RELATIVE COHOMOLOGY OF CUSPIDAL FORMS ON PEL-TYPE SHIMURA VARIETIES

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Abstract. We present a short proof that, for PEL-type Shimura varieties, subcanonical extensions of automorphic bundles, whose global sections over toroidal compactifications of Shimura varieties are represented by cuspidal automorphic forms, have no higher direct images under the canonical morphism to the minimal compactification, in characteristic zero or in positive characteristics greater than an explicitly computable bound.

1. Introduction

The main goal of this article is to present a short proof of Theorem 1.1 below, as an application of certain vanishing theorem of automorphic bundles in mixed characteristics. (We shall refer to [16, 19, 20] for the precise definitions and descriptions of smooth integral models of PEL-type Shimura varieties and their various compactifications, and of the automorphic bundles and their canonical and subcanonical extensions.)

Theorem 1.1. *Let* $\pi : \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma} \to \mathsf{M}^{\mathrm{min}}_{\mathcal{H}}$ *denote the canonical proper morphism from any projective smooth toroidal compactification to the minimal compactification of a* p*-integral model* $\mathsf{M}_{\mathcal{H}}$ *of a PEL-type Shimura variety at a neat level* $\mathcal{H} \subset \mathrm{G}(\hat{\mathbb{Z}}^p)$, *where* p *is* **good** *for the integral PEL datum* $(\mathcal{O}, \star, L, \langle\,\cdot\,,\,\cdot\,\rangle, h_0)$ *defining* $\mathsf{M}_{\mathcal{H}}$, *as in* [20, §4.1] (*and the references there*). *Let* $\underline{W}_{\nu_0,R}$ *be an automorphic bundle over* $\mathsf{M}_{\mathcal{H},R}$ *of weight* $\nu_0 \in \mathrm{X}^{+,<p}_{\mathrm{M}_1}$, *where* R *denotes the coefficient ring, and let* $\underline{W}^{\mathrm{sub}}_{\nu_0,R}$ *denote its subcanonical extension to* $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}$, *as in* [19, §6.3] *and* [20, §7] (*and the references there*). *Then*

$$R^i \pi_* \underline{W}^{\mathrm{sub}}_{\nu_0,R} = 0 \tag{1.2}$$

for all $i > 0$, *provided that the residue characteristics of* R *are zero or* p *greater than a bound* $C(\nu_0)$ *depending only on the integral PEL datum* $(\mathcal{O}, \star, L, \langle\,\cdot\,,\,\cdot\,\rangle, h_0)$ *and the weight* ν_0. (*See Lemma* 3.3 *below for an explicit choice of* $C(\nu_0)$.)

We note that, when $R = \mathbb{C}$, global sections of $\underline{W}^{\mathrm{sub}}_{\nu_0,R}$ over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ can be represented by holomorphic cuspidal automorphic forms. (See, e.g., [11, Prop. 5.4.2]; see also [10] for a survey on how the higher cohomology of $\underline{W}^{\mathrm{sub}}_{\nu_0,R}$ can be represented by

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nonholomorphic automorphic forms. See [15] for the comparison between algebraic and analytic constructions hidden behind this.) Combined with the Leray spectral sequence, Theorem 1.1 allows one to identify the cohomology of $\underline{W}^{\mathrm{sub}}_{\nu_0,R}$ over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ with the cohomology of $\pi_*\underline{W}^{\mathrm{sub}}_{\nu_0,R}$ over $\mathsf{M}^{\mathrm{min}}_{\mathcal{H}}$. Although the coherent sheaf $\pi_*\underline{W}^{\mathrm{sub}}_{\nu_0,R}$ is not locally free in general, there are reasons for $\mathsf{M}^{\mathrm{min}}_{\mathcal{H}}$ to be useful for the construction of p-adic modular forms and p-adic Galois representations.

Special cases of Theorem 1.1 has been (independently) proved in [2, 1] (in the Siegel and Hilbert cases, for trivial weight ν_0) and in [12] (in the unitary case, for all weights ν_0), without the assumption on residue characteristic p. The idea in [12] has also been carried out for all PEL-type cases in [17]. Such results have played crucial roles in positive characteristics in [2, 1, 5, 23], and in characteristic zero in [12, 26]. The proofs in [2, 1] and [12, 17] directly used the toroidal and minimal boundary structures, and hence can be considered more elementary, which is why they work for all residue characteristics p; but they are lengthier and arguably more complicated. It is not easy to see from their proofs why Theorem 1.1 should be true. (It is not even clear how the two strategies in [2, 1] and [12, 17] are related to each other.) Thus it is desirable to find a proof more closely related to other vanishing statements, at least in characteristic zero or sufficiently large characteristics.

It was first observed by the second author that this is indeed possible—in characteristic zero, the trivial-weight case can be deduced from Grauert and Riemenschneider's vanishing theorem [8]; in positive characteristics, under suitable assumptions (involving choices of projective but generally nonsmooth cone decompositions Σ for the toroidal compactification $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$, whose existence is not very clearly documented in the literature), it is also possible to deduce the statement from Deligne and Illusie's and Kato's vanishing theorems [4, 14]. Then the first author made the observations that the assumption on cone decompositions can be relaxed by using Esnault and Viehweg's vanishing theorem [6] as in [18], and that (along similar lines) cases of nontrivial weights can be treated using stronger vanishing theorems in [20]. (In the Siegel case, one can also use [24, 25].)

