We give a proof of Talagrand's fundamental result on Gaussian processes ([a]), proof that the reader should agree to call *simple-minded*, if (s)he masters what was known by 1975 on Gaussian processes, especially Fernique's theorem on stationary processes ([b]). Understanding the definition of $\gamma_2(T)$ ([c]) is also required. Let $(X_t)_{t \in T}$ be a centered Gaussian process; the result to be proved is that

$$\gamma_2(T) \le K\, E \sup\{X_t : t \in T\}$$

for some universal constant K. We may assume that T is a finite or countable ([d]) set (with at least two points), and identify T with a subset of $L_2(\Omega, P)$ by $t \leftrightarrow X_t$; we then let $\mathrm{rco}(T)$ be the set of convex combinations with rational coefficients of the X_t, $t \in T$. Let d denote the L_2-metric and $B(x,r) = \{y \in L_2(\Omega, P) : d(y,x) < r\}$; identifying also $s \leftrightarrow X_s$ for $s \in \mathrm{rco}(T)$, letting $W(x,r) = B(x,r) \cap \mathrm{rco}(T)$, we set for $x \in T$ and $r > 0$

$$\varphi(x,r) = E \sup\{X_s : s \in \mathrm{rco}(T),\ d(x,s) < r\} = E \sup\{X_s - X_x : s \in W(x,r)\}$$

(notice ([e]) that $EX_x = 0$). For fixed x, the function $r \to \varphi(x,r)$ is clearly non-decreasing; if $D > 0$ denotes the diameter of T and $x \in T$, then

$$\varphi(x,r) \le \varphi(x,D) = E \sup\{X_s : s \in T\} =: V_0$$

which we assume finite. For every r fixed, $x \to \varphi(x,r)$ is continuous ([f]) on T: convexity implies that $\varphi(y, \lambda r) \ge \lambda \varphi(y,r)$ if $y \in T$ and λ rational < 1; if $\varepsilon < r$ and $d(y,x) < \varepsilon$, the ball $B(y, r-\varepsilon)$ is contained in $B(x,r)$, thus $(1 - \varepsilon/r)\, \varphi(y,r) \le \varphi(y, r-\varepsilon) \le \varphi(x,r)$. This yields that $|\varphi(x,r) - \varphi(y,r)| \le r^{-1} V_0\, d(x,y)$ for all $x, y \in T$.

If $x_1, \ldots, x_N \in T$ and $d(x_i, x_j) \ge \delta > 4\rho$ for all $i \ne j$, then it follows from the Fernique-Slepian comparison lemma that

$$(FS) \qquad E \sup\Bigl\{X_s : s \in \bigcup_{1 \le i \le N} W(x_i, \rho)\Bigr\} \ge c\,(\delta - 4\rho)\,\sqrt{\log_2 N} + \inf_{1 \le i \le N} \varphi(x_i, \rho)$$

with $c = (2\pi)^{-1/2} > 1/3$ ([g]). This is proved by comparing (X_t) to the process defined on $T' = \bigcup_{1 \le i \le N} W(x_i, \rho)$ by $Y_t = X_t - X_{x_i} + (\delta - 4\rho)\, g_i/\sqrt{2}$ when $t \in W(x_i, \rho)$, where $g_1, \ldots, g_N$ are independent standard Gaussian r.v., independent from the process (X_t).

We may assume without changing the supremum $\sup_t X_t$ that T has no isolated point ([h]). We set $\theta = 256/255$ and introduce a tree $\mathcal{T}$ with the following seven properties.

properties of the tree

1.— each node R of the tree $\mathcal{T}$ is a non-empty open subset of T ([i]); to each node R we assign a *center* $x \in R$, a *radius* r such that $R \subset B(x,r)$, the *control ball* $B(x, \theta r)$ and the *value* $v(R) = \varphi(x, \theta r)$;

2.— each node R is the union of a finite family of *homogeneous zones* ([j]), which we call *R-zones*; each R-zone $Z \subset R$ is the union of a certain number N_Z of sons of R, that we call Z-sons of R; for each Z-son R' of R, we call N_Z the *multiplicity* of R', and the *weight* w' of R' is ([k]) the smallest integer $k \ge 0$ such that $\log_2 N_Z \le 2^{k-1}$;

