NEARBY CYCLES OF AUTOMORPHIC ÉTALE SHEAVES, II

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Dedicated to Joachim Schwermer on the occasion of his 66th birthday

Abstract. We review some recent results of ours on the nearby cycles of automorphic étale sheaves, and record some improvements of the arguments.

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1. Introduction

In the context of the Langlands program, the étale cohomology of Shimura varieties serves as an important source of Galois representations associated with automorphic representations. Concretely, let X be a model of a Shimura variety defined over some number field contained in $\mathbb{C}$, which we temporarily assume to be $\mathbb{Q}$, for simplicity of exposition; let $\bar{\mathbb{Q}}$ denote the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$; and let $X_{\bar{\mathbb{Q}}}$ denote the base extension of X to $\bar{\mathbb{Q}}$. Let ℓ be any prime number. Then the étale cohomology $H^i_{\text{ét}}(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)$ is canonically a representation of the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, and it is desirable to understand such a representation.

For this purpose, it is important to also understand the restrictions of such a representation to the decomposition groups $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$, for all prime numbers p. This can be achieved by considering the canonical action of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ on $H^i_{\text{ét}}(X_{\bar{\mathbb{Q}}_p}, \mathbb{Q}_\ell) \cong H^i_{\text{ét}}(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)$ (see [10, Arcata, V, 3.3]), where $\bar{\mathbb{Q}}_p$ is any algebraic closure of $\mathbb{Q}_p$ containing $\bar{\mathbb{Q}}$. Then we can ask, for example, when $p \neq \ell$, whether $H^i_{\text{ét}}(X_{\bar{\mathbb{Q}}_p}, \mathbb{Q}_\ell)$ is an *unramified* representation of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$.

To answer such a question, a general method is to reduce it to the case where X has some model over $\mathbb{Z}_p$, and consider the so-called *nearby cycles* over the geometric

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special fiber of such a model. Let us explain this in more detail. (From now on, the symbol X will no longer denote a model of Shimura variety over $\mathbb{Q}$.)

Let us consider some more general base rings. Let R be a Henselian discrete valuation ring of residue characteristic $p > 0$, with fraction field $K = \mathrm{Frac}(R)$. Let $\bar{K}$ be an algebraic closure of K, and let $\bar{R}$ be the integral closure of R in $\bar{K}$. Let k denote the residue field of R, and let $\bar{k}$ denote the residue field of $\bar{R}$. Then we have the following commutative diagram

$$\begin{tikzcd}
\bar{s} := \mathrm{Spec}(\bar{k}) \arrow[r, "\bar{i}"] \arrow[d] & \bar{S} := \mathrm{Spec}(\bar{R}) \arrow[d] & \bar{\eta} := \mathrm{Spec}(\bar{K}) \arrow[l, "\bar{j}"'] \arrow[d] \\
s := \mathrm{Spec}(k) \arrow[r, "i"'] & S := \mathrm{Spec}(R) & \eta := \mathrm{Spec}(K) \arrow[l, "j"]
\end{tikzcd}$$

of canonical morphisms. We shall denote pullbacks with subscripts $\bar{s}$ etc as usual.

Suppose ℓ is a prime number different from p, and suppose Λ is a coefficient ring that is either $\mathbb{Z}/\ell^m\mathbb{Z}$ (for some integer $m \geq 1$), $\mathbb{Z}_\ell$, $\mathbb{Q}_\ell$, $\bar{\mathbb{Q}}_\ell$, or a finite extension of any of these. (These are the coefficient rings accepted in, for example, [24, 3.1].) For each scheme X separated and of finite type over S, we denote by $D_c^b(X_\eta, \Lambda)$ the bounded derived category of Λ-étale constructible sheaves over X_η, and by $D_c^b(X_{\bar{s}} \times \bar{\eta}, \Lambda)$ the bounded derived category of Λ-étale constructible sheaves over $X_{\bar{s}}$ with compatible continuous $\mathrm{Gal}(\bar{K}/K)$-actions. (See [12, 1.1] and [14] when Λ is not torsion.) Then we have the functor of *nearby cycles*:

$$R\Psi_X : D_c^b(X_\eta, \Lambda) \to D_c^b(X_{\bar{s}} \times \bar{\eta}, \Lambda) : \mathcal{F} \mapsto \bar{i}^* R\bar{j}_*(\mathcal{F}_{\bar{\eta}}), \tag{1.1}$$

where $\mathcal{F}_{\bar{\eta}}$ denotes the pullback of $\mathcal{F}$ to $X_{\bar{\eta}}$. (See [13, XIII], [10, Th. finitude, Sec. 3], and [24, Sec. 4] for more details.)

Suppose we have a morphism $\varphi : X \to Y$ of schemes of finite type over S. Then, on one hand, we have the adjunction morphisms

$$R\Psi_Y\, R\varphi_{\eta,*}(\mathcal{F}) \to R\varphi_{\bar{s},*}\, R\Psi_X(\mathcal{F}) \tag{1.2}$$

and

$$R\varphi_{\bar{s},!}\, R\Psi_X(\mathcal{F}) \to R\Psi_Y\, R\varphi_{\eta,!}(\mathcal{F}) \tag{1.3}$$

for pushforwards, which are isomorphisms when φ is *proper*, by the proper base change theorem (cf. [2, XII, 5.1] and [13, XIII, (2.1.7.1) and (2.1.7.3)]). On the other hand, we have the adjunction morphism

$$\varphi_{\bar{s}}^*\, R\Psi_Y(\mathcal{F}) \to R\Psi_X\, \varphi_\eta^*(\mathcal{F}) \tag{1.4}$$

for pullbacks, which is an isomorphism when φ is *smooth*, by the smooth base change theorem (see [2, XVI, 1.2] and [13, XIII, (2.1.7.2)]).

To see why these are useful, consider the special case where the structural morphism $X \to S$ is both *proper* and *smooth*, let $Y = S$, and let φ be the above structural morphism. Then we obtain from (1.4) and (1.2) the canonical isomorphisms

$$\Lambda \xrightarrow{\sim} R\Psi_X(\Lambda) \tag{1.5}$$

and

$$H^i(X_{\bar{\eta}}, \Lambda) \xrightarrow{\sim} H^i(X_{\bar{s}}, R\Psi_X(\Lambda)), \tag{1.6}$$

respectively, and hence their combination

$$H^i(X_{\bar{\eta}}, \Lambda) \xrightarrow{\sim} H^i(X_{\bar{s}}, \Lambda), \tag{1.7}$$

which are compatible with actions of $\mathrm{Gal}(\bar{K}/K)$. In particular, the action of $\mathrm{Gal}(\bar{K}/K)$ on the left-hand side $H^i(X_{\bar{\eta}}, \Lambda)$ of (1.7) is *unramified*, because the action of $\mathrm{Gal}(\bar{K}/K)$ on the right-hand side $H^i(X_{\bar{s}}, \Lambda)$ of (1.7) factors through $\mathrm{Gal}(\bar{k}/k)$.

In this article, we shall consider the more general situation where $\varphi : X \to S$ is some integral model of Shimura varieties that is neither proper nor smooth, and we shall also allow the trivial coefficient Λ to be replaced with certain automorphic étale sheaves valued in Λ-modules. (We will make these more precise in Sections 2 and 3.) Certainly, we cannot expect an isomorphism as in (1.7) in such a generality, but we will show that, for most integral models we know, we still have an isomorphism as in (1.6), despite the lack of the properness assumption. Intuitively speaking, this means, at least for studying étale cohomology, the special fibers of these integral models have as many points as there should be—there are no *missing points*.

We will review some results and ideas in our previous work [41], but with some improvements of the statements and proofs. This is partly motivated by some recent developments (such as [30]) after [41] was written. Compared with the corresponding results in [41], the main innovations of this article are the following: (i) a different argument in the proof of the key Theorem 4.1 using torsion automorphic coefficients instead of using Kuga families; and (ii) the inclusion of abelian-type cases in Corollaries 4.6 and 4.10, and in Theorems 4.13, 4.18, and 4.22.

We shall follow [34, Notation and Conventions] unless otherwise specified. We will sometimes use the terminologies introduced in [41] without repeating their definitions in detail, when their meanings can be understood from the context. (Nevertheless, we will still provide references to such definitions.)

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2. Integral models we consider

Let us start by emphasizing that we cannot expect isomorphisms as in (1.6) to be true for all kinds of nonproper integral models. For an extreme example, in the context of Section 1, if we consider the *trivially wrong* model obtained by replacing X with X_η, in which case the whole special fiber X_s is removed, then the right-hand side of (1.6) is always zero, and this cannot be what we want. Thus, if we are to have meaningful generalizations of (1.6) without the properness assumption, we need to be precise about our choices of integral models.

Let us retain the notation K, R, etc in Section 1. Suppose K is of characteristic zero, and suppose $\mathsf{X}_{\mathcal{H}}$ is the pullback to $S = \mathrm{Spec}(R)$ of one of the following models of Shimura varieties (or related moduli problems), where $\mathcal{H}$ is an open compact subgroup of $\mathrm{G}(\mathbb{A}^\infty)$ for some group scheme G over $\mathbb{Z}$ associated with the various constructions, and where R is now also an $\mathcal{O}_{F_0,(p)}$-algebra, with F_0 denoting the reflex field: (For more detailed references in the first four cases, see [41, Sec. 2.1].)

(Sm) A good reduction integral model defined by a smooth PEL moduli problem, as in [31, Sec. 5] and [34, Sec. 1.4.1–1.4.2].

(Nm) A flat integral model defined by taking normalization, as in [37, Sec. 6], of a characteristic zero PEL moduli problem over a product of good reduction integral models in Case (Sm) above. (This includes all normalizations of the PEL moduli defined by multichains of isogenies as in [54, Ch. 3 and 6].)

(Spl) A flat integral model defined by taking normalization, as in [35, Sec. 2.4], of the pullback of a characteristic zero PEL moduli problem over the so-called *splitting models* defined as in [51, Sec. 15].

(Hdg) A flat integral model defined by taking normalization, as in [43, Introduction], of a Hodge-type Shimura variety over some Siegel moduli scheme.

(Ab) A flat integral model defined as in [29], [28], or [30], of an abelian-type Shimura variety.

We shall say that we are in Case (Sm), (Nm), (Spl), (Hdg), or (Ab) depending on the case above from where $\mathsf{X}_{\mathcal{H}}$ is pulled back.

