The (B) conjecture for the Gaussian measure of dilates of symmetric convex sets and related problems

D. Cordero-Erausquin, M. Fradelizi and B. Maurey

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Abstract

We show that given a symmetric convex set $K \subset \mathbb{R}^d$, the function

$$t \longrightarrow \gamma(e^t K)$$

is log-concave on $\mathbb{R}$, where γ denotes the standard d-dimensional Gaussian measure. We also comment on the extension of this property to unconditional log-concave measures and sets, and on the complex case.

1 Introduction

Let us denote by γ the standard Gaussian probability measure on $\mathbb{R}^d$,

$$d\gamma(x) = (2\pi)^{-d/2} e^{-\|x\|^2/2}\, dx$$

where $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and $\|x\|^2 = x \cdot x = \sum_{j=1}^d x_j^2$. The following problem was proposed by W. Banaszczyk and popularized by R. Latała [14] under the name of (B) conjecture: given a (centrally) symmetric convex set $K \subset \mathbb{R}^d$, is it true that the inequality

$$\gamma\big(\sqrt{ab}\, K\big)^2 \geq \gamma(a\, K)\, \gamma(b\, K) \tag{1}$$

holds for every $a, b > 0$? Using a deep result of Latała and Oleszkiewicz [13] on the rate of growth of the Gaussian measure of dilates of symmetric convex sets, Latała [14] noticed that (1) holds if $\gamma(aK), \gamma(bK) \geq 0.85$. As usual, if the inequality (1) is valid in every dimension $d \geq 1$, then it remains valid when γ is a centered Gaussian measure on a separable Banach space F and $K \subset F$ is a convex symmetric Borel set (we refer to [14] for details). The aim of this paper is to prove that (1) is indeed true in the following equivalent form: *If $K \subset \mathbb{R}^d$ is a symmetric convex set, then the function*

$$t \longrightarrow \gamma(e^t K)$$

is log-concave on $\mathbb{R}$.

Actually, a little more is true. If we associate to every $(t_1, \ldots, t_d) \in \mathbb{R}^d$ the diagonal matrix $T = \Delta(t_1, \ldots, t_d)$ with diagonal entries $t_1, \ldots, t_d$, we may introduce the image $e^T K$ of K under the linear operation e^T and consider $\gamma(e^T K)$ as a function of $(t_1, \ldots, t_d)$. We shall prove the following theorem, which is the main result of the present article.

Theorem 1 *If $K \subset \mathbb{R}^d$ is a symmetric convex set, then the function*

$$(t_1, \ldots, t_d) \longrightarrow \gamma(e^{\Delta(t_1,\ldots,t_d)} K)$$

is log-concave on $\mathbb{R}^d$.

Recall that a non-negative function f is said to be *log-concave* if $\log(f)$ is concave. Of course the (B) conjecture is obtained by applying Theorem 1 on the line in $\mathbb{R}^d$ consisting of all points of the form $(t, \ldots, t)$. We may notice that inequality (1) implies in particular that the function

$$0 < \lambda \longrightarrow \gamma(\frac{1}{\lambda} K)\, \gamma(\lambda K)$$

achieves its maximum at $\lambda = 1$. The Gaussian Brunn-Minkowski inequality that follows from the Prékopa-Leindler inequality and the log-concavity of the Gaussian density implies that

$$\gamma\Big(\frac{a+b}{2}\, K\Big)^2 \geq \gamma(a\, K)\, \gamma(b\, K)$$

for all $a, b > 0$. It is clear that this known result is weaker than (1): since $\sqrt{ab} \leq (a+b)/2$, we have $\sqrt{ab}\, K \subset \frac{a+b}{2}\, K$ when K is convex and symmetric.

The function in the theorem above can be written more explicitly as

$$f(t_1, \ldots, t_d) = \gamma(e^{\Delta(t_1,\ldots,t_d)} K) = (2\pi)^{-d/2} \int_K \exp(-e^{2t_1} x_1^2/2 - \cdots - e^{2t_d} x_d^2/2)\, e^{t_1 + \cdots + t_d}\, dx.$$

We see that a special case of our problem is to prove the log-concavity on the real line of functions of the form

$$g(t) := \int_K e^{-e^{2t} \|x\|^2/2}\, dx.$$

We would like to stress the fact that Prékopa's theorem, asserting that marginals of log-concave densities are log-concave, does not give the result, because the function

$$(t, x) \longrightarrow e^{2t} \|x\|^2 \tag{2}$$

is not convex on $\mathbb{R} \times \mathbb{R}^d$.

The next section §2 contains the proof of Theorem 1. It relies on Poincaré-type (or spectral gap) inequalities for γ_K, the normalized restriction of the Gaussian measure to the symmetric convex set K. We shall give two ways of completing the proof. The first one uses an L_2-technique, while the second uses an observation of Caffarelli on the non-expansivity of the Brenier map pushing forward γ to γ_K (precise statements will be given in Remark 1). In section §3, we briefly investigate the extension of the (B) conjecture to the case of non-symmetric convex sets. We shall see that if K has enough symmetries, then inequality (1) holds.

Our proof makes an important use of the Gaussian structure of the problem, but a natural question to ask is whether the inequality (1) remains valid when γ is replaced by some measure μ having a log-concave density, say. We shall say that the measure μ and the set K satisfy the (B) conjecture if the function

$$t \longrightarrow \mu(e^t K)$$

is log-concave on $\mathbb{R}$. In section §4 we study the situation in $\mathbb{C}^d$. There, the problem is naturally linked with complex interpolation, and we will prove that the (B) conjecture is true for a wide class of sets and measures, provided they are *circled* (i.e. $\mathbb{C}$-symmetric). In section §5, we turn back to $\mathbb{R}^d$ and observe in particular that the (B) conjecture holds when the log-concave measure μ and the set K are *unconditional*. These last two sections are independent (in spirit and in content) from the main one devoted to the proof of Theorem 1. We see them as commentary sections whose main interest is to bring together different methods going from complex interpolation to Brunn-Minkowski inequality.

