Nash–Moser iteration and singular perturbations

Benjamin Texier* and Kevin Zumbrun†

July 27, 2009

Abstract

We present a simple and easy-to-use Nash–Moser iteration theorem tailored for singular perturbation problems admitting a formal asymptotic expansion or other family of approximate solutions depending on a parameter \( \varepsilon \to 0 \). The novel feature is to allow loss of powers of \( \varepsilon \) as well as the usual loss of derivatives in the solution operator for the associated linearized problem. We indicate the utility of this theorem by describing sample applications to (i) large-amplitude, high-frequency WKB solutions of quasilinear hyperbolic systems, and (ii) existence of small-amplitude profiles of quasilinear relaxation systems.

Contents

1 Introduction 2
2 A simple Nash–Moser theorem 3
  2.1 Remarks . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 5
3 Proofs of Theorems 2.4 and 2.5 7
4 Application 0: Solutions of quasilinear hyperbolic systems 11
  4.1 Standard Nash–Moser . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 12
  4.2 Nash–Moser with \( \varepsilon \) loss . . . . . . . . . . . . . . . . . . . . . . . . . . . 12
5 Application 1: large-amplitude, high-frequency solutions of quasilinear hyperbolic systems 13
  5.1 Pseudo-differential symbols and associated semiclassical operators . . . . . 13
  5.2 Singular pseudo-differential equations . . . . . . . . . . . . . . . . . . . . . . 14
  5.3 Example: singular limit of a quasilinear Klein-Gordon-wave system . . . . . 17
  5.4 Example: singular limit of the Euler-Maxwell system . . . . . . . . . . . . . 20

*Université Paris Diderot (Paris 7), Institut de Mathématiques de Jussieu, UMR CNRS 7586; texier@math.jussieu.fr: Research of B.T. was partially supported under NSF grant number DMS-0505780.
†Indiana University, Bloomington, IN 47405; kzumbrun@indiana.edu: Research of K.Z. was partially supported under NSF grants no. DMS-0300487 and DMS-0801745.
6 Application 2: small-amplitude shock profiles for quasilinear relaxation equations

6.1 Assumptions ........................................................................................................ 22
6.2 Chapman–Enskog approximation ...................................................................... 22
6.3 Statement of the main theorem .......................................................................... 24
6.4 Functional equation and spaces ......................................................................... 24
6.5 Fréchet bounds .................................................................................................... 26
6.6 Linearized estimates ............................................................................................ 26
  6.6.1 Internal and high frequency estimates .......................................................... 27
  6.6.2 Higher order estimates .................................................................................. 29
  6.6.3 Linearized Chapman–Enskog estimate .......................................................... 30
6.7 Proof of the main theorem .................................................................................. 32
6.8 Why Nash–Moser? .............................................................................................. 33

A Existence of phase conditions ............................................................................. 34

1 Introduction

Because the expansions themselves furnish arbitrarily accurate approximate solutions, and because the associated linearized estimates are often stiff in terms of amplitude and or smoothness, Nash–Moser iteration appears particularly well-adapted to the verification of asymptotic expansions such as arise in various singular perturbation problems depending on a small parameter $\varepsilon \to 0$. However, standard Nash–Moser theorems allow only for loss of derivatives and not loss of powers of $\varepsilon$ in the estimates on the linearized solution operator, so that to apply Nash–Moser iteration to problems that do lose powers of $\varepsilon$ would appear to require a careful accounting of constants throughout the entire Nash–Moser iteration to check that the argument closes.

The purpose of this article therefore is to present a simple and general-purpose theorem carrying out this accounting, which can be applied as an easy-to-use black box to this type of problem. We conclude by presenting two sample applications for which both loss of derivatives and of powers of $\varepsilon$ naturally occur for the linearized problem, one in large-amplitude, high-frequency WKB solutions of quasilinear hyperbolic systems, and one in existence of small-amplitude profiles of quasilinear relaxation systems. The former, due to Texier, was originally solved by this approach but later treated in a different way in [19] using paradifferential calculus. The latter, due to Métivier, Texier and Zumbrun, was treated in [14] by the approach presented here. Though special cases may be treated by other methods [22, 10, 11, 4], we do not know of any other solution in the generality considered there.

Our approach follows a very simple proof given by Xavier Saint-Raymond [21] of a (parameter-independent) Nash–Moser implicit function theorem [9, 15] in a Sobolev space setting. A novel aspect is our treatment of uniqueness, which we have not seen elsewhere—in particular the incorporation of a phase condition in the case that the linearized operator has a kernel. (See Thm 2.5.)
We note that a parameter-dependent Nash–Moser scheme was recently used by Alvarez-Samaniego and Lannes [2] to prove local-in-time well-posedness of model equations in oceanography. Alvarez-Samaniego and Lannes do not allow losses in $\varepsilon$ in the linearized solution operator, which is the main point here. The main reference on Nash–Moser-type theorems is Hamilton [5]. Another good reference is Alinhac and Gérard’s book [1].

**Plan of the paper.** In Section 2, we state carefully the main theorem on $\varepsilon$-dependent Nash–Moser iteration, giving the proof afterward in Section 3. A first application in Section 4 describes classical local-in-time existence results for quasi-linear hyperbolic systems. In Section 5, we describe applications to large-amplitude, high-frequency WKB solutions of quasilinear hyperbolic systems, and in Section 6 to existence of small-amplitude profiles of quasilinear relaxation systems.

## 2 A simple Nash–Moser theorem

Consider two families of Banach spaces $\{E_s\}_{s \in \mathbb{R}}, \{F_s\}_{s \in \mathbb{R}}$, and a family of equations

\begin{equation}
\Phi^\varepsilon(u^\varepsilon) = 0, \quad u^\varepsilon \in E_s,
\end{equation}

indexed by $\varepsilon \in (0, 1)$, where for all $\varepsilon$,

\begin{equation}
\Phi^\varepsilon \in C^2(E_s, F_{s-m}), \quad \text{for all } s \in [s_0, \bar{s}],
\end{equation}

for some $m \geq 0$ and some $s_0 \leq \bar{s} \in \mathbb{R}$.

Let $| \cdot |_s$ denote the norm in $E_s$ and $\| \cdot \|_s$ denote the norm\(^1\) in $F_s$. We assume that the embeddings

\begin{equation}
E_{s'} \hookrightarrow E_s, \quad F_{s'} \hookrightarrow F_s, \quad s \leq s',
\end{equation}

hold, and have norms less than one:

\begin{equation}
| \cdot |_s \leq | \cdot |_{s'}, \quad \| \cdot \|_s \leq \| \cdot \|_{s'}, \quad s \leq s'.
\end{equation}

We assume the interpolation property\(^2\):

\begin{equation}
| \cdot |_{s+\sigma} \lesssim | \cdot |_{\frac{s'}{\sigma}}^{\frac{\sigma}{s'+\sigma}} | \cdot |_{\frac{s'}{\sigma'}}, \quad 0 < \sigma < \sigma'.
\end{equation}

We assume in addition the existence of a family of regularizing operators

\[ S_\theta : E_s \to E_s, \quad \theta > 0, \]

---

\(^1\)The norms $| \cdot |_s$ and $\| \cdot \|_s$ and spaces $E_s$ and $F_s$ are possibly $\varepsilon$-dependent, as in our application in Section 6.

\(^2\)In (2.5) and below, $|u|_s \lesssim |v|_{s'}$ stands for $|u|_s \leq C|v|_{s'}$, for some $C > 0$ depending on $s$ and $s'$ but not on $\varepsilon$, nor on $u$ and $v$. 

---
such that for all $s \leq s'$,
\begin{align}
|S_{\theta}u - u|_s &\lesssim \theta^{s-s'}|u|_{s'}.
\end{align}
\begin{align}
|S_{\theta}u|_{s'} &\lesssim \theta^{s'-s}|u|_s.
\end{align}

Our first main assumption describes the action of $\Phi^\varepsilon$ and its first two derivatives in $|\cdot|_s$ and $\|\cdot\|_s$ norms.

**Assumption 2.1.** For some $\gamma_0 \geq 0$, for all $s \in [s_0, \bar{s} - m]$, for all $u, v, w \in E_{s+m}$,
\begin{align}
\|\Phi^\varepsilon(u)\|_s &\leq C_0(1 + |u|_{s+m} + |u|_{s_0+m}|u|_s),
\end{align}
\begin{align}
\|[(\Phi^\varepsilon)'](u) \cdot v\|_s &\leq C_0(|v|_{s+m} + |v|_{s_0+m}|u|_{s+m}),
\end{align}
\begin{align}
\|[\Phi^\varepsilon]''(u) \cdot (v, w)\|_s &\leq C_0(|v|_{s_0+m}w|_{s+m} + |v|_{s+m}w|_{s_0+m} + |u|_{s+m}v|_{s_0+m}w|_{s_0+m})
\end{align}
where $C_0 = C_0(\varepsilon, |u|_{s_0+m})$ satisfies
\begin{align}
\sup_{\varepsilon} \sup_{|u|_{s_0+m} \lesssim \varepsilon^\gamma} C_0 < +\infty.
\end{align}

Our second and key assumption states that if $u$ is small enough in $|\cdot|_s$ norm, then $\Phi^\varepsilon$ has a right inverse. The right inverse bound (2.13) is stiff with respect to $\varepsilon$ and shows a loss of derivatives.

**Assumption 2.2.** For some $\gamma \geq 0$, $\kappa \geq 0$, $r \geq 0$, $r' \geq 0$, for all $s \in [s_0 + \max(m, r), \bar{s} - \max(r, r')]$, for all $u \in E_{s+r}$ such that
\begin{align}
|u|_{s_0+\max(m, r)} &\lesssim \varepsilon^\gamma,
\end{align}
the map $(\Phi^\varepsilon)'(u) : E_{s+m} \to F_s$ has a right inverse $\Psi^\varepsilon(u) :$
\begin{align}
(\Phi^\varepsilon)'(u)\Psi^\varepsilon(u) = \text{Id} : F_s \to F_s,
\end{align}
satisfying, for all $\phi \in F_{s+r'}$,
\begin{align}
|\Psi^\varepsilon(u)\phi|_s &\leq \varepsilon^{-\kappa}C(|\phi|_{s_0+m+r'}|u|_{s+r} + \|\phi\|_{s+r'}),
\end{align}
where $C = C(\varepsilon, |u|_{s_0+\max(m, r)})$ satisfies
\begin{align}
\sup_{\varepsilon} \sup_{|u|_{s_0+\max(m, r)} \lesssim \varepsilon^\gamma} C < \infty.
\end{align}

Our third assumption states that the equation (2.1) has a family of approximate solutions. (See Remark 2.8.)
Assumption 2.3. There holds the bound

\[ \| \Phi^\epsilon(0) \|_s \lesssim \epsilon^k, \tag{2.14} \]

for some \( k \) and \( s \) satisfying

\[ \max(2\kappa, \kappa + \gamma_0, \kappa + \gamma) < k, \tag{2.15} \]

\[ s \in [s_0 + \max(m + r', r), \bar{s} - \bar{p}), \tag{2.16} \]

where \( \bar{p} \) is the positive function specified in Remark 2.10.

Our main Theorem gives existence in \( E_{s+m} \) of a solution to equation (2.1).

**Theorem 2.4 (Existence).** Under Assumptions 2.1, 2.2 and 2.3, for \( \epsilon \) small enough, there exists a real sequence \( \theta_j^\epsilon \), satisfying \( \theta_j^\epsilon \to +\infty \) as \( j \to +\infty \) and \( \epsilon \) is held fixed, such that the sequence

\[ u_0^\epsilon := 0, \quad u_{j+1}^\epsilon := u_j^\epsilon + S_{\theta_j^\epsilon} v_j^\epsilon, \quad v_j^\epsilon := -\Psi^\epsilon(u_j^\epsilon)\Phi^\epsilon(u_j^\epsilon), \]

is well defined and converges, as \( j \to \infty \) and \( \epsilon \) is held fixed, to a solution \( u^\epsilon \) of (2.1) in \( s + m \) norm, which satisfies the bound

\[ |u^\epsilon|_s \lesssim \epsilon^{k-\kappa}. \tag{2.17} \]

We supplement the above existence result by the following local uniqueness Theorem. Contrary to Theorem 2.4, Theorem 2.5 does not rely on a Nash-Moser iterative scheme.

**Theorem 2.5 (Local uniqueness).** Under Assumptions 2.1, 2.2 and 2.3, for \( \epsilon \) small enough, if \( (\Phi^\epsilon)' \) is invertible, i.e., \( \Psi^\epsilon \) is also a left inverse, then the solution described in Thm 2.4 is unique in a ball of radius \( o(\epsilon^{\max(\kappa, \gamma_0, \gamma)}) \) in \( s_0 + 2m + r' \) norm. More generally, if \( \hat{u}^\epsilon \) is a second solution within this ball, then \( (\hat{u}^\epsilon - u^\epsilon) \) is approximately tangent to \( \text{Ker}(\Phi^\epsilon)'(u^\epsilon) \), in the sense that its distance in \( s_0 \) norm from \( \text{Ker}(\Phi^\epsilon)'(u^\epsilon) \) is \( o(|\hat{u}^\epsilon - u^\epsilon|_{s_0}) \). In particular, if \( \text{Ker}(\Phi^\epsilon)'(u^\epsilon) \) is finite-dimensional, then \( u \) is the unique solution in the ball satisfying the additional “phase condition”

\[ \Pi_{\text{Ker}(\Phi^\epsilon)'(u^\epsilon)}(\hat{u}^\epsilon - u^\epsilon) = 0, \tag{2.18} \]

where \( \Pi_{\text{Ker}(\Phi^\epsilon)'(u^\epsilon)} \) is any uniformly bounded projection onto \( \text{Ker}(\Phi^\epsilon)'(u^\epsilon) \) (in a Hilbert space, any orthogonal projection onto \( \text{Ker}(\Phi^\epsilon)'(u^\epsilon) \)).

**2.1 Remarks**

**Remark 2.6.** Estimates (2.8), (2.9) and (2.10) in Assumption 2.1 can often be made tame with respect to \( m \), as in the standard Moser estimate

\[ \| u \partial^m_x u \|_{L^s} \lesssim \| u \|_{H^{d_0}} \| u \|_{H^{s+m}} + \| u \|_{H^{d_0+m}} \| u \|_{H^s}, \]

\( H^s = H^s(\mathbb{R}^d), \ s > 0, \) for any \( d_0 \) such that \( d/2 < d_0 \leq s \).
Remark 2.7. Estimate (2.13) in Assumption 2.2 is tame with respect to $r$ and $r'$. This is essential: one may check that the proof collapses if it is not. However, an examination of the proofs of Theorems 2.4 and 2.5 reveals that for existence we require estimate (2.13) only for $\phi$ in the image of $\Phi^\varepsilon$ or $(\Phi^\varepsilon)'$, since $\Psi^\varepsilon$ is estimated only in composition with one or the other of these operators, while for uniqueness we need only the estimate for $\Psi^\varepsilon(U)(\Phi^\varepsilon)''(U)$ that would result by composing estimates (2.13) and (2.10). This generalization is important in the application of Section 6.