Remark 2.1. Let us be more precise about the levels allowed in these cases. For simplicity, we shall assume $\mathcal{H}$ to be of the form $\mathcal{H} = \mathcal{H}^p\mathcal{H}_p$ for some neat open compact subgroup $\mathcal{H}^p$ of $\mathrm{G}(\mathbb{A}^{\infty,p})$ and for some open compact subgroup $\mathcal{H}_p$ of $\mathrm{G}(\mathbb{Q}_p)$, and we say $\mathcal{H}^p$ and $\mathcal{H}_p$ are the levels *away from* p and *at* p, respectively. In Case (Sm), the level at p is hyperspecial. (But the prime $p = 2$ is excluded if simple factors of type D are involved, as in [34, Def. 1.2.1.15].) In Cases (Nm) and (Spl), we emphasize that we allow not only the parahoric levels at p as in [54, Ch. 3 and 6] and [51, Sec. 15] defined by certain multichains of isogenies, but also arbitrarily higher levels and also arbitrary collections of isogenies. (Also, $p = 2$ is allowed.) In Case (Hdg), the level at p is exactly the pullback of a hyperspecial level at p of a symplectic similitude group, which can be the hyperspecial levels at p as in [29] and [28], by composing any Siegel embedding as in [29, Sec. 2.3] and [28, Sec. 4] with the embedding given by "Zarhin's trick" as in [37, Lem. 4.9] or [39, Lem. 2.1.1.9]. (In fact, this was first explained in an earlier version of [43], but not in the current version.) However, there is some subtlety for Hodge-type Shimura varieties at parahoric levels, in addition to the requirement in [30] that we need $p > 2$ and the group $\mathrm{G}_{\mathbb{Q}_p}$ to be split over a tamely ramified extension of $\mathbb{Q}_p$, when $\mathcal{H}_p = K_p^\circ \neq K_p$ (in the notation of [30, Sec. 4.3]), in which case we have to defer them to Case (Ab) below. (That is, we have to treat some integral models of Hodge-type Shimura varieties with parahoric levels at p only as integral models of abelian-type Shimura varieties.) In Case (Ab), the level at p has to be either hyperspecial or parahoric at p, and in the latter case there are the above-mentioned requirements in [30] that $p > 2$ and that the group $\mathrm{G}_{\mathbb{Q}_p}$ to be split over a tamely ramified extension of $\mathbb{Q}_p$.

Remark 2.2. None of the three PEL-type cases we consider is completely subsumed by the Hodge-type case, and the Hodge-type case is not subsumed by the abelian-type case either. We emphasize again that this is about the actual choices of integral models, but not about the classification in characteristic zero. As we consider more and more general Shimura varieties in characteristic zero, the integral models that are available to us also become more and more restrictive, and there are some subtleties due to the fact that not everything available in the literature has been written in the most generality or flexibility its arguments allowed.

In Cases (Sm), (Nm), (Spl), and (Hdg), we have a commutative diagram

$$\begin{tikzcd} \mathsf{X}_{\mathcal{H}} \arrow[r, hook, "J^{\mathrm{tor}}"] \arrow[dr, "J^{\mathrm{min}}"'] & \mathsf{X}_{\mathcal{H}}^{\mathrm{tor}} \arrow[d] \\ & \mathsf{X}_{\mathcal{H}}^{\mathrm{min}} \end{tikzcd} \tag{2.3}$$

of canonical morphisms between noetherian normal schemes over $S = \mathrm{Spec}(R)$, where $\mathsf{X}_{\mathcal{H}}^{\mathrm{tor}}$ and $\mathsf{X}_{\mathcal{H}}^{\mathrm{min}}$ denote some projective toroidal and minimal compactifications. In Case (Sm), we use [34, Thm. 6.4.1.1, 7.2.4.1, and 7.3.3.4]. In Case (Nm), we use [36, Thm. 6.1] and [37, Prop. 6.4, and Thm. 12.1 and 12.16]. In Case (Spl), we use [35, Thm. 3.4.1 and 4.3.1]. In Case (Hdg), we use [43, Thm. 4.1.5 and 5.2.11]. (See [41, Prop. 2.2] for a detailed qualitative description of these compactifications, and see the proof there for further references to the literature.)

In Case (Ab), we expect similar results, but they are not yet available in the literature. Nevertheless, we still have the following crude constructions:

Proposition 2.4. *Given any $\mathsf{X}_{\mathcal{H}}$ in Case* (Ab)*, there exists an open immersion*

$$J^{\mathrm{min}} : \mathsf{X}_{\mathcal{H}} \hookrightarrow \mathsf{X}_{\mathcal{H}}^{\mathrm{min}} \tag{2.5}$$

from a normal quasi-projective scheme to a projective scheme over $S = \mathrm{Spec}(R)$, which we consider the minimal compactification of $\mathsf{X}_{\mathcal{H}}$, with the following properties:

(1) There exists a Galois finite étale extension $R \to R^+$ of discrete valuation rings of mixed characteristics $(0,p)$ and an integral model $\widetilde{\mathsf{X}}_{\widetilde{\mathcal{H}}}$ in Case (Hdg) *defined over a subring $\widetilde{R}$ of R^+ such that each connected component $\mathsf{X}_{\mathcal{H},R^+}^+$ of the base extension $\mathsf{X}_{\mathcal{H},R^+} := \mathsf{X}_{\mathcal{H}} \underset{R}{\otimes} R^+$ is noetherian normal and has geometrically connected fiber over K^+, where $K^+ := \mathrm{Frac}(R^+)$, and is isomorphic to the quotient by the free action of a finite group Δ^+ of some (noetherian normal) connected component $\widetilde{\mathsf{X}}_{\widetilde{\mathcal{H}},R^+}^+$ of the base extension $\widetilde{\mathsf{X}}_{\widetilde{\mathcal{H}},R^+} := \widetilde{\mathsf{X}}_{\widetilde{\mathcal{H}}} \underset{\widetilde{R}}{\otimes} R^+$. (The group Δ^+ depends not only on the levels $\mathcal{H}$ and $\widetilde{\mathcal{H}}$, but also on the actual connected components $\mathsf{X}_{\mathcal{H},R^+}$ and $\widetilde{\mathsf{X}}_{\widetilde{\mathcal{H}},R^+}^+$.)*

(2) Let $\widetilde{\mathsf{X}}_{\widetilde{\mathcal{H}}} \hookrightarrow \widetilde{\mathsf{X}}_{\widetilde{\mathcal{H}}}^{\mathrm{min}}$ denote the minimal compactification of $\widetilde{\mathsf{X}}_{\widetilde{\mathcal{H}}}$ as in (2.3)*. Then the action of Δ^+ on $\widetilde{\mathsf{X}}_{\widetilde{\mathcal{H}},R^+}^+$ extends to a (generally non-free) action on the schematic closure $\widetilde{\mathsf{X}}_{\widetilde{\mathcal{H}},R^+}^{\mathrm{min},+}$ of $\widetilde{\mathsf{X}}_{\widetilde{\mathcal{H}},R^+}^+$ in $\widetilde{\mathsf{X}}_{\widetilde{\mathcal{H}},R^+}^{\mathrm{min}} := \widetilde{\mathsf{X}}_{\widetilde{\mathcal{H}}}^{\mathrm{min}} \underset{R}{\otimes} R^+$, and the quotient of $\widetilde{\mathsf{X}}_{\widetilde{\mathcal{H}},R^+}^{\mathrm{min},+}$ by Δ^+ is isomorphic to the schematic closure $\mathsf{X}_{\mathcal{H},R^+}^{\mathrm{min},+}$ of $\mathsf{X}_{\mathcal{H},R^+}^+$ in $\mathsf{X}_{\mathcal{H},R^+}^{\mathrm{min}} := \mathsf{X}_{\mathcal{H}}^{\mathrm{min}} \underset{R}{\otimes} R^+$.*

Remark 2.6. In Proposition 2.4, the quotients of quasi-projective schemes by finite groups are defined by the same argument as in [48, Sec. 7, Thm. and Rem.] (see also [19, V, 1.8]). Moreover, in (1) of Proposition 2.4, when the action of Δ^+ on $\widetilde{\mathsf{X}}_{\widetilde{\mathcal{H}},R^+}^+$ is free, the canonically induced morphism $\widetilde{\mathsf{X}}_{\widetilde{\mathcal{H}},R^+}^+ \to \widetilde{\mathsf{X}}_{\widetilde{\mathcal{H}},R^+}^+/\Delta^+ \cong \mathsf{X}_{\mathcal{H},R^+}^+$ is a *Galois finite étale cover* with Galois group Δ^+ by the same argument as in the proof of the last assertion of [48, Sec. 7, Thm.] (if we use completions of strict local rings instead of complete local rings; see also [19, V, 2.4 and 2.6]).

Proof of Proposition 2.4. Note that (1) follows (up to slight reformulation) from the constructions in [29, Sec. 3.4], [28, Thm. 3.10], and [30, Sec. 4.6]. We note that we need [30, Prop. 4.3.7] (when $\mathcal{H}_p = K_p^\circ \neq K_p$ in the notation there; cf. Remark 2.1) and "Zarhin's trick" (as in [37, Lem. 4.9] or [39, Lem. 2.1.1.9]) to ensure that, when working with connected components over a finite étale base ring extension, we can indeed reduce to Case (Hdg) (where the level at p is exactly the pullback of a hyperspecial level at p of a symplectic similitude group, as explained above).

Our main task is to prove (2). By [53, 12.3], we have a canonical open immersion

$$\mathsf{X}_{\mathcal{H},K} := \mathsf{X}_{\mathcal{H}} \underset{R}{\otimes} K \hookrightarrow \mathsf{X}^{\min}_{\mathcal{H},K}, \tag{2.7}$$

where $\mathsf{X}^{\min}_{\mathcal{H},K}$ denotes the pullback to K of the canonical model of the minimal compactification over the reflex field (which is a subfield of K by assumption), and we would like to extend (2.7) to its analogue (2.5) over R (with the desired properties).

Let us temporarily assume that G^{ad} is simple as an algebraic group over $\mathbb{Q}$. By [30, the proof of Prop. 4.6.28], and by [38, the proof of Thm. 3.8], there exists an ample invertible sheaf $\mathcal{L}_0$ over $\mathsf{X}_{\mathcal{H}}$ whose pullback to $\mathsf{X}_{\mathcal{H}} \underset{R}{\otimes} K$ is isomorphic to a positive tensor power of the canonical bundle $\Omega^d_{(\mathsf{X}_{\mathcal{H}} \underset{R}{\otimes} K)/K}$, where $d := \dim(\mathsf{X}_{\mathcal{H}} \underset{R}{\otimes} K)$. Note that [30, Prop. 4.6.28] assumed that G^{ad} is absolutely simple (not just simple), and that the level at p is very special (as in [52, Sec. 10.3.2]). Let us explain in the next two paragraphs why we can borrow their arguments without these assumptions.