2 Proof of Theorem 1

Without loss of generality we can assume that $\gamma(K) \neq 0$, or equivalently that K has non-empty interior. Let $(b_1, \dots, b_d)$ and $(a_1, \dots, a_d)$ be two arbitrary points in $\mathbb{R}^d$, and B, A the corresponding diagonal matrices. With the notation of the theorem, let us set

$$\forall t \in \mathbb{R}, \quad f_{K,B,A}(t) := \gamma(e^{B+tA}\, K).$$

The proof will be done if we show that $f = f_{K,B,A}$ is log-concave on $\mathbb{R}$ for any choice B, A. The log-concavity of f amounts to proving that $f''(t)/f(t) - f'(t)^2/f(t)^2 \leq 0$. Observe that proving this inequality for every symmetric convex set K, every B, A and every real t is equivalent to proving it for every symmetric convex set K, every A and for $B = 0$, $t = 0$. Indeed, $f_{K,B,A}(s+t) = f_{L,0,A}(t)$ if we define a convex set L by $L := e^{B+sA}\, K$. We shall therefore concentrate on the study at $t = 0$ of

$$f(t) := \gamma(e^{tA}\, K).$$

The reader interested only in the (B) conjecture can think that $A = \mathrm{Id}$. By the simple change of variable $y = e^{tA}x$, we see that

$$f(t) = \gamma(e^{tA}K) = (2\pi)^{-d/2} \int_K e^{-\|e^{tA}x\|^2/2}\, e^{t\,\mathrm{tr}\,A} dx,$$

where $\mathrm{tr}\, A$ denotes the trace of the matrix A. Since $t \to e^{t\,\mathrm{tr}\,A}$ is log-affine, it is enough to prove the log-concavity at $t = 0$ of

$$g(t) := \int_K e^{-\|e^{tA}x\|^2/2}\, dx. \tag{3}$$

Writing $x \cdot y$ for the scalar product on $\mathbb{R}^d$, we have

$$g'(t) = -\int_K e^{tA}x \cdot Ae^{tA}x\, e^{-\|e^{tA}x\|^2/2}\, dx$$

and

$$g''(t) = \int_K \big((e^{tA}x \cdot Ae^{tA}x)^2 - 2e^{tA}x \cdot A^2e^{tA}x\big)\, e^{-\|e^{tA}x\|^2/2}\, dx.$$

We need to show that $g''(0)/g(0) - g'(0)^2/g(0)^2 \le 0$, namely

$$\frac{\int_K (x \cdot Ax)^2\, e^{-\|x\|^2/2}\, dx}{\int_K e^{-\|x\|^2/2}\, dx} - \left(\frac{\int_K x \cdot Ax\, e^{-\|x\|^2/2}\, dx}{\int_K e^{-\|x\|^2/2}\, dx}\right)^2 \le 2\, \frac{\int_K x \cdot A^2x\, e^{-\|x\|^2/2}\, dx}{\int_K e^{-\|x\|^2/2}\, dx}. \tag{4}$$

Introducing the probability measure γ_K on $\mathbb{R}^d$ defined by

$$d\gamma_K(x) = \frac{1_K(x)\, e^{-\|x\|^2/2}}{\int_K e^{-\|y\|^2/2}\, dy}\, dx, \tag{5}$$

which is the Gaussian measure restricted to the set K, we obtain the following compact form for inequality (4)

$$\int (x \cdot Ax)^2\, d\gamma_K(x) - \Big(\int x \cdot Ax\, d\gamma_K(x)\Big)^2 \le 2 \int x \cdot A^2x\, d\gamma_K(x).$$

Setting $M_{2,K} := \int x \cdot Ax\, d\gamma_K(x)$, we see that we have to prove the following inequality:

$$\int \big(x \cdot Ax - M_{2,K}\big)^2\, d\gamma_K(x) \le 2 \int x \cdot A^2x\, d\gamma_K(x). \tag{6}$$

In other words, we need to bound the variance with respect to γ_K of the function f defined by $f(x) = x \cdot Ax$.

The usual Poincaré inequality for the Gaussian measure γ tells us that, for every smooth function $f \in L_2(\mathbb{R}^d, \gamma)$ orthogonal to constant functions, one has

$$\int f(x)^2\, d\gamma(x) \le \int \|\nabla f(x)\|^2\, d\gamma(x). \tag{7}$$

The measure γ_K defined in (5) is log-concave with respect to γ: it belongs to the family of probability measures μ of the form $d\mu(y) = e^{-W(y)}\, dy$, with W convex on $\mathbb{R}^d$, satisfying $\mathrm{Hess}_y W \ge \mathrm{Id}$ on the convex set K where $\{W < +\infty\}$. In fact, it is often technically simpler to assume that W is defined on the whole space $\mathbb{R}^d$ and not just on the convex set K. For this, we consider γ_K as the pointwise limit of a sequence of densities of the form

$$e^{-\|x\|^2/2-\psi(x)}$$

where ψ is a convex function on $\mathbb{R}^d$, constant on K and "big" outside K. We may assume that ψ is even when K is symmetric, and that the Hessian of ψ is smooth and bounded on $\mathbb{R}^d$. Writing $W(x) := \|x\|^2/2 + \psi(x)$, we work with a probability measure μ given by

$$d\mu(x) := e^{-W(x)}\, dx, \quad \text{with } W : \mathbb{R}^d \longrightarrow \mathbb{R} \text{ verifying } \operatorname{Hess} W \geq \mathrm{Id}. \tag{8}$$

In our situation, the function W is furthermore even and the result to be proved (6) is now that, for

$$q(x) := Ax \cdot x - \int Ay \cdot y \, d\mu(y)$$

one has

$$\int q^2 \, d\mu \leq \frac{1}{2} \int \|\nabla q\|^2 \, d\mu. \tag{9}$$

Here we used that $\|\nabla q(x)\|^2 = 4\|Ax\|^2 = 4\, x \cdot A^2 x$.

