Operator theory and exotic Banach spaces

(Banach spaces with small spaces of operators)

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This set of Notes is a largely expanded version of the mini-course "Banach spaces with small spaces of operators", given at the Summer School in Spetses, August 1994. The lectures were based on a forthcoming paper [GM2] with the same title by Tim Gowers and the speaker. A similar series of lectures "Operator theory and exotic Banach spaces" was given at Paris 6 during the spring of '95 as a part of a program of three mini-courses organized by the "Equipes d'Analyse" of the Universities of Marne la Vallée and Paris 6.

We present in section 10, 11 and 12 several examples of Banach spaces which we call "exotic". The first class is the class of Hereditarily Indecomposable Banach spaces (in short H.I. spaces), introduced in [GM1]: a Banach space X is called H.I. if no subspace of X is the topological direct sum of two infinite dimensional closed subspaces. One of the main properties of a H.I. Banach space X is the following: every bounded linear operator T from X to itself is of the form $\lambda I_X + S$, where $\lambda \in \mathbb{C}$, I_X is the identity operator on X and S is strictly singular. It is well known that this implies that the spectrum of T is countable, and it follows easily that a H.I. space is not isomorphic to any proper subspace. More generally, we present in section 11 a class of examples of Banach spaces having "few" operators. The general principle is the following: given a relatively small semi-group of operators on the space of scalar sequences (for example, the semi-group generated by the right and left shifts), we construct a Banach space such that every bounded linear operator on this space is (or is almost) a strictly singular perturbation of an element of the algebra generated by the given semi-group. We obtain in this way in section 12 a new prime space, a space isomorphic to its subspaces with finite even codimension but not isomorphic to its hyperplanes, and a space isomorphic to its cube but not to its square.

We have chosen to present a fairly detailed account of all the tools of general interest that are necessary to the analysis, although these appear already in many classical books (but probably not in the same book); we develop elementary Banach algebra theory in section 2, Fredholm theory in section 4, strictly singular operators and strictly singular perturbations of Fredholm operators in section 6, an incursion into the K-theory for Banach algebras in section 9. Ultraproducts and Krivine's theorem about the finite representability of ℓ_p are also presented in sections 7 and 8, with some emphasis on the operator approach to these questions. The actual construction of our class of examples appears in section 11, and the applications to some specific examples in the last section 12.

1. Notation

We denote by X, Y, Z infinite dimensional Banach spaces, real or complex, and by E, F finite dimensional normed spaces, usually subspaces of the preceding. Subspaces are closed vector subspaces. We write $X = Y \oplus Z$ when X is the topological direct sum of two closed subspaces Y and Z. The unit ball of X is denoted by B_X. We denote by $\mathbb{K}$ the field of scalars for the space in question ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$).

We denote by $\mathcal{L}(X, Y)$ the space of bounded linear operators between two (real or complex) Banach spaces X and Y. When $Y = X$, we simply write $\mathcal{L}(X)$. We denote by S, T, U, V bounded linear operators. Usually, S will be a "small" operator; it could be small in operator norm, or compact, or finite rank or strictly singular... By I_X we denote the identity operator from X to X. An *into isomorphism* from X to Y is a bounded linear operator T from X to Y which is an isomorphism between X and the image TX; this is equivalent to saying that there exists $c > 0$ such that $\|Tx\| \geq c\|x\|$ for every $x \in X$. Let $\mathcal{K}(X, Y)$ denote the closed vector subspace of $\mathcal{L}(X, Y)$ consisting of compact operators (we write $\mathcal{K}(X)$ if $Y = X$).

A *normalized* sequence in a Banach space X is a sequence $(x_n)_{n\geq 1}$ of norm one vectors. The closed linear span of a sequence $(x_n)_{n\geq 1}$ is noted $[x_n]_{n\geq 1}$. A *basic sequence* is a Schauder basis for its closed linear span $[x_n]_{n\geq 1}$. This is equivalent to saying that there exists a constant C such that for all integers $m \leq n$ and all scalars $(a_k)_{k=1}^n$ we have

$$\Big\|\sum_{k=1}^{m} a_k x_k\Big\| \leq C \,\Big\|\sum_{k=1}^{n} a_k x_k\Big\|.$$

The smallest possible constant C is called the *basis constant* of $(x_n)_{n\geq 1}$.

An *unconditional basic sequence* is an (infinite) sequence $(x_n)_{n\geq 1}$ in a Banach space for which there exists a constant C such that for every integer $n \geq 1$, all scalars $(a_k)_{k=1}^n$ and all signs $(\eta_k)_{k=1}^n$, $\eta_k = \pm 1$, we have

$$\Big\|\sum_{k=1}^{n} \eta_k a_k x_k\Big\| \leq C \,\Big\|\sum_{k=1}^{n} a_k x_k\Big\|.$$

The smallest possible constant C is called the *unconditional basis constant* of $(x_n)_{n\geq 1}$. A question that remained open until '91 motivated much of the research contained in these Notes: does every Banach space contain an unconditional basic sequence (in short: UBS). The answer turned out to be negative and lead to the introduction of H.I. spaces.

2. Basic Banach Algebra theory

(For this paragraph, see for example Bourbaki, Théories spectrales, or [DS], or many others.) A *Banach algebra* A is a Banach space (real or complex) which is also an algebra where the product $(a, b) \to ab$ is norm continuous from $A \times A$ to A. This means that the product and the norm are related in the following way: there exists a constant C such that for all $a, b \in A$, we have

$$\|ab\| \leq C\|a\| \, \|b\|.$$

It is then possible to define an equivalent norm on A satisfying the sharper inequality

$$\forall a, b \in A, \;\; \|ab\| \leq \|a\| \|b\|.$$

We shall call a norm satisfying this second property a *Banach algebra norm*. In order to define an equivalent Banach algebra norm from a norm satisfying the first property with a constant C, we may for example consider

$$|||a||| = \sup\{\|ab + \lambda a\| : \|b\| \leq 1, |\lambda| \leq 1\}.$$

We say that A is *unital* if there exists an element $e \in A$ such that $ea = ae = a$ for every $a \in A$; we write usually 1_A for this element e. If A is unital, we get an equivalent Banach algebra norm on A using the formula

$$|||a||| = \sup\{\|ab\| : b \in A, \|b\| \leq 1\}$$

and for this norm $|||1_A||| = 1$. A Banach algebra norm with this additional property will be called *unital Banach algebra norm.*

A C^*-algebra is a complex Banach algebra A with a Banach algebra norm and with an anti-linear involution $a \to a^*$ (i.e. $a^{**} = a$, $(a+b)^* = a^* + b^*$, $(\lambda a)^* = \overline{\lambda} a$, $(ab)^* = b^* a^*$ for every $a, b \in A$ and $\lambda \in \mathbb{C}$) and such that

$$\forall a \in A,\ \|a^* a\| = \|a\|^2.$$

If A is unital, then $1_A^* = 1_A$ and $\|1_A\| = 1$. An element $x \in A$ is Hermitian (or self-adjoint) if $x^* = x$. Every $a \in A$ can be written as $a = x + iy$, where x and y are Hermitian ($x = (a + a^*)/2$, $y = i(a^* - a)/2$). At some point we will need the self-explanatory notion of a C^*-norm on a not complete algebra with involution.

The above definition is not satisfactory in the real case. Indeed, if A is any real unital subalgebra of the complex algebra $C(K)$ of continuous functions on a compact topological space K, and if we define on A the trivial involution $f^* = f$ for every $f \in A$, then all properties of the preceding definition hold (because λ is now restricted to $\mathbb{R}$), but A is not necessarily what we want to call a real C^*-algebra. In order to obtain a reasonable definition for the real case, we need to add an axiom which is consequence of the others in the complex case, but not in the real case, for example

$$\forall a, b \in A, \quad \|a^* a\| \leq \|a^* a + b^* b\|.$$

Adding 1

When A has no unit it is possible to embed A in a larger unital Banach algebra, by considering on $A^+ = A \oplus \mathbb{K}$ the product $(a, \lambda)(b, \mu) = (ab + \lambda b + \mu a, \lambda\mu)$. Then $1_{A^+} = (0, 1)$ is the unit of A^+ and A is a closed two-sided ideal in A^+. When A is a C^*-algebra, it is possible to define on A^+ a C^*-norm.

An important example of unital Banach algebra is $\mathcal{L}(X)$ where X is a (real or complex) Banach space. The operator norm is a unital Banach algebra norm on $\mathcal{L}(X)$. The subspace $\mathcal{K}(X)$ is a non unital closed subalgebra of $\mathcal{L}(X)$, actually a closed two-sided ideal of $\mathcal{L}(X)$. The algebra $(\mathcal{K}(X))^+$ is isomorphic to the subalgebra of $\mathcal{L}(X)$ consisting of all operators of the form $T = \lambda I_X + K$, K compact (recall that X is infinite dimensional). The Calkin algebra $\mathcal{C}(X) = \mathcal{L}(X)/\mathcal{K}(X)$ is another important example. It will play a role for the notion of essential spectrum later in this section.

When H is a Hilbert space, $\mathcal{L}(H)$ is a C^*-algebra. It is a fundamental example since every C^*-algebra can be $*$-embedded in some $\mathcal{L}(H)$. The quotient of a C^*-algebra by a closed two-sided ideal is a C^*-algebra for the quotient norm (it is true but not obvious that for any such ideal I, we have $x \in I \Rightarrow x^* \in I$); in particular the Calkin algebra $\mathcal{C}(H)$ of a Hilbert space H is a C^*-algebra.

Complexification

Most of spectral theory is done when the field of scalars is $\mathbb{C}$. It is therefore important to be able to pass from the real case to the complex case. If X is a real Banach space, the complexified space is $X_{\mathbb{C}} = X \oplus X$ with the rule

$$i(x, y) = (-y, x).$$

In this way we have $(0, y) = i(y, 0)$ and identifying $(x, 0) \in X_{\mathbb{C}}$ with $x \in X$, we see that every $z \in X_{\mathbb{C}}$ can be written as $z = x + iy$ with $x, y \in X$.