Remark 2.8. Assumption 2.3 is satisfied in particular for time-evolution equations which admit approximate solutions up to any order; if $\Phi^\varepsilon = 0$ is such an equation, with approximate solution $u_0^\varepsilon$, then $\|\Phi^\varepsilon(u_0^\varepsilon)\|_s \lesssim \varepsilon^k$ translates into (2.14) by the shift $\Phi^\varepsilon(\cdot) := \Phi^\varepsilon(u_0^\varepsilon + \cdot)$. See Section 5. In this situation, the index $s$ measures the regularity of the approximate solution.

Remark 2.9. For $\gamma_0, \gamma \leq \kappa$, Theorem 2.4 states that a loss of $\varepsilon^{-\kappa}$ in the linear estimates means that, with the notation of Remark 2.8, $\|\tilde{\Phi}^\varepsilon(u_0^\varepsilon)\|_s \leq C\varepsilon^{2\kappa+\eta}$, any $\eta > 0$, is the accuracy needed on the approximate solution.

This condition is sharp even for convergence of a standard Newton iteration scheme $u_{n+1}^\varepsilon = u_n^\varepsilon - \Psi^\varepsilon(u_n^\varepsilon)\Phi^\varepsilon(u_n^\varepsilon)$ for problems with no loss of derivatives, corresponding by the computation

$$|\Phi^\varepsilon(u_1^\varepsilon)| = \Phi^\varepsilon(u_0^\varepsilon) + (\Phi^\varepsilon)'(u_0^\varepsilon)(u_1^\varepsilon - u_0^\varepsilon) + O(|u_1^\varepsilon - u_0^\varepsilon|^2) \lesssim |u_1^\varepsilon - u_0^\varepsilon|^2 \lesssim |\Psi^\varepsilon(u_0^\varepsilon)\Phi^\varepsilon(u_0^\varepsilon)|^2 \lesssim \varepsilon^{-2\kappa}|\Phi^\varepsilon(u_0^\varepsilon)|^2 \lesssim \varepsilon^\eta|\Phi^\varepsilon(u_0^\varepsilon)|$$

to the condition that $\varepsilon^\eta|\Phi^\varepsilon(u_0^\varepsilon)|$ decreases at the first step.

Remark 2.10. Bound (2.16) contains in particular the information that the required regularity of the approximate solution (see Remark 2.8) tends to $+\infty$ as $k \downarrow \max(2\kappa, \kappa+\gamma_0, \kappa+\gamma)$. The proof gives indeed the lower bound (3.5) on $\bar{s}$:

$$s_0 + \max(m + r', r) + \bar{p} < \bar{s},$$

corresponding to the upper bound in $s$ in condition (2.16). From (3.22), we find that $\bar{p}$ satisfies

$$\bar{p} = r' - \max(r, r') + \inf_{N > N_0} \frac{k(N + 1)(M + m + \max(r, r'))}{k - \kappa - kM},$$

with

$$N_0 := \frac{km}{k - 2\kappa}, \quad M := \max\left(\frac{\gamma_0}{k}, \frac{\gamma}{k}, \frac{1}{2}\left(1 + \frac{m'}{N}\right)\right), \quad m' := \max(m + r', r).$$

In the case $\max(\gamma_0, \gamma) \leq \kappa$, $\bar{p}$ blows up as $O(k - 2\kappa)^{-2}$ as $k \downarrow 2\kappa$. In the case $\max(\gamma_0, \gamma) > \kappa$, $\bar{p}$ blows up as $O(k - \kappa - \max(\gamma, \gamma_0))^{-1}$ as $k \downarrow \kappa + \max(\gamma_0, \gamma)$.
Remark 2.11. The distinction between $r$ and $r'$ is somewhat illusory, since we can always redefine $F_s$, $m$, so that $r'=0$. We note that the proof with $r' \neq 0$ is the same as with $r'=0$.

Remark 2.12. If the map $(u,v) \to \Pi_{\ker(\Phi^{\varepsilon})}(u)v$ is continuous in $E_{s_0} \times E_{s_0}$, uniformly in $\varepsilon$, then the implicit phase condition (2.18) can be replaced by the explicit

$$\Pi_{\ker(\Phi^{\varepsilon})}(0)(\hat{u}^\varepsilon - u^\varepsilon) = 0.$$ 

See [20], Section 2, for related discussions of uniqueness up to phase conditions.

3 Proofs of Theorems 2.4 and 2.5

We write $\Phi$ for $\Phi^{\varepsilon}$, $\theta_j$ for $\theta^{\varepsilon}_j$, etc., in this proof. An index $s$, satisfying (2.16), is fixed. Let $\theta_0$ such that

$$(3.1) \quad \theta^{-\alpha}_0 \leq \varepsilon^{\max(\gamma, \gamma_0)},$$

for some $\alpha > 0$ to be chosen later. Let

$$C_1(j; q, \alpha) : \quad |v_j|_{s+q} \lesssim \theta^{-\alpha}_j$$

for some $q \geq m$ to be chosen later.

We assume that Assumptions 2.1, 2.2, and 2.3 hold, and start by proving three Lemmas.

Lemma 3.1. Assume:

- the sequence $u_j$ is well defined,
- $\lim_{j \to +\infty} \|\Phi(u_j)\|_s = 0$,
- Condition $C_1(j, s, q, \alpha)$ holds for all $j$,
- the series $\theta^{-\alpha}_j$ is convergent, with

$$(3.2) \quad \sum_{j=0}^{+\infty} \theta^{-\alpha}_j \lesssim \theta^{-\alpha}_0.$$ 

Then $u_j$ converges, in $s + q$ norm, to a solution of (2.1) which satisfies

$$(3.3) \quad |u|_{s+q} \lesssim \theta^{-\alpha}_0.$$ 

Proof. If $C_1(j)$ holds for all $j$, then the sequence $u_j$ converges, in $s + q$ norm, to $u \in E_{s+q}$, and we have the estimate

$$(3.4) \quad |u_j|_{s+q} \lesssim \sum_{j=0}^{j-1} \theta^{-\alpha}_j,$$
which implies (3.3). Estimates (2.8) and (2.9) then imply
\[
\|\Phi(u)\|_s \leq \|\Phi(u_j)\|_s + \left\| \int_0^1 \Phi'(u_j + t(u - u_j)) \cdot (u - u_j) dt \right\|_s
\]
\[\lesssim \|\Phi(u_j)\|_s + |u - u_j|_{s+m},\]
and, as \( q \geq m \), the upper bound tends to 0 as \( j \to +\infty \), hence \( u \) solves (2.1).

Let \( N \geq 0 \), and \( p \) such that
\[
q + \max(r, r' + m) \leq p, \quad s + p - \max(m + r', r) + r' \leq \bar{s}.
\]
Let
\[
C_2(j; q, \alpha, p, N) : \begin{cases}
|u_j|_{s+q} \lesssim \theta_0^{-\alpha}, \\
\|\Phi(u_j)\|_s \lesssim \theta_j^{-1}, \\
|u_j|_{s+p} \lesssim \theta_j^N.
\end{cases}
\]

**Lemma 3.2.** Assume:

- For all \( j' \leq j \), \( u_{j'} \) is well defined and condition \( C_1(j') \) holds,
- There holds
  \[
  \sum_{j'=0}^j \theta_j^{-\alpha} \lesssim \theta_0^{-\alpha},
  \]
- Condition \( C_2(j) \) holds, with parameters satisfying
  \[
  \theta_j^{m-q-\alpha} + \theta_j^{-2\alpha} \leq \theta_j^{-1},
  \]
  \[
  \varepsilon^{-k} \theta_j^{\max(m+r', r)} \theta_j^N \leq \theta_j^N.
  \]

Then \( v_{j+1} \) is well defined in \( E_{s+q} \) and \( C_2(j + 1) \) holds.

**Proof.** If conditions \( C_1(j') \) hold for all \( j' \leq j \) and if (3.6) holds, then
\[
|u_{j+1}|_{s+q} \lesssim \theta_0^{-\alpha}.
\]
Bound (3.9) is \( C_2(j + 1)(i) \). Besides, (3.9) and (3.1) imply that \( u_{j+1} \) also satisfies (2.12), so that, by \( C_2(j + 1)(iii) \), the first bound in (3.5) and (2.13), \( v_{j+1} \) is defined in \( E_{s+q} \).

To prove \( C_2(j + 1)(ii) \), we use the fact that (2.7) is almost a Newton’s scheme:
\[
\|\Phi(u_{j+1})\|_s \leq E_1 + E_2,
\]
where $E_1$ is the error due to the regularization:

$$E_1 = \|\Phi'(u_j) \cdot (S_{\theta_j} v_j - v_j)\|_s,$$

and $E_2$ is the error due to the scheme:

$$E_2 = \left\| \int_0^1 (1 - t)\Phi''(u_j + tS_{\theta_j} v_j) \cdot (S_{\theta_j} v_j, S_{\theta_j} v_j) \, dt \right\|_s.$$

Conditions $C_1(j')$, $j' \leq j - 1$, together with (3.1) and (3.6), implies that $u_j$ and $S_{\theta_j} v_j$ are bounded, in $s_0 + m$ norm, by $\varepsilon^{\gamma_0}$. Then, bounds (2.6), (2.9) and (2.10) give

$$E_1 \lesssim |S_{\theta_j} v_j - v_j|_{s+m} + |S_{\theta_j} v_j - v_j|_{s_0+m} |u_j|_{s+m} \lesssim \theta_j^{m-q} |v_j|_{s+q} \lesssim \theta_j^{m-q-\alpha},$$

and

$$E_2 \lesssim |S_{\theta_j} v_j|_{s_0+m} (|S_{\theta_j} v_j|_{s+m} + |S_{\theta_j} v_j|_{s_0+m} |u_j|_{s+m}) \lesssim \theta_j^{-2\alpha}.$$

Bounds (3.10), (3.11) and (3.7) imply $C_2(j+1)(i)$. Finally, to prove $C_2(j+1)(ii)$, we remark that, by (2.7),

$$|u_{j+1}|_{s+p} \leq |u_j|_{s+p} + |S_{\theta_j} v_j|_{s+p} \lesssim |u_j|_{s+p} + \theta_j^{\max(m+r',r')} |v_j|_{s+p-\max(m+r',r')} .$$

so that, under (3.5), bounds (2.8) and (2.13) imply

$$|v_j|_{s+p-\max(m+r',r')} \lesssim \varepsilon^{-\kappa} |\Phi(u_j)|_{s_0+m+r'} + |\Phi(u_j)|_{s+p-\max(m+r',r')} + |\Phi(u_j)|_{s_0} \lesssim \varepsilon^{-\kappa} (1 + |u_j|_{s+p}) (1 + |\Phi(u_j)|_{s_0}) \lesssim \varepsilon^{-\kappa} \theta_j^{N} .$$

Bounds (3.12) and (3.13) and (3.8) imply $C_2(j+1)(iii)$. \hfill \square

**Lemma 3.3.** Let $j$ such that

$$\varepsilon^{-\kappa} \theta_j^{-\beta} \leq \theta_j^{-\alpha}$$

where

$$\beta := (p' + \max(r, r'))^{-1} ((p' - q) - N(q + \max(r, r'))), \quad p' := p - \max(m + r', r).$$

Then condition $C_2(j)$ implies $C_1(j)$.

**Proof.** Bound $C_2(j)(i)$, together with (3.1), implies that $u_j$ satisfies (2.12). Then, bound $C_2(j)(ii)$ implies that $v_j$ is well defined in $E_{s+p-\max(m+r',r)}$, and we can check, exactly as in the proof of (3.13) in Lemma 3.2, that the bound

$$|v_j|_{s+p-\max(m+r',r)} \lesssim \varepsilon^{-\kappa} \theta_j^{N} .$$
holds. Besides, by (2.13),
\[
|v_j|_{s-\max(r,r')} \lesssim \varepsilon^{-\kappa}(|u_j|_s \|\Phi(u_j)\|_{s_0+m} + \|\Phi(u_j)\|_s) \\
\lesssim \varepsilon^{-\kappa}(1 + \theta_0^{-\alpha})\|\Phi(u_j)\|_s \\
\lesssim \varepsilon^{-\kappa}\theta_j^{-1}.
\]
(3.16)

Finally, bounds (3.15), (3.16) and the interpolation property (2.5) imply
\[
|v_j|_{s+p} \lesssim |v_j|_{s-r'}^{\frac{p'-q}{p''-r'}}|v_j|_{s+r''}^{\frac{q+r''}{p'+r''}} \\
\lesssim \varepsilon^{-1}\theta_j^{-\beta},
\]
where \(r'' = \max(r, r')\), and the Lemma follows, with (3.14).

\[\square\]

End of proof of Theorem 2.4, existence. Let \(q = m + \alpha\). Define
\[
\theta_0 := \varepsilon^{-k}, \quad \theta_{j+1} := \theta_j^\zeta, \quad j \geq 0,
\]
for some \(\zeta > 1\) to be chosen below. Then (3.1) is satisfied if
\[
\max\left(\frac{\gamma_0}{\alpha}, \frac{\gamma}{\alpha}\right) \leq k,
\]
and (3.6) is satisfied.

By (3.18) and Assumption 2.3, condition \(C_2(0)\) is satisfied. By Lemma 3.3, condition \(C_1(0)\) is satisfied as well if
\[
\frac{k}{\beta - \alpha} \leq k.
\]
(3.20)

With definition (3.18), conditions (3.7) and (3.8) translate respectively into
\[
\zeta \leq 2\alpha, \quad \text{and} \quad \frac{\kappa}{N\zeta - N - \max(m + r', r)} \leq k.
\]
(3.21)

Suppose now that for all \(0 \leq j' \leq j\), \(u_{j'}\) is well defined and \(C_1(j')\) and \(C_2(j')\) hold. Then by Lemma 3.2, condition \(C_1(j+1)\) is satisfied if (3.21) holds, and by Lemma 3.3, condition \(C_2(j+1)\) is satisfied if (3.20) holds.

We just proved that, under (3.19), (3.20) and (3.21), conditions \(C_1(j)\) and \(C_2(j)\) hold for all \(j\).

Now conditions (3.19), (3.20) and (3.21) can be put in the form of a set of lower and upper bounds for \(\alpha\):
\[
\max\left(\frac{\gamma_0}{k}, \frac{\gamma}{k}, \frac{1}{2}\left(1 + \frac{m'}{N}\right)\right) \leq \alpha \leq \left(1 + \frac{1}{p_0}\right)^{-1}\left(1 - \frac{\kappa}{k} - \frac{1}{p_0}(m+r'')\right),
\]
(3.22)
where \( p_0 := \frac{p'' + r''}{N+1} \), \( m' := \max(m + r', r) \), \( p' := p - m' \) and \( r'' := \max(r, r') \). By (2.15), condition (3.22) is not empty. (We can deduce from (3.22) a lower bound for \( p \), which implies a lower bound for the regularity of the approximate solution; see Remark 2.10.)

