). In two space dimensions, blow-up results in finite time for the Zakharov system without the term in \( \partial_z \) were proved by Gangletas and Merle \cite{GM1, GM2}.

The approximate solution is now defined according to the previous set of equations. Set \( u_{0,0} = 0 \). With Proposition 5.1, equations (3.1), (3.2), (3.4), (3.6)-(3.7) for \( p = 0 \) and (5.1) and (5.2) together with the initial datum \( u_{0}(t = 0) = u_0, u_{1}(t = 0) = 0 \), have a regular solution \((\tilde{u}_0, u_{1,0})\) over \([0, t^*]\).

Note that the scaling in (5.1)-(5.2) is different than the scaling of \((Z)\). The density \( n \) in \((Z)\) corresponds to \( \varepsilon n_{i_1,0} \), where \( n_{i_1,0} \) solves (5.2). Proposition 5.1 in particular asserts that \( \varepsilon u_{1,0} \) is bounded with respect to \( \varepsilon \) over any closed subinterval of \([0, t^*]\).

Now suppose that for \( m > 0 \), the equations for the profiles of type \( F_m \) have been uniquely solved over an interval \([0, t^*_{m-1}]\). Then (5.3) and (5.4) determine uniquely \((1 - \Pi_L(\beta))u_{m+1,0} \) and \( u_{|m+1,0}; \) finally (5.6)-(5.7) together with null initial data have a unique regular solution \((\tilde{\Pi}_L(\beta)u_{m+1,0,0})\) with an existence time \( t^*_m \leq t^*_{m-1} \). The above proof for (5.1)-(5.2) adapts indeed for the linearized system (5.6)-(5.7), providing an existence time which is a priori smaller than the existence time of the leading profiles. Again, by Proposition 5.1, the terms \( \varepsilon u_{m+1,0} \) are bounded with respect to \( \varepsilon \) over any closed subinterval of \([0, t^*_m]\). Thus in the generalized WKB expansion, the term \( \varepsilon u_{m+1} \) is eventually negligible to \( u_m \).

Then for \( l \geq 0 \),

\[
\begin{align*}
\varepsilon \sum_{m=0}^{l} \varepsilon^m u_m + \varepsilon^{l+1}((1 - \Pi_L(\beta))u_{l+1,0}, \tilde{u}_{l+1,0}) + \varepsilon^{l+1}u_{l+1,0} \\
+ \varepsilon^{l+2}((1 - \pi_L(0))u_{l+2,0}, (1 - \pi_M(0))u_{l+2,0}).
\end{align*}
\]

defines a formal solution \( u^\varepsilon \) at order \( l \). We show in the following theorem that this approximate solution converges, as \( \varepsilon \to 0 \), towards the exact solution.
with a close initial datum. We call regular profiles that have the same regularity as the initial data, that is $C^\infty(t, H^\infty(x, z, \theta))$.

**Theorem 5.3.** For all $l \geq 2$, given a real initial datum $\varepsilon u^{0,\varepsilon} \in H^\infty(\mathbb{R}^2)$ of the form (2.4) and satisfying the polarization condition (2.5); given a perturbative initial profile $w^0 \in H^\infty(\mathbb{R}^2)$:

- there exists a unique weakly nonlinear formal solution $u^\varepsilon$ at order $l$ of the form (2.9) with the initial datum $\varepsilon u^{0,\varepsilon}$; $u^\varepsilon$ is regular and defined on a time interval $[0, t^*_l]$ independent of $\varepsilon$. The leading terms of $u^\varepsilon$ satisfy the weak Zakharov equations (5.1)-(5.2) of the form $\text{(Z)}$; precisely, $(E^0_x, E^0_y, n^1_0) \text{ solve (5.1)-(5.2)}$ and $E := (E^0_x, E^0_y) \text{ and } n := \varepsilon n^{1,0} \text{ solve (Z)}$. The higher-order terms of $u^\varepsilon$ satisfy the system (5.6)-(5.7);