In order to define a complex norm on $X_{\mathbb{C}}$ it is useful to think about $X_{\mathbb{C}}$ as being $\mathbb{C} \otimes X$; it is clear that any tensor norm on $\mathbb{C} \otimes X$ will give in particular a complex norm on $X_{\mathbb{C}}$ (of course we choose the modulus as norm on the 2 dimensional real space $\mathbb{C}$). We can consider for example the injective tensor norm $\mathbb{C} \otimes_\varepsilon X$,

$$\|x + iy\| = \sup\{|x^*(x) + ix^*(y)| : x^* \in X^*, \|x^*\| \leq 1\}.$$

With this norm $\mathbb{C} \otimes X$ is isometric to (the real space) $\mathcal{L}(X^*, \mathbb{C})$. A (real) linear functional x^* on X is extended to a complex linear functional on $X_{\mathbb{C}}$ by setting simply $x^*(x + iy) = x^*(x) + ix^*(y)$, and it is easy to see that every complex linear functional on $X_{\mathbb{C}}$ can be obtained as $x^* + iy^*$, where $x^*, y^* \in X^*$. Given $T \in \mathcal{L}(X, Y)$, one gets a complex linear operator $T_{\mathbb{C}} \in \mathcal{L}(X_{\mathbb{C}}, Y_{\mathbb{C}})$ by setting $T_{\mathbb{C}} = Id \otimes T$ in the tensor product language. In the previous language we can write $T_{\mathbb{C}}(x + iy) = Tx + iTy$ for all $x, y \in X$.

When A is a real Banach algebra, it can be checked that if we norm $A_{\mathbb{C}}$ by $A_{\mathbb{C}} = \mathbb{C} \otimes_\pi A$, we get a complex Banach algebra norm (if A had one); if $A = \mathcal{L}(X)$, X real, then $A_{\mathbb{C}}$ identifies with $\mathcal{L}(X_{\mathbb{C}})$ and any complex norm on $X_{\mathbb{C}}$ gives a Banach algebra norm on $A_{\mathbb{C}}$; given $V = T + iU$ in $A_{\mathbb{C}}$, $T, U \in A = \mathcal{L}(X)$, we associate the operator on $X_{\mathbb{C}} = X \oplus X$ given by the matrix

$$V = \begin{pmatrix} T & -U \\ U & T \end{pmatrix}.$$

Invertible elements

Let A be a unital Banach algebra over $\mathbb{K}$. We say that $a \in A$ is invertible (in A) if there exists $x \in A$ such that $ax = xa = 1_A$.

Lemma 2.1. *Let A be a unital Banach algebra (with a Banach algebra norm). If $b \in A$ and $\|b\| < 1$ then $1 - b$ is invertible in A.*

Proof. $(1 - b)^{-1} = \sum_{n=0}^{\infty} b^n$.

Corollary 2.1. Let A be a unital Banach algebra. The set of invertible elements is open in A. Furthermore, if $\mathbb{K} = \mathbb{C}$, when $a \in A$ is invertible, the function $f(z) = (a - zb)^{-1}$ is analytic in a neighborhood of 0 in $\mathbb{C}$ for every $b \in A$.

Proof. Write $a - b = a(1 - a^{-1}b)$ and use Lemma 2.1. If $\|b\| < \|a^{-1}\|^{-1}$, then $\|a^{-1}b\| < 1$, and we obtain the following formula for the inverse

$$(a - b)^{-1} = u + ubu + ububu + \cdots,$$

where $u = a^{-1}$. Applying to zb instead of b clearly gives an analytic function of z in a neighborhood of 0 in $\mathbb{C}$,

$$(a - zb)^{-1} = u + z\, ubu + z^2\, ububu + \cdots.$$

Remark 2.1. The same proof works when a is replaced by $T \in \mathcal{L}(X, Y)$ invertible, and b is replaced by a small operator $S \in \mathcal{L}(X, Y)$; the above formulas hold with $u = T^{-1} \in \mathcal{L}(Y, X)$, provided $\|S\| < \|T^{-1}\|^{-1}$.

Spectrum of an element, resolvent set

Let A be unital over $\mathbb{C}$ and let $a \in A$. The *resolvent set* $\rho(a)$ is the set of all $\lambda \in \mathbb{C}$ such that $\lambda 1_A - a$ is invertible in A. The *spectrum* $\sigma(a)$ is the complementary set $\mathbb{C} \setminus \rho(a)$. This set $\rho(a)$ is open by Corollary 2.1 and is clearly a neighborhood of infinity, hence $\sigma(a)$ is a closed and bounded subset of $\mathbb{C}$.

Exercise 2.1. Let $a, b \in A$. Show that $\sigma(ab) \setminus \{0\} = \sigma(ba) \setminus \{0\}$ (hint: if $z(\lambda 1_A - ab) = 1_A$, find $u \in A$ such that $u(\lambda 1_A - ba) = \lambda 1_A$). Why did we exclude 0?

Real spectrum

If A is a real unital Banach algebra and if $a \in A$ we may define a real spectrum by

$$\sigma^{\mathbb{R}}(a) = \{\lambda \in \mathbb{R} : (a - \lambda 1_A) \text{ not invertible in } A\}.$$

The problem is that this spectrum may be empty (contrary to what happens in the complex case: we shall recall that the spectrum is non empty in the complex case). If we extend the scalars and consider the same $a \in A$ as an element of $A_{\mathbb{C}}$, we can work with the complex spectrum $\sigma_{A_{\mathbb{C}}}(a)$.

Exercise. Let A be a real unital Banach algebra and let $a \in A$.

1. Show that $\sigma_{A_{\mathbb{C}}}(a) = \overline{\sigma_{A_{\mathbb{C}}}(a)}$
2. Show that $\sigma^{\mathbb{R}}(a) = \mathbb{R} \cap \sigma_{A_{\mathbb{C}}}(a)$.

In what follows we shall denote by $\sigma^{\mathbb{K}}(a)$ the real spectrum when A is real and the complex spectrum if A is complex.

Changing the ambient algebra

Assume that A is a closed subalgebra of a unital Banach algebra B, with the induced norm and containing 1_B. If $x \in A$ is invertible in A, it is obviously also invertible in B, but it is possible for $x \in A$ to be invertible in B but not in A. It is necessary in this case to distinguish the spectrum of x relative to B or relative to A; we denote the spectrum by $\sigma_A^{\mathbb{K}}(x)$ or $\sigma_B^{\mathbb{K}}(x)$, and similarly for the resolvent sets. If $x \in A$, it is clear that $\sigma_A^{\mathbb{K}}(x) \supset \sigma_B^{\mathbb{K}}(x)$ (equivalently, $\rho_A^{\mathbb{K}}(x) \subset \rho_B^{\mathbb{K}}(x)$). We also have $\partial\sigma_A^{\mathbb{K}}(x) \subset \partial\sigma_B^{\mathbb{K}}(x)$ from the next Lemma.

Lemma 2.2. Let B be a unital Banach algebra, and let A be a closed subalgebra with $1_B \in A$. For every $x \in A$, the resolvent set $\rho_A^{\mathbb{K}}(x)$ is open and closed in $\rho_B^{\mathbb{K}}(x)$. Hence, for every connected component ω of $\rho_B^{\mathbb{K}}(x)$, either $\omega \subset \rho_A^{\mathbb{K}}(x)$ or $\omega \cap \rho_A^{\mathbb{K}}(x) = \emptyset$. It follows that $\partial\rho_A^{\mathbb{K}}(x) = \partial\sigma_A^{\mathbb{K}}(x) \subset \partial\sigma_B^{\mathbb{K}}(x) = \partial\rho_B^{\mathbb{K}}(x)$. If $\rho_B^{\mathbb{K}}(x)$ is connected, then $\sigma_A^{\mathbb{K}}(x) = \sigma_B^{\mathbb{K}}(x)$.

Proof. Let x be fixed in A and write simply ρ_A, ρ_B for $\rho_A^{\mathbb{K}}(x)$ and $\rho_B^{\mathbb{K}}(x)$. We know that ρ_A is open. Let $(\lambda_n) \subset \rho_A$, $\lambda_n \to \lambda \in \rho_B$. There exists $b \in B$ such that $(x-\lambda)b = b(x-\lambda) = 1_B$. On the other hand, for every n there exists $a_n \in A$ such that $(x-\lambda_n)a_n = 1_B$. Multiplying by b we get

$$b = (bx - \lambda_n b)a_n = (1_B + (\lambda - \lambda_n)b)a_n;$$

when $n \to \infty$ we see that $(1_B + (\lambda - \lambda_n)b)^{-1}$ tends to 1_B, thus $a_n \to b$, hence $b \in A$ and $\lambda \in \rho_A$.

Let $\lambda \in \partial\rho_A$. Then $\lambda \notin \rho_A$ since ρ_A is open in $\mathbb{K}$, and therefore $\lambda \notin \rho_B$ since ρ_A is closed in ρ_B; this implies that $\lambda \in \partial\rho_B$.

Remark. When $\sigma_A(x) \neq \sigma_B(x)$, we see that the interior $\mathrm{int}(\sigma_A(x))$ must be non empty.

Examples 2.1.

1. It will be shown later (section 6, Proposition 6.1) that the spectrum (in $B = \mathcal{L}(X)$) of a strictly singular operator S on a complex Banach space X is a countable compact subset of $\mathbb{C}$. This implies that the resolvent set $\rho_B(S)$ is connected, hence the inverse of $\lambda I_X - S$, when it exists, belongs to the closed subalgebra A of $\mathcal{L}(X)$ generated by I_X and the operator S.

2. Suppose that A is a complex unital C^*-subalgebra of $\mathcal{L}(H)$ and let $u \in A$ be hermitian. Since the spectrum of u in $B = \mathcal{L}(H)$ is contained in $\mathbb{R}$ and bounded, it is clear that the resolvent set $\rho_B(u)$ is connected, hence $\sigma_A(u) = \sigma_B(u)$. This remark implies easily that $\sigma_A(x) = \sigma_B(x)$ for every $x \in A$ (it is enough to show that when $x \in A$ is invertible in B its inverse belongs to A; if x is invertible in B, then the hermitian operators x^*x and xx^* are invertible in B, hence in A, therefore x is invertible in A: there exist $y, z \in A$ such that $y(x^*x) = (xx^*)z = 1_A$, and $yx^* = x^*z$ is the inverse of x in A). More generally, if an isomorphism from $H \oplus H$ onto H is given by a matrix (a, b) with $a, b \in A$, the inverse operator from H to $H \oplus H$ is given by a matrix with two entries in A.

3. We can always embed a Banach algebra A in the space of bounded linear operators on some Banach space. Suppose that A is a unital Banach algebra with a unital Banach algebra norm. We simply embed A into $\mathcal{L}(A)$ by mapping each $a \in A$ to the operator M_a of left multiplication by a, defined by $M_a(x) = ax$ for every $x \in A$. It is clear that this gives an isometric embedding from A into $\mathcal{L}(A)$. We show now that the spectrum of $a \in A$ is the same relative to A or to the larger algebra $B = \mathcal{L}(A)$. We only need to show that a is invertible in A iff M_a is invertible in B. One direction is obvious. In the other direction, assume that M_a is invertible in B. Then M_a is onto and there exists $u \in A$ such that $1_A = M_a(u) = au$. We see that $M_a M_u = 1_B$. Since M_a is invertible in B, this implies $M_u M_a = 1_B$ and $ua = au = 1_A$ shows that a is invertible in A.