As for the assumption that the level at p is very special, it was made because the goal of [30, Prop. 4.6.28] was to show that the integral models constructed there are canonical, and detailed properties of local models were used to ensure that the special fiber of each connected component is reduced and irreducible, so that $\mathcal{L}_0$ is the unique extension of its pullback to $\mathsf{X}_{\mathcal{H}} \underset{R}{\otimes} K$—since we have a rather different goal here, we can drop this assumption.

As for the assumption that G^{ad} is absolutely simple, it was made to ensure that, in the Hodge-type case, in the notation of [30, Prop. 4.6.28], the pullback of the Hodge invertible sheaf ω_{GSp} under the Siegel embedding is a positive power of ω_{G}. But we can drop this assumption because we know that a positive tensor power of this pullback is isomorphic to a positive tensor power of the canonical bundle—let us explain why. This pullback is an automorphic line bundle with a canonical model over the reflex field (see [47]), and for our purpose (up to replacing the invertible sheaf with a positive tensor power) we just have to identify its pullback from the reflex field to $\mathbb{C}$, which is associated with some one-dimensional representation of the Levi of a parabolic subgroup P of $\mathrm{G}_{\mathbb{C}}$ defined by the Hodge cocharacter determined by the Shimura datum. As explained in [11, 2.3.7], on each simple factor of $\mathrm{G}_{\mathbb{C}}$, the pullback of the standard representation of the symplectic group under the Siegel embedding is a multiple of the fundamental weight representation associated with some underlined node in [11, Table 1.3.9], and these multiplicities are the same on all the simple factors of $\mathrm{G}_{\mathbb{C}}$. Each such fundamental weight representations has a two-step filtration whose stabilizer is the corresponding factor of P, which corresponds to a direct summand of the pullback of the Hodge filtration on the relative de Rham homology of the universal abelian scheme over the Siegel moduli, and the dual of the top exterior power of the top graded piece corresponds to a tensor factor of the pullback of the Hodge invertible sheaf (as in [43, Def. 5.1.2]).

It can be easily checked (using explicit realizations in example-rich texts such as [16] or [18]) that, as a representation of the Levi of this factor of P, the weight of this tensor factor is exactly the pullback of the corresponding fundamental weight. Hence, it follows from [38, the proof of Thm. 3.8] that $h^\vee$ times this weight, where $h^\vee$ is the dual Coxeter number of the root system of this (and every other) simple factor of $\mathrm{G}_\mathbb{C}$, is the weight of the corresponding factor of the canonical bundle. Then the above assertion follows, because the exponents of the tensor factors from the simple factor of $\mathrm{G}_\mathbb{C}$ are all the same.

By [49, Prop. 3.4 b)] (whose assertion can be formulated in terms of the push-forward of the log canonical bundle of any toroidal compactification, and therefore is compatible with descent), $\mathcal{L}_0 \underset{R}{\otimes} K$ extends to an ample invertible sheaf $\mathcal{L}_1$ over $\mathsf{X}^{\min}_{\mathcal{H},K}$. Let $\mathsf{X}_{\mathcal{H},R^+} = \mathsf{X}_{\mathcal{H},K} \underset{R}{\otimes} R^+$ be as in (1) of Proposition 2.4, and let $\mathsf{X}_{\mathcal{H},K^+} := \mathsf{X}_{\mathcal{H},K} \underset{K}{\otimes} K^+$ and $\mathsf{X}^{\min}_{\mathcal{H},K^+} := \mathsf{X}^{\min}_{\mathcal{H},K} \underset{K}{\otimes} K^+$. Let us denote by U the union of $\mathsf{X}_\mathcal{H}$ and $\mathsf{X}^{\min}_{\mathcal{H},K}$ (glued over the open subscheme $\mathsf{X}_{\mathcal{H},K^+}$), and by $\mathcal{L}$ the invertible sheaf over U whose restrictions to $\mathsf{X}_\mathcal{H}$ and $\mathsf{X}^{\min}_{\mathcal{H},K}$ are $\mathcal{L}_0$ and $\mathcal{L}_1$, respectively. Let $\mathsf{U}^+_{R^+}$ denote the union of $\mathsf{X}^{\min,+}_{\mathcal{H},K^+} := \mathsf{X}^+_{\mathcal{H},R^+} \underset{R^+}{\otimes} K^+$ and $\mathsf{X}^+_{\mathcal{H},R^+}$ (glued over the open subscheme $\mathsf{X}^+_{\mathcal{H},K^+} := \mathsf{X}^+_{\mathcal{H},R^+} \underset{R^+}{\otimes} K^+$), which is noetherian normal and is the connected component of $\mathsf{U}_{R^+} := \mathsf{U} \underset{R}{\otimes} R^+$ containing $\mathsf{X}^+_{\mathcal{H},R^+}$. Let $\mathcal{L}_{R^+}$ and $\mathcal{L}^+_{R^+}$ denote the pullbacks of $\mathcal{L}$ to U_{R^+} and $\mathsf{U}^+_{R^+}$, respectively.

Let $\widetilde{\mathsf{U}}^+_{R^+}$ denote the union of $\widetilde{\mathsf{X}}^{\min,+}_{\widetilde{\mathcal{H}},K^+} := \widetilde{\mathsf{X}}^{\min,+}_{\widetilde{\mathcal{H}},R^+} \underset{R^+}{\otimes} K^+$ and $\widetilde{\mathsf{X}}^+_{\widetilde{\mathcal{H}},R^+}$ in $\widetilde{\mathsf{X}}^{\min,+}_{\widetilde{\mathcal{H}},R^+}$, whose complement has codimension at least two because the morphism $\widetilde{\mathsf{X}}_{\widetilde{\mathcal{H}}} \hookrightarrow \widetilde{\mathsf{X}}^{\min}_{\widetilde{\mathcal{H}}}$ is fiberwise dense over R^+. As explained in [30, the proof of Prop. 4.6.28] (with adjustments as explained above), up to replacing $\mathcal{L}$ with a positive tensor power, we may and we shall assume that the pullback of $\mathcal{L}^+_{R^+}$ to $\widetilde{\mathsf{U}}^+_{R^+}$, which we denote by $\widetilde{\mathcal{L}}^+_{R^+}$, extends to an ample invertible sheaf over the whole $\widetilde{\mathsf{X}}^{\min,+}_{\widetilde{\mathcal{H}},R^+}$, which we abusively still denote by $\widetilde{\mathcal{L}}^+_{R^+}$. Then we have

$$\widetilde{\mathsf{X}}^{\min,+}_{\widetilde{\mathcal{H}},R^+} \cong \operatorname{Proj}\Big(\underset{k\geq 0}{\oplus} \Gamma(\widetilde{\mathsf{X}}^{\min,+}_{\widetilde{\mathcal{H}},R^+}, (\widetilde{\mathcal{L}}^+_{R^+})^{\otimes k})\Big) \cong \operatorname{Proj}\Big(\underset{k\geq 0}{\oplus} \Gamma(\widetilde{\mathsf{U}}^+_{R^+}, (\widetilde{\mathcal{L}}^+_{R^+})^{\otimes k})\Big), \tag{2.8}$$

and the same holds up to replacing $\mathcal{L}$ and $\widetilde{\mathcal{L}}^+_{R^+}$ with the same positive tensor power.

At this point, we can explain how to remove the assumption that G^{ad} is simple over $\mathbb{Q}$, and drop this assumption. Since all we need is the ampleness and extensibility of certain invertible sheaves up to replacing them with the same positive tensor powers, we may work at a higher level defined by a finite index subgroup of $\mathcal{H}$, and pullback the desired sheaves from products of integral models of smaller Shimura varieties and their minimal compactifications—we already have the desired sheaves over the factors, as explained above. (The upshot is that we can use possibly different positive tensor powers of the Hodge invertible sheaves over different factors. See [42, Sec. 7] for a prototypical example of such an argument in PEL-type cases.)

Note that the action of Δ^+ on $\widetilde{\mathsf{X}}^+_{\widetilde{\mathcal{H}},K^+}$ extends to an action on $\widetilde{\mathsf{X}}^{\min,+}_{\widetilde{\mathcal{H}},K^+}$, which induces an isomorphism $\widetilde{\mathsf{X}}^{\min,+}_{\widetilde{\mathcal{H}},K^+}/\Delta^+ \xrightarrow{\sim} \mathsf{X}^+_{\mathcal{H},K^+}$, because $\widetilde{\mathsf{X}}^+_{\widetilde{\mathcal{H}},K^+}$ and $\mathsf{X}^+_{\mathcal{H},K^+}$ have models over some number field whose pullbacks to $\mathbb{C}$ are arithmetic quotients of the same Hermitian symmetric domain, and Δ^+ is just the quotient of one such arithmetic subgroup by another, whose action extends to the minimal compactifications in [3] by their very constructions. (The adelic construction in [53, 6.1–6.2] has to be formulated in terms of finite disjoint unions of Hermitian symmetric domains, but the connected components of the minimal compactifications thus obtained are still the same projective normal varieties in [3]. We need the theory in [11] to relate the constructions in [29, Sec. 3.4], [28, Thm. 3.10], and [30, Sec. 4.6] to the complex analytic construction mentioned above, but we do not need to generalize the theory in [11] to an analogous theory for minimal compactifications.) Hence, the action of Δ^+ on $\widetilde{\mathsf{X}}^+_{\widetilde{\mathcal{H}},R^+}$ extends to an action on $\widetilde{\mathsf{U}}^{\min,+}_{R^+}$, which induces an isomorphism $\widetilde{\mathsf{U}}^+_{R^+}/\Delta^+ \xrightarrow{\sim} \mathsf{U}^+_{R^+}$ extending the isomorphism $\widetilde{\mathsf{X}}^+_{\widetilde{\mathcal{H}},R^+}/\Delta^+ \xrightarrow{\sim} \mathsf{X}^+_{\mathcal{H},R^+}$. Consequently, $\widetilde{\mathcal{L}}^+_{R^+}$ is canonically isomorphic to its pullbacks under the action of Δ^+ on $\widetilde{\mathsf{U}}^+_{R^+}$, and it follows from (2.8) that the action of Δ^+ on $\widetilde{\mathsf{U}}^+_{R^+}$ extends to the whole $\widetilde{\mathsf{X}}^{\min,+}_{\widetilde{\mathcal{H}},R^+}$.