- there exists a unique solution $v^\varepsilon$ to the (EM) system with initial datum $\varepsilon (u^0, \varepsilon + \varepsilon^l w^0, \varepsilon)$, where $w^0$ is represented by the profile $w^0$; the exact solution $v^\varepsilon = (B^\varepsilon, E^\varepsilon, n^\varepsilon, v^\varepsilon, n^\varepsilon)$ is regular and defined on the same time interval as $u^\varepsilon$; it admits a representation by a profile $v^\varepsilon$. For all $0 \leq t_0 < t^*_l$, for all $\sigma$, the asymptotic estimate holds:

$$\sup_{0 \leq t \leq t_0} \|v^\varepsilon - u^\varepsilon\|_{H^\sigma(\mathbb{R}^2 \times \mathbb{T})} = O(\varepsilon^{l+1}), \quad (5.15)$$

and in particular

$$\sup_{0 \leq t \leq t_0} \left( \frac{1}{\varepsilon} \|E^\varepsilon - \varepsilon(E^0 e^{(kz-\omega t)/\varepsilon} + \text{c.c.})\|_{L^\infty(\mathbb{R}^2)} + \frac{1}{\varepsilon} \|n^\varepsilon - \varepsilon(n^{1,0})\|_{L^\infty(\mathbb{R}^2)} \right) = O(\varepsilon). \quad (5.16)$$

**Proof.** The proof follows from Joly, Métivier and Rauch [18]; indeed the manipulations operated at the level of the profile equations do not modify the energy estimates; the regime is really weakly nonlinear. Therefore we only give a sketch of proof.

As in the proof of Theorem 4.2, we look for the exact solution $v^\varepsilon$ in the form of a profile $v^\varepsilon$ of the form (2.9): $v^\varepsilon = \varepsilon [v^\varepsilon]$, where $v^\varepsilon$ is a perturbation of $u^\varepsilon$:

$v^\varepsilon = u^\varepsilon + \varepsilon^l w^\varepsilon$.

With the notations of the proof of Theorem 4.2, the equation satisfied by $u^\varepsilon$ is

$$\mathcal{L}(\partial_t - \frac{\omega}{\varepsilon} \partial_\theta, \partial_x, \partial_z)u^\varepsilon + \mathcal{L}(\varepsilon u^\varepsilon, (\partial_x, 0, \partial_z + \frac{k}{\varepsilon} \partial_\theta)) u^\varepsilon = \mathcal{B}(u^\varepsilon, u^\varepsilon) + \frac{1}{\varepsilon} C(\varepsilon u^\varepsilon, u^\varepsilon) + r^\varepsilon.$$

41
The equation to be satisfied by $w^\varepsilon$ in order that $v^\varepsilon$ be an exact solution is
\[
L w^\varepsilon + L(\varepsilon u^\varepsilon + \varepsilon' w^\varepsilon) w^\varepsilon + L(\varepsilon w^\varepsilon) u^\varepsilon = B_v (u^\varepsilon) w^\varepsilon + \varepsilon' B(w^\varepsilon, w^\varepsilon)
+ \varepsilon^{-(l+1)} (C(\varepsilon u^\varepsilon + \varepsilon' w^\varepsilon, u^\varepsilon + \varepsilon' w^\varepsilon) - C(\varepsilon u^\varepsilon, u^\varepsilon)) + \varepsilon^{-l} r^\varepsilon.
\]

The local existence of $w^\varepsilon$ follows from the standard existence theory for quasilinear symmetrizable hyperbolic systems. The fact that $w^\varepsilon$ is defined over $[0, t^*_1]$ and the estimates (5.15) and (5.16) follow from Joly, Métivier and Rauch [18].

**Remark 5.4.** As stated in Proposition 5.1, $E$ is bounded in $H^s$, uniformly with respect to $\varepsilon$. Hence, letting $\varepsilon \to 0$ in the wave equation in $(Z)$, we seek an approximate solution in the form (5), changing only the amplitude $\varepsilon$ to $\sqrt{\varepsilon}$.

If we impose $r_k^\varepsilon = 0, 0 \leq k \leq m_0$, we are led to an overdetermined system of equations. This is because the rules of construction of the generalized WKB equations make the equations for the profiles more nonlinear.

The terms of the residual are thus modified as follows:
\[
\begin{align*}
\tilde{r}_{\perp m}^\varepsilon := & \quad L(\beta \partial_y) u_{\perp m} + \tilde{N}_L(\beta)L(\partial_t, \partial_z) \tilde{N}_L(\beta) u_{\perp m-2} \\
& + (1 - \tilde{N}_L(\beta)) L(\partial_t, \partial_z)(1 - \tilde{N}_L(\beta)) u_{\perp m-2} \\
& + \sqrt{\varepsilon} \tilde{N}_L(\beta)L(\partial_t, \partial_z)(1 - \tilde{N}_L(\beta)) u_{\perp m-1} \\
& + \sqrt{\varepsilon}(1 - \tilde{N}_L(\beta)) L(\partial_t, \partial_z) \tilde{N}_L(\beta) u_{\perp m-1} \\
& + (1 - \tilde{N}_L(\beta)) C_1 \partial_z u_{\parallel m-2} + \sqrt{\varepsilon} \tilde{N}_L(\beta) C_1 \partial_z u_{\parallel m-1} + C_2 \partial_z u_{\parallel m-2} \\
& + \tilde{L}_{m-1} + \tilde{L}_{m-3} - B_{\perp m-1} - B_{L_{0,\perp m-1}} - R_{\perp m-2}.
\end{align*}
\]
and
\[
\tilde{r}_m^\varepsilon := M(\beta \partial_y)u_m + M(\partial_t, \partial_x)u_{m-2} + C_1^* \partial_x(\sqrt{\varepsilon} \tilde{u}_{\perp m-1} + u_{\perp m-2,0}) \\
+ C_2^* \partial_x u_{\perp m-2} + M_{|m-1} + M_{|m-3} - B_{|m-1} - B_{|m-1} - R_{|m-2}.
\]