4. Suppose that A is a real unital Banach algebra; if $a \in A$ is invertible in $A_{\mathbb{C}}$, then a is invertible in A.

5. Consider the embedding of $\mathcal{L}(X)$ into $\mathcal{L}(X^*)$ given by the adjoint (this is not exactly an algebra morphism since $(UT)^* = T^*U^*$); the spectrum of T^* in $\mathcal{L}(X^*)$ is the same as the spectrum of $T \in \mathcal{L}(X)$ (of course this is totally obvious when X is reflexive).

Spectral radius

Let $a \in A$. The *spectral radius* of a is defined by

$$r(a) = \lim_n \|a^n\|^{1/n}.$$

Exercise. Show that the above limit exists.

Example 2.2. If u is hermitian in a C^*-algebra, we get $\|u^{2n}\| = \|u\|^{2n}$ for every integer $n \geq 1$, thus $r(u) = \|u\|$.

Notice that the spectral radius is not changed if the norm on A is replaced by an equivalent norm. Also, $r(a) \leq \|a\|$ if the norm is a Banach algebra norm.

Proposition 2.1. ($\mathbb{K} = \mathbb{C}$ and $1_A \neq 0$) *The spectrum $\sigma(a)$ is contained in the closed disc $\Delta(0, r(a))$ in $\mathbb{C}$ centered at 0 and of radius $r(a)$, and it intersects the circle of radius $r(a)$. In other words,*

$$r(a) = \max\{|\lambda| : \lambda \in \sigma(a)\}.$$

In particular, $\sigma(a)$ is non empty.

Proof. The function $g(z) = (1_A - za)^{-1}$ is clearly defined and analytic when $|z| < r(a)^{-1}$, hence $\sigma(a) \subset \Delta(0, r(a))$. Since $1_A \neq 0$ we may assume $\|1_A\| = 1$ for some equivalent unital Banach algebra norm. If a is invertible, it is then easy to see that $r(a)r(a^{-1}) \geq 1$, hence $r(a) > 0$. If $r(a) = 0$, we know therefore that a is not invertible, thus $0 \in \sigma(a)$ and finally $\sigma(a) = \{0\}$ in this case.

Assume now $r(a) > 0$. If the inverse $(1_A - za)^{-1}$ exists for every z on the circle $|z| = r(a)^{-1}$, we can show that the function g is analytic in a neighborhood of a closed disc of radius $R > r(a)^{-1}$. It follows then from Cauchy's inequalities that for some constant M, we have

$$\forall n \geq 0, \ \|a^n\| \leq \frac{M}{R^n}$$

yielding $r(a) \leq 1/R$ and contradicting the choice of R.

Resolvent equation. Spectral projections

Let A be a unital Banach algebra over $\mathbb{C}$ and let $a \in A$. The *resolvent operator* of a is defined for $z \in \rho(a)$ by $R(z) = (z1_A - a)^{-1}$. Note that $R(z)$ and $R(z')$ commute and commute with a. We have

$$\frac{R(z') - R(z)}{z' - z} = -R(z')R(z).$$

Let $\lambda \in \mathbb{C}$ be isolated in $\sigma(a)$. Let Δ_r be the closed disc $\Delta(\lambda, r)$ with radius r centered at λ; let $r_0 > 0$ be such that λ is the unique point of $\sigma(a)$ contained in Δ_{r_0}; let γ_r be the boundary of Δ_r, oriented in the counterclockwise direction. Let f be holomorphic in a neighborhood V of λ; for $r > 0$ such that $\Delta_r \subset V$ and $r \leq r_0$ we know that $\gamma_r \subset \rho(a)$ and we set

$$\Phi(f) = \frac{1}{2i\pi} \int_{\gamma_r} f(z)R(z)\, dz \in A.$$

Since f is holomorphic, the result does not depend upon the particular value $r \in (0, r_0]$ such that $\Delta_r \subset V$. Also $\Phi(f)$ commutes with a and with every $R(\mu)$. The mapping $f \to \Phi(f)$ is clearly linear with respect to f. What is more interesting is that

$$\Phi(fg) = \Phi(f) \circ \Phi(g).$$

For the proof let $0 < r < s \le r_0$ be such that f and g are holomorphic in a neighborhood of Δ_s and write

$$\Phi(f) \circ \Phi(g) = \frac{1}{2i\pi} \int_{z \in \gamma_r} \frac{1}{2i\pi} \int_{z' \in \gamma_s} f(z) g(z') R(z) R(z') dz dz';$$

use then the resolvent equation and Cauchy's formula. It follows that for every r with $0 < r \le r_0$

$$p = \Phi(1) = \frac{1}{2i\pi} \int_{\gamma_r} (z 1_A - a)^{-1} dz$$

is an idempotent ($p^2 = p$), commuting with a and with every $R(\mu)$; furthermore for every element $b = \Phi(g) \in A$ we have

$$pb = bp = b = pbp$$

since $p\Phi(g) = \Phi(1) \circ \Phi(g) = \Phi(g)$.

To every idempotent p in A we may associate the Banach algebra $A_p = pAp$ of all elements of the form pap, $a \in A$. As a Banach algebra, the norm and the product in A_p are those of A, but the unit of A_p is p. For example, let p be a bounded projection defined on some Banach space X, and let $Y = p(X)$ be the range of p. We see that $\mathcal{L}(X)_p$ identifies with $\mathcal{L}(Y)$ (with an equivalent norm).

Let us come back to our isolated $\lambda \in \sigma(a)$ and let again $p = \Phi(1)$. The above remark shows that $\Phi(f) \in A_p$ for every f holomorphic in a neighborhood of λ. Also notice that

$$\Phi(z) - ap = \frac{1}{2i\pi} \int_{\gamma_r} (z 1_A - a) R(z) \, dz = 0,$$

so that $ap = \Phi(z)$. Suppose that $f(\lambda) \ne 0$; then $g = 1/f$ is holomorphic in a neighborhood of λ, and $\Phi(f) \circ \Phi(1/f) = \Phi(1) = p$. It follows that $\Phi(f)$ is invertible in A_p when $f(\lambda) \ne 0$. This applies in particular to $f(z) = z - \mu$ when $\mu \ne \lambda$ to show that $\Phi(z - \mu) = ap - \mu p$ is invertible in A_p. This shows that the spectrum of $pa = pap$ in A_p reduces to $\{\lambda\}$. Letting $q = 1_A - p$, the spectrum of qa in A_q does not contain λ. This follows from

$$(\lambda 1_A - a) \Big(\frac{1}{2i\pi} \int_{\gamma_r} \frac{R(z)}{z - \lambda} \, dz \Big) = q.$$

More generally, when the spectrum $\sigma(a)$ can be decomposed into two subsets σ_1 and σ_2 open and closed in $\sigma(a)$, we can construct similarly a spectral projection p by replacing the above circle γ_r by a curve γ around σ_1, such that σ_2 is exterior to γ. We still get $pa = ap$ and $\sigma(pa) = \sigma_1$ (in A_p) and similarly in A_q we have that $\sigma(qa) = \sigma_2$.

Exercise. Suppose that $\|a^2 - a\| < \varepsilon < 1/4$; show that there exists an idempotent p such that $ap = pa$ and $\|p - a\| < f(\varepsilon, \|a\|)$ (prove that $\sigma(a)$ is contained in the union of the interiors of two circles γ_1 and γ_0 with radius 1/2 and centered at 1 and 0, for example by considering $(a-t)(a-(1-t))$ for $|t(1-t)| \geq 1/4$; give an upper bound for $\|(z-a)^{-1}\|$ when z belongs to γ_0 or γ_1; let Φ_0 and Φ_1 be the operators as above associated with the two circles, and consider $p = \Phi_1(1)$; use $b = \Phi_1(1/z)$ for proving that $(p - ap) = (1_A - a)abp$ is small, and similarly for $(1_A - p)a$).

Essential spectrum

Let X be an infinite dimensional Banach space. The ideal $\mathcal{K}(X)$ is then proper and the Calkin algebra $\mathcal{C}(X) = \mathcal{L}(X)/\mathcal{K}(X)$ is not $\{0\}$. For every $T \in \mathcal{L}(X)$, the essential spectrum $\widehat{\sigma}^{\mathbb{K}}(T)$ is the spectrum of the image $\widehat{T}$ of T in $\mathcal{C}(X)$. We also consider the corresponding resolvent set $\widehat{\rho}^{\mathbb{K}}(T)$. A scalar $\lambda \in \mathbb{K}$ belongs to this essential resolvent set iff $T - \lambda I_X$ is invertible modulo compact operators. We shall recall in section 6 that this happens iff $T - \lambda I_X$ is a Fredholm operator on X.

Commutative Banach algebras

A (complex) Banach algebra A which is a (skew-)field is isomorphic to $\mathbb{C}$. Indeed, let $a \in A$ and $\lambda \in \sigma(a)$. Since $a - \lambda 1_A$ is not invertible, we must have $a - \lambda 1_A = 0$. Hence every element of A is of the form $\lambda 1_A$ for some $\lambda \in \mathbb{C}$.

Let A be a unital commutative Banach algebra over $\mathbb{C}$. A maximal ideal I is closed and is an hyperplane: first, I is closed because the set of invertible elements is open, hence the closure of I is still a proper ideal, equal to I since I is maximal; second, A/I is a Banach field, therefore isomorphic to $\mathbb{C}$, and I is thus an hyperplane. The linear functional χ such that $\ker \chi = I$, normalized by the condition $\chi(1_A) = 1$, is called a *character*. A character χ is a non zero bounded linear functional on A which is also multiplicative. Indeed, if $a \in A$ and $\chi(a) \neq 0$, let $g(x) = \chi(ax)/\chi(a)$. Then g vanishes on I, so g is proportional to χ, but $\chi(1_A) = g(1_A) = 1$, thus $\chi = g$ and $\chi(ax) = \chi(a)\chi(x)$ for every $x \in A$.

The set of characters on A is called spectrum of A, and denoted by $\mathrm{Sp}(A)$; suppose that A is equipped with a Banach algebra norm; then $\mathrm{Sp}(A)$ is a subset of the unit sphere of the dual space A^*; to see this, observe that the sequence $(a^n)_{n \geq 0}$ is bounded when $\|a\| \leq 1$, so that $(\chi(a^n)) = (\chi(a)^n)$ is bounded, therefore $|\chi(a)| \leq 1$, and $\|\chi\| = 1$ since $\chi(1_A) = 1$.