In Section 2, we will present the proof of Theorem 1.1 and highlight the main inputs. In Section 3, we will carry out some elementary computations needed in the proof of Theorem 1.1, and find an explicit choice of $C(\nu_0)$. In Section 4, we sketch a logically simpler proof for the trivial-weight case.

2. PROOF OF THE THEOREM

Let $\pi : \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma} \to \mathsf{M}^{\mathrm{min}}_{\mathcal{H}}$, $\underline{W}^{\mathrm{sub}}_{\nu_0,R}$, and $\nu_0 \in \mathrm{X}^{+,<p}_{\mathrm{M}_1}$ be as in Theorem 1.1. Since $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}$ and $\mathsf{M}^{\mathrm{min}}_{\mathcal{H},1}$ are proper over $\mathsf{S}_1 = \mathrm{Spec}(R_1)$ (see [20, §4.1] and the references there for the notation), which are in particular separated and of finite type, for the purpose of proving Theorem 1.1, we may write R as an inductive limit over its sub-R_1-algebras, and assume that R is of finite type over R_1, which is in particular noetherian. Then we may base change to R and (abusively) denote $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R} \to \mathsf{M}^{\mathrm{min}}_{\mathcal{H},R}$ by the same notation π. Our goal is to show that $R^i\pi_*\underline{W}^{\mathrm{sub}}_{\nu_0,R} = 0$ for all $i > 0$.

2.1. **Application of Serre's fundamental theorem.** By [20, Prop. 7.13], there exists some weight $\nu_1 \in \mathrm{X}^{+,<p}_{\mathrm{M}_1}$ such that $W_{\nu_1,R}$ is free of rank-one as an R-module, and such that there exists an *ample line bundle* ω_{ν_1} over $\mathsf{M}^{\mathrm{min}}_{\mathcal{H},R}$ such that

$$\pi^*\omega_{\nu_1} \cong \underline{W}^{\mathrm{can}}_{\nu_1,R}, \tag{2.1}$$

the canonical extension $\underline{W}^{\mathrm{can}}_{\nu_1,R}$ of $\underline{W}_{\nu_1,R}$. Since (by definition)

$$\underline{W}^{\mathrm{sub}}_{\nu_0+N\nu_1,R} \cong \underline{W}^{\mathrm{sub}}_{\nu_0,R} \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}}}{\otimes} (\underline{W}^{\mathrm{can}}_{\nu_1,R})^{\otimes N}, \tag{2.2}$$

for all integer N, by the projection formula [9, 0_{I}, 5.4.10.1], we have

$$R^i\pi_*\underline{W}^{\mathrm{sub}}_{\nu_0+N\nu_1,R} \cong (R^i\pi_*\underline{W}^{\mathrm{sub}}_{\nu_0,R}) \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{min}}_{\mathcal{H},R}}}{\otimes} \omega_{\nu_1}^{\otimes N}. \tag{2.3}$$

Then we have the following:

Lemma 2.4. *There exists some integer $N_1 \geq 0$ such that, for all integers $N \geq N_1$ and all $i \geq 0$, the sheaves $R^i\pi_*\underline{W}^{\mathrm{sub}}_{\nu_0+N\nu_1,R}$ over $\mathsf{M}^{\mathrm{min}}_{\mathcal{H},R}$ are generated by their global sections and satisfy $H^j(\mathsf{M}^{\mathrm{min}}_{\mathcal{H},R}, R^i\pi_*\underline{W}^{\mathrm{sub}}_{\nu_0+N\nu_1,R}) = 0$ for all $j > 0$.*

Proof. Since π is proper and $\mathsf{M}^{\mathrm{min}}_{\mathcal{H},R}$ is noetherian, by the theorem of finiteness [9, III, 3.2.1], $R^i\pi_*\underline{W}^{\mathrm{sub}}_{\nu_0,R}$ are coherent over $\mathsf{M}^{\mathrm{min}}_{\mathcal{H},R}$ for all $i \geq 0$, and are nonzero only for finitely many i. Since ω_{ν_1} is ample over $\mathsf{M}^{\mathrm{min}}_{\mathcal{H},R}$, the lemma follows from (2.3) and Serre's fundamental theorem for projective schemes [9, III, 2.2.1]. $\square$

2.2. **Shifting weights into the holomorphic chamber.** Let w_0 (resp. w_1) be the longest Weyl element in $\mathrm{W}_{\mathrm{M}_1}$ (resp. $\mathrm{W}^{\mathrm{M}_1}$) (see [19, §2.4]), so that $(-w_0)\Phi^+_{\mathrm{M}_1} = \Phi^+_{\mathrm{M}_1}$ and $W_\nu \cong W^\vee_{-w_0(\nu)}$ for all $\nu \in \mathrm{X}^{+,<p}_{\mathrm{M}_1}$, and $l(w_1) = d = \dim_{\mathsf{S}_1}(\mathsf{M}_{\mathcal{H},1})$.