It is well known that probability measures μ of the form (8) verify the Poincaré inequality (7); this fact follows from an inequality of Brascamp and Lieb [5], but there are now several simple proofs, see [15]. So we have, for every smooth function $f \in L_2(\mu)$ orthogonal to constant functions,

$$\int f(x)^2 \, d\mu(x) \leq \int \|\nabla f(x)\|^2 \, d\mu(x). \tag{10}$$

We see that a direct application of (10) to the function q would miss the expected result (9) by a factor 2. One way to cope with this problem is to use the symmetry of the situation in order to relate inequality (9) to a "second eigenvalue problem". Indeed, we are working with a measure μ which is symmetric, and the function q is even. In particular we have that $\int \nabla q \, d\mu = 0$. The following lemma will therefore conclude the proof of Theorem 1.

Lemma 2 *Let μ be a probability measure of the form (8). For every smooth function $f \in L_2(\mu)$, with mean 0 with respect to μ and such that $\int \nabla f \, d\mu = 0$, one has*

$$\int f^2 \, d\mu \leq \frac{1}{2} \int \|\nabla f\|^2 \, d\mu. \tag{11}$$

Proof of the Lemma The proof starts with an L_2-technique rather standard in problems of this kind. The fact that such techniques were related to our present problem became clear to us after reading the papers by Artstein, Ball, Barthe and Naor [1, 2]. In fact, for the study of the function g defined in (3), one can apply the "Basic formula" of [2], which was extended to the d-dimensional case in [1] (although not in the form (13) reproduced here). This formula gives an expression for $(-\log g)''(0)$; using it directly, we could jump faster to the end of the proof below, in order to see that $(-\log g)''(0) \geq 0$, which is the content of Theorem 1, without stating the lemma above. We feel however that Lemma 2 is of independent interest.

We introduce the differential operator $L = \Delta - \nabla W \cdot \nabla$; we shall of course use that

$$\int (Lu)\, v \, d\mu = -\int \nabla u \cdot \nabla v \, d\mu,$$

when $v, \nabla v \in L_2(\mu)$ and when u is C^2-smooth and compactly supported. For a function f verifying the assumptions of the Lemma and $\nabla f \in L_2(\mu)$, we start with the obvious observation that, since $\int f\, d\mu = 0$,

$$\min\Big\{\int (g-f)^2\, d\mu\,;\ g \in L_2(\mu),\ \int g\, d\mu = 0\Big\} = 0.$$

The next classical fact is that the space

$$\{Lu\,; u\ C^2\text{-smooth and with compact support}\}$$

is $L_2(\mu)$-dense in the space $\{g \in L_2(\mu)\,; \int g\, d\mu = 0\}$. For completeness, we shall include a short proof of this fact at the end of this section. As a consequence we have,

$$\inf\Big\{\int (Lu-f)^2\, d\mu\Big\} = 0.$$

where the infimum is taken over all u C^2-smooth and with compact support. Proving the variance inequality (11) reduces to showing that, for every such function u, we have

$$\int \Big((Lu-f)^2 - f^2 + \frac{1}{2}\,\|\nabla f\|^2\Big)\, d\mu \geq 0,$$

that is to say

$$\int \Big((Lu)^2 - 2\,(Lu)\, f + \frac{1}{2}\,\|\nabla f\|^2\Big)\, d\mu \geq 0. \tag{12}$$

We observe first that $-\int (Lu)\, f\, d\mu = \int \nabla u \cdot \nabla f\, d\mu$; next, integration by parts, using the relation $\nabla Lu = L\nabla u - \operatorname{Hess} W(\nabla u)$, shows that

$$\int (Lu)^2\, d\mu = -\int (\nabla Lu)\cdot \nabla u\, d\mu = \int \Big(\|\operatorname{Hess} u\|_2^2 + \operatorname{Hess} W(\nabla u)\cdot \nabla u\Big)\, d\mu,$$

where $\|\operatorname{Hess} u\|_2$ denotes the Hilbert-Schmidt norm of the Hessian matrix of u. Since the Hessian of W is $\geq \mathrm{Id}$, we have $\operatorname{Hess} W(\nabla u)\cdot\nabla u \geq \|\nabla u\|^2$ and from (12) it is therefore enough to show that

$$\int \Big(\|\operatorname{Hess} u\|_2^2 + \|\nabla u\|^2 + 2\,\nabla u \cdot \nabla f + \frac{1}{2}\,\|\nabla f\|^2\Big)\, d\mu \geq 0.$$

We rewrite the preceding inequality as

$$\int \Big(\|\operatorname{Hess} u\|_2^2 - \|\nabla u\|^2 + \frac{1}{2}\,\|2\,\nabla u + \nabla f\|^2\Big)\, d\mu \geq 0. \tag{13}$$

Introducing $c := \int \nabla u\, d\mu$ and u_0 such that $\nabla u_0 = \nabla u - c$, and using the assumption that $\int \nabla f\, d\mu = 0$, we see that (13) is equivalent to

$$\|c\|^2 + \int \Big(\|\operatorname{Hess} u_0\|_2^2 - \|\nabla u_0\|^2 + \frac{1}{2}\,\|2\,\nabla u_0 + \nabla f\|^2\Big)\, d\mu \geq 0.$$

Thus, it is enough to know that

$$\int \left(\|\mathrm{Hess}\, u_0\|_2^2 - \|\nabla u_0\|^2\right) d\mu \geq 0,$$

which is obtained by summing, for $j = 1, \dots, d$, the Poincaré inequality (10) for μ, applied to each mean 0 coordinate function $D_j u_0$ of ∇u_0,

$$\int (D_j u_0)^2 \, d\mu \leq \int \|\nabla D_j u_0\|^2 \, d\mu.$$

This ends the proof of Lemma 2. $\square$

Remark 1 (Alternative proof of Lemma 2) We shall in fact give an alternative proof of the following *weaker* form of Lemma 2, which is however sufficient to complete the proof of (9) and thus of Theorem 1:

Let μ be a probability measure of the form (8) with W even. For every smooth function $f \in L_2(\mu)$, even and with mean 0 with respect to μ, one has

$$\int f^2 \, d\mu \leq \frac{1}{2} \int \|\nabla f\|^2 \, d\mu. \tag{14}$$

The idea is to "transport" from γ to μ the following variant of the Poincaré inequality (7) where the constant 1 is replaced by $1/2$: for every smooth function $g \in L_2(\mathbb{R}^d, \gamma)$ orthogonal to constant and linear functions, one has

$$\int g^2 \, d\gamma \leq \frac{1}{2} \int \|\nabla g\|^2 \, d\gamma. \tag{15}$$

This is nothing else but Lemma 2 for the Gaussian measure γ, since we have in this case that $\int \nabla f \, d\gamma = \int x f(x) \, d\gamma(x)$. But this result for γ is an essentially well known spectral estimate, and it can be proved in an elementary way, by expanding g on the basis of $L_2(\mathbb{R}^d, \gamma)$ formed by the d-dimensional Hermite polynomials.