Let now \( \alpha, N, \) and \( p \) such that (3.22) holds; then any \( \zeta \) satisfying (3.21) is admissible, and \( C_1(j) \) and \( C_2(j) \) hold for all \( j \). By (3.18) and \( \zeta > 1 \), the series \( \theta_j^{-\alpha} \) is convergent and satisfies (3.2). Besides, conditions \( C_2(j) \) imply \( \| \Phi(u_j) \|_s \to 0 \). We can thus apply Lemma 3.1: the sequence \( u_j \) converges to a solution \( u \) of (2.1) in \( s + q \) norm, satisfying (3.3).

Besides, as (3.16) holds for all \( j \),

\[
|u_j|_s \lesssim \varepsilon^{-\kappa} \sum_{j'=0}^j \theta_{j'}^{-\beta} \lesssim \varepsilon^{-\kappa} \theta_0^{-1},
\]

hence (2.17).

\[ \square \]

**Proof of Theorem 2.5, local uniqueness.** Suppressing \( \varepsilon \), let \( \hat{u} \) be a second solution in \( E_{s_0 + m + r'} \) of \( \Phi(u) = 0 \), lying within \( o(\varepsilon^{\max(s_0, r, \gamma)}) \) of \( u \) (and thus of 0). Then, Taylor expanding, and using Assumption 2.1, we have

\[
0 = \Phi(\hat{u}) - \Phi(u) = \Phi'(u)(\hat{u} - u) + \int_0^1 (1 - t)\Phi''(tu + (1 - t)\hat{u}) \cdot (\hat{u} - u, \hat{u} - u) \, dt.
\]

where

\[
\| \Phi(u, \hat{u}) \|_{s_0 + m + r'} \leq C|\hat{u} - u|_{s_0}|\hat{u} - u|_{s_0 + 2m + r'}.
\]

Applying \( \Psi(u) \) and using Assumption 2.2, we thus have

\[
(\hat{u} - u) - \Psi(u)B(u, \hat{u}) \in \text{Ker}\Phi'(u),
\]

where

\[
\| \Psi(u)B(u, \hat{u}) \|_{s_0} \leq C\varepsilon^{-\kappa}(1 + |u|_{s_0 + r})|\hat{u} - u|_{s_0 + m + r'}^2|\hat{u} - u|_{s_0} = o(|\hat{u} - u|_{s_0 + m + r'}).
\]

This verifies tangency. Finally, from \( \hat{u} - u + o(|\hat{u} - u|) \in \text{Ker}(\Phi^\varepsilon)'(u^\varepsilon) \), we have

\[
\hat{u} - u + o(|\hat{u} - u|) = \Pi_{\text{Ker}(\Phi^\varepsilon)'(u^\varepsilon)}(\hat{u}^\varepsilon - u^\varepsilon) + o(\|\Pi_{\text{Ker}(\Phi^\varepsilon)'(u^\varepsilon)}\|\hat{u} - u|),
\]

which, with (2.18) and the assumed uniform boundedness of \( \|\Pi_{\text{Ker}(\Phi^\varepsilon)'(u^\varepsilon)}\| \), gives

\[
\hat{u} - u = o(|\hat{u} - u|) + o(\Pi_{\text{Ker}(\Phi^\varepsilon)'(u^\varepsilon)}\|\hat{u} - u|) = o(|\hat{u} - u|),
\]

and thus \( \hat{u} - u = 0 \).

\[ \square \]

### 4 Application 0: Solutions of quasilinear hyperbolic systems

We first describe a somewhat artificial application for orientation.
4.1 Standard Nash–Moser

Consider a first-order quasilinear symmetric hyperbolic system
\[
\partial_t u + A(u)\partial_x u = 0, \quad u|_{t=0} = u_0,
\]
\(A \in C^\infty \) symmetric, and \(u_0 \in H^s(\mathbb{R}^d), s \gg 1\). Rescaling \((x,t) \to (x/\varepsilon, t/\varepsilon)\), we obtain a sequence of problems
\[
(4.1) \quad \partial_t u + A(u)\partial_x u = 0, \quad u|_{t=0} = u_0^\varepsilon(x) = u_0(\varepsilon x),
\]
with
\[
(4.2) \quad |u_0^\varepsilon|_{H^s} \leq C\varepsilon
\]
for \(| \cdot |_{H^s} := | \cdot |_{L^2} + \varepsilon^{-1}| \cdot |_{H^1} + \cdots + \varepsilon^{-s}| \cdot |_{H^s}|.

By Friedrichs type energy estimates taking the \(L^2\) inner product of \(\partial_t^2 u\) with (4.1), \(k = 0, \ldots, s\) and summing, then applying Gronwall’s inequality, we easily obtain on \(L^\infty([0, 1]; H^s_\varepsilon)\) the tame estimates needed for Nash–Moser iterations, with \(m = 1, r' = 0, r = 1\), but no loss of \(\varepsilon^{-1}\) in (2.13). Applying standard Nash–Moser iteration, therefore, we obtain existence for the problem up to rescaled time 1, for \(\varepsilon\) sufficiently small, and uniqueness on an \(H^s_\varepsilon\) ball of radius \(o(1)\). Converting back to unscaled coordinates, we have short time existence up to time \(t = \varepsilon\) and uniqueness on an \(H^s\) ball of radius \(o(\varepsilon^{-d/2})\) for \(\varepsilon\) sufficiently small.

Note that we could also obtain this result in standard fashion by the simpler iteration \(\partial_t u^{n+1} + A(u^n)\partial_x u^{n+1} = 0\) using boundedness in high norms and contraction in low norms.

Consider now the slight variant of a conservative form equation
\[
\partial_t u + \partial_x (A(u)u) = 0, \quad u|_{t=0} = u_0.
\]
It is readily checked that for this equation the iteration \(\partial_t u^{n+1} + \partial_x (A(u^n)u^{n+1}) = 0\) gives estimates losing one derivative in \(u^n\), so that the classical approach does not yield a solution. This is similar to the situation of the more realistic application described in Section 6 below. However, we readily obtain tame estimates with \(r = m = 1\), and still \(r' = 0\), so again obtain existence, and uniqueness in a large ball, by (standard, no \(\varepsilon\) loss) Nash–Moser iteration.

For high-frequency initial data of the form \(u_0^\varepsilon = v_0(x/\varepsilon) + w_0(x)\), we obtain in the same way an existence time \(O(\varepsilon)\) under the preparation condition \(A(v_0) = 0\).

The same preparation condition \(A(v_0) = 0\) gives an existence time \(O(1)\) for solutions issued from initial data of the form \(u_0^\varepsilon = v_0(x) + \varepsilon w_0(x)\).

4.2 Nash–Moser with \(\varepsilon\) loss

Consider now hyperbolic problems of the form (4.1), in a high-frequency regime and with a source:
\[
(4.3) \quad \varepsilon\partial_t u + \varepsilon A(u)\partial_x u = f, \quad u|_{t=0} = u_0
\]
If \( u_0 \in H^s \), with \( s \) large enough, the Friedrichs type energy estimates of the above Section yield a tame bound of the form (2.13), with \( m = 1 \), \( r = 1 \) and \( r' = 0 \). If there exists an approximate solution \( u_a \) (corresponding to Assumption 2.3 and typically constructed as a WKB-type approximation):

\[
\epsilon \partial_t u_a + \epsilon A(u_a) \partial_x u_a = f + \epsilon^k g, \quad u_a|_{t=0} = u_0 + \epsilon^k v_0,
\]

with \( k > 2 \), \( v_0 \in H^s \), \( g \in H^{s'} \), for some \( s' \) large enough, then Theorem 2.4 gives an existence time \( O(1) \) for the solution of (4.3) and uniqueness in a small ball centered at \( u_a \), of radius \( o(\epsilon) \) in \( H^{s+1} \) norm.

5 Application 1: large-amplitude, high-frequency solutions of quasilinear hyperbolic systems

5.1 Pseudo-differential symbols and associated semiclassical operators

Given \( m, s \in \mathbb{R} \), we define the class \( \Gamma^m_s \) as the space of symbols \( \sigma \) defined on \((0, 1) \times \mathbb{R}_x \times \mathbb{R}_\xi \), such that, for all \( \epsilon \) and all \( k \in \mathbb{N} \), \( \sigma(\epsilon) \in C^k(\mathbb{R}_\xi; H^s(\mathbb{R}_d)) \), and, for all \( \epsilon \),

\[
N^m_{k,s}(\sigma) := \sup_{|\beta| \leq k} \sup_{\xi} (1 + |\xi|^2)^{(|\beta| - m)/2} \| \partial^\beta \sigma(\epsilon, t, \cdot, \xi) \|_{\epsilon, s} < \infty,
\]

and the norm \( \| \cdot \|_{\epsilon, s} \) is the weighted norm in \( H^s \) defined by

\[
\| v \|_{\epsilon, s} := \| (1 + |\epsilon \xi|^2)^{s/2} \hat{v}(\xi) \|_{L^2(\mathbb{R}_\xi)}.
\]

where \( \hat{\cdot} \) denotes Fourier transform from \( \mathbb{R}_x \) to \( \mathbb{R}_\xi \). Define also the norm

\[
M^m_{k,k'}(\sigma) := \sup_{|\beta| \leq k} \sup_{|\beta'| = k'} \sup_{\xi} (1 + |\xi|^2)^{(|\beta| - m)/2} \| \partial^\beta \partial^{\beta'} \sigma(\epsilon, t, \cdot) \|_{L^\infty}.
\]

Given a symbol \( \sigma \in \Gamma^m_s \), if \( s > k' + d/2 \) then by Sobolev’s inequality \( M^m_{k,k'}(\sigma) < \infty \), for all \( 0 \leq k'' \leq k' \).

To a symbol \( \sigma \in \Gamma^m_s \), one associates the pseudo-differential operator \( \text{op}_\epsilon(\sigma) \) defined by its action on \( \mathcal{S}(\mathbb{R}^d) \) as

\[
\text{op}_\epsilon(\sigma)u := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{ix \cdot \xi} \sigma(\epsilon, x, \epsilon \xi) \hat{u}(\xi) \, d\xi.
\]

In the following, \( d_0 \) is a fixed real number, strictly greater than \( d/2 \), and such that \( d_0 - d/2 \) is arbitrarily small. We let

\[
\sigma_1 \hat{\sigma}_2 := \sum_{0 \leq |\alpha| \leq n} \epsilon^{|\alpha|} \frac{(-i)^{|\alpha|}}{\alpha!} \partial^{\alpha} \sigma_1 \partial^{\alpha} \sigma_2.
\]

The following results are based on the bounds of [19], themselves based on the precise bounds of [8]. (See Propositions 1 and 3 in [19], Lemma 19 and Propositions 20 and 23 in [8].)
Proposition 5.1 (Action). Given $m \in \mathbb{R}$, $s > d/2$ and $\sigma \in \Gamma^m_s$, for all $u \in H^{s+m}$,
\[
\|\text{op}_\varepsilon(\sigma)u\|_{\varepsilon,s} \lesssim M^m_{d,0}(\sigma)\|u\|_{\varepsilon,s+m} + \varepsilon^{s-d/2}N_{2[d/2]+2,s}^m(\sigma)\|u\|_{\varepsilon,m+d_0}.
\]

Proposition 5.2 (Composition). Let $s, s_1, s_2, m_1, m_2 \in \mathbb{R}$, $n \in \mathbb{N}$, $s' := s + m_1 + m_2 - n - 1$ and $d' := 2[d/2] + 2$. Given $\sigma_1 \in \Gamma^m_{s_1}$ and $\sigma_2 \in \Gamma^m_{s_2}$, for all $u \in H^{s'}$,
\[
\left\|\left(\text{op}_\varepsilon(\sigma_1)\text{op}_\varepsilon(\sigma_2) - \text{op}_\varepsilon(\sigma_1\sigma_2)\right)u\right\|_{\varepsilon,s'} \lesssim \varepsilon^{n+1} \left(M_{d,0}^{m_1}(\sigma_1)N_{d',s+m_1}^{m_2}(\sigma_2) + M_{d,n+1}^{m_1}(\sigma_1)M_{d,n}^{m_2}m_2(\sigma_2)\right)\|u\|_{\varepsilon,s'},
\]
\[\sup\varepsilon\|u\|_{\varepsilon,m_1+m_2+d_0},\]
where
\[
C := \left(M_{d,0}^{m_1}(\sigma_1)N_{d',s+m_1}^{m_2}(\sigma_2) + N_{d',s}^{m_1}(\sigma_1)M_{d,n}^{m_2}m_2(\sigma_2)\right)
\]
\[\sum_{1 \leq k \leq n} \varepsilon^k \left(N_{d'+k,s}^{m_1}(\sigma_1)M_{d',k}^{m_2}m_2(\sigma_2) + M_{d'+k,n}^{m_1}(\sigma_1)N_{d',s+k}^{m_2}m_2(\sigma_2)\right).
\]

Define $C^\infty\mathcal{M}^m$ as the set of symbols $\Sigma$, defined on $(0, 1)_\varepsilon \times \mathbb{R}_v^n \times \mathbb{R}_\xi^d$, with values $\Sigma(\varepsilon, v, \xi)$ in the $n \times n$ matrices with complex entries, such that for all $\alpha, \beta$ and all $v$,
\[
\sup \varepsilon (1 + |\xi|^2)^{(1/2)\mu v}(\partial^\alpha_{\varepsilon,v}\partial^{\beta}_{\xi}\Sigma(\varepsilon, v, \xi)) \leq C_{\alpha,\beta}(|v|),
\]
for some non-decreasing function $C_{\alpha,\beta}$ independent of $v$.

By Moser’s inequality, given $\Sigma \in C^\infty\mathcal{M}^m$, for all $s > 0$ and all $v \in H^s \cap L^\infty$, the symbol $\Sigma(v) - \Sigma(0)$ belongs to $\Gamma^m_s$, with the bound
\[
N_{k,s}^m(\Sigma(v) - \Sigma(0)) \leq C_{\Sigma,k,s}(|v|_{L^\infty})\|v\|_{\varepsilon,s},
\]
where $C_{\Sigma,k,s}$ is nondecreasing and independent of $\varepsilon$. If $\Sigma \in C^\infty\mathcal{M}^m$ and $v \in W^{k',\infty}$, then
\[
M_{k,k'}^m(\Sigma(v)) \leq C_{\Sigma,k,k'}(|v|_{W^{k',\infty}}),
\]
where $C_{\Sigma,k,k'}$ is nondecreasing and independent of $\varepsilon$.

We will use in the following Propositions 5.1 and 5.2 for the subclass of symbols
\[
\{\Sigma(u) - \Sigma(0), \Sigma \in C^\infty\mathcal{M}^m, u \in H^s \cap L^\infty\} \subset \Gamma^m_s.
\]

### 5.2 Singular pseudo-differential equations

Let $A$ be a symbol in $C^\infty S^1$. We consider the family of initial value problems
\[
\begin{aligned}
\varepsilon^2 \partial_t u + \text{op}_\varepsilon(A(u))u &= 0, \\
u(t=0) &= a + \varepsilon^k \varphi,
\end{aligned}
\]
indexed by $\varepsilon \in (0, 1)$, where $t \in \mathbb{R}_+, x \in \mathbb{R}^d$ ($d \in \mathbb{N}$), $u(\varepsilon, t, x) \in \mathbb{R}^n$ ($n \in \mathbb{N}$). The initial perturbation $\varepsilon^k \varphi$ is such that $k \geq 0$ and $\varphi \in H^s$, with $\varphi$ being possibly $\varepsilon-$dependent and $d/2 < \bar{s}$. We assume

(5.6) $\sup_\varepsilon \| \varphi^0 \|_{\varepsilon, \bar{s}} < +\infty$.