An element $a \in A$ is invertible in A iff $\chi(a) \neq 0$ for every character χ on A. Indeed, it is clear that $\chi(a) \neq 0$ when a is invertible; if a is not invertible, aA is a proper ideal in A, thus contained in a maximal ideal I. If χ is the character such that $I = \ker \chi$ then $\chi(a) = 0$. It follows that for every $a \in A$,

$$\sigma(a) = \{\chi(a) : \chi \in \mathrm{Sp}(A)\};$$

since $\chi(a - \chi(a)1_A) = 0$, we see that $\chi(a) \in \sigma(a)$ for every $\chi \in \mathrm{Sp}(A)$. Conversely, if $a - \lambda 1_A$ is not invertible, there exists a character χ such that $\chi(a - \lambda 1_A) = 0$.

Furthermore $\mathrm{Sp}(A)$ is w^*-closed in the unit sphere; this allows to map A to the space $C(\mathrm{Sp}(A))$ of continuous functions on the compact space $\mathrm{Sp}(A)$; for every $a \in A$, let $j(a)$

be the continuous function on $\mathrm{Sp}(A)$ defined by $j(a)(\chi) = \chi(a)$; this map j need not be injective in general.

When A is a unital (complex) commutative C^*-algebra, this embedding is isometric; we first observe that when u is hermitian, $a = \mathrm{e}^{iu}$ is unitary, i.e. $a^*a = aa^* = 1_A$; then $a^* = a^{-1}$ and $\|a\| = \|a^{-1}\| = 1$; this implies that the sequence $(a^n)_{n\in\mathbb{Z}}$ is bounded, thus $(\chi(a^n))_{n\in\mathbb{Z}}$ is bounded and therefore $1 = |\chi(a)| = |\,\mathrm{e}^{i\chi(u)}\,|$, hence $\chi(u)$ is real for every hermitian u; this implies

$$\forall a \in A, \quad \chi(a^*) = \overline{\chi(a)}.$$

If u is hermitian, we have $r(u) = \|u\|$ by Example 2.2, hence there exists $\chi \in \mathrm{Sp}(A)$ such that $|\chi(u)| = \|u\|$. This implies that for every $a \in A$, there exists χ such that

$$|\chi(a)|^2 = \chi(a^*a) = \|a^*a\| = \|a\|^2,$$

showing that the mapping $j : A \to C(\mathrm{Sp}(A))$ is isometric in this case. The image of A into $C(\mathrm{Sp}(A))$ is now a subalgebra closed under complex conjugation and obviously separating points of $\mathrm{Sp}(A)$, therefore our embedding is onto by Stone-Weierstrass' theorem.

Suppose that φ is a (unital) $*$-homomorphism between two unital commutative C^*-algebras, then $\|\varphi\| \le 1$; by the preceding paragraph we may think that $A = C(K)$ and $B = C(L)$, where K and L are two compact topological spaces. Since φ is a $*$-homomorphism, it sends every function $f \ge 0$ on K to $\varphi(f) \ge 0$ (introduce the hermitian element $g = \sqrt{f}$). If $\|f\| \le 1$, then $f^*f \le 1$, so that $\varphi(f^*f) \le 1$ and $\|\varphi(f)\| \le 1$; furthermore if φ is injective then φ is isometric; this is because the adjoint map $\varphi^* : (C(L))^* \to (C(K))^*$ sends $\mathrm{Sp}(B) \simeq L$ into $\mathrm{Sp}(A) \simeq K$, and must be onto by the preceding result (otherwise we may find a continuous function $f \ne 0$ supported on $K \setminus \varphi^*(\mathrm{Sp}(B))$, and then $\varphi(f) = 0$, contradicting the injectivity). Observe that we don't need B to be commutative in the above argument, because the range $\varphi(A)$ is commutative, so that its closure in B is a commutative unital C^*-algebra.

Proposition 2.2. *Let B be a non necessarily commutative unital C^*-algebra. For every unital C^*-algebra C, every unital $*$-homomorphism $\varphi : B \to C$ satisfies $\|\varphi\| \le 1$. If φ is injective, then φ is isometric.*

Proof. Given an hermitian element $u \in B$, we may consider the unital subalgebra A of B generated by u. This is a commutative C^*-algebra. We have seen that the spectrum of u in A is real, hence $\sigma_A(u) = \sigma_B(u)$ by Lemma 2.2. Suppose that φ is a $*$-homomorphism from B to some C^*-algebra; restricting φ to A, it follows from the preceding remark that $\|\varphi(u)\| \le \|u\|$, and this is true for every hermitian $u \in B$. For a general $b \in B$, we write

$$\|\varphi(b)\|^2 = \|(\varphi(b))^*\varphi(b)\| = \|\varphi(b^*b)\| \le \|b^*b\| = \|b\|^2.$$

Suppose further that φ is injective; then φ is isometric; indeed, the preceding remarks show that $\|\varphi(u)\| = \|u\|$ when u is hermitian; for a general $b \in B$, we write

$$\|\varphi(b)\|^2 = \|(\varphi(b))^*\varphi(b)\| = \|\varphi(b^*b)\| = \|b^*b\| = \|b\|^2.$$

Wiener's algebra

The Wiener algebra W is the algebra of continuous (complex) functions on $\mathbb{T}$ with absolutely summable Fourier coefficients; the product is the pointwise product and the norm is the ℓ_1-norm of Fourier coefficients. This algebra W is clearly isometric to $\ell_1(\mathbb{Z})$ with its usual norm and the convolution as product.

A function f in W is invertible in W iff f does not vanish on $\mathbb{T}$. This amounts to showing that the characters on W reduce to evaluation at points of $\mathbb{T}$. Let χ be a character on W and let $\lambda = \chi(\mathrm{e}^{i\theta})$. Since the family $(\mathrm{e}^{in\theta})_{n\in\mathbb{Z}}$ is bounded in W, it follows that the sequence $(\lambda^n = \chi(\mathrm{e}^{in\theta}))_{n\in\mathbb{Z}}$ is bounded, hence $|\lambda| = 1$. For every function $f(\theta) = \sum_{n\in\mathbb{Z}} a_n\, \mathrm{e}^{in\theta}$ in W,

$$\chi(f) = \sum_{n\in\mathbb{Z}} a_n \lambda^n = f(\lambda).$$

If f does not vanish on $\mathbb{T}$, we see that $\chi(f) \neq 0$ for every character χ, hence f is invertible in W.

C^*-algebra summary

(see first pages of Pedersen's book [Pd])

Let A be a unital (complex) C^*-algebra. For every hermitian element $b \in A$, we may consider the unital subalgebra B generated by b; it is a commutative C^*-algebra. We know that $j_B : B \to C(\mathrm{Sp}(B))$ is an onto isomorphism. For example, when $b = b^*$ and $\sigma_A(b) \subset [0,\infty)$, then $\sigma_B(b) = \sigma_A(b)$ by Lemma 2.2; the function $j_B(b)$ is a non-negative continuous function on $\mathrm{Sp}(B)$, therefore there exists $c \in B$ which is the "square root" of b, i.e. $c = c^*$, $b = c^2$ and $\sigma(c) \subset [0,\infty)$. Consider

$$C = \{a \in A : a = a^* \text{ and } \sigma(a) \subset [0,\infty)\}.$$

If $b = b^*$ and $a = b^2$, then $a \in C$; this follows from the commutative theory, but also simply from the fact that $\sigma(b) \subset \mathbb{R}$ which implies that for every $t > 0$, $b^2 + t1_A = (b + i\sqrt{t}1_A)(b - i\sqrt{t}1_A)$ is invertible.

If $a = a^*$ and $\|t1_A - a\| \le t$ for some $t \ge 0$, then $a \in C$. This is clear when $t = 0$; when $t > 0$, we have $\|1_A - a/t\| \le 1$ and we may define $b = \sqrt{1_A - (1_A - a/t)} = b^*$ from its Taylor series. Then $a = (\sqrt{t}\, b)^2 \in C$. Conversely, if $a \in C$ and $t = \|a\|$, then $\|t1_A - a\| \le t$, so that we got a characterization of C; this last fact is because we know for every hermitian element a that $r(a) = \|a\| = t$, and

$$\|t1_A - a\| = r(t1_A - a) = \max\{|t - \lambda| : \lambda \in \sigma(a) \subset [0,t]\} \le t.$$

It follows immediately from this characterization that $a_1, a_2 \in C$ implies $a_1 + a_2 \in C$, so that C is a closed convex cone in A, with $C \cap (-C) = \{0\}$.

The next essential step is to prove that $a^*a \in C$ for every $a \in A$. This is easy once we know that we may embed A as a $*$-subalgebra of some $\mathcal{L}(H)$, but it is not obvious from the abstract definition, and it is a main ingredient for the proof of the representation of an abstract C^*-algebra as a subalgebra of some $\mathcal{L}(H)$.

Observe first that for every $b = r + is \in A$, r, s hermitian, we have that $b^*b + bb^* = 2(r^2 + s^2) \in C$. Let $a \in A$. The commutative theory implies that we can write $a^*a = u - v$, with $u, v \in C$ and $uv = vu = 0$. Let $b = a\sqrt{v}$; then $b^*b = -v^2 \in -C$, and $bb^* \in C$ since $b^*b + bb^* \in C$; by Exercise 2.1, this implies that $\sigma(b^*b) = \{0\}$, hence $r(b^*b) = \|b^*b\| = \|b\|^2 = 0$, thus $v^2 = 0$, and finally $v = 0$, $a^*a = u \in C$.

For every $u \neq 0$, we know that $-u^*u \notin C$. We may therefore find by Hahn-Banach a linear functional ξ on A such that $-\xi(u^*u) \leq \inf \xi(C)$; this yields that $\xi \geq 0$ on C. we define a scalar product on A by

$$\langle a, b \rangle = \xi(a^*b).$$

To every $a \in A$ we associate the operator $T_a(b) = ab$. Let $a \in A$ with $\|a\| \leq 1$. We have

$$\langle T_a(b), T_a(b) \rangle = \xi(b^*a^*ab);$$

Since $\|a\| \leq 1$, we know that $1_A - a^*a = c^2 \in C$ and $b^*b - b^*a^*ab = b^*c^2b \in C$, therefore $\xi(b^*a^*ab) \leq \xi(b^*b) = \langle b, b \rangle$. This shows that

$$\forall a \in A, \quad \|T_a\|_{\mathcal{L}(H)} \leq \|a\|,$$

where H is the Hilbert space obtained from the above scalar product. It is easy to check that $(T_a)^* = T_{a^*}$, so that we have a $*$-homomorphism from A to $\mathcal{L}(H)$. We may improve the argument and obtain $\|T_u\| = \|u\|$ for the given u and then find an isometric embedding of A into some $\mathcal{L}(\mathcal{H})$ using a direct sum of such embeddings (if $\|u\| = 1$, observe that the closed convex cone $u^*u + C$ is disjoint from the open unit ball, and use the separation theorem as before to obtain ξ such that $\xi(u^*u) = \|\xi\| = 1$).