In order to pass from γ to μ, we will use optimal transportation (we refer to [19] for background and proofs). If $F : \mathbb{R}^d \to \mathbb{R}$ and $G : \mathbb{R}^d \to \mathbb{R}$ are two probability densities, it follows from Brenier [6] and McCann [16] that there exists a convex function $\varphi : \mathbb{R}^d \to \mathbb{R}$ such that the measure $G(y)\, dy$ is the image measure of $F(x)\, dx$ under the gradient map $\nabla\varphi$. We say that the map $T := \nabla\varphi$ *transports* $F(x)\, dx$ onto $G(y)\, dy$. The map T is uniquely determined and it is often called the *Brenier map*. We will use a crucial observation about T due to Caffarelli [7]: this map T is non-expansive when we transport the Gaussian measure γ to a probability measure μ of the form (8). The non-expansivity of $T = \nabla\varphi$ can be rephrased as $0 \leq \mathrm{Hess}\, \varphi \leq \mathrm{Id}$. Of course, it was already noticed by Caffarelli that the non-expansivity of T can be used to "transport" inequalities from γ to the measure

μ. We can also point out that Caffarelli's result was used to recover Gaussian correlation inequalities in [9].

For μ verifying (8) with W even, we introduce the Brenier map $T = \nabla\varphi$ transporting γ onto μ. By Caffarelli's result, this map is non-expansive. Also note that $T(-x) = -T(x)$ for every $x \in \mathbb{R}^d$, due to the symmetry of μ and the uniqueness of the Brenier map. For a smooth even function $f \in L_2(\mu)$ with $\int f\,d\mu = 0$, we set $g(x) := f(T(x))$. Since the function f is even and since $T(-x) = -T(x)$, the function g is even, hence orthogonal to linear functions in $L_2(\mathbb{R}^d, \gamma)$, and g is orthogonal to constant functions since $\int g\,d\gamma = \int f\,d\mu = 0$ by transportation. Next, the gradient $\nabla g(x)$ can be written as $T'(x)(\nabla f(Tx))$, where $T'(x)$ is the Hessian of φ at x; it follows that $\|\nabla g(x)\| \le \|\nabla f(Tx)\|$. Finally, using (15)

$$\int f(y)^2\,d\mu(y) = \int g(x)^2\,d\gamma(x) \le \frac{1}{2}\int \|\nabla g(x)\|^2\,d\gamma(x) \le \frac{1}{2}\int \|\nabla f(y)\|^2\,d\mu(y).$$

This gives inequality (14).

The reader will observe that this second form of Lemma 2 is valid for any symmetric probability measure on $\mathbb{R}^d$ which is a contractive image of γ.

$\square$

Finally, we include for completeness a proof of the following known fact that was used in our "L_2-proof" of Lemma 2.

Lemma 3 *Let W be a real C^2-smooth function on $\mathbb{R}^n$ such that $d\mu(x) = e^{-W(x)}\,dx$ is a probability measure on $\mathbb{R}^n$, and let L denote the differential operator $\Delta - \nabla W \cdot \nabla$, acting on the vector space D of all C^2-smooth functions on $\mathbb{R}^n$ with compact support. The range $L(D)$ of L is dense in the subspace H_0 of $L_2(\mu)$ orthogonal to constant functions.*

Proof. Let U be the unitary operator from $L_2(\mu)$ onto $L_2(\mathbb{R}^n)$ defined by $Uf = e^{-W/2}f$; formally U transforms the operator L into a Schrödinger operator of the form $-\Delta + V$. To be precise, if $\mathcal{D}(\mathbb{R}^n)$ denotes as usual the space of C^∞-smooth functions with compact support, we have, for $g = Uf$ and $\psi \in \mathcal{D}(\mathbb{R}^n)$,

$$\langle -L(U^*\psi), f\rangle_{L_2(\mu)} = \int_{\mathbb{R}^n} \Big(-\Delta\psi(x) + V(x)\psi(x)\Big) g(x)\,dx,$$

where $V = \frac{1}{4}|\nabla W|^2 - \frac{1}{2}\Delta W$. If $f \in L_2(\mu)$ is orthogonal to $L(D)$, it follows that $g = Uf$ satisfies $\Delta g = Vg$ in $\mathcal{D}'(\mathbb{R}^n)$, which yields that $g \in W^{2,2}_{\mathrm{loc}}(\mathbb{R}^n)$ by the classical theory. If $\theta \in \mathcal{D}(\mathbb{R}^n)$, then $\theta g \in W^{2,2}(\mathbb{R}^n)$ and using $\displaystyle\int \operatorname{div}(\theta^2 g \nabla g) = 0$ we get

$$\int_{\mathbb{R}^n} \Big(|\nabla(\theta g)|^2 + (\theta g)^2\,V\Big) = \int_{\mathbb{R}^n} \Big(|\nabla(\theta g)|^2 + \theta^2 g \Delta g\Big) = \int_{\mathbb{R}^n} |\nabla\theta|^2 g^2.$$

Note that for every $h \in W^{2,2}(\mathbb{R}^n)$ with compact support, we have, by integrations by parts,

$$\int_{\mathbb{R}^n} |\nabla(he^{W/2})|^2 \, e^{-W} = \int_{\mathbb{R}^n} \left(|\nabla h|^2 + h^2 V\right).$$