Remark 2.5. When $R = \mathbb{C}$, for any $\mu \in \mathrm{X}^+_{\mathrm{G}_1}$, sections in $H^0(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}, (\underline{W}^\vee_{w_1\cdot\mu,R})^{\mathrm{sub}})$ are represented by holomorphic cusp forms of weight $(-w_0)(w_1\cdot\mu) \in \mathrm{X}^+_{\mathrm{M}_1}$, which contribute via the dual BGG spectral sequence to $H^d_{\text{log-dR}}(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}, (\underline{V}^\vee_{[\mu],R})^{\mathrm{sub}}) \cong H^d_{\mathrm{dR,c}}(\mathsf{M}_{\mathcal{H},R}, \underline{V}^\vee_{[\mu],R})$ (compactly supported of middle degree), compatible with their contribution to the (better understood) L^2 cohomology of $\mathsf{M}_{\mathcal{H},R}$. (For more explanations, see [7, Thm. 9], [10, §2], and [11, Prop. 5.4.2]; see also the comparisons with transcendental results in [19, 20] and the references there.) Thus we consider weights of the form $(-w_0)(w_1\cdot\mu) = (-w_0w_1)(\mu) + (-w_0)(w_1\cdot 0)$ *holomorphic*, which is a translation of the dominant chamber $\mathrm{X}^+_{\mathrm{G}_1}$ (because $(-w_0w_1)$ preserves $\mathrm{X}^+_{\mathrm{G}_1}$).

Proposition 2.6. *There exists an integer N_2, a positive parallel weight $\nu_2 \in \mathrm{X}^+_{\mathrm{M}_1}$, and a weight $\mu_0 \in \mathrm{X}^+_{\mathrm{G}_1}$, all of which can be explicitly determined, such that*

$$\nu_0 + N_2\nu_1 - \nu_2 = -w_0(w_1\cdot\mu_0) \tag{2.7}$$

This proposition is elementary in nature. One can prove Proposition 2.6 using general principles that also work for all reductive groups defining Shimura varieties. However, we shall spell out a (less elegant) case-by-case argument, which has the advantage of giving explicit choices of N_2, ν_2, and μ_0 of small sizes.

We will assume Proposition 2.6 in the remainder of this section, and postpone its proof to Section 3.1. In Lemma 3.3, we will give an explicit choice of $C(\mu_0)$, depending only on $(\mathcal{O}, \star, L, \langle\,\cdot\,,\,\cdot\,\rangle, h_0)$ and the weight ν_0, such that $C(\nu_0) \geq |\mu_0|_{\mathrm{re}}$ (see [19, Def. 3.9]) for some triple (N_2, ν_2, μ_0) as in Proposition 2.6.

2.3. **Application of automorphic vanishing.**

Corollary 2.8. *Let (N_2, ν_2, μ_0) be any triple as in Proposition 2.6. Suppose that $p > |\mu_0|_{\mathrm{re}}$ and that N is any integer satisfying $N \geq N_2$. Then we have*

$$H^i(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}, \underline{W}^{\mathrm{sub}}_{\nu_0+N\nu_1,R}) = 0$$

for every $i > 0$

Proof. The assertion follows from [20, Thm. 8.13(2)], because $\nu := \nu_0 + N\nu_1$ and $\nu_+ := (N - N_2)\nu_1 + \nu_2$ satisfy the condition there, with $\mu(\nu - \nu_+) = \mu_0 \in \mathrm{X}_{\mathrm{G}_1}^{+,<_{\mathrm{re}}p}$ and $w(\nu) = w_1$ (so that $d - l(w(\nu)) = d - l(w_1) = 0$). $\square$

Remark 2.9 (Erratum). There are typos in [20, Thm. 8.13]: Both instances of $\mathrm{X}_{\mathrm{G}_1}^{+,<_{\mathrm{W}}p}$ there should be $\mathrm{X}_{\mathrm{G}_1}^{+,<_{\mathrm{re}}p}$, which is what was used in [20, Cor. 7.24], on which the theorem depends.

Remark 2.10. When $R = \mathbb{C}$, it seems plausible that Corollary 2.8 may have a real analytic proof (which should work for all Shimura varieties, including exceptional cases). But we cannot find such a statement in the literature.

2.4. **End of the proof of Theorem 1.1.**

Let N_1 be as in Lemma 2.4, and let (N_2, ν_2, μ_0) be any triple as in Proposition 2.6 satisfying $C(\nu_0) \geq |\mu_0|_{\mathrm{re}}$ for some $C(\nu_0)$ (which will be given in Lemma 3.3 below). Suppose that $p > C(\nu_0)$ and that N is any integer satisfying $N \geq N_1$ and $N \geq N_2$. By Lemma 2.4 and by the Leray spectral sequence, and by Corollary 2.8, we have

$$H^0(\mathsf{M}_{\mathcal{H},R}^{\mathrm{min}}, R^i\pi_*\underline{W}_{\nu_0+N\nu_1,R}^{\mathrm{sub}}) \cong H^i(\mathsf{M}_{\mathcal{H},\Sigma,R}^{\mathrm{tor}}, \underline{W}_{\nu_0+N\nu_1,R}^{\mathrm{sub}}) = 0 \tag{2.11}$$

for all $i > 0$. Since $R^i\pi_*\underline{W}_{\nu_0+N\nu_1,R}^{\mathrm{sub}}$ is generated by its global sections (by Lemma 2.4), it follows that

$$R^i\pi_*\underline{W}_{\nu_0+N\nu_1,R}^{\mathrm{sub}} = 0 \tag{2.12}$$

for all $i > 0$. By combining (2.3) and (2.12), we obtain the desired vanishing (1.2) for ail $i > 0$ (under the assumption that $p > C(\nu_0) \geq |\mu_0|_{\mathrm{re}}$).