Suppose now that I is a closed two-sided ideal in A, and let $a \in I$. Let $|a| = \sqrt{a^*a}$; then $|a| \in I$; indeed, there exists a sequence (P_n) of polynomials with real coefficients such that $P_n(0) = 0$ and such that $(P_n(t))$ converges to $\sqrt{t}$ uniformly on any compact interval $[0, T]$, so that for every $x \in C \cap I$, we have $\sqrt{x} \in I$. For every $\varepsilon > 0$, $\varepsilon^2 1_A + a^*a$ is invertible, thus there exists $u_\varepsilon \in A$ such that

$$a = \sqrt{\varepsilon^2 1_A + a^*a}\; u_\varepsilon.$$

We obtain from the commutative theory

$$\|u_\varepsilon\|^2 = \|(\varepsilon^2 1_A + a^*a)^{-1/2} a^*a (\varepsilon^2 1_A + a^*a)^{-1/2}\| \leq 1.$$

When $\varepsilon \to 0$ we obtain that $a = \lim_{\varepsilon \to 0} |a| u_\varepsilon$ hence

$$a^* = \lim_{\varepsilon} u_\varepsilon^* |a| \in I.$$

Finally, let us say some words about the real case. Let A be a real Banach algebra with involution and with a Banach algebra norm such that

$$\forall a, b \in A, \quad \|a\|^2 \leq \|a^*a + b^*b\|.$$

This implies as in the complex case that every hermitian element $x \in A$ as a real spectrum in $A_{\mathbb{C}}$ (we don't claim so far that $A_{\mathbb{C}}$ can be equipped with a C^*-norm). Indeed, $a = \cos tx$ and $b = \sin tx$ are hermitian for every real t, and $a^2 + b^2 = 1_A$, therefore by our hypothesis $\|\cos tx\| \le 1$ and $\|\sin tx\| \le 1$ for every real t, which implies that e^{itx} is bounded in $A_{\mathbb{C}}$, from which it follows that the spectrum of x is real. With this information, it is possible to reproduce the arguments from the beginning of this paragraph and to $*$-embed A in $\mathcal{L}(H)$, for some real Hilbert space H.

Exercise. Complete the details. Show first that if B is the (real) subalgebra generated by an hermitian element $a \in A$, and if $x, y \in B$,

$$\|x + iy\| = \|x^2 + y^2\|_A^{1/2}$$

is a C^*-norm on $B_{\mathbb{C}}$. If $v \in A$ is anti-hermitian, i.e. $v^* = -v$, observe that $\mathrm{e}^{v^*}\,\mathrm{e}^{v} = 1_A$ and $\|\,\mathrm{e}^{v}\,\| = 1$. It follows that $\sigma(v) \subset i\mathbb{R}$, and $v^2 - t^2 1_A$ is invertible for t real, $t \ne 0$, thus $\sigma(-v^2) \subset [0, +\infty)$.

3. Some operator theory: finitely singular operators

See for example [LT1], section 2.c.

Lemma 3.1. *Let X and Y be real or complex Banach spaces.*

1. Trivial principle: if T is an isomorphism from X into Y, a small norm perturbation $T + S$ of T is still an into isomorphism.

2. Fundamental principle: if T is an isomorphism from X onto Y, a small norm perturbation $T + S$ is still an isomorphism from X onto Y (this is Remark 2.1).

Lemma 3.2. *Let $T \in \mathcal{L}(X, Y)$ be an into isomorphism and $k \ge 0$ an integer.*

1. Suppose that $\operatorname{codim} TX \ge k$. There exists $c > 0$ such that $\operatorname{codim}(T + S)X \ge k$ whenever $\|S\| < c$.

2. Suppose that $\operatorname{codim} TX = k$. There exists $d > 0$ such that $\operatorname{codim}(T + S)X = k$ whenever $\|S\| < d$.

Proof. If $\operatorname{codim} TX \ge k$ we can find a subspace $F \subset Y$ such that $\dim F = k$ and $TX \cap F = \{0\}$. Let π_F denote the quotient map $Y \to Y/F$. Then $\pi_F \circ T$ is an isomorphism from X into Y/F. If $c > 0$ is small enough and $\|S\| < c$, $\pi_F \circ (T + S)$ is also an into isomorphism by Lemma 3.1. This implies that $F \cap (T + S)X = \{0\}$, hence $\operatorname{codim}(T + S)X \ge k$.

In the second case the proof is similar, but we can now select a subspace F such that $Y = TX \oplus F$, $\dim F = k$. Then $\pi_F \circ T$ is an onto isomorphism, hence there is $d > 0$ (we may choose $d < c$), such that $\pi_F \circ (T + S)$ is an onto isomorphism when $\|S\| < d$. By the above argument we already know that $F \cap (T + S)X = \{0\}$; furthermore, for every $y \in Y$, there exists $x \in X$ such that $\pi_F(y) = \pi_F((T + S)x)$, so $y - (T + S)x \in F$, showing that $Y = F + (T + S)X$. Finally $Y = F \oplus (T + S)X$ and $\operatorname{codim}(T + S)X = k$.

Proposition 3.1. *Let $T \in \mathcal{L}(X, Y)$ be an into isomorphism. Then $T + S$ is an into isomorphism and $\operatorname{codim}(T+S)X = \operatorname{codim} TX$ (finite or $+\infty$) for every S in a neighborhood of 0 in $\mathcal{L}(X, Y)$.*

Proof. Let $c > 0$ be such that $T + S$ is an into isomorphism whenever

$$S \in B_c = \{U \in \mathcal{L}(X, Y) : \|U\| < c\}.$$

Observe that $D_k = \{U \in B_c : \operatorname{codim}(T+U)X = k\}$ is open and closed in B_c for every integer $k \geq 0$. Indeed, the set $\{W \in B_c : \operatorname{codim}(T+W)X \geq k+1\}$ is open by Lemma 3.2, part 1, while each D_j, $j = 0, 1, \ldots, k$ is open by part 2 of the same Lemma. Since B_c is connected, each D_k is empty or equal to B_c. The result follows.

Boundary of spectrum lemma

Lemma 3.3. *If $U \in \mathcal{L}(X)$ is an into isomorphism but is not invertible in $\mathcal{L}(X)$, then 0 belongs to the interior $\operatorname{int}(\sigma^{\mathbb{K}}(U))$ (relative to $\mathbb{K}$) of $\sigma^{\mathbb{K}}(U)$.*

Proof. Since U is an into isomorphism but not invertible, U is not onto and $\operatorname{codim} UX \geq 1$. This remains true under small perturbation by Lemma 3.2, part 1: there exists $\varepsilon_0 > 0$ such that $U - \varepsilon I_X$ is not onto, therefore not invertible, for every $\varepsilon \in \mathbb{K}$ such that $|\varepsilon| < \varepsilon_0$. This shows that $B(0, \varepsilon_0) \cap \mathbb{K} \subset \sigma^{\mathbb{K}}(U)$.

Corollary 3.1. *Let $T \in \mathcal{L}(X)$. If $\lambda \in \partial\sigma^{\mathbb{K}}(T)$ (of course this boundary is relative to $\mathbb{K}$), there exists a normalized sequence (x_n) in X such that $(T - \lambda I_X)x_n \to 0$ (this sequence is possibly constant).*

Proof. What we want to prove is equivalent to saying that $U = T - \lambda I_X$ is not an into isomorphism. By our assumption, we have that U is not invertible, but 0 is not interior to $\sigma^{\mathbb{K}}(U)$, hence U is not an into isomorphism by Lemma 3.3.

Exercise. If X is real, $T \in \mathcal{L}(X)$ and if $\lambda = r(\cos\theta + i\sin\theta)$, $r\sin\theta \neq 0$, belongs to the boundary of $\sigma(T_{\mathbb{C}})$, then there exist two sequences (x_n) and (y_n) in X such that $\|x_n\| + \|y_n\| = 1$, $Tx_n - r(\cos\theta\, x_n - \sin\theta\, y_n) \to 0$ and $Ty_n - r(\sin\theta\, x_n + \cos\theta\, y_n) \to 0$.

Remark 3.1. Let A be a unital Banach algebra. If λ belongs to the boundary of $\sigma^{\mathbb{K}}(a)$, there exists a normalized sequence (b_n) in A such that $ab_n - \lambda b_n$ tends to 0 in A. This follows from section 2, example 2.1,3 (we could also get a normalized sequence (c_n) such that $c_n a - \lambda c_n$ goes to 0).

Definition 3.1. Let $T \in \mathcal{L}(X, Y)$; we say that T is *finitely singular* if there exists $c > 0$ and a (closed) finite codimensional subspace $X_0 \subset X$ such that

$$\|Tx\| \geq c\|x\|$$

for every $x \in X_0$. In other words the restriction of T to X_0 is an into isomorphism.

Let $c(T)$ denote the supremum of all $c > 0$ for which the above property holds, and set $c(T) = 0$ if T is not finitely singular.

Our terminology is not classical and perhaps a little strange, since we call a non singular operator, for instance an onto isomorphism, "finitely singular"; it would be better to say "at most finitely singular", but this is definitely too long.

Remark 3.2. Suppose that $T \in \mathcal{L}(X, Y)$ is finitely singular. It is clear that a small norm perturbation of T is still finitely singular (precisely, $T + S$ is finitely singular if $\|S\| < c(T)$; actually it is enough that $\|S_{|X_1}\| < c(T)$ for some finite codimensional subspace X_1 of X). It is also clear that the restriction of T to any infinite dimensional subspace Z of X is finitely singular.

If $T \in \mathcal{L}(X, Y)$ and if UT is finitely singular for some $U \in \mathcal{L}(Y, Z)$, then T is finitely singular.

Exercise 3.1. Suppose that $T \in \mathcal{L}(X, Y)$ is finitely singular. Show that

1. $\ker T$ is finite dimensional.
2. For every (closed) subspace Z of X, $T(Z)$ is a closed subspace of Y.
3. If (x_n) is a bounded sequence in X and if (Tx_n) converges in Y, then there exists a norm-converging subsequence (x_{n_k}); in other words, the restriction of T to any bounded subset of X is a proper map.
4. One can choose X_0 in Definition 3.1 in such a way that $X = \ker T \oplus X_0$. Hence, if $\ker T = \{0\}$, then T is an isomorphism from X into Y.
5. Let $T \in \mathcal{L}(X, Y)$. Show that T is finitely singular if and only if TX is closed and $\dim \ker T < \infty$.
6. If T is finitely singular from X to Y and $\mathbb{K} = \mathbb{R}$, the complexified operator $T_{\mathbb{C}}$ is finitely singular from $X_{\mathbb{C}}$ to $Y_{\mathbb{C}}$.