If $\theta \in \mathcal{D}(\mathbb{R}^n)$ is such that $\theta = 1$ in a neighborhood of 0, then, setting $\theta_k(x) := \theta(x/k)$ for $k \geq 1$,

$$\int |\nabla(\theta_k f)|^2 \, d\mu = \int_{\mathbb{R}^n} \Big(|\nabla(\theta_k g)|^2 + (\theta_k g)^2 \, V\Big) = \int_{\mathbb{R}^n} |\nabla \theta_k|^2 g^2 \longrightarrow 0,$$

when $k \to +\infty$. Hence f is constant; this shows that $L(D)$ is dense in H_0 (this argument is taken from Simader's paper [17, page 49]). □

3 A remark on the non-symmetric case

It is not clear whether the symmetry of K is compulsory for the inequality (1) to be true. We will not investigate here the general case of Theorem 1, but the simpler case $T = t\,\mathrm{Id}$ corresponding to the (B)-conjecture. Let us go through the proof of the previous section (in the case $A = \mathrm{Id}$) and see where the symmetry assumption was used. As explained above, one has to prove the log-concavity of the function

$$g(t) := \int_K e^{-\|e^t x\|^2/2} \, dx.$$

The log-concavity at $t = 0$ followed from inequality (9) applied to $q(x) = \|x\|^2 - \int \|x\|^2 \, d\mu(x)$ and $\mu = \gamma_K$. The measure $\mu = \gamma_K$ satisfies the assumption of the Lemma 2 as soon as K is convex (after the approximation procedure described above). Thus, in order to apply Lemma 2 to q, we only need to check that $\int \nabla q \, d\gamma_K = 2 \int x \, d\gamma_K(x) = 0$, which of course holds when K is symmetric. In other words, for the log-concavity of g to hold at $t = 0$, it suffices that the Gaussian barycenter of K be 0,

$$\int_K x \, d\gamma(x) = 0.$$

But this does not mean that the log-concavity of g at all t's follows from this assumption. Indeed, in order to "shift" the problem from t to 0, one has to apply to K the dilation $x \longrightarrow e^t x$. Thus, we see that our proof gives that the function g is log-concave on $\mathbb{R}$ as soon as, for every $\lambda > 0$, the Gaussian barycenter of λK is 0. We shall see that this condition is true when K has enough symmetries.

Let K denote a convex subset of $\mathbb{R}^d$, containing 0 in its interior. We introduce the group $G(K)$ of isometries leaving K globally invariant,

$$G(K) := \{R \in \mathcal{O}_d \,;\; R(K) = K\}.$$

We now introduce the set of points fixed by the action of $G(K)$,

$$\mathrm{Fix}(K) := \{x \in \mathbb{R}^d;\ R(x) = x,\ \forall R \in G(K)\}.$$

Since the Gaussian measure γ is invariant under isometries, the Gaussian barycenter of K belongs to $\mathrm{Fix}(K)$. Furthermore, one has that for every $\lambda > 0$,

$$\mathrm{Fix}(\lambda K) = \lambda\,\mathrm{Fix}(K).$$

If $\mathrm{Fix}(K) = \{0\}$, we deduce that the Gaussian barycenter of λK is 0 for every $\lambda > 0$; we have therefore

Theorem 4 *Let $K \subset \mathbb{R}^d$ be a convex set such that $\mathrm{Fix}(K) = \{0\}$. Then the function*

$$t \longrightarrow \gamma(e^t K)$$

is log-concave on $\mathbb{R}$.

A symmetric set of course verifies that $\mathrm{Fix}(K) = \{0\}$, since $-\mathrm{Id} \in G(K)$ in this case. As an example of a non-symmetric convex set K verifying $\mathrm{Fix}(K) = \{0\}$, one can take the (centered) regular simplex in $\mathbb{R}^d$.

4 The complex case

We identify $\mathbb{C}^d$ with the Euclidean space $(\mathbb{R}^{2d}, \|\cdot\|)$ and we denote by $d\mathrm{vol}$ the Lebesgue measure on $\mathbb{C}^d$. The natural notions replacing convex sets and convex functions in the complex setting are pseudo-convex sets and plurisubharmonic functions. We refer to Hörmander's book [12] for precise definitions. It is worth recalling that convexity implies plurisubharmonicity (and therefore convexity for a set implies pseudo-convexity). Let us point out the following difference between the real and the complex situation: the function

$$(z, w) \longrightarrow e^{2z}\|w\|^2$$

is plurisubharmonic on $\mathbb{C} \times \mathbb{C}^d$. Comparing this fact with (2), we might expect that a complex version of Prékopa's theorem would be of some help here. But first, let us connect our problem with complex interpolation.

Recall that a set $K \subset \mathbb{C}^d$ is the unit ball of a complex norm on $\mathbb{C}^d$ if and only if K is a convex body and is *circled* in the sense that

$$\forall w \in K, \forall \theta \in \mathbb{R}, \quad e^{i\theta} w \in K.$$

If K and L are two circled convex bodies, we denote by $[K, L]_\theta$ the unit ball of the complex interpolated space at $\theta \in [0, 1]$ between the normed spaces whose unit balls are K and L, respectively. In the very simple case where the unit balls are homothetic, one has $[aK, bK]_\theta = a^{1-\theta} b^\theta K$ for every $\theta \in [0, 1]$. Thus the (B) conjecture for K amounts to the log-concavity of

$$[0, 1] \ni \theta \longrightarrow \gamma([aK, bK]_\theta).$$

For this we can use the following general result

Theorem 5 ([10]) *Let K and L be two circled convex bodies in $\mathbb{C}^d$ and let $\varphi : \mathbb{C}^d \longrightarrow \mathbb{R}\cup\{-\infty\}$ be a plurisubharmonic function such that for all $w \in \mathbb{C}^d$, for all $\theta \in \mathbb{R}$, $\varphi(e^{i\theta}w) = \varphi(w)$ (we say that φ is* circled*). If μ is the measure on $\mathbb{C}^d$ given by $d\mu(w) = e^{-\varphi(w)}\mathrm{dvol}(w)$, then the function*

$$\theta \longrightarrow \mu([K, L]_\theta)$$

is log-concave on $[0, 1]$.