Suppose that the residue characteristics of R are all zero. By shrinking R and enlarging R by flat descent, we may replace the setup with a different one in which $p > C(\nu_0) \geq |\mu_0|_{\mathrm{re}}$, and obtain the desired vanishing from the above.

Thus Theorem 1.1 follows. $\square$

3. ELEMENTARY COMPUTATIONS

We shall freely use the notation in [19, §2 and §7]. The materials in this section can be read without any knowledge of algebraic geometry or Shimura varieties.

3.1. **Proof of Proposition 2.6.**

We can rewrite (2.7) as

$$\nu_0 + N_2\nu_1 - \nu_2 = -w_0(w_1\mu_0 + w_1\rho - \rho) = \mu_0' + (-w_0)(w_1 \cdot 0),$$

where $\mu_0' = -(w_0w_1)(\mu_0) \in \mathrm{X}_{\mathrm{G}_1}^{+}$ satisfies $V_{[\mu_0']} \cong V_{[\mu_0]}^{\vee}$, because w_0w_1 is the longest Weyl element in $\mathrm{W}_{\mathrm{G}_1}$. Hence it suffices to find N_2 and ν_2 such that

$$\mu_0' = \nu_0 + N_2\nu_1 - \nu_2 - (-w_0)(w_1 \cdot 0) \in \mathrm{X}_{\mathrm{G}_1}^{+}. \tag{3.1}$$

Let us write $\nu_j = ((\nu_{j,\tau})_{\tau\in\Upsilon/c}; \nu_{j,0}) = (((\nu_{j,\tau,i_\tau})_{1\leq i_\tau\leq r_\tau})_{\tau\in\Upsilon/c}; \nu_{j,0}) \in \mathrm{X}_{\mathrm{M}_1}^{+}$, for $j = 0, 1, 2$. We shall also denote by ρ_τ (resp. $w_{0,\tau}$, resp. $w_{1,\tau}$) the corresponding factors of ρ (resp. w_0, resp. w_1). Then we need

$$\mu_{0,\tau}' = \nu_{0,\tau} + N_2\nu_{1,\tau} - \nu_{2,\tau} - (-w_{0,\tau})(w_{1,\tau} \cdot 0) \in \mathrm{X}_{\mathrm{G}_\tau}^{+} \tag{3.2}$$

for each factor G_τ of G_1. There are two cases:

(1) If $\tau = \tau \circ c$, then $\mathrm{G}_\tau \cong \mathrm{Sp}_{2r_\tau} \underset{\mathbb{Z}}{\otimes} R_1$ or $\mathrm{O}_{2r_\tau} \underset{\mathbb{Z}}{\otimes} R_1$, and $\mathrm{M}_\tau \cong \mathrm{GL}_{r_\tau} \underset{\mathbb{Z}}{\otimes} R_1$. Set $d_\tau = \frac{1}{2} r_\tau (r_\tau + 1)$ (resp. $\frac{1}{2} r_\tau (r_\tau - 1)$), $e_\tau = (1, 1, \ldots, 1)$, and $r'_\tau = r_\tau + 1$ (resp. r_τ) if $\mathrm{G}_\tau \cong \mathrm{Sp}_{2r_\tau} \underset{\mathbb{Z}}{\otimes} R_1$ (resp. $\mathrm{O}_{2r_\tau} \underset{\mathbb{Z}}{\otimes} R_1$). If $d_{[\tau]_\mathbb{Q}} = \sum\limits_{\tau' \in [\tau]_\mathbb{Q}} d_{\tau'} = 0$, then we must have $\mathrm{O}_{2r_\tau} \underset{\mathbb{Z}}{\otimes} R_1$ and $r_\tau = 1$, in which case (3.2) is trivially true if we take $\mu'_{0,\tau} = \nu_{0,\tau}$, any $N_2 \in \mathbb{Z}$, and $\nu_{2,\tau} = N_2 \nu_{1,\tau} - (-w_{0,\tau})(w_{1,\tau} \cdot 0)$. Hence we may assume that $d_{[\tau]_\mathbb{Q}} > 0$. By assumption, we know that $\nu_{0,\tau,1} \geq \nu_{0,\tau,2} \geq \ldots \geq \nu_{0,\tau,r_\tau}$, and that $\nu_{1,\tau} = k_{1,\tau} e_\tau$, where $k_{1,\tau} > 0$ depends only on the equivalence class $[\tau]_\mathbb{Q}$ of τ (see [19, Def. 7.12]). Also, we have $\rho_\tau = (r'_\tau, r'_\tau - 1, \ldots, r'_\tau - r_\tau)$ and $(-w_{0,\tau})(w_{1,\tau} \cdot 0) = r'_\tau e_\tau$. Thus, in order for (3.2) to hold, we need

$$\nu_{0,r_\tau} + N k_{1,\tau} - k_{2,\tau} \geq r_\tau + 1 = r'_\tau$$

if $\mathrm{G}_\tau \cong \mathrm{Sp}_{2r_\tau} \underset{\mathbb{Z}}{\otimes} R_1$, or

$$\nu_{0,r_\tau - 1} + N k_{1,\tau} - k_{2,\tau} - r_\tau \geq |\nu_{0,r_\tau} + N k_{1,\tau} - k_{2,\tau} - r_\tau|$$

if $\mathrm{G}_\tau \cong \mathrm{O}_{2r_\tau} \underset{\mathbb{Z}}{\otimes} R_1$. We may take:

   (a) $\mu'_{0,\tau} := \nu_{0,\tau} - \nu_{0,[\tau]_\mathbb{Q}} e_\tau$, where $\nu_{0,[\tau]_\mathbb{Q}} := \min\limits_{\tau' \in [\tau]_\mathbb{Q}} (\nu_{0,\tau',r_\tau})$;

   (b) $\mu_{0,\tau} := -(w_{0,\tau} w_{1,\tau})(\mu'_{0,\tau}) = \mu'_{0,\tau}$; and

   (c) N_τ is any integer satisfying $\nu_{0,[\tau]_\mathbb{Q}} + N_\tau k_{1,\tau} > r'_\tau$, so that $\nu_{0,\tau} + N \nu_{1,\tau} - \mu'_{0,\tau} - (-w_{0,\tau})(w_{1,\tau} \cdot 0) = (\nu_{0,[\tau]_\mathbb{Q}} + N k_{1,\tau} - r'_\tau)\, e_\tau$, with a positive coefficient $\nu_{0,[\tau]_\mathbb{Q}} + N k_{1,\tau} - r'_\tau > 0$ for every $N \geq N_\tau$.

(2) If $\tau \neq \tau \circ c$, then $\mathrm{G}_\tau \cong \mathrm{GL}_{r_\tau} \underset{\mathbb{Z}}{\otimes} R_1$ and $\mathrm{M}_\tau \cong (\mathrm{GL}_{q_\tau} \times \mathrm{GL}_{p_\tau}) \underset{\mathbb{Z}}{\otimes} R_1$. Set $d_\tau = p_\tau q_\tau$, $e_\tau = (1, 1, \ldots, 1, 0, 0, \ldots, 0)$ with 1's in the first q_τ entries, and $e'_\tau = (0, 0, \ldots, 0, -1, -1, \ldots, -1)$ with -1's in the last p_τ entries. If $d_{[\tau]_\mathbb{Q}} = \sum\limits_{\tau' \in [\tau]_\mathbb{Q}/c} d_{\tau'} = 0$, then we must have $p_\tau q_\tau = 0$ for all $\tau \in [\tau]_\mathbb{Q}$, in which case (3.2) is trivially true if we take $\mu'_{0,\tau} = \nu_{0,\tau}$, any $N_2 \in \mathbb{Z}$, and $\nu_{2,\tau} = N_2 \nu_{1,\tau} - (-w_{0,\tau})(w_{1,\tau} \cdot 0)$. Hence we may assume that $d_{[\tau]_\mathbb{Q}} > 0$. By assumption, we know that $\nu_{0,\tau,1} \geq \nu_{0,\tau,2} \geq \ldots \geq \nu_{0,\tau,q_\tau}$ and $\nu_{0,\tau,q_\tau+1} \geq \nu_{0,\tau,q_\tau+2} \geq \ldots \geq \nu_{0,\tau,r_\tau}$, and that $\nu_{1,\tau} = k_{1,\tau} e_\tau + k_{1,\tau \circ c} e'_\tau$, where $[k_1]_\tau = k_{1,\tau} + k_{1,\tau \circ c} > 0$ depends only on the equivalence class $[\tau]_\mathbb{Q}$ of τ (see [19, Prop. 7.15]). Also, we have $\rho_\tau = \frac{1}{2}(r_\tau - 1, r_\tau - 3, \ldots, -r_\tau + 1)$ and $(-w_{0,\tau})(w_{1,\tau} \cdot 0) = p_\tau e_\tau + q_\tau e'_\tau$. Thus, in order for (3.2) to hold, we need

$$\nu_{0,q_\tau} + N k_{1,\tau} - k_{2,\tau} - p_\tau \geq \nu_{0,q_\tau+1} - N k_{1,\tau \circ c} + k_{2,\tau \circ c} + q_\tau,$$

or equivalently

$$(\nu_{0,q_\tau} - \nu_{0,q_\tau+1}) + N [k_1]_\tau - [k_2]_\tau \geq p_\tau + q_\tau = r_\tau.$$

We may take:

   (a) $\mu'_{0,\tau} := \nu_{0,\tau} - \nu_{0,[\tau]_\mathbb{Q}} e_\tau - (\nu'_{0,\tau,1} - \nu_{0,[\tau]_\mathbb{Q}})(e_\tau - e'_\tau)$, where $\nu_{0,[\tau]_\mathbb{Q}} := \min\limits_{\tau' \in [\tau]_\mathbb{Q}, d_{\tau'} \neq 0} (\nu_{0,\tau',q_{\tau'}} - \nu_{0,\tau',q_{\tau'}+1})$, and where $\nu'_{0,\tau,1} = \nu_{0,\tau,1}$ (resp. $\nu_{0,\tau,1} + \nu_{0,[\tau]_\mathbb{Q}}$) if $q_\tau > 0$ (resp. $q_\tau = 0$);

   (b) $\mu_{0,\tau} := -(w_{0,\tau} w_{1,\tau})(\mu'_{0,\tau})$, which ends with $\mu_{0,\tau,r_\tau} = 0$ because $\mu'_{0,\tau}$ starts with $\mu'_{0,\tau,1} = 0$; and

(c) N_τ is any integer satisfying $\nu_{0,[\tau]_{\mathbb{Q}}}+N_\tau[k_1]_\tau > r_\tau$, so that $\nu_{0,\tau}+N\nu_{1,\tau}-\mu'_{0,\tau}-(-w_{0,\tau})(w_{1,\tau}\cdot 0) = (\nu_{0,\tau,1}+Nk_{1,\tau}-p_\tau)\,e_\tau+(\nu_{0,[\tau]_{\mathbb{Q}}}-\nu_{0,\tau,1}+Nk_{1,\tau\circ c}-q_\tau)\,e_\tau$, with sum of coefficients $(\nu_{0,\tau,1}+Nk_{1,\tau}-p_\tau)+(\nu_{0,[\tau]_{\mathbb{Q}}}-\nu_{0,\tau,1}+Nk_{1,\tau\circ c}-q_\tau)=\nu_{0,[\tau]_{\mathbb{Q}}}+N[k_1]_\tau-r_\tau>0$ for every $N\geq N_\tau$.