Let us form the quotient $\mathsf{X}^{\min,+}_{\mathcal{H},R^+} := \widetilde{\mathsf{X}}^{\min,+}_{\widetilde{\mathcal{H}},R^+}/\Delta^+$, which is a noetherian normal projective scheme over R^+ containing $\mathsf{U}^+_{R^+}$ as an open dense subscheme whose complement has codimension at least two. By locally forming Δ^+-invariant sections by taking norms, and by replacing $\widetilde{\mathcal{L}}^+_{R^+}$ and $\mathcal{L}^+_{R^+}$ with their $|\Delta^+|$-th tensor powers, we obtain an ample invertible sheaf over $\mathsf{X}^{\min,+}_{\mathcal{H},R^+}$ extending $\mathcal{L}^+_{R^+}$, which we abusively still denote by $\mathcal{L}^+_{R^+}$, whose pullback to $\widetilde{\mathsf{X}}^{\min,+}_{\widetilde{\mathcal{H}},R^+}$ necessarily coincides with $\widetilde{\mathcal{L}}^+_{R^+}$. The disjoint union of such quotients $\mathsf{X}^{\min,+}_{\mathcal{H},R^+}$, which we abusively denote by $\mathsf{X}^{\min}_{\mathcal{H},R^+}$, is a noetherian normal projective scheme carrying an ample invertible sheaf extending $\mathcal{L}_{R^+}$, which we abusively still denote by $\mathcal{L}_{R^+}$, whose pullback to each $\widetilde{\mathsf{X}}^{\min,+}_{\widetilde{\mathcal{H}},R^+}$ as above is isomorphic to $\widetilde{\mathcal{L}}^+_{R^+}$ (up to replacement with a positive tensor power, to account for the replacements of invertible sheaves with positive tensor powers on the connected components thus far—this is feasible because there are only finitely many connected components). Then, as in the case of (2.8) above, we have

$$\mathsf{X}^{\min}_{\mathcal{H},R^+} \cong \operatorname{Proj}\Big(\underset{k\geq 0}{\oplus} \Gamma(\mathsf{X}^{\min}_{\mathcal{H},R^+}, (\mathcal{L}_{R^+})^{\otimes k})\Big) \cong \operatorname{Proj}\Big(\underset{k\geq 0}{\oplus} \Gamma(\mathsf{U}_{R^+}, (\mathcal{L}_{R^+})^{\otimes k})\Big). \tag{2.9}$$

Since the pair $(\mathsf{U}_{R^+}, \mathcal{L}_{R^+})$ of a quasi-projective scheme and an ample invertible sheaf has a model $(\mathsf{U}, \mathcal{L})$ over R, it carries descent data with respect to the finite étale morphism $R \to R^+$. Hence, it follows from (2.9) that $\mathsf{X}^{\min}_{\mathcal{H},R^+}$ carries induced descent data and (by the theory of descent—see [19, VIII, 7.8]) has a model $\mathsf{X}^{\min}_{\mathcal{H}}$ over R, together with the open immersion (2.5) extending (2.7), as desired. □

Remark 2.10. There is also some crude construction of (integral models of) toroidal compactifications in Case (Ab), if we combine the constructions in [43] and [36] in Case (Hdg) and pursue a similar strategy as in the proof of Proposition 2.4. But we have decided to omit it at the expense of excluding Case (Ab) in Theorem 4.1 below, not just because the details are more tedious to write up, but also because it would be more desirable to have a construction of toroidal compactifications which we can describe in as much detail as in other cases, and because we are still able to include Case (Ab) in Corollary 4.6 below. On the other hand, while it would also be more desirable to have a construction of minimal compactifications which we can better describe, we do need some crude construction as in Proposition 2.4 to at least define the intersection complexes of nearby cycles as in Theorem 4.13

and Corollary 4.15 below. This is admittedly a compromise, but it still allows the applications to the intersection cohomology in Theorems 4.18 and 4.22 below.

3. Automorphic étale sheaves

Let $\ell > 0$ be a rational prime number. Let us fix the choice of an algebraic closure $\bar{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$. For simplicity of notation, let us assume the following:

(1) In Case (Sm), we have $\ell \notin \Box$ and $\mathcal{H} = \mathcal{H}^\ell \mathcal{H}_\ell$ in $\mathrm{G}(\hat{\mathbb{Z}}^\Box)$ for some open compact subgroups $\mathcal{H}^\ell \subset \mathrm{G}(\hat{\mathbb{Z}}^{\Box \cup \{\ell\}})$ and $\mathcal{H}_\ell \subset \mathrm{G}(\mathbb{Z}_\ell)$.

(2) In Cases (Nm), (Spl), (Hdg), and (Ab), we have $\mathcal{H} = \mathcal{H}^\ell \mathcal{H}_\ell$ in $\mathrm{G}(\hat{\mathbb{Z}})$ for some open compact subgroups $\mathcal{H}^\ell \subset \mathrm{G}(\hat{\mathbb{Z}}^\ell)$ and $\mathcal{H}_\ell \subset \mathrm{G}(\mathbb{Z}_\ell)$.

As in [47, Ch. III], let us denote by G^c the quotient of $\mathrm{G}_\mathbb{Q}$ by the maximal $\mathbb{Q}$-anisotropic $\mathbb{R}$-split subtorus of the center of $\mathrm{G}_\mathbb{Q}$ (as algebraic groups over $\mathbb{Q}$). (In Cases (Sm), (Nm), (Spl), and (Hdg), we have $\mathrm{G} \cong \mathrm{G}^c$, but in Case (Ab) this is not true in general.) For any subgroup of $\mathrm{G}(\mathbb{A})$ (including those of $\mathrm{G}(\mathbb{Q})$, $\mathrm{G}(\mathbb{A}^\infty)$, $\mathrm{G}(\hat{\mathbb{Z}})$, $\mathrm{G}(\mathbb{Z}_\ell)$, etc), we shall denote its image in $\mathrm{G}^c(\mathbb{A})$ with an additional superscript c. (We are not introducing a model of G^c over $\mathbb{Z}$.) Therefore, for example, we have an open compact subgroup $\mathcal{H}^c$ of $\mathrm{G}^c(\mathbb{A}^\infty)$, which is of the form $\mathcal{H}^c = \mathcal{H}^{\ell,c}\mathcal{H}^c_\ell$.

For each integer $r > 0$, let $\mathcal{U}_\ell(\ell^r) := \ker(\mathrm{G}(\mathbb{Z}_\ell) \to \mathrm{G}(\mathbb{Z}/\ell^r\mathbb{Z}))$, and consider $\mathcal{H}(\ell^r) := \mathcal{H}^\ell \mathcal{U}_\ell(\ell^r)$, which is contained in $\mathcal{H}$ when r is sufficiently large. For such sufficiently large r, in all cases considered above, we have a finite cover

$$\mathsf{X}_{\mathcal{H}(\ell^r)} \to \mathsf{X}_\mathcal{H}, \tag{3.1}$$

which induces a Galois finite étale cover

$$\mathsf{X}_{\mathcal{H}(\ell^r)} \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \to \mathsf{X}_\mathcal{H} \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \tag{3.2}$$

with Galois group $\mathcal{H}^c_\ell / \mathcal{U}_\ell(\ell^r)^c$ (cf. [47, Ch. III, Sec. 6, Rem. 6.1]), where $\mathsf{X}_{\mathcal{H}(\ell^r)}$ is defined as in the case of $\mathsf{X}_\mathcal{H}$ but with $\mathcal{H}$ replaced with its normal subgroup $\mathcal{H}(\ell^r)$. If $\ell \neq p$, then the finite cover (3.1) is étale over all of S. (In Cases (Sm), (Nm), (Spl), and (Hdg), this is because (3.1) relatively represents a functor of level structures at $\ell \neq p$. In Case (Ab), this is because (3.1) can be étale locally identified with analogous morphisms in Case (Hdg); cf. (1) of Proposition 2.4 and Remark 2.6.)

For any finite-dimensional algebraic representation V of G^c over $\Lambda = \bar{\mathbb{Q}}_\ell$, which we also view as an algebraic representation of G by pullback (whose restriction to the maximal $\mathbb{Q}$-anisotropic $\mathbb{R}$-split subtorus of the center of $\mathrm{G}_\mathbb{Q}$ is trivial), there exists a canonical étale sheaf $\mathcal{V}$ over $\mathsf{X}_\mathcal{H}$ (with stalks isomorphic to V). Let us briefly review the construction, because we will need it in our later argument.

As explained in [22, Sec. III.2], by the Baire category theorem (see, for example, the proof of [6, 2.2.1.1] or the beginning of [56, Sec. 2]), there exists a finite extension E of $\mathbb{Q}_\ell$ in $\bar{\mathbb{Q}}_\ell$, and an $\mathcal{O}_E$-lattice V_0 with a continuous action of $\mathrm{G}(\mathbb{Z}_\ell)$ (with respect to the ℓ-adic topology), such that $V \cong V_0 \underset{\mathcal{O}_E}{\otimes} \bar{\mathbb{Q}}_\ell$ as continuous representations of $\mathrm{G}(\mathbb{Q}_\ell)$. For each $m > 0$, by the continuity of the action of $\mathrm{G}(\mathbb{Z}_\ell)$ on V_0, there exists an integer $r(m) > 0$ such that $\mathcal{H}(\ell^{r(m)}) \subset \mathcal{H}$ and $\mathcal{U}_\ell(\ell^{r(m)})$ acts trivially on the finite quotient $V_{0,\ell^m} := V_0 \underset{\mathbb{Z}_\ell}{\otimes} (\mathbb{Z}/\ell^m\mathbb{Z})$. By abuse of notation, let us also denote by $\underline{V}_{0,\ell^m}$ the constant group scheme over $\mathbb{Z}$, which then carries an action of $\mathcal{H}^c_\ell / \mathcal{U}_\ell(\ell^{r(m)})^c$ induced by that of $\mathcal{H}_\ell / \mathcal{U}_\ell(\ell^{r(m)})$. Let us define $\mathcal{V}_{0,\ell^m}$ to be the torsion étale sheaf of sections over $\mathsf{X}_{\mathcal{H}} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ of the contraction product

$$(\mathsf{X}_{\mathcal{H}(\ell^{r(m)})} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}) \overset{\mathcal{H}_\ell^c/\mathcal{U}_\ell(\ell^{r(m)})^c}{\times} \underline{V}_{0,\ell^m}, \tag{3.3}$$

whose pullback to $\mathsf{X}_{\mathcal{H}(\ell^{r(m)})} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ is isomorphic to $\underline{V}_{0,\ell^m}$ because (3.2) is a Galois finite étale cover with Galois group $\mathcal{H}_\ell^c/\mathcal{U}_\ell(\ell^r)^c$; and define the étale sheaves

$$\mathcal{V}_0 := \varprojlim_m \mathcal{V}_{0,\ell^m} \tag{3.4}$$

and

$$\mathcal{V} := \mathcal{V}_0 \underset{\mathcal{O}_E}{\otimes} \bar{\mathbb{Q}}_\ell \tag{3.5}$$

over $\mathsf{X}_{\mathcal{H}} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ as usual. Then it is elementary (though tedious) to verify that such a construction is independent of the various choices, is functorial in various natural senses, and allows us to define the Hecke actions on the cohomology groups if we take the limit over $\mathcal{H}$. (See [31, Sec. 6] and [22, Sec. III.2] for more details.)