Applying this theorem to two homothetic images of a circled convex body K, we deduce that the (B) conjecture holds in $\mathbb{C}^d$ for (circled) measures whose density is $-\log$ plurisubharmonic (this of course includes the $2d$-dimensional Gaussian measure) and for circled convex bodies. Actually, we can prove a little more, namely that the result also holds for circled pseudo-convex subsets. To see this, one goes back to the central tool used in [10], which is Berndtsson's [3] complex version of Prékopa's theorem.

Theorem 6 (Berndtsson) *Let Ω be a pseudo-convex domain in $\mathbb{C}^d \times \mathbb{C}$, and let $\varphi : \Omega \longrightarrow \mathbb{R} \cup \{-\infty\}$ be a plurisubharmonic function. For every $z \in \mathbb{C}$, define the set $\Omega(z) := \{w \in \mathbb{C}^d\,;\ (w, z) \in \Omega\}$. Assume that one of the following conditions holds*
i) For all $z \in \mathbb{C}$, $\Omega(z)$ is a circled domain, $0 \in \Omega(z)$ and $\varphi(\cdot, z)$ is circled on $\Omega(z)$.
ii) The set $\Omega(z)$ is a connected Reinhardt domain and for all $w = (w_1, \ldots, w_d) \in \mathbb{C}^d$, $\varphi(w, z) = \varphi(|w_1|, \ldots, |w_d|, z)$.
Then the function Φ defined on $\mathbb{C}$ by

$$e^{-\Phi(z)} = \int_{\Omega(z)} e^{-\varphi(w,z)}\,\mathrm{dvol}(w)$$

is subharmonic.

We recall that a set $V \subset \mathbb{C}^d$ is a Reinhardt set if for every $w \in \mathbb{C}^d$,

$$(w_1, \ldots, w_d) \in V \Longleftrightarrow (|w_1|, \ldots, |w_d|) \in V.$$

Using the theorem of Berndtsson, we can show that the (B) conjecture holds for some classes of pseudo-convex sets and of $-\log$-plurisubharmonic measures.

Proposition 7 *Let $K \subset \mathbb{C}^d$ be a pseudo-convex set, let $\varphi : \mathbb{C}^d \longrightarrow \mathbb{R} \cup \{-\infty\}$ be a plurisubharmonic function, and let μ be the measure given by $d\mu(w) = e^{-\varphi(w)}\mathrm{dvol}(w)$. Assume that one of the following conditions holds*
i) K is a circled domain, $0 \in K$ and φ is circled.
ii) K is a connected Reinhardt domain and for all $w = (w_1, \ldots, w_d) \in \mathbb{C}^d$, $\varphi(w_1, \ldots, w_d) = \varphi(|w_1|, \ldots, |w_d|)$.
Then the function

$$t \longrightarrow \mu(e^t K)$$

is log-concave on $\mathbb{R}$.

Proof We apply Berndtsson's theorem to the pseudo-convex set $\Omega := \{(w, z)\,;\ e^{-z}w \in K\}$ and to the plurisubharmonic function $\varphi(z, w) = \varphi(w)$. Then the hypotheses are satisfied and the theorem gives that the function

$$z \longrightarrow -\log \mu(e^z K)$$

is subharmonic. Since this function does not depend upon the imaginary part of z, it follows that it is convex. $\square$

It would be interesting to know if such a general result can be true in a real setting. In the sequel, we shall concentrate on the unconditional case.

5 The real case for unconditional measures and sets

A function $\varphi : \mathbb{R}^d \to \mathbb{R}$ is said to be *unconditional* if

$$\forall (x_1, \dots, x_d) \in \mathbb{R}^d, \quad \varphi(x_1, \dots, x_d) = \varphi(|x_1|, \dots, |x_d|).$$

In the same way, a set $K \subset \mathbb{R}^d$ is *unconditional* if its characteristic function is unconditional, i.e.:

$$(x_1, \dots, x_d) \in K \Longleftrightarrow (|x_1|, \dots, |x_d|) \in K.$$

An *unconditional log-concave* measure is a measure μ on $\mathbb{R}^d$ of the form $d\mu(x) = e^{-\varphi(x)}\,dx$ where $\varphi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is convex and unconditional. For example, the Gaussian measure is obviously an unconditional log-concave measure.

We present two ways of proving the (B) conjecture in the case of unconditional measures and sets in $\mathbb{R}^d$ (Proposition 9). The first one consists in using a complexification method, in order to deduce the result from the complex case. This approach can also be understood from an interpolation point of view, since the unconditional case corresponds to the case of (finite dimensional) Banach lattices, where the interpolation is explicit and can be performed in the real case. The second approach is much simpler and relies on a version of Prékopa-Leindler's inequality due to Borell and Uhrin. The idea is that after a logarithmic change of variable, we are back to the classical Brunn-Minkowski theory.

5.1 Complexification

The strongest result will be obtained by using Proposition 7. But we find interesting to explore first the connection with complex interpolation and with Theorem 5. If K and L are unconditional sets in $\mathbb{R}^d$, we define an analogue of the interpolation between K and L in the following way. For $\theta \in [0, 1]$, we introduce the unconditional set

$$K^{1-\theta}L^{\theta} := \{w \in \mathbb{R}^d\ ;\ \exists x \in K,\ \exists y \in L,\ |w_j| = |x_j|^{1-\theta}|y_j|^{\theta}, \forall j = 1, \dots, d\}.$$

If K and L are convex bodies, then $K^{1-\theta}L^{\theta}$ is a convex body of $\mathbb{R}^d$. We have:

Proposition 8 *Let K and L be unconditional convex bodies in $\mathbb{R}^d$. Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be an unconditional function such that $(r_1, \ldots, r_d) \longrightarrow \varphi(e^{r_1}, \ldots, e^{r_d})$ is convex and let μ be the measure on $\mathbb{R}^d$ given by $d\mu(x) = e^{-\varphi(x)}dx$. Then the function $\theta \longrightarrow \mu(K^{1-\theta}L^\theta)$ is* log-*concave. In particular:*

$$\mu(K^{1-\theta}L^\theta) \geq \mu(K)^{1-\theta}\mu(L)^\theta.$$

For a justification of the condition on φ, see Remark 7.