3.— the root R_0 of $\mathcal{T}$ is the set T itself; its center $x_0 \in T$ is arbitrary and the radius is $r_0 = D$, the diameter of T; the root R_0 is declared to be of multiplicity $N_0 = 1$ and weight $w_0 = 0$; all sons of R_0 have the same radius $r_1 = r_0/16$;

4.— let $R \in \mathcal{T}$, let w be the weight of R and r its radius, let R' be a son of R with weight w' and radius r'; then([l])

$$\textbf{4a:}\ r' \le r/16\,; \quad \textbf{4b:}\ w' > w\,; \quad \textbf{4c:}\ (r' = r/16) \text{ or } (w' = w+1)\,;$$

5.— if w is the weight of $R \in \mathcal{T}$, the number of R-zones is $< 2^{2^w}$;

6.— let $R \in \mathcal{T}$ and let Z be a R-zone; all Z-sons have the same radius r_Z; if R', R'' are two distinct Z-sons of R and x', x'' their centers, then $d(x', x'') \ge r_Z/2$;

7.— let $R \in \mathcal{T}$, w its weight and r its radius; if Z is a R-zone, then([m])

$$\sup_{y \in Z} \varphi(y, r/255) - \inf_{y \in Z} \varphi(y, r/255) \le 2^{-w}\, V_0.$$

We can now tell the impatient reader what the sets (T_p) of ([c]) are going to be: $T_0 = \{x_0\}$ and T_p for $p > 0$ will be the family of centers of all elements in $\mathcal{T}$ with weight $< p$.

constructing the tree

Let $R \in \mathcal{T}$ have weight $w > 0$ and radius r; set $A_w = 2^{2^w} - 1$, and for $j = 1, \ldots, A_w$ let

$$Z_j = \bigl\{ y \in R : \, |\varphi(y, r/255) - (j - 1/2)\, A_w^{-1} V_0| < 2^{-w-1} V_0 \bigr\}.$$

The R-zones are the non-empty Z_j s. These open sets cover R because $A_w > 2^w$ (condition **2**, first part), their number is $< 2^{2^w}$, giving thus **5**, and they satisfy condition **7**. When $w = 0$, we define a unique zone $Z_1 = T = R_0$, and the conditions **5**, **7** are clear.

Next, we cover each R-zone Z with a family of Z-sons of R, as asked by **2**. We describe a filling process for Z, constructing by induction finite subsets (C_α) of Z for even indices α, and radii $(\rho_{\alpha+1})$ for odd indices: we start with $C_0 = \emptyset$ and $\rho_1 = r/16$; we choose an arbitrary point z_2 in Z and define $C_2 = \{z_2\}$. Suppose C_α is already defined; if the balls $B(z, \rho_{\alpha-1})$ centered at all $z \in C_\alpha$ do not cover Z, we keep $\rho_{\alpha+1} = \rho_{\alpha-1}$; if not, let $\rho = \rho_{\alpha-1}$ and([n]) replace ρ by $\rho/2$, as many times as necessary, until the balls $B(z, \rho)$ centered at the $z \in C_\alpha$ do not cover Z; this will eventually happen since Z is infinite, as non-empty open subset of T; then set $\rho_{\alpha+1} = \rho$, choose a point $z_{\alpha+2} \in Z$ such that the distance of $z_{\alpha+2}$ to C_α is $\ge \rho_{\alpha+1}$ and set $C_{\alpha+2} = C_\alpha \cup \{z_{\alpha+2}\}$.

We make a stop when $\rho_{\alpha+1} < \rho_1 = r/16$ for the first time. Then, the balls $B(z, r/16)$ centered at all $z \in C_\alpha$ cover Z; if the cardinality of C_α is $\ge 2^{2^w}$, we introduce the family of Z-sons $Z \cap B(z, r/16)$, $z \in C_\alpha$; they have center z and radius $r_Z = r/16$. Since the multiplicity is $\ge 2^{2^w}$, the weight w_Z is $\ge w + 1$; this fits([o]) with condition **4**.

Otherwise([p]), we go on, until the cardinality of C_α is equal to 2^{2^w}. We know here that $\rho = \rho_{\alpha-1} < \rho_1 = r/16$, so a change of ρ as occurred since the start; let β be the last index such that $\rho_{\beta-1} = 2\rho$. Then the balls $B(z, 2\rho)$ centered at all $z \in C_\beta \subset C_\alpha$ already cover Z; we introduce the family of Z-sons $Z \cap B(z, 2\rho)$, $z \in C_\alpha$; they have center z and radius $r_Z = 2\rho$, and distinct points in C_α have distance $\ge \rho$ (condition **6**). Here the weight w_Z is exactly $w + 1$, and $r_Z = \rho_{\beta-1} \le \rho_1 = r/16$; again, this agrees with **4**.