The leading term in the initial datum $a$ is assumed to be independent of $\varepsilon$, and to belong to $H^{s+O(k)}$. We introduce the spaces, indexed by $\sigma \leq \bar{s}$ and $t \in \mathbb{R}_+$,

$E(\sigma, t) = C^0([0, t], H^\sigma(\mathbb{R}^d)) \cap C^1([0, t], H^{\sigma-1}(\mathbb{R}^d))$,

$F(\sigma, t) = C^0([0, t], H^\sigma(\mathbb{R}^d)) \times H^\sigma(\mathbb{R}^d)$,

and associated $\varepsilon$-dependent norms (based on the weighted norms (5.1))

$|u|_{\varepsilon, \sigma, t} := \sup_{0 \leq t' \leq t} (\| \varepsilon^2 \partial_t u(t') \|_{\varepsilon, \sigma-1} + \| u(t') \|_{\varepsilon, \sigma})$,

$\| f, \varphi \|_{\varepsilon, \sigma, t} := \sup_{0 \leq t' \leq t} \| f(t') \|_{\varepsilon, \sigma} + \| \varphi \|_{\varepsilon, \sigma}$.

These spaces satisfy (2.3), (2.4), (2.5). A family of regularizing operators in $E(\sigma, t)$ is given by

$S_\theta : u \rightarrow S_\theta(u) := \mathcal{F}^{-1}(\chi(\theta^{-1} \xi) \hat{u})$, $\theta > 0$,

where $\mathcal{F}^{-1}$ denotes inverse Fourier transform, and $\chi : \mathbb{R}^d \rightarrow \mathbb{R}_+$ is a smooth truncation function, identically equal to 1 for $|\xi| \leq 1$, and identically equal to 0 for $|\xi| \geq 2$. The family $\{S_\theta\}_{\theta > 0}$ satisfies (2.6) and (2.7).

We assume that the initial value problems (5.5) have a family of approximate solutions:

**Assumption 5.3.** There exists $t_\ast > 0$, independent of $\varepsilon$, and a family $\{u_\varepsilon\}_\varepsilon$ of approximate solutions of (5.5), with $u_\varepsilon \in E(\bar{s}, t_\ast)$ for all $\varepsilon$, satisfying

(5.7) $\begin{cases} \varepsilon^2 \partial_t u_\varepsilon + \text{op}_\varepsilon(A(u_\varepsilon))u_\varepsilon = \varepsilon^k r_\varepsilon, \\
\quad u_\varepsilon|_{t=0} = a, \end{cases}$

with the bounds

(5.8) $\sup_\varepsilon (|u_\varepsilon(t)|_{\varepsilon, \bar{s}, t_\ast} + |r_\varepsilon(t)|_{\varepsilon, \bar{s}, t_\ast}) < +\infty$.

The linearized equations about $u_\varepsilon + u$ are

(5.9) $\begin{cases} \varepsilon^2 \partial_t u + \text{op}_\varepsilon(A(u_\varepsilon + u))u = -\text{op}_\varepsilon(\partial_u A(u_\varepsilon + u) \cdot u)(u_\varepsilon + u) + f, \\
\quad u|_{t=0} = \varphi. \end{cases}$

The following Assumption, in which $d_0 > d/2$, with $d_0 - d/2$ arbitrarily small, and in which $\tilde{p}$ is defined by (2.19) with $m = 1$, gives tame linearized estimates for (5.9):
Assumption 5.4. There exists $r \geq 0$, $r' \geq 0$ and $s \in [s_0 + \max(1 + r', r), \bar{s} - \bar{p}]$, such that, given $u \in E(s + r, t_*)$, for all $(f, \varphi) \in F(s + r', t_*)$, the initial value problem (5.9) has a unique solution $u$ that satisfies the tame estimate

\begin{equation}
|u|_{\varepsilon, s, t_*} \lesssim \varepsilon^{-2} C(\|f, \varphi\|_{\varepsilon, d_0 + 1 + r', t_*}, \|u|_{\varepsilon, s + r, t_*} + \|f, \varphi\|_{\varepsilon, s + r', t_*}),
\end{equation}

where $C : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing function of $\sup_{0 \leq |\rho| \leq \max(1, r)} |\partial_\rho x (u_a + u)|_{L^\infty}$.

Theorem 5.5. Under Assumptions 5.3 and 5.4, if

\begin{equation}
\sup_{\varepsilon} \sup_{0 \leq |\rho| \leq \max(1, r)} |\partial_\rho x u_a|_{L^\infty} < \infty,
\end{equation}

and if

\begin{equation}
k > \max \left(4, 2 + \max(1, r) + \frac{d}{2} \right),
\end{equation}

then the initial value problem (5.5) has a solution $u \in E(s, t_*)$, $s$ as in Assumption 5.4, which satisfies

\begin{equation}
|u - u_a|_{\varepsilon, s, t_*} \leq C \varepsilon^{k - 2}.
\end{equation}

Proof. We apply Theorem 2.4 to the perturbation equations $\Phi^\varepsilon := (\Phi_1^\varepsilon, \Phi_2^\varepsilon) = 0$,

\begin{align*}
\Phi_1^\varepsilon(u) &:= \varepsilon^2 \partial_t u + \text{op}_\varepsilon(A(u_a + u))u + \text{op}_\varepsilon(A(u_a + u) - A(u_a))u_a - \varepsilon^k r_a, \\
\Phi_2^\varepsilon(u) &:= u|_{t=0} - \varepsilon^k \varphi.
\end{align*}

By Assumption 5.3, Proposition 5.1, and the Sobolev embedding

\begin{equation}
H^{d_0 + \kappa_0} \hookrightarrow W^{\kappa, \infty}, \quad |v|_{W^{\kappa, \infty}} \lesssim \varepsilon^{-d/2 - \kappa_0} \|v\|_{\varepsilon, d_0 + \kappa},
\end{equation}

applied with $\kappa_0 = 0$, the map $\Phi^\varepsilon$ is $C^2$ from $E(s, t_*)$ to $F(\bar{s} - 1, t_*)$, for all $s \leq \bar{s}$, and satisfies Assumption 2.1, with $s_0 = d_0$, $\gamma_0 = d/2$, and $m = 1$.

The linearized initial value problem (5.9) is $((\Phi^\varepsilon)'(u)u = (f, \varphi)$. Assumption 5.4, together with (5.14) applied with $\kappa_0 = d/2 + \max(1, r)$, implies that Assumption 2.2 is satisfied, with $\gamma := d/2 + \max(1, r)$, and $\kappa = 2$.

Finally, $\Phi^\varepsilon(0) = (-\varepsilon^k r_a, -\varepsilon^k \varphi)$, so that, under (5.6) and (5.12), Assumption 2.3 is satisfied.

By Theorem 2.4, there exists a solution $u \in E(s + 1, t^*)$ to $\Phi^\varepsilon(u) = 0$. By definition of $\Phi^\varepsilon$, this implies that $u_a + u$ solves the initial value problem (5.5); bound (2.17) translates to (5.13).
5.3 Example: singular limit of a quasilinear Klein-Gordon-wave system

Consider a quasilinear system in $u = (v, w) \in \mathbb{R}^3 \times \mathbb{R}^2$, of the form (5.5), in which $A$ decomposes as the sum of a Klein-Gordon-wave operator with a lower-order (in $\varepsilon$) quasilinear part and a large bilinear term:

\[
op_{\varepsilon}(A(u)) = \begin{pmatrix} L(\omega, \varepsilon \partial_x) & 0 \\ 0 & \varepsilon M(\varepsilon \partial_x) \end{pmatrix} u + \varepsilon^3 A(u, \varepsilon \partial_x) u - \varepsilon B(u, u),
\]

where

\[L(\omega, \partial_x) := \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \sqrt{\omega} \\ 0 & -\sqrt{\omega} \end{pmatrix}, \quad M(\partial_x) := \begin{pmatrix} 0 & \alpha \partial_x \\ \alpha \partial_x & 0 \end{pmatrix},\]

with a fixed frequency $\omega > 0$ and sound velocity $\alpha > 0$, and $A$ is a coupling term of convective type, such that $A(u, \xi)$ is symmetric for all values of $u \in \mathbb{R}^5$ and $\xi \in \mathbb{R}^d$, and linear in $u$ and $\xi$. The bilinear source term $B$ is for instance

\[
B(u, u) = \begin{pmatrix} v_3w_1 \\ 0 \\ 0 \\ 0 \\ v_2v_3 \end{pmatrix}^T.
\]

We show in this Section that Theorem 5.5 is applicable to (5.5) when $A$ is defined by (5.15)-(5.16). Also, in Remarks 5.8 and 5.13, we give compatibility (transparency, in the sense of Joly, Métivier and Rauch [7]) conditions on general sources $B$, under which Theorem 5.5 applies to (5.5)-(5.15). (The case $B = O(\varepsilon)$, $\phi \equiv 0$ is covered by classical hyperbolic local-in-time theory in $H^s$ spaces; in the case $B = O(1)$ and $\phi \equiv 0$, the existence time given by the classical hyperbolic theory is $O(\varepsilon)$.)

**Remark 5.6.** System (5.5), with $A$ of the form (5.15), are quasilinear extensions of the semilinear Klein-Gordon-wave systems of [3] (corresponding to $\underline{A} \equiv 0$), and simplified versions of the Euler-Maxwell system (see Section 5.4).

**Lemma 5.7 (WKB solution).** The initial value problem for (5.5), with $A$ defined by (5.15)-(5.16), has a family of WKB approximate solutions $u_a$, with initial datum $u_a|_{t=0} = a$, with $a$ as in Section 5.2, which satisfies Assumption 5.3, and (5.11) for $r \leq k - d/2$.

**Proof.** The ansatz

\[
u_a(\varepsilon, t, x) \sim \sum_{n \geq 0} \varepsilon^n \sum_{p \in \mathbb{Z}} \left[ e^{i\theta} u_{np}(t, x) \right]_{\theta = \omega_t / \varepsilon^2}, \quad u_{np} \in H^s, \]

leads to a cascade of WKB equations as in [7, 3, 19]. With the choice (5.16), the limit system in $(v_{0\pm}, w_{10})$ is a Zakharov system to which the local-in-time existence result of Ozawa and Tsutsumi applies [16]. Higher-order terms satisfy a linearized Zakharov system around $u_0$, with extra source terms in $\underline{A}$, which involve only lower-order terms.
Remark 5.8. Let $\Pi_p$ be the projector onto $\ker(ip\omega + \mathcal{L}(\omega, 0))$, for $p \in \{-1, 0, 1\}$. The bilinear source term $B = (B_1, B_2) \in \mathbb{R}^3 \times \mathbb{R}^2$, $B_1 = B_1(v, w)$, $B_2 = B_2(v, v)$, defined in (5.16), satisfies, for all vectors $(v, w) \in \mathbb{R}^3 \times \mathbb{R}^2$,

\begin{equation}
(5.18) \quad \Pi_{\pm} B_1(\Pi_{\pm} v, w) = 0 \iff w = 0, \quad \text{and} \quad B_2(\Pi_{\pm} v, \Pi_{\pm} v) = 0.
\end{equation}

These two properties are crucial in the proof of the above Lemma, see [7, 3, 19] for similar WKB analysis. More generally, any source $B$ satisfying (5.18) leads to a cascade of compatible (but not necessarily well-posed) WKB equations.

Remark 5.9. The kernel of $ip\omega + \mathcal{L}(\omega, 0)$ is trivial if $p \notin \{-1, 0, 1\}$. As a consequence, the leading term $u_0$ in the approximate solution, such that $u_a = u_0 + O(\varepsilon)$ (in $L^\infty$), satisfies $u_0 = e^{-i\omega/t/\varepsilon^2}u_{\pm 1} + u_{00} + e^{i\omega/t/\varepsilon^2}u_{01}$; by the first property in (5.18), there also holds $u_{00} = 0$.

We now work on the linearized equations (5.9) and undertake to verify Assumption 5.4. Let $\mathcal{L} := \begin{pmatrix} L & 0 \\ 0 & \varepsilon M \end{pmatrix}$, and $B(u)u' := B(u, u') + B(u', u)$. The symbol $\mathcal{L}(\omega, \xi) + \varepsilon^3 A(u, \xi)$ is diagonalizable for all $u, \xi$; given $M > 0$, such that $|u| \leq M$, for $\varepsilon$ small enough, the eigenvalues satisfy, by standard perturbation theory,

\begin{equation}
(5.19) \quad \lambda_{\pm} = \pm \sqrt{\omega^2 + \xi^2 + \varepsilon^3 \lambda_{\mp}(u, \xi)}, \quad \lambda_0 = \varepsilon^3 \lambda_0(u, \xi), \quad \mu_{\pm} = \varepsilon (a|\xi| + \varepsilon^2 \mu_{\mp}(u, \xi)),
\end{equation}

with $\lambda_{\pm}, \lambda_0, \mu_{\pm} \in C^\infty M^1$. (Actually, the eigenvalues $\lambda_0$ and $\mu_{\pm}$ cross above $\xi = 0$, hence may not be smooth at $\xi = 0$; this can be remedied by using a low-frequency truncation in the definition of $\Gamma^m$ and $C^\infty M^m$, as in [8], or by restricting the class of quasilinear perturbations to which $A$ belongs). The associated eigenprojectors are $P_{\pm}, P_0, Q_{\pm} \in C^\infty M^0$, with

\[ A = \sum_{\pm} \lambda_{\pm} P_{\pm} + \mu_{\pm} Q_{\pm}. \]

Let $u \in E(s, t_s)$, where $t_s$ is the existence time of $u_a$, and $\sigma \in [d_0 + 2, \sigma - \bar{p}]$, with $\bar{p}$ as in (2.19), with $r' = 0, r = 2$, and $m = 1$. We define high-frequency and low-frequency variables by

\begin{equation}
(5.20) \quad u_h := \text{op}_\varepsilon((P_+ + P_-)(u_a + u))u, \quad u_\ell := \text{op}_\varepsilon((Q_+ + Q_- + P_0)(u_a + u))u,
\end{equation}

where $u \in E(s, t^*(\varepsilon))$ is the local-in-time solution of the linearized equations at $u_a + u$ with a given source $f$, such that $(f, \varphi) \in F(s, t_0)$, for some large $t_0$, independent of $\varepsilon$. Our aim is to prove that $t^*(\varepsilon) = O(1)$. We denote in the following by $R(\rho), \rho \in \mathbb{N}$, any linear operator $H^s \rightarrow H^s$, with $1 + d_0 \leq s \leq \sigma$, such that, for all $v \in H^s$, for all $t \in [0, t^*(\varepsilon)]$,

\begin{equation}
(5.21) \quad \|R(\rho)v\|_{\varepsilon, s} \leq C|u_a + u|_{W^{\max(0, \rho - 1), \infty}} \left( \|v\|_{\varepsilon, s} + \varepsilon^{s-d/2-\rho}\|u_a + u\|_{\varepsilon, s+\rho}\|v\|_{\varepsilon, 1+d_0} \right),
\end{equation}

where $C$ is non-decreasing. We also let $\tilde{u} := (u_h, u_\ell)$ and $P := \begin{pmatrix} P_h \\ P_\ell \end{pmatrix}$. 