When $\ell \neq p$, the same construction defines the étale sheaf extensions of $\mathcal{V}_0$ and $\mathcal{V}$ to all of $\mathsf{X}_{\mathcal{H}}$. This construction also works if we replace V with a continuous representation of $\mathrm{G}(\mathbb{Z}_\ell)^c$ on a (possibly torsion) finite $\mathbb{Z}_\ell$-module (or a $\bar{\mathbb{Z}}_\ell$-module if one prefers), without referring to any representation over $\bar{\mathbb{Q}}_\ell$.

For simplicity, we shall often denote the pullbacks of $\mathcal{V}$ by the same symbol.

Let us record the following observation, based on the above construction:

Lemma 3.6. *For any finite-dimensional algebraic representation V of G^c as above over $\Lambda = \bar{\mathbb{Q}}_\ell$, there exists a finite extension E of $\mathbb{Q}_\ell$ in $\bar{\mathbb{Q}}_\ell$, together with a projective system of torsion étale sheaves $\mathcal{V}_{0,m}$ of $\mathcal{O}_E$-modules indexed by integers $m > 0$ such that $\mathcal{V} \cong \mathcal{V}_0 \underset{\mathcal{O}_E}{\otimes} \bar{\mathbb{Q}}_\ell$ for $\mathcal{V}_0 := \varprojlim_m \mathcal{V}_{0,m}$ over $\mathsf{X}_{\mathcal{H}} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$, and such that, for each $m > 0$, there exist some integer $r(m) > 0$ such that $\mathcal{H}(\ell^{r(m)}) \subset \mathcal{H}$ and such that the pullback of $\mathcal{V}_{0,m}$ from $\mathsf{X}_{\mathcal{H}} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ to $\mathsf{X}_{\mathcal{H}(\ell^{r(m)})} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ is a constant étale sheaf. When $\ell \neq p$, the same assertions are true with all sheaves $\mathcal{V}$, $\mathcal{V}_0$, and $\mathcal{V}_{0,m}$, for all $m > 0$, defined over all of $\mathsf{X}_{\mathcal{H}}$. If we replace V with a continuous representations of $\mathrm{G}(\mathbb{Z}_\ell)^c$ on a (possibly torsion) finite $\mathbb{Z}_\ell$-module, then there exist a projective system of torsion étale sheaves $\mathcal{V}_{0,m}$ of $\mathbb{Z}_\ell$-modules indexed by integers $m > 0$ such that $\mathcal{V} \cong \varprojlim_m \mathcal{V}_{0,m}$ and such that the remaining assertions as above concerning $\mathcal{V}_{0,m}$ hold verbatim.*

4. Main results

Let $\mathsf{X}_{\mathcal{H}}$ be as in Section 2, with the toroidal and minimal compactifications $J^{\mathrm{tor}} : \mathsf{X}_{\mathcal{H}} \hookrightarrow \mathsf{X}_{\mathcal{H}}^{\mathrm{tor}}$ and $J^{\mathrm{min}} : \mathsf{X}_{\mathcal{H}} \hookrightarrow \mathsf{X}_{\mathcal{H}}^{\mathrm{min}}$ as in (2.3), in Cases (Sm), (Nm), (Spl), and (Hdg); or with the compactification $J^{\mathrm{min}} : \mathsf{X}_{\mathcal{H}} \hookrightarrow \mathsf{X}_{\mathcal{H}}^{\mathrm{min}}$ as in Proposition 2.4, in Case (Ab). Let $\mathcal{V}$ be as in Section 3, which is associated with either a finite-dimensional algebraic representation V of G^c over $\Lambda = \bar{\mathbb{Q}}_\ell$, or with a continuous representations of $\mathrm{G}(\mathbb{Z}_\ell)^c$ on a (possibly torsion) finite $\mathbb{Z}_\ell$-module with $\Lambda = \mathbb{Z}_\ell$ (or $\bar{\mathbb{Z}}_\ell$, if one prefers). By abuse of language, we shall indicate the type of $\mathcal{V}$ we consider by simply specifying the type of the corresponding coefficient ring Λ.

Then we have the following results:

Theorem 4.1 (cf. [41, Thm. 5.15])**.** *In Cases* (Sm)*,* (Nm)*,* (Spl)*, and* (Hdg)*, for all $\mathcal{V}$ over $\Lambda = \bar{\mathbb{Q}}_\ell$ or $\mathbb{Z}_\ell$ (or $\bar{\mathbb{Z}}_\ell$), the adjunction morphisms*

$$R\Psi_{\mathsf{X}^{\mathrm{tor}}_{\mathcal{H}}} RJ^{\mathrm{tor}}_{\eta,*}(\mathcal{V}) \to RJ^{\mathrm{tor}}_{\bar{s},*} R\Psi_{\mathsf{X}_{\mathcal{H}}}(\mathcal{V}) \tag{4.2}$$

and

$$J^{\mathrm{tor}}_{\bar{s},!} R\Psi_{\mathsf{X}_{\mathcal{H}}}(\mathcal{V}) \to R\Psi_{\mathsf{X}^{\mathrm{tor}}_{\mathcal{H}}} J^{\mathrm{tor}}_{\eta,!}(\mathcal{V}) \tag{4.3}$$

(*see* (2.3) *for the notation*) *are isomorphisms in $D^b_c((\mathsf{X}^{\mathrm{tor}}_{\mathcal{H}})_{\bar{s}} \times \bar{\eta}, \Lambda)$.*

Remark 4.4. Compared with [41, Thm. 5.15], we note that, in the case of integral or torsion coefficients, we no longer require $\mathcal{V}$ to be of the form $\mathcal{W}_{0,M}$ over $\Lambda = \mathbb{Z}_\ell$ as in [41, Prop. 3.4] for some $\ell > c_W$. That is, roughly speaking, we no longer require the integral or torsion coefficients to be defined by representations of ℓ-small weights.

Remark 4.5. Theorem 4.1 is the logical foundation of all our other results in this article. We will explain the proof of this key theorem in Section 5. As we shall see, the main reason to consider the four kinds of integral models in Cases (Sm), (Nm), (Spl), and (Hdg) as in Section 2 is that they are known to have good toroidal compactifications, and Case (Ab) is not included in this theorem exactly because good integral models of toroidal compactifications of abelian-type Shimura varieties (with properties we need) are not yet available in the literature (cf. Remark 2.10).

Corollary 4.6 (cf. [41, Cor. 5.20])**.** *In all Cases* (Sm)*,* (Nm)*,* (Spl)*,* (Hdg)*, and* (Ab)*, for all $\mathcal{V}$ over $\Lambda = \bar{\mathbb{Q}}_\ell$ or $\mathbb{Z}_\ell$ (or $\bar{\mathbb{Z}}_\ell$), the canonical adjunction morphisms*

$$H^i_{\text{ét}}((\mathsf{X}_{\mathcal{H}})_{\bar{\eta}}, \mathcal{V}) \to H^i((\mathsf{X}_{\mathcal{H}})_{\bar{s}}, R\Psi_{\mathsf{X}_{\mathcal{H}}}(\mathcal{V})) \tag{4.7}$$

and

$$H^i_{\text{ét},c}((\mathsf{X}_{\mathcal{H}})_{\bar{s}}, R\Psi_{\mathsf{X}_{\mathcal{H}}}(\mathcal{V})) \to H^i_{\text{ét},c}((\mathsf{X}_{\mathcal{H}})_{\bar{\eta}}, \mathcal{V}) \tag{4.8}$$

of $\mathrm{Gal}(\bar{K}/K)$*-modules are isomorphisms, for all i.*

Proof. In Cases (Sm), (Nm), (Spl), and (Hdg), this follows from Theorem 4.1 and the proper base change theorem, as in the proof of [41, Cor. 5.20].

In Case (Ab), the question is whether the morphisms (4.7) and (4.8) are isomorphisms, and we can ignore the $\mathrm{Gal}(\bar{K}/K)$-module structures when answering such a question. Also, by first reducing to the torsion case by Lemma 3.6, and by duality (see [24, 4.2]), it suffices to answer the question for the morphism (4.7). Hence, we are free to replace K with a finite extension inside $\bar{K}$ (and accordingly R with its integral closure in this finite extension), and to replace $\mathsf{X}_{\mathcal{H}}$ with its connected components. Because the construction of integral models in Case (Ab) (see (1) of Proposition 2.4 and Remark 2.6) was achieved by descent and by taking quotients of connected components of some integral models of Hodge-type Shimura varieties by the free actions of some finite groups (which is compatible with the construction of $\mathcal{V}$ in Section 3 when restricted to subgroups stabilizing the connected components), by using the Hochschild–Serre spectral sequence for étale cohomology (see, for example, [46, Ch. III, Sec. 2, Thm. 2.20] or [15, Ch. 9, Sec. 9.1, p. 501]), the isomorphism assertion is reduced to the known one in Case (Hdg), as desired. □

Remark 4.9 (cf. [41, Rem. 5.42]). There are closely related results in [25, Thm. 4.2] and [26, Cor. 7.3] for the supercuspidal parts of cohomology. Concretely, although they have not shown that (4.7) and (4.8) (in the PEL-type cases they consider) are isomorphisms, they showed that the kernels and cokernels do not contain supercuspidal representations. Their method based on the consideration of adic spaces is quite flexible and of some independent interest.

Corollary 4.10 (cf. [41, Cor. 5.23]). *In all Cases* (Sm)*,* (Nm)*,* (Spl)*,* (Hdg)*, and* (Ab)*, for all $\mathcal{V}$ over $\Lambda = \bar{\mathbb{Q}}_\ell$ or $\mathbb{Z}_\ell$ (or $\bar{\mathbb{Z}}_\ell$), the adjunction morphisms*

$$R\Psi_{\mathsf{X}_{\mathcal{H}}^{\min}} RJ_{\eta,*}^{\min}(\mathcal{V}) \to RJ_{\bar{s},*}^{\min} R\Psi_{\mathsf{X}_{\mathcal{H}}}(\mathcal{V}) \tag{4.11}$$

and

$$J_{\bar{s},!}^{\min} R\Psi_{\mathsf{X}_{\mathcal{H}}}(\mathcal{V}) \to R\Psi_{\mathsf{X}_{\mathcal{H}}^{\min}} J_{\eta,!}^{\min}(\mathcal{V}) \tag{4.12}$$

are isomorphisms in $D_c^b((\mathsf{X}_{\mathcal{H}}^{\min})_{\bar{s}} \times \bar{\eta}, \Lambda)$.

Proof. In Cases (Sm), (Nm), (Spl), and (Hdg), this follows from Theorem 4.1 and the proper base change theorem, as in the proof of [41, Cor. 5.23].