Proof For any unconditional set $A \subset \mathbb{R}^d$ we define

$$A_{\mathbb{C}} := \{w = (w_1, \ldots, w_d) \in \mathbb{C}^d \ ; \ (|w_1|, \ldots, |w_d|) \in A\} \subset \mathbb{C}^d. \tag{16}$$

Note that $K_{\mathbb{C}}$ and $L_{\mathbb{C}}$ are circled convex bodies in $\mathbb{C}^d$. We also define the function $\psi : \mathbb{C}^d \to \mathbb{R} \cup \{-\infty\}$ by

$$\psi(w_1, \ldots, w_d) := \varphi(|w_1|, \ldots, |w_d|) + \sum_{j=1}^{d} \log |w_j|$$

and the measure $\mu_{\mathbb{C}}$ on $\mathbb{C}^d$ by

$$d\mu_{\mathbb{C}}(w) := e^{-\psi(w)}d\mathrm{vol}(w) = e^{-\varphi(|w_1|,\ldots,|w_d|)}\frac{d\mathrm{vol}(w)}{\prod_{j=1}^d |w_j|} \tag{17}$$

We have $\psi(e^{i\theta}w) = \psi(w)$, for all $w \in \mathbb{C}^d$ and for all $\theta \in \mathbb{R}$. Moreover the hypothesis on φ implies that ψ is plurisubharmonic on $\mathbb{C}^d$ (see remark 7). Hence, from Theorem 5 we get that $\theta \longrightarrow \mu_{\mathbb{C}}([K_{\mathbb{C}}, L_{\mathbb{C}}]_\theta)$ is log-concave. Since from Calderon [8], $[K_{\mathbb{C}}, L_{\mathbb{C}}]_\theta = (K^{1-\theta}L^\theta)_{\mathbb{C}}$, we find that

$$\theta \longrightarrow \mu_{\mathbb{C}}((K^{1-\theta}L^\theta)_{\mathbb{C}}) \tag{18}$$

is log-concave. For any compact unconditional set A in $\mathbb{R}^d$, let us calculate $\mu_{\mathbb{C}}(A_{\mathbb{C}})$. By writing all w_j in polar coordinates and using the invariance properties of A and φ, we get

$$\mu_{\mathbb{C}}(A_{\mathbb{C}}) = \int_{A_{\mathbb{C}}} e^{-\varphi(|w_1|,\ldots,|w_d|)}\frac{d\mathrm{vol}(w)}{\prod_{j=1}^d |w_j|} = (2\pi)^d \int_{A\cap(\mathbb{R}_+)^d} e^{-\varphi(r_1,\ldots,r_d)}\, dr_1 \ldots dr_d.$$

Therefore $\mu_{\mathbb{C}}(A_{\mathbb{C}}) = \pi^d\mu(A)$ for every unconditional compact set $A \subset \mathbb{R}^d$. Combining this with (18) we conclude that the function $\theta \longrightarrow \mu(K^{1-\theta}L^\theta)$ is log-concave. □

Remark 2 (Log-concave measures) An unconditional log-concave measure satisfies the hypothesis of the preceding theorem since

$$\varphi \text{ convex unconditional } \Longrightarrow (r_1, \ldots, r_d) \longrightarrow \varphi(e^{r_1}, \ldots, e^{r_d}) \text{ convex }.$$

Indeed, a convex unconditional function φ satisfies $\varphi(x) \leq \varphi(y)$ when $|x| \leq |y|$, where $|x|$ denotes the vector such that $|x|_j = |x_j|$ for $j = 1, \ldots, d$; this is true because x is then a convex combination of a family of vectors z such that $|z| = |y|$.

If we apply the preceding theorem to two homothetic images of K we see that the (B) conjecture holds in $\mathbb{R}^d$ for measures whose density satisfy the hypothesis of Proposition 8 and for unconditional convex bodies. Actually, we can prove a little more: the result holds also if K is an unconditional *logarithmically convex* body in $\mathbb{R}^d$, in the sense that

$$\log(K) := \{(r_1, \dots, r_d) \in \mathbb{R}^d \; ; \; (e^{r_1}, \dots, e^{r_d}) \in K\}$$

is convex. Note that for every body $K \subset \mathbb{R}^d$,

$$K \text{ unconditional convex } \Longrightarrow K \text{ unconditional logarithmically convex.}$$

Proposition 9 *Let K be an unconditional logarithmically convex body in $\mathbb{R}^d$. Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be an unconditional function such that $(r_1, \dots, r_d) \longrightarrow \varphi(e^{r_1}, \dots, e^{r_d})$ is convex, and let μ be the measure on $\mathbb{R}^d$ given by $d\mu(x) = e^{-\varphi(x)}dx$. Then the function*

$$t \mapsto \mu(e^t K)$$

is log-concave on $\mathbb{R}$.

Proof We define the set $K_{\mathbb{C}} \subset \mathbb{C}^d$ (16) and the complex measure $\mu_{\mathbb{C}}$ (17) as in the proof of Proposition 8. We still have $\mu_{\mathbb{C}}(K_{\mathbb{C}}) = \pi^d \mu(K)$. Note that $K_{\mathbb{C}}$ is a connected Reinhardt pseudo-convex set and that $\mu_{\mathbb{C}}$ satisfies the hypothesis ii) of Proposition 7. Hence it follows that

$$t \longrightarrow \mu_{\mathbb{C}}(e^t K_{\mathbb{C}}) = \pi^d \mu(e^t K)$$

is log-concave on $\mathbb{R}$.