Lemma 3.4. *Let $T \in \mathcal{L}(X, Y)$. Then T^* is finitely singular from Y^* to X^* if and only if TX is closed and finite codimensional in Y.*

Proof. Suppose that T^* is finitely singular. Since $\ker T^* = (\overline{TX})^\perp$ is finite dimensional, we know that $\overline{TX}$ is finite codimensional. It is enough to show that for some $c > 0$ and for every $y \in \overline{TX}$, there exists $x \in X$ with $\|y - Tx\| \le \|y\|/2$ and $\|x\| \le \|y\|/c$ (the end of the proof is by iteration: one constructs a convergent series $x' = \sum x_n$ in X such that $y = Tx'$). If the preceding claim is not true, we can find for every integer $n \ge 1$ a vector $y_n \in \overline{TX}$ such that $\|y_n\| = 1$ and

$$y_n \notin nT(B_X) + \frac{1}{2} B_Y.$$

By Hahn-Banach there exists $y_n^* \in Y^*$ such that $y_n^*(y_n) = 1$ and $\|y_n^*\| \le 2$, $\|T^*(y_n^*)\| \le 1/n$. Since T^* is finitely singular, we know from Exercise 3.1,3 that there exists a subsequence $(y_{n_k}^*)$ converging to some y^*; it follows then that $T^* y^* = 0$, thus $y^* \in \ker T^*$, which implies that $y^*(y_n) = 0$ for every n, contradicting $y_n^*(y_n) = 1$ and $y_{n_k}^* \to y^*$.

Conversely, assume TX closed and finite codimensional in Y. By the open mapping theorem, T induces an isomorphism from $X/\ker T$ onto TX. Let $Y_0 = TX$ and let $c > 0$ be such that for every $y_0 \in Y_0$, there exists $x \in X$ such that $y_0 = Tx$ and $\|y_0\| \ge c\|x\|$. Let $Y = Y_0 \oplus F$ and let Q be the projection from $Y_0 \oplus F$ onto Y_0. Given $y^* \in F^\perp$, $\|y^*\| = 1$, there exists $y = y_0 + f \in Y_0 \oplus F$ such that $\|y_0 + f\| \le 1$ and $y^*(y_0 + f) = y^*(y_0) > 1/2$. Then, since $\|y_0\| = \|Qy\| \le \|Q\|$ there exists $x \in X$ such that $y_0 = Tx$ and $\|x\| \le \|Q\|/c$, hence

$$\frac{\|Q\|}{c} \|T^* y^*\| \ge T^*(y^*)(x) = y^*(y_0) > 1/2,$$

showing that $\|T^* y^*\| \ge c\|y^*\|/(2\|Q\|)$ for every y^* in the finite codimensional subspace $F^\perp$ of Y^*, hence T^* is finitely singular.

We say that T is *infinitely singular* if it is not finitely singular.

Proposition 3.2. *Let $T \in \mathcal{L}(X, Y)$. Then T is infinitely singular if and only if for every $\varepsilon > 0$, there exists an infinite dimensional subspace $Z \subset X$ such that $\|T_{|Z}\| < \varepsilon$.*

Furthermore, we may assume that this subspace Z has a Schauder basis $(z_n)_{n\geq 1}$ and that the norm of the restriction of T to $[z_n, z_{n+1}, \ldots]$ tends to 0 when $n \to \infty$; in particular we may assume that $T_{|Z}$ is compact.

Proof. Suppose that T is infinitely singular. We construct a normalized basic sequence (z_n) in X such that $\|Tz_n\| < \varepsilon' 2^{-n}$ for every $n \geq 1$, where $0 < \varepsilon' < \varepsilon/4$. If $z_1, \ldots, z_n$ are already selected, let A_n be a finite subset of B_{X^*} which is 1/2-norming for the linear span $[z_1, \ldots, z_n]$, that is

$$\forall x \in [z_1, \ldots, z_n], \quad \|x\| \leq 2 \max_{x^* \in A_n} |x^*(x)|.$$

We may assume that $A_n \supset A_{n-1}$. Let $X_0 = \bigcap_{x^* \in A_n} \ker x^*$; since X_0 is finite codimensional and T infinitely singular, we may find $z_{n+1} \in X_0$ such that $\|z_{n+1}\| = 1$ and $\|Tz_{n+1}\| < \varepsilon' 2^{-n-1}$. We let Z be the closed linear span of the sequence $(z_n)_{n=1}^{\infty}$; it is easy to show that this sequence is a Schauder basis with constant 2 for Z; indeed, if $m < n$, since $z_{m+1}, \ldots, z_n$ were chosen in the kernel of all $x^* \in A_m$, we obtain for all scalars $(a_k)_{k=1}^n$

$$\|\sum_{k=1}^m a_k z_k\| \leq 2 \max_{x^* \in A_m} \bigl|x^*(\sum_{k=1}^m a_k z_k)\bigr| = 2 \max_{x^* \in A_m} \bigl|x^*(\sum_{k=1}^n a_k z_k)\bigr| \leq 2\|\sum_{k=1}^n a_k z_k\|.$$

For $z = \sum_{k\geq n} a_k z_k$ this implies that $|a_k| \leq 4\|z\|$ for every k, hence

$$\|Tz\| \leq 4\|z\| \sum_{k\geq n} \varepsilon' 2^{-k} \leq 8\, 2^{-n} \varepsilon' \|z\| < \varepsilon \|z\|.$$

We obtain that $T_{|Z}$ is compact and that $\|T_{|Z}\| < \varepsilon$. The other direction is clear.

Exercise. T is finitely singular iff the image of every closed subspace $Z \subset X$ is closed.

Remark. What we have showed is that when $c(T) = 0$, there exists a subspace $Z \subset X$ such that $\|T_{|Z}\|$ is small; it does not seem possible to obtain in general a quantitative result of the form $\|T_{|Z}\| \leq Mc(T) + \varepsilon$ for some universal constant M when $c(T) > 0$. This is however true with $M = 1$ in a Hilbert space or in ℓ_p, $1 \leq p < \infty$ (or in c_0).

4. Basic Fredholm theory

Definition 4.1. We say that $T \in \mathcal{L}(X, Y)$ is *a Fredholm operator* from X to Y if there exists a (closed) finite codimensional subspace X_0 of X such that $T_{|X_0}$ is an isomorphism from X_0 onto some finite codimensional subspace $Y_0 = TX_0$ of Y.

In particular T is finitely singular. We know by Exercise 3.1, part 4, that one can choose X_0 such that $X = \ker T \oplus X_0$. In this case $Y_0 = TX_0 = TX$.

Exercise 4.1.

1. Let T be a Fredholm operator from X to Y, and let X_0, Y_0 be as above. Prove that

$$\operatorname{codim}_X X_0 - \operatorname{codim}_Y Y_0 = \dim \ker T - \operatorname{codim}_Y TX.$$

This integer is called *index of* T, and denoted $\operatorname{ind}(T)$.

Hint. Write $TX = Y_0 \oplus F$, $X = X_0 \oplus T^{-1}F$, $T^{-1}F = \ker T \oplus E$ and count dimensions.

2. Show that $T \in \mathcal{L}(X, Y)$ is Fredholm if and only if $\ker T$ is finite dimensional, and TX closed and finite codimensional in Y (this is the usual definition).

3. If T is Fredholm on a real Banach space X, then $T_{\mathbb{C}}$ is Fredholm on $X_{\mathbb{C}}$, with the same index (counting of course complex dimensions for $T_{\mathbb{C}}$).

4. Direct sums; if T_1, T_2 are Fredholm from X_1 to Y_1 and from X_2 to Y_2, then $T_1 \oplus T_2$ is Fredholm from $X_1 \oplus X_2$ to $Y_1 \oplus Y_2$. Check that $\operatorname{ind}(T_1 \oplus T_2) = \operatorname{ind}(T_1) + \operatorname{ind}(T_2)$.

5. When $U_1 T$ and $T U_2$ are Fredholm, then T is Fredholm.

6. If $Q : X \to Y$ is a quotient map with finite dimensional kernel E, then Q is Fredholm and $\operatorname{ind}(Q) = \dim E$. If $T : Z \to X$ is such that QT is Fredholm, show that T is Fredholm.

Perturbation by a small norm operator or a finite rank operator

Proposition 4.1. *Let $T \in \mathcal{L}(X, Y)$ be Fredholm. There exists $d > 0$ such that $\|S\| < d$ implies that $T + S$ is Fredholm and* $\operatorname{ind}(T + S) = \operatorname{ind}(T)$.

Proof. Let X_0, Y_0 be as in Definition 4.1. The result follows immediately from Lemma 3.2, part 2, applied to the operator $T_0 = T_{|X_0}$, considered as operator from X_0 to Y. This operator is an into isomorphism, hence for $d > 0$ small and $\|S\| < d$ we know that $(T + S)_{|X_0}$ is an into isomorphism and that $\operatorname{codim}(T + S)X_0 = \operatorname{codim} TX_0$.

Lemma 4.1. If $T \in \mathcal{L}(X, Y)$ is Fredholm and if S has finite rank, then $T + S$ is Fredholm and $\operatorname{ind}(T + S) = \operatorname{ind}(T)$.

Proof. This is because we may choose X_0 contained in the finite codimensional subspace $\ker S$ of X. Then $T + S$ and T coincide on X_0.

Exercise 4.2.

1. If $T \in \mathcal{L}(X, Y)$ is Fredholm, there exists $U \in \mathcal{L}(Y, X)$ such that $UT - I_X$ and $TU - I_Y$ have finite rank.

2. If $U_1 T - I_X$ and $T U_2 - I_Y$ have finite rank, then T is Fredholm. Hence T is Fredholm iff it is invertible modulo finite rank operators.

Proposition 4.2. Composition formula. If $T : X \to Y$ and $U : Y \to Z$ are Fredholm, then UT is Fredholm from X to Z and

$$\operatorname{ind}(UT) = \operatorname{ind}(T) + \operatorname{ind}(U).$$

Proof. We can find X_0, Y_0, Y_1, Z_1 finite codimensional such that T defines an isomorphism from X_0 onto Y_0 and U an isomorphism from Y_1 onto Z_1. We simply replace these finite codimensional subspaces by smaller finite codimensional subspaces given by $Y_2 = Y_0 \cap Y_1$, $X_2 = X_0 \cap T^{-1}Y_2$ and $Z_2 = UY_2$; now $T_{|X_2}$ is an isomorphism from X_2 onto Y_2 and $U_{|Y_2}$ an isomorphism from Y_2 onto Z_2 and we compute

$$\operatorname{ind}(T) + \operatorname{ind}(U) = (\operatorname{codim} X_2 - \operatorname{codim} Y_2) + (\operatorname{codim} Y_2 - \operatorname{codim} Z_2) = \operatorname{ind}(UT).$$

Duality

Lemma 4.2. *An operator $T \in \mathcal{L}(X, Y)$ is Fredholm if and only if T and T^* are finitely singular.*

Proof. If T is Fredholm, we know that T is finitely singular, and that TX is closed and finite codimensional, so that T^* is finitely singular by Lemma 3.4. Conversely, when T and T^* are finitely singular, we have $\dim \ker T < \infty$ by Exercise 3.1,1 and TX closed and finite codimensional by Lemma 3.4.