Then one defines generalized WKB solutions at all order by
\[
\begin{cases}
\tilde{r}_m^\varepsilon &= 0, \\
\tilde{r}_{0,m+1}^\varepsilon &= 0,
\end{cases}
\]
for all \( m \geq 0 \). In the second equation, the tilde means that in the nonlinear terms coming from the nonlinear interaction of oscillating terms, we eliminate the terms of size \( \varepsilon \).

Formal computations lead to the nonlinear Schrödinger equation
\[
(T_\beta(\partial) + iR(\partial, \partial))\tilde{\Pi}_L(\beta)u_{\perp 0} = \tilde{\Pi}_L(\beta)B_{\perp}(u_{|1,0}, \tilde{\Pi}_L(\beta)u_{\perp 0}),
\]
and explicitly
\[
(T_\beta(\partial) + iR(\partial, \partial)) \left( \frac{E_{0x,p}}{E_{0y,p}} \right) = \frac{\omega_p^2}{ip\omega}n_{c1,0} \left( \frac{E_{0x,p}}{E_{0y,p}} \right).
\]

The nonlinear wave equation for \( u_{\perp 0}^\varepsilon \) is again (5.2). The system (6.1)-(5.2) is the announced Zakharov equation (Z)\(_\varepsilon\). One can adapt the proof of Proposition 5.1 to obtain an existence, regularity and uniqueness result for the initial value problem for (Z)\(_\varepsilon\) with an existence time \( t^*(\varepsilon) = O(\sqrt{\varepsilon}) \). One can again construct approximate solutions to arbitrary order.

**Proposition 6.1.** For all \( l \geq 2 \), for \( \varepsilon \) small enough, there exists a unique formal solution \( u^\varepsilon \) at order \( l \) with the initial datum \( \sqrt{\varepsilon}u_{\perp 0}^\varepsilon \) satisfying (2.4) and (2.5), defined and regular on a time interval of size \( O(\sqrt{\varepsilon}) \). The leading terms of \( u^\varepsilon \) satisfy the Zakharov system (6.1)-(5.2) of the form \( (Z)_{\varepsilon} \); and \( u^\varepsilon \) satisfies
\[
L^\varepsilon(u^\varepsilon, \partial)u^\varepsilon - \frac{1}{\varepsilon} B(u^\varepsilon) = O(\varepsilon),
\]
in \( L^\infty([0, t_0] \times \mathbb{R}^2) \), for \( t_0 = O(\sqrt{\varepsilon}) \).

Note that the above result is only formal, in the sense that we do not prove that the above estimate (6.2) is valid over time intervals independent of \( \varepsilon \). As noted above, a straightforward adaptation of Proposition 5.1 to (Z)\(_\varepsilon\) would indeed only yield a time existence \( O(\varepsilon) \), and standard energy estimates would yield an exact solution defined over a time interval of size
only $O(\sqrt{\varepsilon})$. Both the questions of the existence of solutions to $(Z)_\varepsilon$ with an existence time independent of $\varepsilon$ and of their convergence towards the exact solutions of $(EM)$ are interesting questions. A recent result by Colin and Métivier gives a hint that the solution to $(Z)_\varepsilon$ can probably not be defined over time intervals $O(1)$ – see also the discussion below.