When applying the filling process to the unique R_0-zone equal to $Z = T$, we start with $\rho_1 = r/16 = r_0/16 = D/16$ and the process will certainly define two successive points z_2, z_4 with distance $\ge \rho_1$; since $w = 0$ and $2^{2^w} = 2$ in this case, the filling process ends at the first stop, and we get $r_1 = r_Z = r_0/16$, as required by condition **3**.

Let $R \in \mathcal{T}$ have radius r, multiplicity N, weight w and center x; let Z be a R-zone, containing a number N_Z of Z-sons and let C_Z be the family of their centers. We first assume that $r_Z = r/16$, and we apply inequality (FS) to the family of balls $B(x', r/255)$ in $\operatorname{rco}(T)$, $x' \in C_Z$: by condition **6**, the distance between centers is $\geq \delta := r_Z/2$, and setting $\rho = r/255 = 16\, r_Z/255$, we have $\delta - 4\rho = r_Z/2 - 64\, r_Z/255 > 0$; also,

$$\bigcup_{x' \in C_Z} B(x', r/255) \cap \operatorname{rco}(T) = \bigcup_{x' \in C_Z} W(x', \rho) \subset B(x, \theta r)$$

(the control ball for R), because $C_Z \subset R \subset B(x, r)$ and $r + r/255 = \theta r$; by (FS),

$$v(R) = \varphi(x, \theta r) \geq c\,(1/2 - 64/255)\, r_Z \sqrt{\log_2 N_Z} + \inf_{x' \in C_Z} \varphi(x', r/255).$$

If R'' is a son of a Z-son R', its radius r'' is $\leq r/256$, by applying the condition **4a** twice. If x'' is the center of R'', the control ball $B(x'', \theta r'')$ for R'' is contained in $B(x'', \theta r/256) = B(x'', r/255)$; since $x'' \in Z$, condition **7** implies that

$$v(R'') = \varphi(x'', \theta r'') \leq \varphi(x'', r/255) \leq \inf_{x' \in C_Z} \varphi(x', r/255) + 2^{-w} V_0$$

and thus, introducing $\kappa = c\,(1/2 - 64/255) > c/5$, we get

$$v(R) - v(R'') \geq \kappa\, r_Z \sqrt{\log_2 N_Z} - 2^{-w} V_0.$$

Assume next that $r_Z < r/16$; then $w_Z = w+1$ by **4c** and $R \neq R_0$ by **3**, hence $w > w_0 = 0$; now $\log_2 N_Z \leq 2^{w_Z - 1} = 4\,.\,2^{w-2} \leq 4 \log_2 N$ (by definition of a weight $w > 0$), and

$$(SP) \qquad r_Z \sqrt{\log_2 N_Z} < (r/16)\bigl(2\sqrt{\log_2 N}\bigr) = 2^{-3}\, r \sqrt{\log_2 N}.$$

Thus, in all cases we have between a node R and a son R'' of a Z-son of R

$$v(R) - v(R'') + 2^{-3}\, \kappa\, r \sqrt{\log_2 N} + 2^{-w}\, V_0 \geq \kappa\, r_Z \sqrt{\log_2 N_Z}.$$

Given $t \in T$, we define $(R_j)_{j \geq 0} \subset \mathcal{T}$ as follows: R_0 is the root, R_{j+1} is a son of R_j containing t; let r_j be the radius, w_j the weight and N_j the multiplicity of R_j. We have just seen that

$$v(R_j) - v(R_{j+2}) + 2^{-3}\, \kappa\, r_j \sqrt{\log_2 N_j} + 2^{-w_j}\, V_0 \geq \kappa\, r_{j+1} \sqrt{\log_2 N_{j+1}}$$

for every $j \geq 0$. Summing separately on even and odd integers, using $r_0 \sqrt{\log_2 N_0} = 0$ (because $N_0 = 1$) and the fact that the weights (w_j) are distinct integers ≥ 0, we get

$$2\, V_0 + 2^{-3}\, \kappa \sum_{j > 0} r_j \sqrt{\log_2 N_j} + 2\, V_0 \geq \kappa \sum_{j > 0} r_j \sqrt{\log_2 N_j}$$