Now set:

(1) $N_2:=\max\limits_{\tau\in\Upsilon/c}(N_\tau)$;
(2) $\mu_0:=((\mu_{0,\tau})_{\tau\in\Upsilon/c};\mu_{0,0})$ with any value of $\mu_{0,0}$;
(3) $\mu'_0:=(-w_0w_1)(\mu_0)$; and
(4) $\nu_2:=\nu_0+N_2\nu_1-\mu'_0-(-w_0)(w_1\cdot 0)$.

Then the triple (N_2,ν_2,μ_0) satisfies (3.1) and hence also (2.7), as desired, because each of its factors $(N_2,\nu_{2,\tau},\mu_{0,\tau})$ satisfies (3.2) by the above. $\square$

3.2. Explicit choice of $C(\nu_0)$.

Lemma 3.3. *The minimal size* $|\mu_0|_{\mathrm{re}}$ (*see* [19, Def. 3.9]) *among all* μ_0 *appearing in some* (N_2,ν_2,μ_0) *satisfying* (2.7) *in Proposition* 2.6 *is smaller than or equal to*

$$C(\nu_0):=\sum_{\tau\in\Upsilon/c}C_\tau(\nu_{0,\tau}),\tag{3.4}$$

where each $C_\tau(\nu_{0,\tau})$ *is defined as follows:*

(1) *If* $\tau=\tau\circ c$*, then we set* $d_\tau=\frac{1}{2}r_\tau(r_\tau+1)$ (*resp.* $\frac{1}{2}r_\tau(r_\tau-1)$) *if* $\mathrm{G}_\tau\cong\mathrm{Sp}_{2r_\tau}\underset{\mathbb{Z}}{\otimes}R_1$ (*resp.* $\mathrm{O}_{2r_\tau}\underset{\mathbb{Z}}{\otimes}R_1$)*,* $\nu_{0,[\tau]_{\mathbb{Q}}}:=\min\limits_{\tau'\in[\tau]_{\mathbb{Q}}}(\nu_{0,\tau',r_\tau})$*, and*

$$C_\tau(\nu_{0,\tau})=d_\tau+\sum_{1\leq i_\tau\leq r_\tau}(\nu_{0,\tau,i_\tau}-\nu_{0,[\tau]_{\mathbb{Q}}}).\tag{3.5}$$

(2) *If* $\tau\neq\tau\circ c$*, then we set* $d_\tau=p_\tau q_\tau$*,* $\nu_{0,[\tau]_{\mathbb{Q}}}:=\min\limits_{\tau'\in[\tau]_{\mathbb{Q}},d_{\tau'}\neq 0}(\nu_{0,\tau',q_{\tau'}}-\nu_{0,\tau',q_{\tau'}+1})$*,* $\nu'_{0,\tau,1}=\nu_{0,\tau,1}$ (*resp.* $\nu_{0,\tau,1}+\nu_{0,[\tau]_{\mathbb{Q}}}$) *if* $q_\tau>0$ (*resp.* $q_\tau=0$)*, and*

$$C_\tau(\nu_{0,\tau})=d_\tau+\sum_{1\leq i_\tau\leq q_\tau}(\nu'_{0,\tau,1}-\nu_{0,\tau,i_\tau})+\sum_{q_\tau<i_\tau\leq r_\tau}(\nu'_{0,\tau,1}-\nu_{0,[\tau]_{\mathbb{Q}}}-\nu_{0,\tau,i_\tau}).\tag{3.6}$$

Proof. These follow from the definition of $|\mu_0|_{\mathrm{re}}=d+\sum\limits_{\tau\in\Upsilon/c}\Big(\sum\limits_{1\leq i_\tau\leq r_\tau}\mu_{0,\tau,i_\tau}\Big)$ and the explicit choices of $\mu_{0,\tau}$ in the proof of Proposition 2.6. $\square$

Remark 3.7. By using [20, (7.9) and (7.11)], it is possible to reduce the proof of Theorem 1.1 to the case where the integral PEL datum is $\mathbb{Q}$-simple, and replace (3.4) with

$$C'(\nu_0):=\max_{[\tau]_{\mathbb{Q}}}\big(C_{[\tau]_{\mathbb{Q}}}(\nu_{0,[\tau]_{\mathbb{Q}}})\big),\tag{3.8}$$

where:

(1) $C_{[\tau]_{\mathbb{Q}}}(\nu_{0,[\tau]_{\mathbb{Q}}})=0$, if $d_{[\tau]_{\mathbb{Q}}}=\sum\limits_{\tau'\in[\tau]_{\mathbb{Q}}/c}d_\tau\leq 1$;
(2) $C_{[\tau]_{\mathbb{Q}}}(\nu_{0,[\tau]_{\mathbb{Q}}})=\sum\limits_{\tau'\in[\tau]_{\mathbb{Q}}/c}C_\tau(\nu_{0,\tau})$, where $C_\tau(\nu_{0,\tau})$ are as in (3.5) and (3.6), if otherwise.