### 6.2 Cold ions

It is common [11] to suppose that the ions are cold. In our context, this means that $\theta_i$ is small compared with the others parameters of the system. We examine the regime $\theta_i = \sqrt{\varepsilon}$. The system takes the form

$$
\begin{align*}
(EM)' \left\{ \begin{array}{l}
\partial_t B + \nabla \times E = 0, \\
\partial_t E - \nabla \times B &= \frac{1}{\varepsilon}(1 + n_e + f(n_e))v_e - \frac{1}{\theta_e \sqrt{\varepsilon}}(1 + n_i + f(n_i))v_i, \\
\partial_t v_e + \theta_e (v_e \cdot \nabla)v_e &= -\theta_e \nabla n_e - \frac{1}{\varepsilon}(E + \theta_e v_e \times B), \\
\partial_t n_e + \theta_e \nabla \cdot v_e + \theta_e (v_e \cdot \nabla)n_e &= 0 \\
\partial_t v_i + \sqrt{\varepsilon}(v_i \cdot \nabla)v_i &= -\alpha^2 \sqrt{\varepsilon} \nabla n_i + \frac{1}{\theta_e \sqrt{\varepsilon}}(E + \sqrt{\varepsilon} v_i \times B), \\
\partial_t n_i + \sqrt{\varepsilon} \nabla \cdot v_i + \sqrt{\varepsilon}(v_i \cdot \nabla)n_i &= 0,
\end{array} \right.
\end{align*}
$$

For a formal solution $u^\varepsilon$, at order $l$ arbitrarily large and of the form (2.10), we sketch the corresponding WKB expansion.

The equation $\bar{R}_0 = 0$ gives the polarization equations (3.1) and (3.2). The polarization for the mean mode is $E_{0,0} = 0$, $v_{e,0,0} = 0$.

Because of the large amplitude of the solution, nonlinear terms appear in the equation $\bar{R}_1 = 0$. One has

$$
\Pi_L(\beta) L_0 = \Pi_L(\beta)(B_{1,0} + B_{L,0}).
$$

With the transparency equality (t1) from Proposition 2.1 and the polarization condition for the leading mean mode, this gives

$$
\Pi_L(\beta) B_{1,0} = 0.
$$

The non transparency equality (t4) from Proposition 2.1 implies that $n_{e,0,0} = 0$. Then with the equation $\bar{R}_{1,0} = 0$, one obtains $u_{0,0} = 0$.

As in [2, 13], the ansatz allows the derivation of a Schrödinger equation in geometric optics regime. The nonlinear Schrödinger equation is given by
the equation \( \pi_L(p\beta)\tilde{R}_{12,p} = 0 \). It is explicitly

\[
2(\partial_t + \frac{k}{\omega} \partial_z)E_{0x,p} - \frac{i}{p\omega} \partial^2_X E_{0x,p} + \frac{i}{p\omega \theta_e^2} E_{0x,p} = -\frac{i}{p\omega} n_{e1,0} E_{0x,p}. \tag{6.3}
\]

With the change of variable \( \tilde{E} := e^{it/(p\omega \theta_e^2)} E_{0x,p} \), the equation satisfied by \( \tilde{E} \) is a nonlinear Schrödinger equation as in \((Z)_c\).

The crucial equation that gives the nonlinear coupling in the Zakharov equation is \( \tilde{R}_{2,0} = 0 \). It implies in particular

\[
\theta_e(v_{e1,z}k \partial \theta v_{e0,x} + v_{e0,x} \partial_X v_{e0,x})_0 = -\theta_e \partial_X n_{e1,0} - E_{2x,0}.
\]

Finally, \( \tilde{R}_{3,0} = 0 \) gives the wave equation for \( u_{1,0} \):

\[
\begin{align*}
\partial_t v_{i1,x,0} + (a^2 + 1) \partial_X n_{i1,0} &= -\frac{2}{\omega^2} \partial_X |E_{0x,p}|^2 \\
\partial_t n_{i1,0} + \partial_X v_{i1,x,0} &= 0.
\end{align*}
\tag{6.4}
\]

The system (6.3)-(6.4) is the announced Zakharov system \((Z)_c\). The proof of Proposition 5.1 does not apply to this system because of a lack of regularity in \( z \). Colin and Métivier actually recently proved [8] that when \( c \neq 0 \), \((Z)_c\) is ill-posed in the sense of Hadamard in \( L^\infty \). We have the formal result:

**Proposition 6.2.** For all \( l \geq 2 \), there exists \( \varepsilon_0 > 0 \) such that for all \( 0 < \varepsilon < \varepsilon_0 \), one can write a cascade of WKB equations to order \( l \), starting from \((EM)^\prime\) with the highly nonlinear ansatz (2.10); the limit system is a Zakharov system of the form \((Z)_c\).

With the result of Colin and Métivier, this result ought to be seen as stating an absence of transparency for the Euler-Maxwell equations. This indicates two interesting directions of research. First, investigate the behaviour of large solutions corresponding to \( c = 0 \), that is to a well-posed Zakharov system. This is the object of the papers [5, 29]. Second, investigate the behaviour of large solutions corresponding to \( c \neq 0 \). With Colin and Métivier’s result, one can then expect strong instabilities to develop in small time.

**References**