18
Lemma 5.10 (Diagonalization). The equation in \( \tilde{u} := (u_h, u_\ell) \) is
\[
(5.22) \quad (\varepsilon^2 \partial_t + \text{op}_\varepsilon(iA)) \tilde{u} = \varepsilon \text{op}_\varepsilon(B)\tilde{u} + \varepsilon^3 R(2)\tilde{u} + \varepsilon \text{op}_\varepsilon(P)f,
\]
where \( A \in C^\infty \mathcal{M}^1, \ B \in C^\infty \mathcal{M}^0 \), with
\[
A(u_a + u, \xi) := \text{diag}(A_h, \varepsilon A_\ell)(u_a + u, \xi) = \text{diag}(\lambda_+, \lambda_-, \mu_+, \mu_-, 0)(u_a + u, \xi),
\]
and
\[
\text{op}_\varepsilon(B)u := \text{op}_\varepsilon \left( (P\mathcal{B}\mathcal{P}^T + (P\mathcal{h}_{\sharp}B)\mathcal{P}^T)(u_a + u, \xi) \right) u - \varepsilon^2 \text{op}_\varepsilon(\partial_u A_\ell(u_a + u, \xi) \cdot u)(u_a + u).
\]

Proof. Symbolic computations, based on Propositions 5.1 and 5.2. We have for instance
\[
\text{op}_\varepsilon(\partial_t P_h) = \varepsilon R(1),
\]
and
\[
\text{op}_\varepsilon(P_h)B = \text{op}_\varepsilon(P_hB + \varepsilon(P_h\mathcal{h}_{\sharp}B)) + \varepsilon^2 R(2),
\]
\[
\text{op}_\varepsilon(P_h)A = \text{op}_\varepsilon(\text{diag}(i\lambda_+, i\lambda_-)) + \varepsilon^4 R(1),
\]
and, by orthogonality and self-adjointness of the eigenprojectors, \( \text{op}_\varepsilon(P_h)\text{op}_\varepsilon(P_\ell) = \varepsilon^4 R(0) \), \( \text{op}_\varepsilon(P_h)\text{op}_\varepsilon(P_\ell) = \text{op}_\varepsilon(P_h) + \varepsilon^4 R(1) \).

Next we let \( v := (u_h, \frac{1}{\varepsilon}u_\ell) \). This change of variable is motivated by the first property in (5.18). The key is that, by Lemma 5.10, the components of \( \tilde{u} \) are coupled by order-one terms, so that this singular change of variable does not induce a loss of hyperbolicity.

Lemma 5.11 (Blow-up). The equation in \( v \) is
\[
(\varepsilon^2 \partial_t + \text{op}_\varepsilon(iA)) v = \text{op}_\varepsilon \left( \begin{array}{cc} \varepsilon P_h\mathcal{B}P_h & 0 \\ P_t\mathcal{B}P_h + \varepsilon(P_t\mathcal{h}_{\sharp}B)P_h & \varepsilon P_t\mathcal{B}P_\ell \end{array} \right) v + \varepsilon^2 R(2)v + \tilde{f},
\]
where \( \tilde{f} := (\text{op}_\varepsilon(P_h)f, \frac{1}{\varepsilon}\text{op}_\varepsilon(P_\ell)f) \).

Proof. Straightforward block-matrix computation from (5.22).

The last step is a normal form reduction procedure:

Lemma 5.12 (Normal form reduction). There exists \( M \in C^\infty \mathcal{M}^{-1} \), such that
\[
(5.23) \quad \varepsilon^2 \partial_t \text{op}_\varepsilon(M) + [\text{op}_\varepsilon(iA), \text{op}_\varepsilon(M)] = \text{op}_\varepsilon \left( \begin{array}{cc} \varepsilon P_h\mathcal{B}P_h & 0 \\ P_t\mathcal{B}P_h + \varepsilon(P_t\mathcal{h}_{\sharp}B)P_h & \varepsilon P_t\mathcal{B}P_\ell \end{array} \right) + \varepsilon^2 R(2).
\]

Proof. We look for \( M \) in the form \( M = \begin{pmatrix} \varepsilon M_h & 0 \\ M_{th} & \varepsilon M_\ell \end{pmatrix} \), where \( M_* = \sum \pm \varepsilon \pm i\omega/\varepsilon^2 M_\pm \) (see Remark 5.9). By Propositions 5.1 and 5.2,
\[
[\text{op}_\varepsilon(iA), \text{op}_\varepsilon(M)] = \text{op}_\varepsilon \left( \begin{array}{cc} \varepsilon[A_h, M_h] & 0 \\ \varepsilon A_\ell M_{th} - M_{th}A_h & \varepsilon[A_\ell, M_\ell] \end{array} \right) + \varepsilon^2 R(2),
\]

\[19\]
so that (5.23) reduces to the system of homological equations

\[(5.24) \begin{align*}
\Phi_{\pi ij}(M_i)_{i,j} &= (P_*B_*P_{i,j})_{i,j}, \quad i,j \in I_*, \quad * = h, \ell, \\
\Phi_{\pi ij}(M_{i\ell})_{i,j} &= (P_*B_{i\ell}P_{i,j})_{i,j}, \quad (i,j) \in I_{i\ell},
\end{align*}\]

where \(\Phi_{\pi ij} = i(p\omega + \lambda_i(0, \xi) - \lambda_j(0, \xi))\), the \((\lambda_i)_{1 \leq i \leq 5}\) being the renamed entries of \(A\), \(I_h = \{1, 2\}\), \(I_\ell = \{3, 4, 5\}\), \(I_{i\ell} = \{3, 4, 5\} \times \{1, 2\}\), \(p \in \{-1, 1\}\). The first equation (5.24)(i) is trivially solved since the corresponding resonance equation \(\Phi_{\pi ij} = 0\) has no solution for \(i,j \in I_h, I_\ell\). On the contrary, the set of solutions to the resonance equation \(\Phi_{\pi ij} = 0\), \((i,j) \in I_{i\ell}\), is not empty, and the resolution of (5.24)(ii) requires a compatibility condition on the right-hand side. This right-hand side of (5.24)(ii) can be made explicit, up to a symbolic remainder whose operator belongs to the class \(\varepsilon^2 R(1)\). Indeed, the eigenvectors of \(L\) can indeed be exactly computed, and \(B\), hence \(B\), is made explicit by (5.16). The compatibility condition is

\[(5.25) \quad \left| (P_*B_{i\ell} + \varepsilon(P_{i\ell1}B)P_{i,j})(u_a + u, \xi) \right| \lesssim \varepsilon^2 + |\Phi_{\pi ij}(\xi)|, \quad (i,j) \in I_{i\ell}, \]

By (5.25), (5.24)(ii) has a unique solution \(M_{i\ell} \in C^\infty \mathcal{M}^{-1}\). Then \(M\) solves (5.23). 

**Remark 5.13.** The above three Lemmas hold more generally for any source term \(B\) that satisfies (5.18) and the compatibility condition (5.25).

Let finally \(\tilde{v} = (\text{Id} + \text{op}_\varepsilon(M))^{-1}v\). The equation in \(\tilde{v}\) is

\[(5.26) \quad \varepsilon \partial_t \tilde{v} + \text{op}_\varepsilon(iA)\tilde{v} = \varepsilon^2 R(2) \tilde{v} + (\text{Id} + \text{op}_\varepsilon(M))^{-1} \tilde{f}.\]

**Corollary 5.14.** Assumption 5.4 is satisfied by (5.5)-(5.15)-(5.16), with \(r' = 0\), \(r = 2\), and \(\kappa = 2\).

**Proof.** By skew-self-adjointness of \(iA\), \(\text{op}_\varepsilon(iA) + \text{op}_\varepsilon(iA)^* = \varepsilon^4 R(1)\), so that, by (5.21), bound (5.10) holds with \(r' = 0\), \(r = 2\) and \(\kappa = 2\). 

The existence result for (5.5)-(5.15)-(5.16) is then a direct consequence of Theorem 5.5:

**Theorem 5.15.** If \(k > 4 + \frac{d}{2}\), the initial-value problem (5.5)-(5.15)-(5.16) has a solution \(u \in \mathcal{E}(s, t_*)\), with \(t_*\) as in Assumption 5.3 and \(s\) as in Assumption 5.4; this solution satisfies (5.13), where \(u_a\) is the WKB approximate solution of Lemma 5.7.
5.4 Example: singular limit of the Euler-Maxwell system

The result of the above Section carries over to the full Euler-Maxwell system, the following non-dimensional form of which was introduced in [18]:

\[
\begin{aligned}
\frac{\partial}{\partial t} B + \nabla \times E &= 0, \\
\frac{\partial}{\partial t} E - \nabla \times B &= \frac{1}{\varepsilon} \left( e^{n_e} v_e - \frac{\theta_i}{\theta_e} e^{n_i} v_i \right), \\
\frac{\partial}{\partial t} v_e + \theta_e (v_e \cdot \nabla) v_e &= -\theta_e \nabla n_e - \frac{1}{\varepsilon} (E + \theta_e v_e \times B), \\
\frac{\partial}{\partial t} n_e + \theta_e \nabla \cdot v_e &+ \theta_e (v_e \cdot \nabla) n_e = 0, \\
\frac{\partial}{\partial t} v_i + \theta_i (v_i \cdot \nabla) v_i &= -\alpha^2 \theta_i n_i + \frac{\theta_i}{\varepsilon \theta_e} (E + \theta_i v_i \times B), \\
\frac{\partial}{\partial t} n_i + \theta_i \nabla \cdot v_i &+ \theta_i v_i \cdot \nabla n_i = 0.
\end{aligned}
\]

(EM)

In (EM), the unknown is \( \tilde{u} := (B, E, v_e, n_e, v_i, n_i) \), where \((B, E) \in \mathbb{R}^{3+3}, (v_e, v_i) \in \mathbb{R}^{3+3}, \) and \((n_e, n_i) \in \mathbb{R}^{1+1} \). The parameters \( \theta_e, \theta_i \) and \( \alpha \) are fixed and have no dimension.

Considered as a system in the variable \( u \), with \( u(t, x) := \varepsilon \tilde{u}(\varepsilon t, x) \), system (EM) belongs to the class (5.5)(i). The verification of Assumption 5.3, existence of an approximate solution, is only a matter of formal WKB computations; the relevant ansatz in the \( u \) variable is (5.17). The verification of Assumption 5.4 is done in detail at the level of the nonlinear equations in [19]. The key arguments (diagonalization, blow-up and reduction) are sketched in the above Section. The key is that the characteristic variety of the Euler-Maxwell system is an union of Klein-Gordon (high-frequency) and wave (low-frequency) branches. The crucial high-frequency/low-frequency interactions appear in the right-hand side of (5.22). In (EM), these interactions come from both the convection and nonlinear Lorentz force terms.

6 Application 2: small-amplitude shock profiles for quasilinear relaxation equations

We consider finally the problem of existence of relaxation profiles

\[
U(x, t) = \bar{U}(x - st), \quad \lim_{z \to \pm \infty} \bar{U}(z) = U_\pm
\]

of a relaxation system \( U_t + A(U)U_x = Q(U) \),

\[
U = \begin{pmatrix} u \\ v \end{pmatrix}, \quad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad Q = \begin{pmatrix} 0 \\ q \end{pmatrix},
\]

in one spatial dimension, \( u \in \mathbb{R}^n, v \in \mathbb{R}^r \), where, for some smooth \( v_\ast \) and \( f \),

\[
q(u, v_\ast(u)) \equiv 0, \quad \Re \sigma(\partial_t q(u, v_\ast(u))) \leq -\theta, \; \theta > 0,
\]
\( \sigma(\cdot) \) denoting spectrum, and

\[
(A_{11} \quad A_{12}) = (\partial_u f \quad \partial_v f).
\]

Here, we are thinking particularly of the case \( n \) bounded and \( r \gg 1 \) arising through discretization or moment closure approximation of the Boltzmann equation or other kinetic models; that is, we seek estimates and proof independent of the dimension of \( v \).

For fixed \( n, r \), the existence problem was treated in [22, 10] under the additional assumption \( \det(A - sI) \neq 0 \) corresponding to nondegeneracy of the traveling-wave ODE, using standard center-manifold techniques for amplitudes \( U_+ - U_- \) sufficiently small. However, as pointed out in [10, 11], this assumption is unrealistic for large models, and in particular is not satisfied for the Boltzmann equations, for which the eigenvalues of \( A \) are constant particle speeds of all values, hence cannot be uniformly satisfied for discrete velocity or moment closure approximations. Moreover, the region of validity for such center manifold arguments may shrink to zero as the number of modes goes to infinity.

A different argument for small-amplitude stability based on Chapman–Enskog expansion and Picard iteration was presented in [12] for the semilinear case \( A \equiv \text{constant} \). This yields results independent of dimension; indeed, with slight modifications, it has been applied to the infinite-dimensional Boltzmann equation itself [13]. However, in the quasilinear case, there seems to be an unavoidable loss of derivatives in the iteration process, and so the argument of [12] does not close. This has been remedied in [14] using the Nash–Moser iteration of the present paper. We describe this application here in a simplified case that illustrates the main issues while avoiding technical details; for the general case, see [14].

### 6.1 Assumptions

Let \( f, A, Q \in C^\infty \) and \( f \) scalar, \( n = 1 \). We take \( A \) symmetric, \( Q = \begin{pmatrix} 0 & 0 \\ 0 & Q_{22} \end{pmatrix} \) block diagonal, with \( \Re Q_{22} := \frac{1}{2}(Q_{22} + Q_{22}^T) \) negative definite and \( v_\ast(u) \equiv 0 \). (In the general case, this structure may be achieved by coordinate transformations [14]...) We assume also the Kawashima genuine coupling condition, which in this case is just \( A_{12} \) nonvanishing. A consequence is that the skew matrix \( K := \begin{pmatrix} 0 & A_{12} \\ -A_{21} & 0 \end{pmatrix} \) satisfies

\[
\Re(KA - Q) \leq -\theta,
\]

for some uniform \( \theta > 0 \). Associated with (6.2) is a scalar viscous conservation law

\[
u_t + f_\ast(u)_x = (b_\ast(u)u_x)_x,
\]

obtained by Chapman–Enskog expansion (described partly below), with

\[
\begin{align*}
  f_\ast(u) &:= f(u, 0), \\
b_\ast(u) &:= -A_{12}Q_{22}^{-1}A_{21}(u, 0).
\end{align*}
\]

By our structural assumptions,

\[
\Re b_\ast \geq \theta > 0.
\]
We assume further that $f_s$ is genuinely nonlinear in the sense of Lax, that is, $d^2 f_s(u) \neq 0$.