We leave the details to the interested readers.

3.3. **Some examples.**

To help the reader understand the notation and formulas, we include some examples of familiar special cases.

Example 3.9 (trivial weight). If $\nu_0 = 0$, then (2.7) holds for $\mu_0 = 0$ and any sufficiently large N_2, and we have $C(\nu_0) = \sum\limits_{\tau \in \Upsilon/c} C_\tau(\nu_{0,\tau}) = \sum\limits_{\tau \in \Upsilon/c} d_\tau = d$ in (3.4).

Example 3.10 (Siegel case). Suppose $(\mathcal{O}, \star, L, \langle\,\cdot\,,\,\cdot\,\rangle, h_0)$ is given with $\mathcal{O} = \mathbb{Z}$ with trivial $\star$, with $(L, \langle\,\cdot\,,\,\cdot\,\rangle)$ given by $\mathbb{Z}^{\oplus 2r}$ with some standard self-dual symplectic pairing, and with any conventional choice of h_0. Then we are in the so-called *Siegel case*. There is a unique $\tau \in \Upsilon$ with $\tau = \tau \circ c$, which we can suppress in our notation, and each $\nu_0 \in \mathrm{X}^+_{\mathrm{M}_1}$ can be represented by a tuple $((\nu_{0,1}, \nu_{0,2}, \ldots, \nu_{0,r}); \nu_{0,0})$, where $\nu_{0,1} \geq \nu_{0,2} \geq \ldots \geq \nu_{0,r}$ are integers. Then μ_0 can be chosen to be $\nu_0 - \nu_{0,r}((1, 1, \ldots, 1, 1); 0) = ((\nu_{0,1} - \nu_{0,r}, \ldots, \nu_{0,r-1} - \nu_{0,r}, 0); \nu_{0,0})$ (where the last entry is irrelevant), and we have $C(\nu_0) = \frac{1}{2}r(r+1) + \sum\limits_{1 \leq i < r} (\nu_{0,i} - \nu_{0,r})$ (see (3.5)).

Example 3.11 ("$\mathbb{Q}$-similitude Hilbert case"). Suppose $(\mathcal{O}, \star, L, \langle\,\cdot\,,\,\cdot\,\rangle, h_0)$ is given with $\mathcal{O} = \mathcal{O}_F$ with trivial $\star$, where F is a totally real number field, with $(L, \langle\,\cdot\,,\,\cdot\,\rangle)$ given by $\mathcal{O}_F^{\oplus 2}$ with some standard symplectic pairing defined by trace, and with any conventional choice of h_0; and if p is any prime number unramified in $\mathcal{O}_F$. Then we are essentially in the so-called *Hilbert case*, although we only consider elements in $\mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_2$ with similitudes in $\mathbb{Q}$. There are d elements $\tau \in \Upsilon$ corresponding to the $d = [F : \mathbb{Q}]$ homomorphisms from $\mathcal{O}_F$ to an algebraic closure of $\mathbb{Q}_p$, which all satisfy $\tau = \tau \circ c$ and determine a unique equivalence class $[\tau]_\mathbb{Q}$ (of Galois orbits of τ), and our coefficient ring R is chosen to contain the images of all these homomorphisms, over which all linear algebraic data are split. Each $\nu_0 \in \mathrm{X}^+_{\mathrm{M}_1}$ can be represented by a tuple $((\nu_{0,\tau})_{\tau \in \Upsilon}; \nu_{0,0})$, where each $\nu_{0,\tau} = (\nu_{0,\tau,1})$ consists of just one integer $\nu_{0,\tau,1}$. Then $\nu_{0,[\tau]_\mathbb{Q}} = \min\limits_{\tau \in \Upsilon}(\nu_{0,\tau,1})$, and μ_0 can be chosen to be $\nu_0 - \nu_{0,[\tau]_\mathbb{Q}}((1)_{\tau \in \Upsilon}; 0) = ((\nu_{0,\tau,1} - \nu_{0,[\tau]_\mathbb{Q}})_{\tau \in \Upsilon}; \nu_{0,0})$, and we have $C(\nu_0) = d + \sum\limits_{\tau \in \Upsilon} (\nu_{0,\tau,1} - \nu_{0,[\tau]_\mathbb{Q}})$ (see (3.5)).