Lemma 4.3. *Let $T \in \mathcal{L}(X, Y)$. Then T is Fredholm iff T^* is Fredholm, and in this case we have* $\operatorname{ind}(T^*) = -\operatorname{ind}(T)$.

Proof. If T is Fredholm, T^* is finitely singular by the preceding Lemma, hence $\dim \ker T^* < \infty$ and T^*Y^* is closed. Furthermore, since $\ker T = (T^*Y^*)^\perp$, it follows that T^*Y^* is finite codimensional and T^* is Fredholm. Conversely, if T^* is Fredholm, then T^* is finitely singular, hence TX is closed and finite codimensional by Lemma 3.4, and $\dim \ker T < \infty$ for the same reason as before.

Let X_0 and Y_0 be finite codimensional subspaces of X and Y such that $T_0 = T_{|X_0}$ is an isomorphism from X_0 onto Y_0. Let $i : X_0 \to X$ and $j : Y_0 \to Y$ be the natural inclusion maps. Now $j^* : Y^* \to Y_0^*$ and $i^* : X^* \to X_0^*$ are the natural projections. They are clearly Fredholm (quotient maps with finite dimensional kernel; $\ker i^* = X_0^\perp$, $\operatorname{ind}(i^*) = \operatorname{codim} X_0$). By the composition formula

$$\operatorname{ind}(T_0^*) + \operatorname{ind}(j^*) = \operatorname{ind}(i^*) + \operatorname{ind}(T^*)$$

so that

$$\operatorname{ind}(T^*) = \operatorname{ind}(j^*) - \operatorname{ind}(i^*) = \operatorname{codim} Y_0 - \operatorname{codim} X_0 = -\operatorname{ind}(T).$$

When T is Fredholm, it follows that T^{**} is Fredholm and $\operatorname{ind}(T^{**}) = \operatorname{ind}(T)$. We can deduce it from the duality statement but it is actually clear directly.

Lemma 4.4. *The set of $T \in \mathcal{L}(X, Y)$ which are finitely singular and not Fredholm is open in $\mathcal{L}(X, Y)$, as well as the set of $T \in \mathcal{L}(X, Y)$ such that T^* is finitely singular and not Fredholm from Y^* to X^*.*

Proof. Let $T \in \mathcal{L}(X, Y)$ be finitely singular and not Fredholm. We can find a finite codimensional subspace X_0 of X such that T is an isomorphism from X_0 into Y. If T is not Fredholm, it implies that $\operatorname{codim} TX_0 = \infty$. By Proposition 3.1 this remains true in a neighborhood of T, hence $\operatorname{codim}(T + S)X_0 = +\infty$ and $\operatorname{codim}(T + S)X = +\infty$ also for $T + S$ in this neighborhood. The second part is similar.

Semi-Fredholm operators

If T is finitely singular from X to Y, there exists a finite codimensional subspace X_0 of X such that T is an isomorphism from X_0 into Y. Either $\operatorname{codim} TX_0 < \infty$ and T is Fredholm, or $\operatorname{codim} TX_0 = \infty$ (and thus $\operatorname{codim} TX = +\infty$) and we define the *generalized index* by $\operatorname{ind}(T) = -\infty$.

More generally, an operator $T : X \to Y$ is called *semi-Fredholm* if TX is closed and if the kernel or the cokernel is finite dimensional; the generalized index is defined as before, with the possible values $+\infty$ and $-\infty$. An operator T is semi-Fredholm iff T or T^* is finitely singular by Lemma 3.4 and Exercise 3.1, 5. Observe that a Fredholm operator is precisely a semi-Fredholm operator with finite index.

Lemma 4.5. *The set of semi-Fredholm operators is open in $\mathcal{L}(X, Y)$, and the generalized index is locally constant.*

Proof. By Lemma 4.4 and Proposition 4.1.

Lemma 4.6. *If T_t is a continuous path in $\mathcal{L}(X, Y)$, $t \in [0, 1]$, such that T_t is semi-Fredholm for every $t \in [0, 1]$, then $\operatorname{ind}(T_1) = \operatorname{ind}(T_0)$. In particular, under the same assumptions, if T_0 is Fredholm then T_1 is Fredholm with same index.*

Proof. We know that the generalized index is locally constant.

Corollary 4.1. *If T is finitely singular and $\|S\| < c(T)$, then $T + S$ is finitely singular and the generalized index satisfies $\operatorname{ind}(T + S) = \operatorname{ind}(T)$. In particular, if T is Fredholm and $\|S\| < c(T)$, then $T + S$ is Fredholm and the index satisfies $\operatorname{ind}(T + S) = \operatorname{ind}(T)$.*

Proof. When $\|S\| < c(T)$, we see that $\|tS\| < c(T)$ for every $t \in [0, 1]$, therefore $T + tS$ is a continuous path consisting of finitely singular operators by Remark 3.2. The result follows from Lemma 4.6.

Corollary 4.2. *If T is Fredholm and K compact from X to Y, then $T + K$ is Fredholm and $\operatorname{ind}(T + K) = \operatorname{ind}(T)$.*

Proof. It is enough to show that $T + K$ is finitely singular for every compact operator K (because then $T + tK$ will be finitely singular for every $t \in [0, 1]$); this will be a consequence of Lemma 6.1, but we give here a different proof: for every $\varepsilon > 0$, there exists a finite codimensional subspace X_0 such that $\|K_{|X_0}\| < \varepsilon$; indeed, there exists a finite set $x_1^*, \ldots, x_n^*$ in X^* such that

$$K^*(B_{Y^*}) \subset \bigcup_{j=1}^{n} (x_j^* + \varepsilon B_X).$$

This implies that $\|Kx\| \le \varepsilon \|x\|$ when $x \in \bigcap_{j=1}^n \ker x_j^*$. If we choose $\varepsilon < c(T)$, we see that $T + K$ is finitely singular (Remark 3.2).

Corollary 4.3. *Let $T \in \mathcal{L}(X, Y)$. Then T is Fredholm if and only if T is invertible modulo compact operators, i.e. iff there exists $U \in \mathcal{L}(Y, X)$ such that $TU - I_Y$ and $UT - I_X$ are compact.*

Proof. This condition is necessary by Exercise 4.2,1. Conversely, if $TU - I_Y$ is compact, we know that TU is Fredholm by Corollary 4.2. In the same way, UT is Fredholm and we know then that T is Fredholm by Exercise 4.2,2.

5. More on operators

See [LT1], section 2.c; Kato, [Kt1], [Kt2].

We prove now another version of Lemma 3.2. Suppose that $T \in \mathcal{L}(X, Y)$ has closed range. Then T induces an isomorphism between $X/\ker T$ and TX, and there exists a constant $c > 0$ such that for every $y \in TX$, we can find $x \in X$ such that $y = Tx$ and $\|y\| \geq c\|x\|$.

Lemma 5.1. *Let $T_1, T_2 \in \mathcal{L}(X, Y)$. Suppose that for every $y \in T_1X$, there exists $x \in X$ such that $y = T_1x$ and $\|y\| \geq c\|x\|$; if $\|T_1 - T_2\| < c$ and if T_2X is closed, then $\operatorname{codim} T_2X \leq \operatorname{codim} T_1X$, finite or infinite.*

Remark. We can recover from the above Lemma the fact proved in Proposition 3.1 that $\operatorname{codim} T_1X$ remains constant in a neighborhood of an into isomorphism T_1. Indeed, if T_1 is an into isomorphism from X into Y, with $\|T_1x\| \geq c\|x\|$ for every $x \in X$, and if we assume $\|T_2 - T_1\| < c/2$, we have $\|T_2x\| \geq (c/2)\|x\|$ for every $x \in X$; this shows that T_2 is an into isomorphism, therefore T_2X is closed; we obtain $\operatorname{codim} T_2X \leq \operatorname{codim} T_1X$ by the above Lemma; since $\|T_2 - T_1\| < c/2$, we can exchange the roles of T_1 and T_2 and conclude that $\operatorname{codim} T_2X = \operatorname{codim} T_1X$, finite or infinite.

Proof of the Lemma. If $\operatorname{codim} T_2X > \operatorname{codim} T_1X$, there exists by the next sublemma, applied to $Z = T_1X$, $Y = T_2X$ and $\varepsilon = 1 - \|T_1 - T_2\|/c > 0$ a vector $z \in T_1X$, $\|z\| = 1$ such that $\operatorname{dist}(z, T_2X) > \|T_1 - T_2\|/c$. Then there exists $x \in X$ such that $z = T_1x$ and $\|x\| \leq 1/c$; now $\|z - T_2x\| \leq \|T_1 - T_2\|\,\|x\|$ gives a contradiction.

Sublemma. Let Y and Z be two finite codimensional subspaces of X. If $\operatorname{codim} Y > \operatorname{codim} Z$, there exists for every $\varepsilon > 0$ a vector $z \in Z$ such that $\|z\| = 1$ and $\operatorname{dist}(z, Y) > 1-\varepsilon$. This is also valid if Y is closed, $\operatorname{codim} Y = \infty$ and $\operatorname{codim} Z < \infty$.

Proof. Let ρ be a continuous lifting (not necessarily linear!) from the unit sphere $S(X/Y)$ of X/Y to the ball of radius $(1 - \varepsilon)^{-1}$ in X. We may assume that $\rho(-x) = -\rho(x)$. Then $\pi_Z \circ \rho$ is an odd mapping from the sphere of X/Y to X/Z and $\dim X/Y > \dim X/Z$. By Borsuk's antipodal mapping theorem, there exists $\overline{x} \in S(X/Y)$ such that $\pi_Z(\rho(\overline{x})) = 0$, i.e. $\rho(\overline{x}) \in Z$. Take $z' = \rho(\overline{x})$ and $z = z'/\|z'\|$. When $\operatorname{codim} Y = \infty$, it is enough to select $F \subset X/Y$ such that $\dim F > \dim X/Z$ and to apply the same reasoning.

Corollary 5.1. Let $T_1, T_2 \in \mathcal{L}(X, Y)$. If $\|T_1x\| \geq c\|x\|$ for every $x \in X$ and if $\|T_1 - T_2\| < c$, then $\operatorname{codim} T_2X = \operatorname{codim} T_1X$, finite or infinite.