Taking without loss of generality $s = 0$, we study the traveling-wave ODE

$$A(U)U' = Q(U).$$

6.2 Chapman–Enskog approximation

Integrating the first equation of (6.9) and noting that $f(u, v)_{\pm} = f_s(u_{\pm})$, we obtain

$$f(u, v) = f_s(u_{\pm}),$$

$$A_{21}(u, v)u' + A_{22}(u, v)v' = q(u, v).$$

Taylor expanding the first equation, we obtain $f(u, 0) + f_v(u, 0)v + O(v^2) = f_s(u_{\pm})$, or

$$f_s(u) + f_v(u, 0)v + O(v^2) = f_s(u_{\pm}).$$

Taylor expanding the second equation, we obtain $A_{21}(u, 0)u' + O(|v||u'|) + O(|v'|) = \partial_v q(u, 0)v + O(|v|^2)$, or, inverting $\partial_v q$,

$$v = \partial_v q(u, 0)^{-1} A_{21}(u, 0)u' + O(|v|) + O(|v'|) + O(|v'|).$$

Substituting (6.12) into (6.11) and rearranging, we obtain the approximate viscous profile ODE

$$b_s(u)u' = f_s(u) - f_s(u_{\pm}) + O(v^2) + O(|v||u'|) + O(|v'|).$$

Motivated by (6.12)–(6.13), we define an approximate solution $(\bar{u}_{CE}, \bar{v}_{CE})$ of (6.10) by choosing $\bar{u}_{CE}$ as a solution of

$$b_s(\bar{u}_{CE})\bar{u}_{CE}' = f_s(\bar{u}_{CE}) - f_s(u_{\pm}),$$

and $\bar{v}_{CE}$ as the first approximation given by (6.12)

$$\bar{v}_{CE} = c_s(\bar{u}_{CE})\bar{u}_{CE}'. $$

Here, (6.14) can be recognized as the traveling-wave ODE associated with approximating scalar viscous conservation law (6.6), with $s = 0$. From standard scalar ODE considerations (normal forms), we obtain the following description of solutions.

Proposition 6.1. Under the assumptions of Section 6.1, for $u_0$ such that $df_s(u_0) = 0$, in a neighborhood of $(u_0, u_0)$ in $\mathbb{R}^1 \times \mathbb{R}^1$, there is a smooth curve $S$ passing through $(u_0, u_0)$, such that for $(u_-, u_+) \in S$ with amplitude $\varepsilon := |u_+ - u_-| > 0$ sufficiently small, the zero speed shock profile equation (6.14) has a unique (up to translation) solution $\bar{u}_{CE}$ local to $u_0$.

The shock profile is necessarily of Lax type: i.e., with $df_s(u_-) > 0 > df_s(u_+)$. Moreover, there is $\theta > 0$ and for all $k$ there is $C_k$ independent of $(u_-, u_+)$ and $\varepsilon$, such that

$$|\partial_x^k(\bar{u}_{CE} - u_{\pm})| \leq C_k \varepsilon^{k+1} e^{-\theta \varepsilon |x|}, \quad x \geq 0.$$
We denote by $S_+$ the set of $(u_-, u_+) \in S$ with amplitude $\varepsilon := |u_+ - u_-| > 0$ sufficiently small that the profile $\bar{u}_{CE}$ exists. Given $(u_-, u_+) \in S_+$ with associated profile $\bar{u}_{CE}$, we define $\bar{v}_{CE}$ by (6.15) and
\begin{equation}
\bar{U}_{CE} := (\bar{u}_{CE}, \bar{v}_{CE}).
\end{equation}

It is an approximate solution of (6.10) in the following sense:

**Corollary 6.2.** For fixed $u_-$ and amplitude $\varepsilon := |u_+ - u_-|$ sufficiently small,

\begin{equation}
\mathcal{R}_u := f(\bar{u}_{CE}, \bar{v}_{CE}) - f_*(u \pm) = O(|\bar{u}_{CE}'|^2) = O(\varepsilon^4 e^{-\theta \varepsilon |x|}),
\end{equation}

\begin{equation}
\mathcal{R}_v := g(\bar{u}_{CE}, \bar{v}_{CE}) - g(\bar{u}_{CE}, \bar{v}_{CE}) = O(|\bar{u}_{CE}''|) = O(\varepsilon^3 e^{-\theta \varepsilon |x|})
\end{equation}
satisfy
\begin{equation}
|\partial_x^k \mathcal{R}_u(x)| \leq C_k \varepsilon^{k+4} e^{-\theta \varepsilon |x|},
\end{equation}
\begin{equation}
|\partial_x^k \mathcal{R}_v(x)| \leq C_k \varepsilon^{k+3} e^{-\theta \varepsilon |x|}, \quad x \geq 0,
\end{equation}
where $C_k$ is independent of $(u_-, u_+)$ and $\varepsilon = |u_+ - u_-|$.

**Proof.** For $k = 0$, bounds (6.19) follow by expansions (6.11) and (6.12), definitions (6.14) and (6.15), and bounds (6.16). Bounds for $k > 0$ follow similarly. \hfill \Box

**Remark 6.3.** One may continue this process to obtain Chapman–Enskog approximations $(\bar{u}_{CE}^N, \bar{v}_{CE}^N)$ to all orders, with truncation errors $(\partial_x^k \mathcal{R}_{u}^N, \partial_x^k \mathcal{R}_{v}^N) \sim (\varepsilon^{N+k+4}, \varepsilon^{N+k+3})$ [14].

### 6.3 Statement of the main theorem

We are now ready to state the main result. Define a base state $U_0 = (u_0, 0)$ and a neighborhood $\mathcal{U} = \mathcal{U}_* \times \mathcal{V}$, with $df_*(u_0) = 0$.

**Theorem 6.4.** Under the assumptions of Section 6.1, there are $\varepsilon_0 > 0$ and $\delta > 0$ such that for $(u_-, u_+) \in S_+$ with amplitude $\varepsilon := |u_+ - u_-| \leq \varepsilon_0$, the standing-wave equation (6.9) has a solution $U$ in $\mathcal{U}$, with associated Lax-type equilibrium shock $(u_-, u_+)$, satisfying for all $k$:
\begin{equation}
|\partial_x^k (\bar{U} - \bar{U}_{CE})| \leq C_k \varepsilon^{k+2} e^{-\delta \varepsilon |x|},
\end{equation}
\begin{equation}
|\partial_x^k (\bar{u} - u_\pm)| \leq C_k \varepsilon^{k+1} e^{-\delta \varepsilon |x|}, \quad x \geq 0,
\end{equation}
\begin{equation}
|\partial_x^k (\bar{v} - v_\pm(\bar{u}))| \leq C_k \varepsilon^{k+2} e^{-\delta \varepsilon |x|},
\end{equation}
where $\bar{U}_{CE} = (\bar{u}_{CE}, \bar{v}_{CE})$ is the approximating Chapman–Enskog profile defined in (6.14), and $C_k$ is independent of $\varepsilon$. Moreover, up to translation, this solution is unique within a ball of radius $c \varepsilon$ about $\bar{U}_{CE}$ in norm $\varepsilon^{-1/2} \| \cdot \|_{L^2} + \varepsilon^{-3/2} \| \partial_x \cdot \|_{L^2} + \cdots + \varepsilon^{-11/2} \| \partial_x^2 \cdot \|_{L^2}$, for $c > 0$ sufficiently small and $K$ sufficiently large. (For comparison, $\bar{U}_{CE} - U_\pm$ is order $\varepsilon$ in this norm, by (6.20)(ii)–(iii).)

That is, behavior of profiles is well-described by Chapman–Enskog approximation.
6.4 Functional equation and spaces

Defining the perturbation variable $U := \bar{U} - \bar{U}_{CE}$, where $\bar{U}_{CE}$ is as in (6.17), we obtain from (6.10) the nonlinear perturbation equations $\Phi^\varepsilon(U) = 0$, where

$$\Phi^\varepsilon(U) := \left( f_1(\bar{U}_{CE}^\varepsilon + U) - f_*(U) \right) + \left( A_{21}(\bar{U}_{CE}^\varepsilon + U)(U_{CE}^\varepsilon + U)' - q(\bar{U}_{CE}^\varepsilon + U) \right).$$

Formally linearizing $\Phi^\varepsilon$ about an approximate solution $\tilde{U}$, we obtain

$$\Phi^\varepsilon(U) = \left( A_{11}u + A_{12}v \right) + \left( A_{21}u' + A_{22}v' - Q_{22}v - bU \right),$$

where

$$A = df(\bar{U}_{CE}^\varepsilon + \bar{U}), \quad Q_{22} = \partial_v q(\bar{U}_{CE}^\varepsilon + \bar{U}),$$

and

$$bU = (d(A_{21}, A_{22})(\bar{U}_{CE}^\varepsilon + \bar{U})U)(\bar{U}_{CE}^\varepsilon + \bar{U})'.$$

The associated linearized equation for a given forcing term $F$ is

$$\Phi^\varepsilon(U) = F = \begin{pmatrix} f \\ g \end{pmatrix}.$$

We have also

$$\Phi^\varepsilon(U) = \begin{pmatrix} N_1(U, \bar{U}) \\ N_2(U, \bar{U})' + N_3(U, \bar{U}) \end{pmatrix},$$

where $N_j(U)$ are quadratic forms depending smoothly on $\bar{U}$.

We introduce weighted spaces and norms, which encounter for the exponential decay of the source and solution: introduce the notations.

$$\|f\|_{H^s_\varepsilon} = \varepsilon^{\frac{1}{2}}\|f\|_{L^2} + \varepsilon^{-\frac{1}{2}}\|\partial_x f\|_{L^2} + \cdots + \varepsilon^{\frac{1}{2} - s}\|\partial_x^s f\|_{L^2}.$$

We introduce weighted spaces and norms, which encounter for the exponential decay of the source and solution: introduce the notations.

$$\langle x \rangle := (x^2 + 1)^{1/2}$$

For $\delta \geq 0$ (sufficiently small), we denote by $H^s_{\varepsilon, \delta}$ the space of functions $f$ such that $e^{\delta \varepsilon x} f \in H^s$ equipped with the norm

$$\|f\|_{H^s_{\varepsilon, \delta}} = \varepsilon^{\frac{1}{2}}\sum_{k \leq s} \varepsilon^{-k}\|e^{\delta \varepsilon x} \partial_x^k f\|_{L^2}.$$
For fixed $\delta$, introduce spaces $E_s := H_{\varepsilon,\delta}^s$ with norm $\| \cdot \|_s = \| \cdot \|_{H_{\varepsilon,\delta}^s}$ and $F_s := \left( \frac{H_{\varepsilon,\delta}^{s+1}}{H_{\varepsilon,\delta}^s} \right)$

with norm $\| \left( \begin{array}{c} f \\ g \end{array} \right) \|_s = \| f \|_{H_{\varepsilon,\delta}^{s+1}} + \| g \|_{H_{\varepsilon,\delta}^s}$.

### 6.5 Fréchet bounds

**Lemma 6.5.** $|\Phi(0)|_{H_{\varepsilon,\delta}^s} \leq C\varepsilon^{N+2}$ for all $0 \leq s \leq \bar{s}$, some $C > 0$.

**Proof.** Immediate from (6.19) and (6.27). \[\square\]

**Lemma 6.6.** $\Phi^s$ is Fréchet differentiable from $H_{\varepsilon,\delta}^{s+1}$ into $H_{\varepsilon,\delta}^s$, for all $s \geq 0$, $\varepsilon > 0$, $\delta \geq 0$, and, for $s_0 \geq 1$, all $s$ such that $s_0 + 1 \leq s + 1 \leq \bar{s}$, and all $U, V, W \in H_{\varepsilon,\delta}^{s+1}$,

\[
|\Phi^s(U)_s| \leq C_0(1 + |U|_{H_{\varepsilon,\delta}^{s+1}} + |U|_{H_{\varepsilon,\delta}^{s_0+1}}|U|_{H_{\varepsilon,\delta}^s}), \\
|\Phi^s(U) \cdot V_s| \leq C_0(|V|_{H_{\varepsilon,\delta}^{s+1}} + |V|_{H_{\varepsilon,\delta}^{s_0+1}}|U|_{H_{\varepsilon,\delta}^{s+1}}),
\]

and

\[
|\Phi^s(U) \cdot (V, W)_s| \leq C_0(|V|_{H_{\varepsilon,\delta}^{s_0+1}} |W|_{H_{\varepsilon,\delta}^{s+1}} + |V|_{H_{\varepsilon,\delta}^{s+1}} |W|_{H_{\varepsilon,\delta}^{s_0+1}} + |U|_{H_{\varepsilon,\delta}^{s+1}} |V|_{H_{\varepsilon,\delta}^{s_0+1}} |W|_{H_{\varepsilon,\delta}^{s_0+1}}),
\]

where $C$ is uniformly bounded for $|U|_{H_{\varepsilon,\delta}^{s_0+1}} \leq C$, for any fixed value of $\delta$.

**Proof.** Standard, using Moser’s inequality, definition (6.22), the fact that $| \cdot |_{H_{\varepsilon,\delta}^s}$ is a fixed weighted norm in coordinates $\tilde{x} = \varepsilon x$, and working in $\tilde{x}$ coordinates, with $\partial_x = \varepsilon \partial_{\tilde{x}}$. \[\square\]

### 6.6 Linearized estimates

The key step in the argument is to obtain the following linearized stability estimates.

**Proposition 6.7.** Under the assumptions of Section 6.1, there are $\varepsilon_0 > 0$ and $\delta > 0$ such that for all $\varepsilon \in [0, \varepsilon_0]$, $\delta \in [0, \delta_0]$, equation (6.25) has a solution operator $\Psi^s(\tilde{U})$ (i.e., there exists a formal right inverse for $(\Phi^s)'(\tilde{U})$), such that, for all $s$ such that $s_0 + 2 \leq s + 1 \leq \bar{s}$, $s_0 = 3$, $F = \left( \begin{array}{c} f \\ g \end{array} \right) \in F_s$, and $U \in H_{\varepsilon,\delta}^{s_0+r}$ such that

\[
|\tilde{U}|_{H_{\varepsilon,\delta}^{s_0+2}} \leq C\varepsilon,
\]

there holds the estimate

\[
\|\Psi^s(\tilde{U})F\|_{H_{\varepsilon,\delta}^s} \leq C\varepsilon^{-1}(\|\tilde{U}\|_{H_{\varepsilon,\delta}^{s+1}}|F|_{s_0+2} + |F|_{s+1})
\]

\[
= C\varepsilon^{-1}(\|\tilde{U}\|_{H_{\varepsilon,\delta}^{s+1}}(|f|_{H_{\varepsilon,\delta}^{s_0+3}} + |g|_{H_{\varepsilon,\delta}^{s_0+2}}) + (|F|_{H_{\varepsilon,\delta}^{s+2}} + |g|_{H_{\varepsilon,\delta}^{s+1}})),
\]

where $C = C(|\tilde{U}|_{H_{\varepsilon,\delta}^{s_0+2}})$ is a non-decreasing function of $|\tilde{U}|_{H_{\varepsilon,\delta}^{s_0+2}}$.  