(notice that $v(R_1) \leq v(R_0) = V_0$) and thus, using again the definition of weights ([q])

$$(M) \qquad 5\, V_0 \geq 32\, V_0/7 \geq \kappa \sum_{j > 0} r_j \sqrt{\log_2 N_j} \geq 2^{-1} \kappa \sum_{j > 0} r_j\, 2^{w_j/2}.$$

bounding $\gamma_2(T)$

Let $T_0 = \{x_0\}$ and for $k > 0$, let $T_k \subset T$ be the set of all centers of nodes in $\mathcal{T}$ with weight $< k$. Let us check the cardinality condition of ($^{\mathbf{c}}$); we see that $T_1 = T_0$, hence $|T_1| = 1$. For $k > 0$, every element of $T_{k+1} \setminus T_k$ has weight k, a father R with weight $j < k$ and center $x \in T_k$; we cut R into homogeneous zones, whose number is $< 2^{2^j} \le 2^{2^{k-1}}$ by **5**. Next, every R-zone Z with weight k was cut into at most $2^{2^{k-1}}$ pieces; thus, every element x in T_k generates at most $2^{2^{k-1}+2^{k-1}}$ elements of T_{k+1} (including x itself), and we get $|T_{k+1}| \le 2^{2^k} |T_k|$. By induction, $|T_k| < 2^{2^k}$ for every $k > 0$, and $|T_0| < 2^{2^0}$ also.

We go on($^{\mathbf{r}}$), with the same $t \in T$ and family $(R_j)_{j \ge 0}$ as before. We want to bound

$$\gamma(t) := \sum_{p \ge 0} 2^{p/2} d(t, T_p).$$

We have $D = r_0 = 16\, r_1$ by **3**, hence for $p = 0$,

$$d(t, T_0) = d(t, x_0) \le D = r_0 < 16\, r_1\, 2^{w_1/2}.$$

Let j be ≥ 0, let x_j be the center of R_j and suppose that $p > w_j$; then $x_j \in T_p$ and

$$d(t, T_p) \le d(t, x_j) < r_j$$

since $t \in R_j$ and since $R_j \subset B(x_j, r_j)$ by **1**. It follows that

$$(G) \qquad s_j := \sum_{w_j < p \le w_{j+1}} 2^{p/2} d(t, T_p) \le r_j \sum_{w_j < p \le w_{j+1}} 2^{p/2} < 4\, r_j\, 2^{w_{j+1}/2}$$

because $\sum_{i \ge 0} 2^{-i/2} = 2 + \sqrt{2} < 4$. If $w_{j+1} - w_j > 1$, we know that $r_j = 16\, r_{j+1}$ by **4c** and in the opposite case, we have $w_{j+1} = w_j + 1$ by **4b**, giving always

$$r_j\, 2^{w_{j+1}/2} \le 16\, r_{j+1}\, 2^{w_{j+1}/2} + \sqrt{2}\, r_j\, 2^{w_j/2}.$$

Summing in $j \ge 0$ we get

$$\sum_{j \ge 0} s_j = \sum_{p \ge 1} 2^{p/2} d(t, T_p) < 4 \Big(16 \sum_{j \ge 0} r_{j+1}\, 2^{w_{j+1}/2} + 2 \sum_{j > 0} r_j\, 2^{w_j/2} + 2\, r_0 \Big).$$

We obtain a control for $\gamma(t) = d(t, T_0) + \sum_{p \ge 1} 2^{p/2} d(t, T_p)$,

$$\gamma(t) \le 16\, r_1\, 2^{w_1/2} + 72 \sum_{j > 0} r_j\, 2^{w_j/2} + 8\, r_0 < 216 \sum_{j > 0} r_j\, 2^{w_j/2},$$

since $r_0 < 16\, r_1\, 2^{w_1/2}$. Finally, using inequality (M) we have($^{\mathbf{s}}$)

$$\gamma_2(T) \le 2160\, \kappa^{-1} V_0 \le 10800\, c^{-1} V_0 \le 32400\, V_0.$$

 ($^{\mathbf{t}}$)

notes ———

([a]) M. Talagrand, The generic chaining, Springer Verlag, 2005
———, *Majorizing measures without measures*, Annals of Probability **29** (2001), 411–417
———, *Majorizing measures: the generic chaining*, Annals of Probability **24** (1996), 1049–1103
———, *A simple proof of the majorizing measure theorem*, Geometric and Functional Analysis **2** (1992), 119–125
———, *Regularity of Gaussian processes*, Acta Math. **159** (1987), 99–149.