Example 3.12 (simplest unitary case). Suppose $(\mathcal{O}, \star, L, \langle\,\cdot\,,\,\cdot\,\rangle, h_0)$ is given with $\mathcal{O} = \mathcal{O}_F$, where F is an imaginary quadratic extension of $\mathbb{Q}$ with an embedding $F \hookrightarrow \mathbb{C}$, with $\star$ given by complex conjugation, with $(L, \langle\,\cdot\,,\,\cdot\,\rangle)$ given by a Hermitian module over $\mathcal{O}_F^{\oplus r}$ with signature $(r-q, q)$ at ∞ (using the given $F \hookrightarrow \mathbb{C}$), and with any conventional choice of h_0 (respecting the signature); and if p is any prime number unramified in $\mathcal{O}_F$. Then we obtain the simplest (nontrivial) *unitary case*. There is a unique representative τ of orbits in Υ/c such that ($\tau \neq \tau \circ c$ and) $(p_\tau, q_\tau) = (r-q, q)$, matching the signatures at ∞ and at p; hence we shall always choose this τ and suppress τ from the notation. Each $\nu_0 \in \mathrm{X}^+_{\mathrm{M}_1}$ can be represented by a tuple $((\nu_{0,1}, \nu_{0,2}, \ldots, \nu_{0,q}, \nu_{0,q+1}, \ldots, \nu_{0,r}); \nu_{0,0})$, where $\nu_{0,1} \geq \nu_{0,2} \geq \ldots \geq \nu_{0,q}$ and $\nu_{0,q+1} \geq \ldots \geq \nu_{0,r}$ are integers. If $q > 0$, then μ_0 can be chosen to be $(\nu_{0,1} + \nu_{0,q} - \nu_{0,q+1} - \nu_{0,r}, \ldots, \nu_{0,1} - \nu_{0,q}, \nu_{0,1} - \nu_{0,q}, \ldots, \nu_{0,1} - \nu_{0,2}, 0; \nu_{0,0})$ (note the reversed order and the repeated term $\nu_{0,1} - \nu_{0,q}$), and we have $C(\nu_0) = (r-q)q + \sum\limits_{1 \leq i \leq q} (\nu_{0,1} - \nu_{0,i}) + \sum\limits_{q < i \leq r} (\nu_{0,1} + \nu_{0,q} - \nu_{0,q+1} - \nu_{0,i})$. If $q = 0$, then μ_0 can be chosen to be $(\nu_{0,1} - \nu_{0,r}, \ldots, \nu_{0,1} - \nu_{0,2}, 0; \nu_{0,0})$ and we have $C(\nu_0) = \sum\limits_{1 \leq i \leq r} (\nu_{0,\tau,1} - \nu_{0,i})$; but $d = 0$ and the map π is trivial—$C(\nu_0) = 0$ suffices. (See (3.6) and Remark 3.7.)

4. Simpler proof for the trivial-weight case

In this final section, we sketch a logically simpler proof for the trivial-weight case $\nu_0 = 0$, which does not require the various advanced technical inputs in [20, §§1–3] (such as the theory of F-spans in [22]). The key is to give a simpler proof of the vanishing statement in Corollary 2.8 when $\nu_0 = 0$ (with suitable choice of (N_2, ν_2, μ_0)). By standard arguments as in the proof of [20, Thm. 8.2], we may and we shall assume that R is a perfect field extension of the residue field of R_1.

Using the extended Kodaira–Spencer isomorphism (see [16, Thm. 6.4.1.1(4)]) and the very construction of canonical extensions of automorphic bundles using the relative Lie algebra of the universal abelian scheme, one can show that

$$\underline{W}^{\mathrm{can}}_{(-w_0)(w_1\cdot 0)} \cong (\underline{W}^{\vee}_{w_1\cdot 0})^{\mathrm{can}} \cong \Omega^d_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}/\mathsf{S}_1}(\log\infty) := \wedge^d(\Omega^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}/\mathsf{S}_1}(\log\infty))$$

as line bundles over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}$ (ignoring Tate twists). (The proof is left to the interested readers.) Moreover, the proof of Proposition 2.6 in Section 3.1 shows that we can take $\mu_0 = 0$ in Proposition 2.6, with some integer N_2 such that the weight $\nu_2 = N_2\nu_1 - (-w_0)(w_1 \cdot 0)$ is positive and parallel. Then we have

$$\underline{W}^{\mathrm{sub}}_{N\nu_1} \cong \underline{W}^{\mathrm{sub}}_{\nu_2} \underset{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}}{\otimes} \underline{W}^{\mathrm{can}}_{(-w_0)(w_1\cdot 0)} \cong \underline{W}^{\mathrm{sub}}_{\nu_2} \underset{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}}{\otimes} \Omega^d_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}/\mathsf{S}_1}(\log \mathsf{D}),$$

where D is the boundary divisor $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1} - \mathsf{M}_{\mathcal{H},1}$ (with reduced subscheme structure).

By [20, Prop. 4.2(5) and Cor. 7.14], there exists a (usually nonreduced) divisor D' with $\mathsf{D}'_{\mathrm{red}} = \mathsf{D}$, and some $r_0 > 0$, such that the line bundle $(\underline{W}^{\mathrm{can}}_{\nu_2})^{\otimes r}(-\mathsf{D}')$ is ample for all integers $r \geq r_0$. (This follows from [16, Thm. 7.3.3.4], which implies that there exists some D' as above such that $\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}}(-\mathsf{D}')$ is relatively ample over $\mathsf{M}^{\min}_{\mathcal{H},1}$.) By base change from R_1 to R, this is exactly the condition $(*)$ needed in [6, Thm. 11.5]. Then, by [6, Thm. 11.5] and by Serre duality, we obtain

$$H^i(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}, \underline{W}^{\mathrm{sub}}_{N\nu_1,R}) = H^i(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}, \underline{W}^{\mathrm{sub}}_{\nu_2,R} \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}}}{\otimes} \Omega^d_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}/\mathsf{S}_1}(\log \mathsf{D})) = 0$$

for all $i > 0$. (This is the same approach taken in [18].) This gives the desired vanishing statement in Corollary 2.8 when $\nu_0 = 0$, and we can conclude as in Section 2.4. This argument does not depend on [20, Thm. 8.13(2)], and hence not on the various advanced technical inputs in [20, §§1–3].

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