26
We here carry out the main step in the proof of obtaining corresponding A Priori estimates; see 6.13 below. The remaining step of demonstrating existence for the linearized problem can be carried out by the vanishing viscosity method as in [13], with viscosity coefficient $\eta > 0$, obtaining existence for each positive $\eta$ by standard boundary-value theory, and noting that the A Priori bounds (6.59) of Proposition 6.13 persist under regularization for sufficiently small viscosity $\eta > 0$, so that we can obtain a weak solution in the limit by extracting a weakly convergent subsequence. We omit this step, referring the reader to Section 8, [12], for details. The asserted estimates then follow in the limit by continuity.

The rest of this subsection is devoted to establishing the asserted A Priori estimates.

### 6.6.1 Internal and high frequency estimates

#### The basic $H^1$ estimate. We consider the equation

\[
\mathcal{L}_\varepsilon U := \begin{pmatrix} A_{11} u + A_{12} v \\ A_{21} u' + A_{22} v' + b U - Q_{22} v \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}
\]

and its differentiated form:

\[
(AU)' - QU = \begin{pmatrix} f' \\ g \end{pmatrix},
\]

where $A$, $Q$, $b$ are smooth functions of $\bar{U}_{CE} + \tilde{U}$, with $\|\tilde{U}\|_4$, $\|\tilde{U}_{CE}\|_{s+1}$ both order $\varepsilon$ (the first by assumption, the second by estimates (6.16)). We shall freely use below the resulting coefficient bounds

\[
|\partial_x^{k+1} A|, |\partial_x^{k+1} Q|, |\partial_x^{k+1} K|, \leq C\varepsilon^{2+k}, \quad |\partial_x^k b| \leq C\varepsilon^{2+k}
\]

for $0 \leq k \leq 3$ and

\[
|\partial_x^{j+1} A|_{L^2}, |\partial_x^{j+1} Q|_{L^2}, |\partial_x^{j+1} K|_{L^2} \leq C\varepsilon^{j+1/2}(\varepsilon + \|\tilde{U}\|_{s+1}), \quad |\partial_x^j b| \leq C\varepsilon^{j+1/2}(\varepsilon + \|\tilde{U}\|_{s+1})
\]

for $0 \leq j \leq s$, where $K$ is the Kawashima multiplier (a smooth function of $A$). The internal variables are $U' = (u', v')$ and $v$.

**Proposition 6.8.** Under the assumptions of Section 6.1, there are constants $C$, $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that for $0 < \varepsilon \leq \varepsilon_0$ and $0 \leq \delta \leq \delta_0$, $f \in H^2_{\varepsilon,\delta}$, $g \in H^1_{\varepsilon,\delta}$ and $U = (u, v) \in H^1_{\varepsilon,\delta}$ satisfying (6.35), one has

\[
\|U''\|_{L^2_{\varepsilon,\delta}} + \|v\|_{L^2_{\varepsilon,\delta}} \leq C(\|(f, f', f'', g, g')\|_{L^2_{\varepsilon,\delta}} + \varepsilon\|u\|_{L^2_{\varepsilon,\delta}}).
\]

We have an ODE

\[
AU' - QU + CU = F,
\]
where $A$ is symmetric, $Q = \begin{pmatrix} 0 & 0 \\ 0 & Q_{22} \end{pmatrix}$, with $\Re Q_{22}$ negative definite, and

$$
C = O(\tilde{u}_{CE}') \hat{C} = O(\varepsilon^2) \hat{C}.
$$

We first prove the estimate (6.39) for $\delta = 0$. Dropping hats and tildes, the ODE reads

$$
AU' - QU + \varepsilon^2 CU = F,
$$

(6.42)

$A$ symmetric and $\Re Q_{22}$ negative definite, and there is a smooth $K = \tilde{K}(\tilde{u}_{CE}) = -\tilde{K}^*$ such that $\Re(KA - SQ)$ is definite positive. Therefore, there is $c > 0$ such that for all $\varepsilon \leq \varepsilon_0$ and $x \in \mathbb{R}$:

$$
\tilde{q} \leq -c \Id, \quad \Re(KA - SQ) \geq c \Id.
$$

**Lemma 6.9.** There is a constant $C$ such that for $\varepsilon$ sufficiently small, $f \in H^2$, $\tilde{U} \in H^2$, $g \in H^1$, and $U \in H^1$ satisfying (6.42), with $\|\tilde{U}\|_2 \leq C\varepsilon$, one has

$$
\|U'\|_{L^2} + \|v\|_{L^2} \leq C(\|f\|_{H^2} + \|g\|_{H^1} + \varepsilon\|u\|_{L^2}).
$$

**Proof.** Introduce the symmetrizer

(6.45)

$$
S = \partial_x^2 + \partial_x \circ K - \lambda.
$$

One has

$$
\Re \partial_x^2 \circ (A\partial_x - Q) = \frac{1}{2} \partial_x \circ A' \circ \partial_x - \partial_x \circ Q \circ \partial_x - \Re \partial_x \circ Q'
$$

$$
\Re \partial_x \circ K(A\partial_x - Q) = \partial_x \circ \Re KA \circ \partial_x - \Re \partial_x \circ KQ
$$

$$
\Re(A\partial_x - Q) = \frac{1}{2} A' - Q.
$$

Thus

$$
\Re S \circ (A\partial_x - Q) = \partial_x \circ (\Re KA - Q) \circ \partial_x + \lambda Q
$$

$$
+ \frac{1}{2} \partial_x \circ A' \circ \partial_x - \frac{1}{2} \lambda A' - \Re \partial_x \circ Q' - \Re \partial_x \circ KQ.
$$

Therefore, for $U \in H^2(\mathbb{R})$, (6.43) implies that

$$
\Re(SF, U)_{L^2} \geq c\|\partial_x U\|_{L^2}^2 + \lambda c\|v\|_{L^2}^2
$$

$$
- \frac{1}{2}\|(A')\|_{L^\infty}(\|\partial_x U\|_{L^2}^2 + \lambda\|U\|_{L^2}^2)
$$

$$
- \|(Q')\|_{L^\infty} \|U\|_{L^2} \|\partial_x U\|_{L^2} - \|K\|_{L^\infty} \|\partial_x U\|_{L^2} \|v\|_{L^2}
$$

$$
- \varepsilon^2(\|C\|_{L^\infty}\|U\|_{H^1}^2 + \|C'\|_{L^2}\|U\|_{L^\infty}^2).
$$

Taking

$$
\lambda = \frac{2}{c} \|K\|_{L^\infty}^2 \|q\|_{L^\infty},
$$

28
and using that

\[ (6.46) \| (A')'\|_{L^\infty} + \| (Q')'\|_{L^\infty} = O(\varepsilon^2), \quad \| (C')'\|_{L^2} \sim \| (A'')'\|_{L^2} \sim \varepsilon^{3/2}(\varepsilon + \| \tilde{U} \|_2) = O(\varepsilon^{5/2}) \]

and \( |U|_{L^\infty} \leq \| U \|_1 = \varepsilon^{-1/2}\| U \|_{L^2} + \varepsilon^{1/2}\| U \|_{H^1} \), yields

\[ \| U'' \|_{L^2}^2 + \| v \|_{L^2}^2 \lesssim \Re(SF, U)_{L^2} + \varepsilon^2 (\| U \|_{L^2}^2 + \| U' \|_{L^2}^2). \]

In the opposite direction,

\[ \Re(SF, U)_{L^2} \leq \| \partial_x U \|_{L^2} (\| \partial_x (F) \|_{L^2} + \| K \|_{L^\infty} \| F \|_{L^2}) + \lambda (\| (u')' \|_{L^2} \| f \|_{L^2} + \| v \|_{L^2} \| g \|_{L^2}). \]

Using again that the derivatives of the coefficients are \( O(\varepsilon^2) \), this implies that

\[ \Re(SF, U)_{L^2} \lesssim (\| f \|_{H^2} + \| g \|_{H^1}) \| U' \|_{L^2} + \varepsilon^2 (\| f \|_{L^2} \| u \|_{L^2} + \| g \|_{L^2} \| v \|_{L^2}), \]

The estimate (6.44) follows provided that \( \varepsilon \) is small enough.

This proves the lemma under the additional assumption that \( U \in H^2 \). When \( U \in H^1 \), the estimates follow using Friedrichs mollifiers.

**Proof of Proposition 6.8.** This follows similarly as in the proof of Lemma 6.9, making the change of variables \( U \to \varepsilon^{\delta}x \| U \) and absorbing commutators. See the proof of Proposition 6.1, [13].

**6.6.2 Higher order estimates**

**Proposition 6.10.** There are constants \( C, \varepsilon_0 > 0, \delta_0 > 0 \) and for all \( k \geq 2 \), there is \( C_k \), such that \( 0 < \varepsilon \leq \varepsilon_0, \delta \leq \delta_0, U \in H^s_{\varepsilon, \delta}, \tilde{U} \in H^{s+1}_{\varepsilon, \delta}, f \in H^{s+1}_{\varepsilon, \delta}, g \in H^s_{\varepsilon, \delta} \) satisfying

\[ (6.42) \] \[ \| U \|_{H^s_{\varepsilon, \delta}} \leq C \varepsilon, \]

holds

\[ (6.47) \]

\[ \| \partial_x^k U'' \|_{L^2} + \| \partial_x^k v \|_{L^2} \leq C \| \partial_x^k (f, f', f'', g, g') \|_{L^2} \]

\[ + \varepsilon^k C_k (\| U' \|_{H^{k-1}_{\varepsilon, \delta}} + \varepsilon \| v \|_{H^{k-1}_{\varepsilon, \delta}} + \varepsilon \| u \|_{L^2}) \]

\[ + C_k \varepsilon^{k+1} \| \tilde{U} \|_{H^{k+2}_{\varepsilon, \delta}} (\| v \|_{H^1_{\varepsilon, \delta}} + \varepsilon \| U \|_{H^2_{\varepsilon, \delta}}). \]

**Proof.** Differentiating (6.35) \( k \) times, yields

\[ (6.48) \]

\[ A \partial_x^k U - Q \partial_x^k U = \begin{pmatrix} \partial_x^k f' \\ \partial_x^k g + r_k \end{pmatrix}, \]

where

\[ r_k = -\partial_x^{k-1}((\partial_x Q_{22})v) - \partial_x^{k-1}((\partial_x A) \partial_x U) - \partial_x^{k-1}((\partial_x C) U). \]
The $H^1$ estimate yields
\[
\|\partial_x^k U'\|_{L^2_{\varepsilon,\delta}} + \|\partial_x^k v\|_{L^2_{\varepsilon,\delta}} \leq C(\|\partial_x^k (f, f', f'', g, g')\|_{L^2_{\varepsilon,\delta}} + \varepsilon\|\partial_x^k u\|_{L^2_{\varepsilon,\delta}} + \|\partial_x r_k\|_{L^2_{\varepsilon,\delta}} + \|r_k\|_{L^2_{\varepsilon,\delta}}),
\]
for $0 \leq k \leq s$, with $r_0 = 0$ when $k = 0$.

Using Moser’s inequality together with (6.37) and (6.38), we may estimate
\[
\|r_k\|_{L^2_{\varepsilon,\delta}} \leq C_k(|\partial_x Q|_{L^\infty} \|\partial_x^{k-1} v\|_{L^2} + |\partial_x^k Q|_{L^2}\|v\|_{L^\infty}) \\
+ |\partial_x A|_{L^\infty} \|\partial_x^k U\|_{L^2} + |\partial_x^k A|_{L^2}\|\partial_x U\|_{L^\infty}) \\
+ |\partial_x C|_{L^\infty} \|\partial_x^{k-1} U\|_{L^2} + |\partial_x^k C|_{L^2}\|U\|_{L^\infty}) \\
\leq C_k(\varepsilon^{k+1}\|v\|_{H^k_{\varepsilon,\delta}} + \varepsilon^{k+2}\|U\|_{H^k_{\varepsilon,\delta}} + \varepsilon^2\|\partial_x^k U\|_{L^2_{\varepsilon,\delta}}) \\
+ C_k(\varepsilon^{k+1}\|\tilde{U}\|_{H^k_{\varepsilon,\delta}}\|v\|_{H^1_{\varepsilon,\delta}} + \varepsilon^{k+2}\|\tilde{U}\|_{H^k_{\varepsilon,\delta}}\|U\|_{H^2_{\varepsilon,\delta}} + \varepsilon^{k+2}\|\tilde{U}\|_{H^{k+1}_{\varepsilon,\delta}}\|U\|_{H^1_{\varepsilon,\delta}}),
\]
obtaining the result by absorbing (smaller) highest-order terms from $\|\partial_x r_k\|_{L^2_{\varepsilon,\delta}}$ on the left-hand side.

\[\square\]

### 6.6.3 Linearized Chapman–Enskog estimate

**The approximate equations.** It remains only to estimate $\|u\|_{L^2_{\varepsilon,\delta}}$ in order to close the estimates and establish (6.39). To this end, we work with the first equation in (6.35) and estimate it by comparison with the Chapman-Enskog approximation (see the computations Section 6.2).

From the second equation $A_{21} u' + A_{22} v' - g = dq_v$, we find
\[
(6.49) \quad v = \partial_v q^{-1}(A_{21} + A_{22} \partial_v d\nu_*(\bar{u}_{CE}))u' + A_{22} v' - g.
\]

Introducing $v$ in the first equation, yields $(A_{11} + A_{12} d\nu_*(\bar{u}_{CE}))u + A_{12} v = f$, thus
\[
(A_{11} + A_{12} d\nu_*(\bar{u}_{CE}))u' = f' - A_{12} v' - d^2\nu_*(\bar{u}_{CE})(\bar{u}_{CE}^' , u).
\]

Therefore, (6.49) can be modified to $v = c_* (\bar{u}_{CE}) u' + r$ with
\[
\begin{align*}
    r &= d_v^{-1} q(\bar{u}_{CE}, v_*(\bar{u}_{CE})) \left( A_{22}(v' - g + d\nu_*(\bar{u}_{CE})(f' - A_{12} v' - d^2\nu_*(\bar{u}_{CE})(\bar{u}_{CE}^' , u)) \right).
\end{align*}
\]

This implies that $u$ satisfies the linearized profile equation
\[
(6.50) \quad \tilde{b}_s u' - \tilde{d}_s u = A_{12} r - f
\]
where $\tilde{b}_s = b_*(\bar{u}_{CE})$ and $\tilde{d}_s := d\nu_*(\bar{u}_{CE}) = A_{11} + A_{12} d\nu_*(\bar{u}_{CE})$.
$L^2$ estimates and proof of the main estimates.

**Proposition 6.11.** For $\|\hat{U}\|_4 \leq C\varepsilon$, the operator $(b_s\partial_x - \hat{d}f_s)(\hat{U})$ has a right inverse $(b_s\partial_x - df^*)^\dagger$ satisfying

$$
(6.51) \quad \|(b_s\partial_x - \hat{d}f_s)^\dagger h\|_{L^2_{\varepsilon,\delta}} \leq C\varepsilon^{-1}\|h\|_{L^2_{\varepsilon,\delta}},
$$

uniquely specified by the property that the solution $u = (b_s\partial_x - df^*)^\dagger h$ satisfies $u(0) = 0$.