□

Remark 3 (An example of application) One can take as measure μ the restricted Lebesgue measure on an unconditional logarithmically convex body. Thus, if A and B are two unconditional logarithmically convex bodies of $\mathbb{R}^d$, one has, for every $\lambda > 0$,

$$\mathrm{vol}(\lambda A \cap B)\, \mathrm{vol}(\frac{1}{\lambda} A \cap B) \leq \mathrm{vol}(A \cap B)^2.$$

Remark 4 We can note that by a suitable application of Berndtsson's theorem, it is possible to assume in Proposition 8 that K and L are simply unconditional logarithmically convex. But we shall see in Proposition 10 that in fact the result holds for arbitrary unconditional sets (without any convexity condition). This does not mean that Proposition 9 is valid when $K \subset \mathbb{R}^d$ is an arbitrary unconditional set, since it is not true in general that $(aK)^{1-\theta}(bK)^\theta = a^{1-\theta} b^\theta K$. However, we see that for every unconditional set K,

$$K \text{ logarithmically convex } \Longleftrightarrow K^{1-\theta} K^\theta = K, \quad \forall \theta \in (0, 1). \tag{19}$$

5.2 Application of Brunn-Minkowski's functional inequality in $\mathbb{R}_+^d$

First, let us recall the classical Prékopa-Leindler theorem which is a functional form of Brunn-Minkowski's inequality (see [11, 15]). If $\theta \in [0,1]$ and if $f,g,h : \mathbb{R}^d \longrightarrow \mathbb{R}_+$ are three Borel functions verifying $h((1-\theta)r+\theta s) \geq f(r)^{1-\theta} g(s)^\theta$ for all $r,s \in \mathbb{R}^d$, then

$$\int_{\mathbb{R}^d} h \geq \left(\int_{\mathbb{R}^d} f\right)^{1-\theta} \left(\int_{\mathbb{R}^d} g\right)^\theta .$$

In this section we will use a multiplicative form of the Prékopa-Leindler inequality for functions on $\mathbb{R}_+^d := (\mathbb{R}_+)^d = \{x \in \mathbb{R}^d;\ x_i \geq 0\}$ which was proved by Uhrin[18] (see also [4]). This form asserts that for $\theta \in [0,1]$, if $f,g,h : \mathbb{R}_+^d \longrightarrow \mathbb{R}_+$ are three Borel functions verifying for all $x = (x_1,\ldots,x_d) \in \mathbb{R}_+^d$ and $y = (y_1,\ldots,y_d) \in \mathbb{R}_+^d$

$$h(x_1^{1-\theta} y_1^\theta, \ldots, x_d^{1-\theta} y_d^\theta)) \geq f(x)^{1-\theta} g(y)^\theta, \tag{20}$$

then

$$\int_{\mathbb{R}_+^d} h \geq \left(\int_{\mathbb{R}_+^d} f\right)^{1-\theta} \left(\int_{\mathbb{R}_+^d} g\right)^\theta . \tag{21}$$

One can note that this result can easily be deduced from Prékopa-Leindler's theorem in the following way. To any measurable function $v :\ \mathbb{R}_+^d \longrightarrow \mathbb{R}_+$, we associate the function $\tilde{v} :\ \mathbb{R}^d \longrightarrow \mathbb{R}_+$ defined by

$$\tilde{v}(t_1,\ldots,t_d) = v(e^{t_1},\ldots,e^{t_d})\, e^{\sum_{j=1}^d t_j}.$$

Using the change of variable $x_j = e^{t_j}$ for $j = 1,\ldots,d$, we get

$$\int_{\mathbb{R}_+^d} v(x)\, dx = \int_{\mathbb{R}^d} \tilde{v}(t)\, dt.$$

The assumptions on f, g and h (20) imply that the functions $\tilde{f}, \tilde{g}$ and $\tilde{h}$ satisfy the hypothesis of the Prékopa-Leindler theorem and thus we recover (21).

Remark 5 It is also possible to prove directly (20)-(21) by using mass transport, in the same way that Prékopa-Leindler's theorem can be proved (see e.g. [11]). In the computation of $\int_{\mathbb{R}_+^d} h(z)\, dz$, the usual change of variable $z = (1-\theta)x + \theta T(x)$, where $T = \nabla\varphi$ is the Brenier map (see Remark 1) between the measures $f(x)\, dx / \int f$ and $g(y)\, dy / \int g$, has to be replaced by the change of variable $z_j = x_j^{1-\theta}(T(x)_j)^\theta,\ j = 1,\ldots,d$.

We finally observe that (20)-(21) gives that Proposition 8 holds without the convexity assumption on K and L.

Proposition 10 *Let $K, L \subset \mathbb{R}^d$ be two closed unconditional sets. Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be an unconditional function such that $(r_1, \ldots, r_d) \longrightarrow \varphi(e^{r_1}, \ldots, e^{r_d})$ is convex and let μ be the measure on $\mathbb{R}^d$ given by $d\mu = e^{-\varphi(x)}dx$. We have, for every $\theta \in [0,1]$,*

$$\mu(K^{1-\theta}L^{\theta}) \geq \mu(K)^{1-\theta}\mu(L)^{\theta}.$$

Proof We define

$$f = e^{-\varphi}1_{K\cap\mathbb{R}^d_+}, \quad g = e^{-\varphi}1_{L\cap\mathbb{R}^d_+} \quad \text{and} \quad h = e^{-\varphi}1_{K^{1-\theta}L^{\theta}\cap\mathbb{R}^d_+}$$

and apply (21) to f, g and h.

$\square$

Remark 6 If we apply the preceding Theorem to two dilates of the same unconditional logarithmically convex body we get, in view of (19), an alternative, elementary proof of Proposition 9.

Remark 7 The following remark enlightens the link between (20)-(21) and Berndtsson's theorem and justifies the conditions made on the function φ: for every function φ on $(\mathbb{R}_+)^d$, the function $(r_1, \ldots, r_d) \longrightarrow \varphi(e^{r_1}, \ldots, e^{r_d})$ is convex *if and only if* the function $(w_1, \ldots, w_d) \longrightarrow \varphi(|w_1|, \ldots, |w_d|)$ is plurisubharmonic on $\mathbb{C}^d$.

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Laboratoire d'Analyse et de Mathématiques Appliquées (UMR 8050)
Université de Marne la Vallée
77454 Marne la Vallée Cedex 2, France
{cordero}{fradeliz}{maurey}@math.univ-mlv.fr