In Case (Ab), by similar reduction steps as in the proof of Corollary 4.6, by Proposition 2.4 and Remark 2.6, and by using the Hochschild–Serre spectral sequence (see, for example, [15, Ch. 9, Sec. 9.1, p. 501]) for the (derived) direct images towards $\mathsf{X}_{\mathcal{H},R^+}^{\min,+}$ (which is now viewed as a base scheme for all other schemes), the desired isomorphism assertion is reduced to the known one in Case (Hdg). □

Theorem 4.13 (cf. [41, Thm. 5.26]). *In all Cases* (Sm)*,* (Nm)*,* (Spl)*,* (Hdg)*, and* (Ab)*, for all $\mathcal{V}$ over $\Lambda = \bar{\mathbb{Q}}_\ell$ and $d = \dim((\mathsf{X}_{\mathcal{H}})_\eta)$, we also have a canonically induced isomorphism*

$$R\Psi_{\mathsf{X}_{\mathcal{H}}^{\min}} J_{\eta,!*}^{\min}(\mathcal{V}[d]) \xrightarrow{\sim} J_{\bar{s},!*}^{\min} R\Psi_{\mathsf{X}_{\mathcal{H}}}(\mathcal{V}[d]) \tag{4.14}$$

in the category of perverse sheaves over $(\mathsf{X}_{\mathcal{H}}^{\min})_{\bar{s}}$ with continuous $\mathrm{Gal}(\bar{K}/K)$*-actions.*

Proof. This follows from Corollary 4.10 by the same argument as in the proof of [41, Thm. 5.26], using the t-exactness of nearby cycle functors (as in [24, 4.5]). □

Corollary 4.15 (cf. [41, Cor. 5.31]). *In all Cases* (Sm)*,* (Nm)*,* (Spl)*,* (Hdg)*, and* (Ab)*, for all $\mathcal{V}$ over $\Lambda = \bar{\mathbb{Q}}_\ell$, we have a canonical isomorphism*

$$H_{\text{ét}}^i((\mathsf{X}_{\mathcal{H}}^{\min})_{\bar{\eta}}, (J_{\bar{\eta},!*}^{\min}(\mathcal{V}[d]))[-d]) \xrightarrow{\sim} H^i((\mathsf{X}_{\mathcal{H}}^{\min})_{\bar{s}}, (J_{\bar{s},!*}^{\min}(R\Psi_{\mathsf{X}_{\mathcal{H}}}(\mathcal{V}[d])))[-d]) \tag{4.16}$$

for the **intersection cohomology***, for all i.*

Proof. This follows from Theorem 4.13 and the proper base change theorem, by the same argument as in the proof of [41, Cor. 5.31]. □

Remark 4.17. For the compatibility of the isomorphisms in Theorem 4.1, Corollaries 4.6 and 4.10, Theorem 4.18, and Corollary 4.15 with Hecke actions (when they are defined), we refer the readers to [41, Rem. 5.35 and 5.41].

As an application, let us answer the question we raised in Section 1 about cases where the étale cohomology of (possibly nonproper) Shimura varieties is unramified:

Theorem 4.18 (cf. [41, Thm. 6.1 and 6.7]). *Suppose we are in Case* (Sm)*, or in Case* (Hdg) *and* (Ab) *when the level at p is hyperspecial. Then $\mathcal{V} \xrightarrow{\sim} R\Psi_{\mathsf{X}_{\mathcal{H}}}(\mathcal{V})$, for all $\mathcal{V}$ over $\Lambda = \bar{\mathbb{Q}}_\ell$ or $\mathbb{Z}_\ell$ (or $\bar{\mathbb{Z}}_\ell$), because $\mathsf{X}_{\mathcal{H}} \to S$ is a smooth morphism. Moreover, we have the following canonical isomorphisms of* $\mathrm{Gal}(\bar{K}/K)$*-modules, for each i: for the usual cohomology,*

$$H_{\text{ét}}^i((\mathsf{X}_{\mathcal{H}})_{\bar{\eta}}, \mathcal{V}) \xrightarrow{\sim} H_{\text{ét}}^i((\mathsf{X}_{\mathcal{H}})_{\bar{s}}, \mathcal{V}); \tag{4.19}$$

for the compactly supported cohomology,

$$H^i_{\text{ét},c}((\mathsf{X}_{\mathcal{H}})_{\bar{s}}, \mathcal{V}) \xrightarrow{\sim} H^i_{\text{ét},c}((\mathsf{X}_{\mathcal{H}})_{\bar{\eta}}, \mathcal{V}); \tag{4.20}$$

and, when $\Lambda = \bar{\mathbb{Q}}_\ell$, *for the intersection cohomology (of the minimal compactification),*

$$H^i_{\text{ét}}((\mathsf{X}^{\min}_{\mathcal{H}})_{\bar{\eta}}, (J^{\min}_{\bar{\eta},!*}(\mathcal{V}[d]))[-d]) \xrightarrow{\sim} H^i_{\text{ét}}((\mathsf{X}^{\min}_{\mathcal{H}})_{\bar{s}}, (J^{\min}_{\bar{s},!*}(\mathcal{V}[d]))[-d]) \tag{4.21}$$

where $d = \dim((\mathsf{X}_{\mathcal{H}})_\eta)$. *In particular, these* $\mathrm{Gal}(\bar{K}/K)$*-modules are* **unramified** *(namely, the inertia subgroup* $I_K := \ker(\mathrm{Gal}(\bar{K}/K) \to \mathrm{Gal}(\bar{k}/k))$ *acts trivially on them). If* $\Lambda = \bar{\mathbb{Q}}_\ell$, *and if* $\mathcal{V}$ *is pointwise pure of weight* m *(which is known, for example, in the cases considered in* [41, Prop. 3.2]*), then both sides of* (4.19) *(resp.* (4.20)*) are mixed of weights* $\geq i+m$ *(resp.* $\leq i+m$*), and both sides of* (4.21) *are pure of weight* $i+m$.

Proof. As in the proofs of [41, Thm. 6.1 and 6.7], these follow from the smooth base change theorem, from Corollaries 4.6 and 4.15, and from [12, 3.3.4, 3.3.5, and 6.2.6] and [4, 5.3.2]. □

Theorem 4.22 (cf. [41, Thm. 6.8 and 6.13])**.** *Suppose we are in Cases* (Nm), (Spl), (Hdg), *and* (Ab) *where the levels at* p *are* **parahoric**, *and suppose* $p > 2$. *(Note that, as explained in Remark* 2.1, *we have to treat some integral models of Hodge-type Shimura varieties with parahoric levels at* p *only as abelian-type ones in Case* (Ab), *although this is harmless for our purpose.) In Cases* (Nm) *or* (Spl), *we consider the same cases as in* [41, Thm. 6.8 and 6.13], *with a field extension* $\widetilde{K}$ *of* K *in* $\bar{K}$ *defined there. In Cases* (Hdg) *and* (Ab), *we assume that* $\widetilde{K}$ *is a tamely ramified extension of* K *in* $\bar{K}$ *over which* $\mathrm{G}_{\mathbb{Q}_p}$ *is split. Then, for each* i, *we have the following canonical isomorphisms of* $\mathrm{Gal}(\bar{K}/K)$*-modules: for the usual cohomology,*

$$H^i_{\text{ét}}((\mathsf{X}_{\mathcal{H}})_{\bar{\eta}}, \mathcal{V}) \xrightarrow{\sim} H^i_{\text{ét}}((\mathsf{X}_{\mathcal{H}})_{\bar{s}}, R\Psi_{\mathsf{X}_{\mathcal{H}}}(\mathcal{V})); \tag{4.23}$$

for the compactly supported cohomology,

$$H^i_{\text{ét},c}((\mathsf{X}_{\mathcal{H}})_{\bar{s}}, R\Psi_{\mathsf{X}_{\mathcal{H}}}(\mathcal{V})) \xrightarrow{\sim} H^i_{\text{ét},c}((\mathsf{X}_{\mathcal{H}})_{\bar{\eta}}, \mathcal{V}); \tag{4.24}$$

and, when $\Lambda = \bar{\mathbb{Q}}_\ell$, *for the intersection cohomology (of the minimal compactification),*

$$H^i_{\text{ét}}((\mathsf{X}^{\min}_{\mathcal{H}})_{\bar{\eta}}, (J^{\min}_{\bar{\eta},!*}(\mathcal{V}[d]))[-d]) \xrightarrow{\sim} H^i_{\text{ét}}((\mathsf{X}^{\min}_{\mathcal{H}})_{\bar{s}}, (J^{\min}_{\bar{s},!*} R\Psi_{\mathsf{X}_{\mathcal{H}}}(\mathcal{V}[d]))[-d]), \tag{4.25}$$

where $d = \dim((\mathsf{X}_{\mathcal{H}})_\eta)$. *Moreover, the restrictions of the* $\mathrm{Gal}(\bar{K}/K)$*-actions of these modules to* $I_{\widetilde{K}} := \ker(\mathrm{Gal}(\bar{K}/\widetilde{K}) \to \mathrm{Gal}(\bar{k}/k))$ *(but not* I_K*) are all* **unipotent**, *and even* **trivial** *when the level at* p *is* **very special** *(as in* [52, Sec. 10.3.2]*).*

Proof. As in the proofs of [41, Thm. 6.8 and 6.13], these follow from Corollaries 4.6 and 4.15, and from the results of local models in [52, Thm. 1.4, and more detailed results in Sec. 10.3] and [51, Thm. 13.1 and Rem. 13.2] (see also [50, Rem. 7.4]) in Cases (Nm) and (Spl); and in [30, Cor. 0.5 and 4.7.3] in Cases (Hdg) and (Ab). □

Remark 4.26 (cf. [41, Rem. 6.15]). The isomorphism (4.24) in Theorem 4.22 established [21, Conj. 10.3] for all integral models of PEL-type Shimura varieties (with parahoric levels at p) considered in [52] and [51]. Also, we have established its analogous for all integral models of Hodge-type and abelian-type Shimura varieties (with parahoric levels at p) considered in [30].