([b]) X. Fernique, *Régularité des trajectoires des fonctions aléatoires gaussiennes*, Lecture Notes in Math. **480** (1975), 1–96; see chapter 8 (and chapter 7).

([c]) The value $\gamma_2(T)$ is the infimum of

$$\sup_{t\in T}\Bigl(\sum_{p\geq 0} 2^{p/2} d(t, T_p)\Bigr)$$

over all sequences $(T_p)_{p\geq 0}$ of finite subsets of T such that $|T_p| \leq 2^{2^p}$ for all $p \geq 0$.

([d]) It is usual, when dealing with a process that has a continuous parameter set T, to restrict first to a dense countable subset and prove some sort of continuity, before being able to define reasonable *trajectories* $t \in T \to X_t(\omega)$ for almost every $\omega \in \Omega$; this gives one possibility for defining $\sup_{t\in T} X_t$; another one is to use the *essential supremum*, which is also defined through countable subsets of T.

([e]) Without all this identifying-blabla, we could just define $\varphi(x, r)$ directly as

$$E\Bigl(\sup\bigl\{Y : Y = \sum_i c_i X_{s_i},\ s_i \in T,\, c_i \geq 0,\, \sum_i c_i = 1,\, c_i \in \mathbb{Q},\ \|Y - X_x\|_2 < r\bigr\}\Bigr).$$

([f]) We are going quite a bit out of our way, in order to get this continuity; it allows to define *open* subsets of T. Together with the assumption that T has no isolated point, this ensures that the tree $\mathcal{T}$ constructed in the sequel has only infinite branches, because all non-empty open subsets of T are then infinite. We could forget about $\mathrm{rco}(T)$, continuity and open sets, but that would force us to admit possible singleton nodes, and to provide a special treatment for them: see EZGGC V2.x for a development of this approach.

([g]) It is known that $E \sup_{1\leq i\leq N} g_i \geq (\ln N)^{1/2}/(\pi \ln 2)^{1/2} = \pi^{-1/2}(\log_2 N)^{1/2}$, when the (g_i) are independent standard Gaussian; see in Fernique's book *Fonctions aléatoires gaussiennes, vecteurs aléatoires gaussiens*, Centre de Recherches Mathématiques, 1997, page 27, inequality (1.7.1); this inequality is sharp, and the given proof uses Ehrhard's results from 1983 (*Symétrisation dans l'espace de Gauss*, Math. Scand. **53**, 281–301), but the weaker inequality with $c/2$ uses only classical computations. Ehrhard's paper is freely available on the web, thanks to the Göttinger DigitalisierungsZentrum, at

`http://www-gdz.sub.uni-goettingen.de/cgi-bin/digbib.cgi?PPN35397434X`

([h]) Fix a point $x_0 \in T$ and draw all (rational) segments from the points of T to this x_0. This larger set T has no isolated point, and it gives the same $\sup X_t$.

([i]) This is not strictly correct, as we give no guarantee that the same *set* R will not appear at two places that we *consider different* in the tree. To be more correct, we should introduce an abstract tree, with nodes *labeled by* subsets R of T. We won't do it here; I tried in EZGGC V2.1 but didn't like the result.

($^{\mathbf{j}}$) We call them homogeneous because the function $\varphi(\cdot, \rho)$ is almost constant on them, for a suitable value of ρ (condition **7**). The fact that $\varphi(\cdot, \rho)$ is constant is the crucial point in Fernique's proof; we are trying to re-create this situation, to some extent. There is a clear limit to what we can do: we could stabilize several functions $\varphi(\cdot, \rho_i)$ instead of one, as we did in EZGGC V1.1, but certainly only a *finite number* of functions. Somehow we are trying to foresee in **7** that the *radius* will be smaller, two or more steps later (down the tree), namely less than one quarter of the radius r_Z at the *next* step, in order to be able to apply (FS). When we stabilize, we of course don't know yet the next radius r_Z. If this new radius r_Z happens to be less than the smallest ρ_i in our preparation, our forecasting work will be useless, see Note ($^{\mathbf{p}}$) and the middle of Note ($^{\mathbf{l}}$) about (SP).