**Proof.** Working in $\tilde{x} = \varepsilon x$ coordinates, and noting that $\varepsilon^{-1}|df_s(\hat{U}) - df_s(u_\pm)| \sim e^{-\theta|\tilde{x}|}$, by (6.16), we obtain using $\partial_x = \varepsilon \partial_{\tilde{x}}$ the equation

$$
(6.52) \quad (b_s\partial_{\tilde{x}} - \varepsilon^{-1}df^*)u = \varepsilon^{-1}h, \quad u(0) = 0.
$$

This is a rather standard boundary-value ODE problem with exponentially convergent coefficients at spatial infinity. Using the extra condition $u(0) = 0$, we may break it into a pair of boundary values problems on $(-\infty,0]$ and $[0,\infty)$, each of which, by the Lax condition $df_s(u_-) > 0 > df_s(u_+)$, implying that there is a one-dimensional manifold of decaying solutions as $\tilde{x} \to -\infty$ or as $\tilde{x} \to +\infty$, is well-posed, from $H^s_{\varepsilon,\delta}$ to itself, so long as $\delta$ is strictly smaller that $\varepsilon^{-1}$ min $|df_s(u_\pm)|$. Taking account of the $\varepsilon^{-1}$ factor in the righthand side of (6.52), we obtain the result. \qed

**Proposition 6.12.** There are constants $C$, $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that for $\varepsilon \in [0,\varepsilon_0]$, $\delta \in [0,\delta_0]$, $f \in H^3_{\varepsilon,\delta}$, $g \in H^2_{\varepsilon,\delta}$ and $U \in H^2_{\varepsilon,\delta}$ satisfying (6.25) and $u(0) = 0$,

$$
(6.53) \quad \|U\|_{H^2_{\varepsilon,\delta}} \leq C\varepsilon^{-1}(\|f\|_{H^3_{\varepsilon,\delta}} + \|g\|_{H^2_{\varepsilon,\delta}}).
$$

**Proof.** Going back now to (6.50), $u$ satisfies

$$
\bar{b}_s u' - \bar{d}f_s u = O(|v'| + |g| + |f'| + \varepsilon^2|u|) - f,
$$

If in addition $u$ satisfies the condition $u(0) = 0$, then

$$
(6.54) \quad \|u\|_{L^2_{\varepsilon,\delta}} \leq C\varepsilon^{-1}(\|v'\|_{L^2_{\varepsilon,\delta}} + \|(f, f', g)\|_{L^2_{\varepsilon,\delta}} + \varepsilon^2\|u\|_{L^2_{\varepsilon,\delta}}).
$$

By Proposition 6.8 and Proposition 6.10 for $k = 1$, we have

$$
(6.55) \quad \|U''\|_{L^2_{\varepsilon,\delta}} + \|v\|_{L^2_{\varepsilon,\delta}} \leq C\varepsilon\|(f, f', f'', g, g')\|_{L^2_{\varepsilon,\delta}} + \varepsilon\|u\|_{L^2_{\varepsilon,\delta}}.
$$

$$
(6.56) \quad \|U''\|_{L^2_{\varepsilon,\delta}} + \|v'\|_{L^2_{\varepsilon,\delta}} \leq C\varepsilon\|(f', f'', f''', g', g'')\|_{L^2_{\varepsilon,\delta}} + \varepsilon\|U''\|_{L^2_{\varepsilon,\delta}} + \varepsilon^2\|u\|_{L^2_{\varepsilon,\delta}}.
$$

Combining these estimates, this implies

$$
\|v'\|_{L^2_{\varepsilon,\delta}} \leq C\varepsilon\|(f', f'', f''', g', g'')\|_{L^2_{\varepsilon,\delta}} + \varepsilon\|(f, f', f'', g, g')\|_{L^2_{\varepsilon,\delta}} + \varepsilon^2\|u\|_{L^2_{\varepsilon,\delta}}
$$

$$
\leq C\varepsilon\|(f, f', f'', g, g')\|_{H^1_{\varepsilon,\delta}} + \varepsilon^2\|u\|_{L^2_{\varepsilon,\delta}}.
$$
Substituting in (6.54), yields \( \varepsilon \|u\|_{L^2_{\varepsilon, \delta}} \leq C(\|(f, f', g)\|_{L^2_{\varepsilon, \delta}} + \varepsilon \|(f, f', f'', g, g')\|_{H^1_{\varepsilon, \delta}} + \varepsilon^2 \|u\|_{L^2_{\varepsilon, \delta}}) \).

Hence for \( \varepsilon \) small, \( \varepsilon \|u\|_{L^2_{\varepsilon, \delta}} \leq C(\|(f, f', g)\|_{L^2_{\varepsilon, \delta}} + \varepsilon \|(f, f', f'', g, g')\|_{H^1_{\varepsilon, \delta}}) \).

Plugging this estimate in (6.55) yields

\[
(6.57) \quad \|U'\|_{L^2_{\varepsilon, \delta}} + \|v\|_{L^2_{\varepsilon, \delta}} + \varepsilon \|u\|_{L^2_{\varepsilon, \delta}} \leq C(\|(f, f', f'', g, g')\|_{L^2_{\varepsilon, \delta}} + \varepsilon \|(f, f', f'', g, g')\|_{H^1_{\varepsilon, \delta}}).
\]

Hence, with (6.56), one has

\[
(6.58) \quad \|U''\|_{H^s_{\varepsilon, \delta}} + \|v\|_{L^2_{\varepsilon, \delta}} + \varepsilon \|u\|_{L^2_{\varepsilon, \delta}} \leq C(\|(f', f'', f''', g', g'')\|_{L^2_{\varepsilon, \delta}} + \varepsilon \|(f', f'', f''', g', g'')\|_{H^1_{\varepsilon, \delta}}).
\]

Therefore, \( \|U''\|_{H^1_{\varepsilon, \delta}} + \|v\|_{L^2_{\varepsilon, \delta}} + \varepsilon \|u\|_{L^2_{\varepsilon, \delta}} \leq C\|f, f', f'', g, g'\|_{H^s_{\varepsilon, \delta}} \) The left hand side dominates \( \|U''\|_{H^1_{\varepsilon, \delta}} + \varepsilon \|U'\|_{L^2_{\varepsilon, \delta}} = \varepsilon \|U'\|_{H^2_{\varepsilon, \delta}} \) and the right hand side is smaller than or equal to \( \|f\|_{H^2_{\varepsilon, \delta}} + \|g\|_{H^1_{\varepsilon, \delta}} \). The estimate (6.53) follows.

Knowing a bound for \( \|u\|_{L^2_{\varepsilon, \delta}} \), Proposition 6.10 implies by induction the following final result.

**Proposition 6.13.** There are constants \( C, \varepsilon_0 > 0 \) and \( \delta_0 > 0 \) and for \( s \geq 3 \) there is a constant \( C_s \) such that for \( \varepsilon \in [0, \varepsilon_0] \), \( \delta \in [0, \delta_0] \), \( f \in H^{s+1}_{\varepsilon, \delta} \), \( g \in H^s_{\varepsilon, \delta} \), \( \bar{U} \in H^{s+1}_{\varepsilon, \delta} \), and \( U \in H^s_{\varepsilon, \delta} \) satisfying (6.25), (2.18), and (6.33), one has

\[
(6.59) \quad \|U\|_{H^s_{\varepsilon, \delta}} \leq C\varepsilon^{-1}(\|\bar{U}\|_{H^{s+1}_{\varepsilon, \delta}} |F|_{s_0+2} + \|F\|_{s+1})
\]

\[
= C\varepsilon^{-1}(\|\bar{U}\|_{H^{s+1}_{\varepsilon, \delta}} (\|F\|_{H^{s_0+3}_{\varepsilon, \delta}} + \|g\|_{H^{s_0+2}_{\varepsilon, \delta}}) + (\|F\|_{H^{s+2}_{\varepsilon, \delta}} + \|g\|_{H^{s+1}_{\varepsilon, \delta}})).
\]

Proposition 6.13 can be used to establish Proposition 6.7 by a vanishing viscosity argument; see [12].

**6.7 Proof of the main theorem**

**Proof of Theorem 6.4 (Existence).** The profile \( \bar{U}_{CE} \) exists if \( \varepsilon \) is small enough. Comparing, we find that Lemma 6.6, Proposition 6.7, and Lemma 6.5 verify, respectively, Assumptions 2.1, 2.2, and 2.3 of our Nash–Moser iteration scheme, with \( s_0 = 3, \gamma_0 = 0, \gamma = 1, k = 3, m = r = 1, r' = 0, \) and arbitrary \( \bar{s} \). Taking \( \bar{s} \) sufficiently large, and applying the Nash Moser Theorem 2.4, we thus obtain existence of a solution \( \bar{U}^\varepsilon \) of (6.21) with \( \|\bar{U}^\varepsilon\|_{H^{s+1}_{\varepsilon, \delta}} \leq C\varepsilon^2 \).

Defining \( \bar{U}^\varepsilon := \bar{U}^\varepsilon_{CE} + U^\varepsilon \), and noting by Sobolev embedding that \( |h|_{H^{s+1}_{\varepsilon, \delta}} \) controls \( \|e^{\delta\xi|x|}h\|_{L^\infty} \), we obtain the result.

**Proof of Theorem 6.4 (Uniqueness).** Applying Theorem 2.5 for \( s_0 = 3, \gamma_0 = 0, \gamma = 1, k = 3, m = r = 1, r' = 0, \) we obtain uniqueness in a ball of radius \( c\varepsilon \) in \( H^1_{\varepsilon, 0} \), \( c > 0 \) sufficiently small, under the additional phase condition (2.18). We obtain unconditional
uniqueness from this weaker version by the observation that phase condition (2.18) may be 
achieved for any solution \( \bar{U} = \bar{U}_{CE} + U \) with 
\[
\|U'\|_{L^\infty} \leq c\varepsilon^2 \ll \bar{U}_{CE}'(0) \sim \varepsilon^2
\]
by translation in \( x \), yielding \( U_a(x) := \bar{U}(x + a) = \bar{U}_{CE}(x) + U_a(x) \) with 
\[
U_a(x) := \bar{U}_{CE}(x + a) - \bar{U}_{CE}(x) + U(x + a)
\]
so that, defining \( \phi := \bar{U}'/|\bar{U}'| \), we have \( \partial_a \langle \phi, U_a \rangle \sim \langle \phi, U_a \rangle \) \( \sim (1 + o(1)) \bar{U}' + U' \rangle \) = \( (1 + o(1))|\bar{U}'| \sim \varepsilon^2 \) and so (by the Implicit Function Theorem applied to \( h(a) := \varepsilon^{-2} \langle \phi, U_a \rangle \), together with the fact that \( \langle \phi, U_0 \rangle = o(\varepsilon) \) and that \( \langle \phi, U_{NS}' \rangle \sim |U_{NS}'| \sim \varepsilon^2 \) the inner product \( \langle \phi, U_a \rangle \), hence also \( IIU_a \) may be set to zero by appropriate choice of \( a = o(\varepsilon^{-1}) \) leaving \( U_a \) in the same \( o(\varepsilon) \) neighborhood, by the computation \( U_a - U_0 \sim \partial_a U \cdot a \sim o(\varepsilon^{-1}) \varepsilon^2 \).

6.8 Why Nash–Moser?

We conclude by discussing why we seem to need Nash–Moser to close the argument. Recall 
the standard proof of existence for quasilinear symmetric hyperbolic systems \( u_t + A(u)u_x = S \) 
using energy estimates. One writes an iteration scheme 
\[
u^{n+1}_t + A(u^n)u_x^{n+1} = S,
\]
which gives \( H^s \) bounds \( |u^{n+1}|_{H^s} \leq C|g|_{H^s} \) so long as \( |u^n|_{H^s} \) is small, and contraction in lower norms on small time intervals, giving the result.

But, it is easily checked that this does not work for equations in conservative form 
\( u_t + (A(u)u)_x = S \), for which 
\[
u^{n+1}_t + (A(u^n)u^{n+1}_x) = S,
\]
gives \( H^s \) bounds \( |u^{n+1}|_{H^s} \leq C|S|_{H^s} \) rather for \( |u^n|_{H^{s+1}} \) small, hence involves loss of derivatives.

Usually, for a conservative equation \( u_t + f(u)_x = S \), this is no problem, since we are free to write it in nonconservative form \( u_t + df(u)u_x = S \). In the present case, however, it is essential for the key Chapman–Enskog estimation of the macroscopic variable \( u \) that we write the first row of our equation in integrated form \( f(u, v) = s \), enforcing a linearization \( A_{11}u + A_{12}v = \tilde{s} \). But, in the part of our argument in which we control microscopic variables by energy estimates, we differentiate this equation and group it with the second row, thus leading to a partially conservative form in which the energy estimates lose a derivative.

That is, the Chapman–Enskog part of our argument does not seem to be compatible with the nonconservative form needed to close energy estimates without losing a derivative. We have not been able to find a direct way around this (using some alternative scheme), and so for the moment Nash–Moser iteration appears essential for the argument.
A Existence of phase conditions

Lemma A.1. There exists a bounded projection $\Pi$ onto any finite-dimensional subspace of a Banach space.

Proof. An application of the Hahn–Banach Theorem; see, e.g., [17].

Lemma A.2. A projection $\Pi$ onto a subspace $S$ of a Banach space is bounded if and only if the distance from $s \in S$ to $\ker \Pi$ is greater than or equal to $|s|/C$ for some uniform $C > 0$.

Proof. $|s| = |\Pi(s-t)| \leq C|s-t|$ for all $s \in S$, $t \in \ker \Pi$ is equivalent to the statement that $\Pi$ is bounded, since $s-t$ runs over the entire Banach space as $s$ and $t$ are varied.

Corollary A.3. Suppose that there is an isometry between spaces $F_s^\varepsilon$ and a common set of spaces $F_s^0$, and suppose that $\ker (\Phi^\varepsilon)'(u^\varepsilon)$, considered (under mapping by this isometry) as a subset of $F_s^0$ is finite-dimensional, with a limit as $\varepsilon \to 0$. Then, there exist a family of projections $\Pi^\varepsilon$ onto $\ker (\Phi^\varepsilon)'(u^\varepsilon)$ that are uniformly bounded with respect to $\varepsilon$ in each $F_s^\varepsilon$, for $\varepsilon$ sufficiently small.

Proof. By Lemma A.1, there exists a bounded projection $\Pi^0$ onto the limit as $\varepsilon \to 0$ of $\ker (\Phi^\varepsilon)'(u^\varepsilon)$. Denote by $\tilde{F} := \ker \Pi^0$ the associated complementary subspace. Defining $\Pi^\varepsilon$ to be the projection along $\tilde{F}$ onto $\ker (\Phi^\varepsilon)'(u^\varepsilon)$, we find by Lemma A.2, compactness of the intersection of the unit ball with $\ker (\Phi^\varepsilon)'(u^\varepsilon)$, and continuity, that $\Pi^\varepsilon$ is bounded for $\varepsilon$ sufficiently small.

References