Remark 4.27. Theorems 4.18 and 4.22 might convey the impression that our main results are only useful at hyperspecial and parahoric levels, in which case we have good theories of local models, but this is not true. In fact, our initial motivation was to generalize Mantovan's formula (describing the cohomology of Shimura varieties in terms of the cohomology of Igusa varieties and Rapoport–Zink spaces) in [44] and [45], and also Scholze's formula (in the context of Langlands–Kottwitz method, describing the cohomology of Shimura varieties in terms of twisted orbital integrals) in [55], to the noncompact case (i.e., removing the assumption that the relevant Shimura varieties are compact from their works). In these works, as explained in the introduction of [41], the analysis of the cohomology of nearby cycles were carried out without the compactness assumption. It is only in their initial steps—or final steps, depending on one's viewpoint—that they used the compactness assumption and the proper base change theorem to relate their results to the étale cohomology in characteristic zero, and our main results above removed the need of the compactness assumption. We emphasize that these generalizations do require our main results in Case (Nm) with arbitrarily high levels at p. See [41, Sec. 6.3] (for the compactly supported cohomology) and [40, Sec. 4.4] (for the usual cohomology, with boundary terms) for more details concerning the generalizations of Mantovan's formula, and see [41, Sec. 6.4] for more details concerning the generalization of Scholze's formula. Moreover, a combination of the generalizations of Mantovan's formula in [40, Sec. 4.4] and of Morel's formula in [40, Sec. 4.5] provides a formula for certain intersection complexes over the partial minimal compactifications of Newton strata, which is useful for Caraiani and Scholze's work in progress on extending their results in [7] to the noncompact case.

5. Proof of the key theorem

In this section, we explain how to prove Theorem 4.1. It suffices to show that the globally defined morphisms (4.2) and (4.3) are isomorphisms *étale locally.*

The rough idea is that any toroidal compactification $\mathsf{X}_{\mathcal{H}} \hookrightarrow \mathsf{X}_{\mathcal{H}}^{\mathrm{tor}}$ we consider in Cases (Sm), (Nm), (Spl), and (Hdg) is étale locally at each point a product of some affine toroidal embedding $E \hookrightarrow E(\sigma)$, which we know everything about, and the identity morphism Id_C of some scheme C, which we do not need to know anything about. Then we can try to reduce the problem to its analogues over the individual factors $E \hookrightarrow E(\sigma)$ and $\mathrm{Id}_C : C \hookrightarrow C$, by using the Künneth isomorphisms as in [2, XVII, 5.4.3] and [4, 4.2.7], and by using Gabber's theorem (see [24, 4.7]) on nearby cycles over products of schemes of finite type over S. (We will explain these in more details. This idea can be traced back to the lemma [17, 7.1.4] due to Laumon. See also [41, Rem. 5.33]. Now we can do more because we have a much better understanding of integral models of Shimura varieties and their toroidal compactifications.) To carry out this idea, we need to have a better control on the étale sheaf $\mathcal{V}$. (What we are about to do is different from what we did in [41, Sec. 4–5]. We will comment on the difference in Remark 5.14 below.)

By Lemma 3.6, we may assume that $\mathcal{V}$ is torsion and associated with some finite $\mathbb{Z}_\ell$-module (with $\Lambda = \mathbb{Z}_\ell$), and that there exists some $r > 0$ such that $\mathcal{H}(\ell^r) \subset \mathcal{H}$ and such that the pullback of $\mathcal{V}$ from $\mathsf{X}_{\mathcal{H}}$ to $\mathsf{X}_{\mathcal{H}(\ell^r)}$ is a constant étale sheaf. Let $\mathsf{X}_{\mathcal{H}(\ell^r)}^{\mathrm{tor}}$ be defined using the collection of cone decompositions induced by that for $\mathsf{X}_{\mathcal{H}}^{\mathrm{tor}}$, so that $\mathsf{X}_{\mathcal{H}(\ell^r)}^{\mathrm{tor}} \to \mathsf{X}_{\mathcal{H}}^{\mathrm{tor}}$ is *finite*. (In Case (Sm), this might move us from the context of [34] into that of [37], because the former assumed that the cone decompositions are always smooth, but this is harmless in practice.)

To understand this finite morphism better, we have the following proposition, in which we shall freely use the notation in [41, Prop. 2.2]: (First-time readers might assume that $\mathcal{V} = \Lambda$ is trivial, and skip all materials in Propositions 5.1 and 5.3 below involving objects at level $\mathsf{X}^{\mathrm{tor}}_{\mathcal{H}(\ell^r)}$, which are denoted with a prime.)

Proposition 5.1. *Let x be a point of a stratum $\mathsf{Z}_{[\sigma]}$ of $\mathsf{X}^{\mathrm{tor}}_{\mathcal{H}}$. Then there exists an étale neighborhood $\overline{U} \to \mathsf{X}^{\mathrm{tor}}_{\mathcal{H}}$ of x such that, if we denote by $\overline{U}'$ the pullback of $\overline{U}$ to $\mathsf{X}^{\mathrm{tor}}_{\mathcal{H}(\ell^r)}$, then there is a commutative diagram*

$$\begin{tikzcd} \mathsf{X}^{\mathrm{tor}}_{\mathcal{H}(\ell^r)} \arrow[d] & \overline{U}' \arrow[l] \arrow[r] \arrow[d] & \coprod\limits_{\mathsf{Z}'_{[\sigma']} \text{ lying above } \mathsf{Z}_{[\sigma]}} \big(E'(\sigma') \underset{S}{\times} C'\big) \arrow[d] \\ \mathsf{X}^{\mathrm{tor}}_{\mathcal{H}} & \overline{U} \arrow[l] \arrow[r] & E(\sigma) \underset{S}{\times} C \end{tikzcd} \tag{5.2}$$

with the following properties:

1. *All objects denoted with a prime are at level $\mathcal{H}(\ell^r)$.*
2. *The vertical morphisms are equipped with compatible actions of $\mathcal{H}/(\mathcal{H}(\ell^r)) \cong \mathcal{H}_\ell/\mathcal{U}_\ell(\ell^r)$ (see Section 3—note that $\mathrm{G} \cong \mathrm{G}^c$ now) that are trivial on the targets and induce isomorphisms from quotients of the sources to the targets.*
3. *The horizontal morphisms are étale, and the squares are Cartesian.*
4. *The morphism $C' \to C$ is finite étale.*
5. *The $E(\sigma)$ is defined by an affine toroidal embedding $E \hookrightarrow E(\sigma)$ for a split torus E, and the $E'(\sigma')$'s are all defined by similar affine toroidal embeddings $E' \hookrightarrow E'(\sigma')$ for a split tori E'. (For simplicity, we only define the tori E and E' over S here.) The morphisms $E'(\sigma') \to E(\sigma)$ are finite flat and tamely ramified, extending isogenies $E' \to E$ of tori of ℓ-power degrees.*
6. *The preimages of $\mathsf{X}_{\mathcal{H}}$ and $E \underset{S}{\times} C$ in $\overline{U}$ are the same open subscheme U, and the preimages of U, $\mathsf{X}_{\mathcal{H}(\ell^r)}$, and $\coprod\limits_{\mathsf{Z}'_{[\sigma']} \text{ lying above } \mathsf{Z}_{[\sigma]}} \big(E' \underset{S}{\times} C'\big)$ in $\overline{U}'$ are the same subscheme U'.*

Consequently, the pullbacks of $\mathcal{V}$ to U descends to an étale sheaf over $E \underset{S}{\times} C$, which we abusively still denote by $\mathcal{V}$, and the pullback of $\mathcal{V}$ to $\coprod\limits_{\mathsf{Z}'_{[\sigma']} \text{ lying above } \mathsf{Z}_{[\sigma]}} (E' \underset{S}{\times} C')$ is constant, whose further pullback to U' is the same pullback of $\mathcal{V}$ from $\mathsf{X}_{\mathcal{H}}$.

Proof. First note that the morphisms $C' \to C$, $E' \to E$, and $E'(\sigma') \to E(\sigma)$ are defined regardless of the étale neighborhood $\overline{U}$. The morphisms $C' \to C$ are finite étale because we can define them alternatively as morphisms relatively representing functors defining certain level structures at $\ell \neq p$, which are finite étale and hence must coincide with the construction by normalizations, by Zariski's main theorem (see [20, III-1, 4.4.3, 4.4.11]), since they already agree in characteristic zero (see [34, Sec. 6.2.4; see also the errata] and the reinterpretations in [39, Sec. 1.3.2] in Case (Sm); see [37, Sec. 8] in Case (Nm); see [35, Sec. 3.2] in Case (Spl); and see [43, Sec. 2.1.7 and 4.1–4.2] in Case (Hdg)). The morphisms $E' \to E$ are isogenies of tori of

ℓ-power degrees, which are dual to homomorphisms $\mathbf{S} \to \mathbf{S}'$ of character groups that are in turn dual to inclusions of lattices of ℓ-power indices induced by $\mathcal{H}(\ell^r) \hookrightarrow \mathcal{H}$ (see [34, Lem. 6.2.4.4] and [32, Cor. 3.6.10] in Cases (Sm), (Nm), and (Spl); and see [43, Sec. 2.1.11] in Case (Hdg)). Consequently, because of the collection of cone decompositions for $\mathsf{X}^{\mathrm{tor}}_{\mathcal{H}(\ell^r)}$ is induced by that for $\mathsf{X}^{\mathrm{tor}}_{\mathcal{H}}$, the induced morphisms $E'(\sigma') \to E(\sigma)$ are finite flat and tamely ramified. The rest of the proposition then follows from an analogue of the approximation argument in the proofs of [41, Prop. and Cor. 2.4], by also approximating the objects at level $\mathcal{H}(\ell^r)$ and the action of the finite group $\mathcal{H}/(\mathcal{H}(\ell^r)) \cong \mathcal{H}_\ell/\mathcal{U}_\ell(\ell^r)$ on these objects. $\square$

Proposition 5.3. *In the context of Proposition* 5.1*, we have the following commutative diagram*

$$\begin{tikzcd}
\coprod\limits_{\mathsf{Z}'_{[\sigma']} \text{ lying above } \mathsf{Z}_{[\sigma]}} \big(E' \underset{S}{\times} C'\big) \arrow[r, hook] \arrow[d] & \coprod\limits_{\mathsf{Z}'_{[\sigma']} \text{ lying above } \mathsf{Z}_{[\sigma]}} \big(E'(\sigma') \underset{S}{\times} C'\big) \arrow[d] \\
\coprod\limits_{\mathsf{Z}'_{[\sigma']} \text{ lying above } \mathsf{Z}_{[\sigma]}} \big(E \underset{S}{\times} C'\big) \arrow[r, hook] \arrow[d] & \coprod\limits_{\mathsf{Z}'_{[\sigma']} \text{ lying above } \mathsf{Z}_{[\sigma]}} \big(E(\sigma) \underset{S}{\times} C'\big) \arrow[d] \\
E \underset{S}{\times} C \arrow[r] & E(\sigma) \underset{S}{\times} C
\end{tikzcd} \tag{5.4}$$

of canonical morphisms, with the following properties:

1. *(1) All squares are Cartesian, and the composition of the vertical morphisms at the right-hand side is the canonical morphism in* (5.2)*.*
2. *(2) The bottom two vertical morphisms are pullbacks of the finite étale morphism* $\coprod\limits_{\mathsf{Z}'_{[\sigma']} \text{ lying above } \mathsf{Z}_{[\sigma]}} C' \to C$*, and the top two morphisms are pullbacks of the morphisms* $\coprod\limits_{\mathsf{Z}'_{[\sigma']} \text{ lying above } \mathsf{Z}_{[\sigma]}} E'(\sigma') \to \coprod\limits_{\mathsf{Z}'_{[\sigma']} \text{ lying above } \mathsf{Z}_{[\sigma]}} E(\sigma)$*.*
3. *(3) For each particular* $\mathsf{Z}'_{[\sigma']}$ *lying above* $\mathsf{Z}_{[\sigma]}$*, there exist finite groups*

$$1 \subset H_{-2} \subset H_{-1} \subset H := \mathcal{H}_\ell/\mathcal{U}_\ell(\ell^r) \tag{5.5}$$

such that $C' \to C$ *is a Galois finite étale cover with Galois group* H_{-1}/H_{-2}*, and such that* $E' \to E$ *is a Galois finite étale cover with Galois group* H_{-2}*. Consequently, the pullback of the étale sheaf* $\mathcal{V}$ *under the finite étale morphism* $E \underset{S}{\times} C' \to E \underset{S}{\times} C$*, which we abusively denote by the same symbol* $\mathcal{V}$*, descends to an étale sheaf* $\mathcal{V}_E$ *over* E *(defined by the action of* H_{-2} *only). That is,* $\mathcal{V} \cong \mathcal{V}_E \underset{S}{\boxtimes} \Lambda$ *over* $E \underset{S}{\times} C'$*, and the pullback of* $\mathcal{V}_E$ *to* E' *is constant.*

Proof. These are because of the very constructions of these boundary objects, which can be compatibly realized as quotients of objects of higher principal levels. (See the same references given in the proof of Proposition 5.1.) $\square$

Thus, in order to prove Theorem 4.1, it suffices to show that, for any point x of $\mathsf{X}^{\mathrm{tor}}_{\mathcal{H}}$ as in Proposition 5.1, together with the commutative diagrams (5.2) and (5.4) of morphisms (with properties as in Propositions 5.1 and 5.3), if we denote by

$$J_{E(\sigma)\underset{S}{\times}C'} = J_{E(\sigma)} \times \mathrm{Id}_{C'} : E \underset{S}{\times} C' \hookrightarrow E(\sigma) \underset{S}{\times} C' \tag{5.6}$$

the canonical open immersion, where $J_{E(\sigma)} : E \hookrightarrow E(\sigma)$ is the affine toroidal embedding and $\mathrm{Id}_{C'}$ is the identity morphism on C', then the adjunction morphisms

$$R\Psi_{E(\sigma)\underset{S}{\times}C'}\, RJ_{E(\sigma)\underset{S}{\times}C',\eta,*}(\mathcal{V}) \to RJ_{E(\sigma)\underset{S}{\times}C',\bar{s},*}\, R\Psi_{E\underset{S}{\times}C'}(\mathcal{V}) \tag{5.7}$$

and

$$J_{E(\sigma)\underset{S}{\times}C',\bar{s},!}\, R\Psi_{E\underset{S}{\times}C'}(\mathcal{V}) \to R\Psi_{E(\sigma)\underset{S}{\times}C'}\, J_{E(\sigma)\underset{S}{\times}C',\eta,!}(\mathcal{V}) \tag{5.8}$$

are isomorphisms.

Since $\mathcal{V} \cong \mathcal{V}_E \underset{S}{\boxtimes} \Lambda$ over $E \underset{S}{\times} C'$ for some étale sheaf $\mathcal{V}_E$ over E whose pullback to E' is constant (see Proposition 5.3), by the Künneth isomorphisms (see [2, XVII, 5.4.3] and [4, 4.2.7]), and by Gabber's theorem (see [24, 4.7]) on nearby cycles over products of schemes of finite type over S, we have canonical isomorphisms

$$\begin{aligned}
(R\Psi_{E(\sigma)}\, RJ_{E(\sigma),\eta,*}(\mathcal{V}_E)) \overset{L}{\underset{\bar{s}}{\boxtimes}} (R\Psi_{C'}(\Lambda)) &\xrightarrow{\sim} R\Psi_{E(\sigma)\underset{S}{\times}C'}((RJ_{E(\sigma),\eta,*}(\mathcal{V}_E)) \overset{L}{\underset{\eta}{\boxtimes}} \Lambda) \\
&\xrightarrow{\sim} R\Psi_{E(\sigma)\underset{S}{\times}C'}\, RJ_{E(\sigma)\underset{S}{\times}C',\eta,*}(\mathcal{V}_E \underset{\eta}{\boxtimes} \Lambda),
\end{aligned}$$

$$\begin{aligned}
(RJ_{E(\sigma),\bar{s},*}\, R\Psi_E(\mathcal{V}_E)) \overset{L}{\underset{\bar{s}}{\boxtimes}} (R\Psi_{C'}(\Lambda)) &\xrightarrow{\sim} RJ_{E(\sigma)\underset{S}{\times}C',\bar{s},*}((R\Psi_E(\mathcal{V}_E)) \overset{L}{\underset{\bar{s}}{\boxtimes}} (R\Psi_{C'}(\Lambda))) \\
&\xrightarrow{\sim} RJ_{E(\sigma)\underset{S}{\times}C',\bar{s},*}\, R\Psi_{E\underset{S}{\times}C'}(\mathcal{V}_E \underset{\eta}{\boxtimes} \Lambda),
\end{aligned}$$

$$\begin{aligned}
(J_{E(\sigma),\bar{s},!}\, R\Psi_E(\mathcal{V}_E)) \overset{L}{\underset{\bar{s}}{\boxtimes}} (R\Psi_{C'}(\Lambda)) &\xrightarrow{\sim} J_{E(\sigma)\underset{S}{\times}C',\bar{s},!}((R\Psi_E(\mathcal{V}_E)) \overset{L}{\underset{\bar{s}}{\boxtimes}} (R\Psi_{C'}(\Lambda))) \\
&\xrightarrow{\sim} J_{E(\sigma)\underset{S}{\times}C',\bar{s},!}\, R\Psi_{E\underset{S}{\times}C'}(\mathcal{V}_E \underset{\eta}{\boxtimes} \Lambda),
\end{aligned}$$

and

$$\begin{aligned}
(R\Psi_{E(\sigma)}\, J_{E(\sigma),\eta,!}(\mathcal{V}_E)) \overset{L}{\underset{\bar{s}}{\boxtimes}} (R\Psi_{C'}(\Lambda)) &\xrightarrow{\sim} R\Psi_{E(\sigma)\underset{S}{\times}C'}((J_{E(\sigma),\eta,!}(\mathcal{V}_E)) \overset{L}{\underset{\eta}{\boxtimes}} \Lambda) \\
&\xrightarrow{\sim} R\Psi_{E(\sigma)\underset{S}{\times}C'}\, J_{E(\sigma)\underset{S}{\times}C',\eta,!}(\mathcal{V}_E \underset{\eta}{\boxtimes} \Lambda),
\end{aligned}$$

which are compatible with each other under the adjunction morphisms

$$R\Psi_{E(\sigma)}\, RJ_{E(\sigma),\eta,*}(\mathcal{V}_E) \to RJ_{E(\sigma),\bar{s},*}\, R\Psi_E(\mathcal{V}_E), \tag{5.9}$$

$$J_{E(\sigma),\bar{s},!}\, R\Psi_E(\mathcal{V}_E) \to R\Psi_{E(\sigma)}\, J_{E(\sigma),\eta,!}(\mathcal{V}_E), \tag{5.10}$$

(5.7), and (5.8) (and the identity morphism on $R\Psi_{C'}(\Lambda)$). Hence, in order to show that the adjunction morphisms (5.7) and (5.8) are isomorphisms, it suffices to show that the simpler adjunction morphisms (5.9) and (5.10) are.

We can complete the collection consisting of σ and its faces into a cone decomposition Σ of $\mathbf{S}^\vee_{\mathbb{R}} = \mathrm{Hom}_{\mathbb{Z}}(\mathbf{S}, \mathbb{R})$, which defines a toroidal embedding

$$J_{\overline{E}} : E \hookrightarrow \overline{E} \tag{5.11}$$

over S, such that $\overline{E}$ is *proper* over S (see [27, Ch. I, Sec. 2, Thm. 8]), which contains $E(\sigma)$ as an open subscheme. Then it suffices to show that the corresponding adjunction morphisms

$$R\Psi_{\overline{E}}\, RJ_{\overline{E},\eta,*}(\mathcal{V}_E) \to RJ_{\overline{E},\bar{s},*}\, R\Psi_E(\mathcal{V}_E) \tag{5.12}$$

and

$$J_{\overline{E},\bar{s},!}\,R\Psi_E(\mathcal{V}_E) \to R\Psi_{\overline{E}}\,J_{\overline{E},\eta,!}(\mathcal{V}_E) \tag{5.13}$$

are isomorphisms. Since smooth refinements of cone decompositions induce *proper morphisms* between the corresponding toroidal embeddings (again, see [27, Ch. I, Sec. 2, Thm. 8]), by the proper base change theorem, we may and we shall assume in the above that $\overline{E}$ is proper and *smooth* over S, and that the boundary $\overline{E} - E$ with its reduced subscheme structure is a simple normal crossings divisor on $\overline{E}$.

Finally, let us consider the étale sheaf $\mathcal{V}_E$ over E, whose pullback under the isogeny $E' \to E$ of ℓ-power degree is constant, which is then *tamely ramified* along the boundary $\overline{E} - E$. Therefore, the adjunction morphisms (5.12) and (5.13) are isomorphisms, by [13, XIII, 2.1.9]. The proof of Theorem 4.1 is now complete. □

Remark 5.14. Compared with the proof of [41, Thm. 5.15], we have not used the Kuga families and their toroidal compactifications as in [41, Sec. 4] (which were based on ideas in [33]). As a result, we do not have any restriction in the case of integral or torsion coefficient. (In [41, Thm. 5.15], we assumed that $\mathcal{V}$ is either over $\Lambda = \bar{\mathbb{Q}}_\ell$, or of the form $\mathcal{W}_{0,M}$ over $\Lambda = \mathbb{Z}_\ell$ as in [41, Prop. 3.4] for some $\ell > c_M$.) Our argument here can be considered as a fulfilment of the strategy in [41, Rem. 5.35], although its actual execution here is subtler than suggested there.

Remark 5.15. We remarked in the introduction of [41] that the argument in [23, Sec. 7], for the usual and compactly supported cohomology in Case (Sm), is unfortunately incomplete, because the first step in the proof of [23, Lem. 7.1] should require some tameness assumption as in [13, XIII, 2.1.9]. Nevertheless, our proof in this section shows that the tameness assumption can indeed be verified.

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