($^{\mathbf{k}}$) The weight shouldn't be an important parameter of the problem, as it is essentially equal to the iterated logarithm of the multiplicity; its relevance comes from the particular way in which $\gamma_2(T)$ is defined. We use several times the following obvious fact: when $w_Z > 0$, we have $2^{w_Z - 2} < \log_2 N_Z \le 2^{w_Z - 1}$.

($^{\mathbf{l}}$) Fernique's proof goes from r to $r/4$, when passing to a "son". Here, following Talagrand, we use (FS) and we need more room; hence we shall go from r to $r' = r/16$, or even $r' < r/16$ but in this latter case we shall necessarily have $w' = w + 1$. The important case is when $r' = r/16$; the other case $r' < r/16$, $w' = w+1$ will be treated as a small perturbation, see equation (SP). One can also observe that the general term in the series $\sum r_j \sqrt{\log_2 N_j}$ (that appears later in the proof) may behave like $j^{-\alpha}$, $\alpha > 1$, thus the quotient of two successive terms $r\sqrt{\log_2 N}$ may tend to 1, and in this case the weight will typically have jumps of size $8 = \log_2 256$, when passing to a son!

($^{\mathbf{m}}$) In the stationary case, $\varphi(x, \rho) = \psi(\rho)$ does not depend upon the point $x \in T$, hence condition **7** is void; furthermore, each node R is itself homogeneous: we may have a unique R-zone Z for every R (or better, not mention zones at all), and condition **5** disappears. Finally, the value of R is a function of r alone, $v(R) = \psi(\theta r)$, r being the radius of R. However, if we specialize to the stationary case the proof that follows, we don't get the usual Fernique's proof, that does not use inequality (FS): using (FS) is due to Talagrand.

($^{\mathbf{n}}$) We could instead replace ρ by $\lambda\rho$, for some fixed $\lambda < 1$, close to 1; that would improve condition **6** to $d(x', x'') \ge \lambda r_Z$, and would allow to replace $r/16$ in **4** by $r/8$ for example, giving a better final constant K.

($^{\mathbf{o}}$) Here, it is possible for the number of points in C_α to be much larger than 2^{2^w}, and thus w_Z can be larger than $w + 1$, see the end of Note ($^{\mathbf{l}}$). If we think about the sets (T_p), this means that we may skip several steps in the definition of these sets: starting from the center x of R, that belongs to T_{w+1}, we jump directly to defining points in T_{w_Z+1}, namely the points in C_α. We don't introduce points in the sets T_p between (other than the points already in T_{w+1}, namely the only point x for the whole region R): this gap will be taken into account in the equation (G) later, where we will group the terms $2^{p/2} d(t, T_p)$ for $w < p \le w_Z$. All we can do there is to use the radius r of R for bounding $d(t, T_p)$ when $p \ge w + 1$, and $\sqrt{\log_2 N_Z}$ for estimating $2^{p/2}$ when $p \le w_Z$.

($^{\mathbf{p}}$) Let us say that a zone Z is *rich* when the filling process ends at the first stop, and *poor* otherwise. We are not going to use the homogeneity of the poor zones; actually, we could decide to put them together to form a unique exceptional zone, that is not to be treated as the rich zones are. Also, it would be more satisfactory at the conceptual level to state condition **7** for rich zones only, but this would take more paper space.

(**q**) We use several times the following obvious fact: let N be a multiplicity and w the associated weight; if $w > 0$, we have $2^{w-2} < \log_2 N \leq 2^{w-1}$.

(**r**) From this point on, all we do is translating to the $\gamma_2(T)$-language the information contained in the main inequality (M). This is routine matter, that can be given as (rather soft) punishment to the student you don't like.

(**s**) In this version we hope to have gained some simplicity in the argument, but we ruined the constants, compared to the preceding version EZGGC V1.71. In EZGGC V1n.x, implementing the improvement suggested in Note (**n**) and a few more, we get $K < 645$, probably still far from the optimum.

(**t**) I love silly jokes, and I don't intend to publish this "work" that doesn't contain much novelty. These lines are copied and adapted from a popular TeX-related free software. I first called my "software" EZGC, but I found out that there was already on the web a software named ezGC!

things to do

— I should stop making new versions. I should rather rebuild the whole thing completely. But thinking is much too hard, compared to rewriting. . .

acknowledgement

I thank all the friends who didn't read this Note and couldn't therefore make any negative comment yet.