Proof. It follows from the hypothesis that every T on the segment $[T_1, T_2]$ is an into isomorphism, and we know that $\operatorname{codim} TX$ is locally constant by Proposition 3.1 or the Remark following the above Lemma.

Lemma 5.2. *Let $T \in \mathcal{L}(X)$; if T or T^* is finitely singular (in other words if T is semi-Fredholm) and if $0 \in \partial\sigma^{\mathbb{K}}(T)$, then T is Fredholm with index 0.*

Proof. Since $0 \in \partial\sigma^{\mathbb{K}}(T)$, we can find invertible operators, in particular Fredholm operators with index 0, arbitrarily close to T, hence T is Fredholm with index 0 by the continuity of the index of semi-Fredholm operators (Lemma 4.5).

Remark. We obtain a slightly different proof for the "Boundary of spectrum lemma": if $T \in \mathcal{L}(X)$ and $\lambda \in \partial\sigma^{\mathbb{K}}(T)$, then there exists a (possibly constant) normalized sequence

(x_n) in X such that $(T-\lambda I_X)x_n \to 0$. Indeed, let $U = T-\lambda I_X$. If U is infinitely singular, we know the result by Proposition 3.2. If U is finitely singular and $0 \in \partial\sigma^{\mathbb{K}}(U)$, then U is Fredholm with index 0 by Lemma 5.2. Since $0 \in \sigma^{\mathbb{K}}(U)$, it follows that U is not invertible, thus $\ker U \neq \{0\}$.

Lemma 5.3. *If $T \in \mathcal{L}(X)$ is finitely singular, then $Y = \bigcap_{n=0}^{\infty} T^n X$ is closed and $TY = Y$. Furthermore $T_{|Y} : Y \to Y$ is Fredholm and the constant value of $\operatorname{ind}(T_{|Y} - \lambda I_Y)$ for λ in a neighborhood of 0 (in $\mathbb{K}$) is the (constant and finite) dimension of the kernel of $T - \lambda I_X$ for small $\lambda \neq 0$. Precisely, there exists $\varepsilon > 0$ such that*

$$\forall \lambda \in \mathbb{K},\ 0 < |\lambda| < \varepsilon \Rightarrow \dim\ker(T-\lambda I_X) = \operatorname{ind}(T_{|Y}).$$

Proof. We prove that $T^n X$ is closed by induction using Exercise 3.1, part 2, hence Y is closed. It is clear that $TY \subset Y$. We also know that $N_1 = \ker T$ is finite dimensional. It follows that there exists an integer k such that, setting $Y_j = T^j X$, we have $N_1 \cap Y_k = N_1 \cap Y$. Let $y \in Y$; for every integer j, there exists a vector $z_j \in Y_j$ such that $y = Tz_j$. For $j > k$ we get $z_j - z_k \in N_1 \cap Y_k = N_1 \cap Y$. In particular $z_j - z_k \in Y \subset Y_j$ and thus $z_k \in Y_j$. Since this is true for every $j > k$, we obtain $z_k \in Y$, and finally $y = Tz_k \in TY$. We have that $\ker T_{|Y} \subset \ker T$ is finite dimensional; since $TY = Y$, it follows that $T_{|Y}$ is Fredholm from Y to Y.

Suppose that $\lambda \neq 0$ and $(T-\lambda I_X)x = 0$. We have $x = T^n x/\lambda^n$ for every integer $n \geq 1$, yielding $x \in Y$. Hence the kernel of $T-\lambda I_X$ coincides with the kernel of $T_{|Y} - \lambda I_Y$. Furthermore, if we choose Y_0 finite codimensional in Y such that $Y = \ker T_{|Y} \oplus Y_0$, we see that $T_{|Y_0}$ is an isomorphism from Y_0 onto Y. This remains true for $T - \lambda I_X$ for small λ, say $|\lambda| < \varepsilon$, hence $T_{|Y} - \lambda I_Y$ remains onto and for $0 < |\lambda| < \varepsilon$ we obtain

$$\dim\ker(T-\lambda I_X) = \dim\ker(T_{|Y} - \lambda I_Y) = \operatorname{ind}(T_{|Y} - \lambda I_Y) = \operatorname{ind}(T_{|Y}).$$

Remark. Suppose that $\ker T = \{0\}$ and $\ker(T - \lambda_n I_X) \neq \{0\}$ for a sequence (λ_n) tending to 0. Then T is infinitely singular. Indeed, there exists a normalized sequence (x_n) such that $Tx_n = \lambda_n x_n$ tends to 0; if T is finitely singular, there exists by Exercise 3.1,3 a subsequence (x_{n_k}) converging to some x; then $x \neq 0$ and $Tx = 0$, a contradiction. More generally, we see that when T is finitely singular, $\dim\ker(T-\lambda I_X) \leq \dim\ker T$ when λ is small (we may apply Lemma 5.1 to T^* and $T^* - \lambda I_{X^*}$).

Proposition 5.1. *Let $T \in \mathcal{L}(X)$; if T or T^* is finitely singular and if $0 \in \partial\sigma^{\mathbb{K}}(T)$, there exists an integer $k \geq 1$ such that $\ker T^k = \ker T^{k+1}$ and $T^k X = T^{k+1} X$. The space X is then the direct sum of two invariant subspaces for T, $Y = T^k X$ and the finite dimensional subspace $N = N_k = \ker T^k \neq \{0\}$. The operator $T_{|Y}$ is an isomorphism from Y onto Y. Furthermore 0 is isolated in $\sigma^{\mathbb{K}}(T)$.*

Proof. If T or T^* is finitely singular and $0 \in \partial\sigma^{\mathbb{K}}(T)$ then T is Fredholm with index 0 by Lemma 5.2; also $\ker T \neq \{0\}$ since $0 \in \sigma^{\mathbb{K}}(T)$. Furthermore by Lemma 5.3, there exists $\varepsilon_0 > 0$ such that $\dim\ker(T-\lambda I_X)$ is constant for all $\lambda \in \mathbb{K}$ such that $0 < |\lambda| < \varepsilon_0$. Since $0 \in \partial\sigma^{\mathbb{K}}(T)$, this constant dimension of $\ker(T-\lambda I_X)$ must be 0; we may assume that ε_0

is so small that $T - \lambda I_X$ is still Fredholm with index 0 when $0 < |\lambda| < \varepsilon_0$. These two facts imply that $(T - \lambda I_X)X = X$ for such λ, hence $T - \lambda I_X$ is invertible when $0 < |\lambda| < \varepsilon_0$. Therefore 0 is isolated in the spectrum of T.

We also know by Lemma 5.3 that $\operatorname{ind}(T_{|Y}) = \dim\ker(T - \lambda I_X) = 0$ when $0 < |\lambda| < \varepsilon_0$ and since $TY = Y$, it yields that $T_{|Y}$ is an isomorphism and so $\ker T \cap Y = \{0\}$; there exists therefore an integer k such that $\{0\} = \ker T \cap Y = \ker T \cap T^k X$, and this yields that $\ker T^k = \ker T^{k+1}$ (if $T^{k+1}x = 0$, then $T^k x \in \ker T \cap T^k X$, hence $T^k x = 0$); it follows that $T^k X = T^{k+1} X = Y$, because T^k and T^{k+1} are Fredholm with index 0 by Proposition 4.2. We obtain a decomposition of the space into two invariant subspaces, Y and $N_k = \ker T^k$ (this one is finite dimensional). Indeed, let $V \in \mathcal{L}(Y)$ be the inverse of $T_{|Y}$ and set $Q = V^k T^k$, considered as a map from X to X. Then Q is a projection from X onto Y and $\ker Q = \ker T^k = N$.

Remark. In the complex case, the restriction of T to N decomposes into a finite number of Jordan cells with 0 on the diagonal. The spectral projection defined in section 2 has Y as kernel and N for range.

Exercise 5.1. If $K \in \mathcal{L}(X)$ is compact, we know that $T = I_X - K$ is finitely singular by the proof of Corollary 4.2. Find a direct proof that $N_k = \ker T^k$ stabilizes. If $0 \in \sigma(T)$, show that 0 is isolated in $\sigma(T)$.

Hint. If N_k does not stabilize, let $x_k \in N_k$ be such that $1 = \|x_k\| = \operatorname{dist}(x_k, N_{k-1})$. If $y_k \in N_{k-1}$ is arbitrary, observe that $\|(x_l - y_l) - (x_k - y_k)\| \ge 1$ when $l \ne k$. Apply to $y_k = T x_k$ to obtain a contradiction to the compactness of $K = I_X - T$.

Boundary of essential spectrum lemma

Recall that the Calkin algebra of a real or complex infinite dimensional Banach space X was defined by $\mathcal{C}(X) = \mathcal{L}(X)/\mathcal{K}(X)$. Let $T \in \mathcal{L}(X)$. It follows from Corollary 4.3 that $\lambda \in \widehat{\rho}^{\mathbb{K}}(T)$ iff $T - \lambda I_X$ is Fredholm. Let

$$\sigma_\infty^{\mathbb{K}}(T) = \{\lambda \in \mathbb{K} : T - \lambda I_X \text{ infinitely singular}\},$$

$$\sigma_\infty^{*\mathbb{K}}(T) = \{\lambda \in \mathbb{K} : T^* - \lambda I_{X^*} \text{ infinitely singular}\}.$$

We know that $T - \lambda I_X$ is semi-Fredholm if and only if $\lambda \notin \sigma_\infty^{\mathbb{K}}(T) \cap \sigma_\infty^{*\mathbb{K}}(T)$; the two sets $\sigma_\infty^{\mathbb{K}}(T)$ and $\sigma_\infty^{*\mathbb{K}}(T)$ are compact, and non-empty in the complex case (see Lemma 5.4 below); we know that $T - \lambda I_X$ is Fredholm iff $\lambda \notin \widehat{\sigma}^{\mathbb{K}}(T)$, and by Lemma 3.4 $T - \lambda I_X$ is Fredholm iff $T - \lambda I_X$ and $(T - \lambda I_X)^*$ are finitely singular, therefore

$$\widehat{\sigma}^{\mathbb{K}}(T) = \sigma_\infty^{\mathbb{K}} \cup \sigma_\infty^{*\mathbb{K}}.$$

Example. Let R be the right shift on $\ell_2(\mathbb{N})$ (complex case). The spectrum of R is the closed unit disc; $\sigma_\infty(R) = \sigma_\infty^*(R) = \mathbb{T}$.

Exercise. Suppose that $\mathbb{K} = \mathbb{C}$ and let T be an isometry from X into X. Show that $\sigma_\infty(T) \cap \sigma_\infty^*(T) \subset \mathbb{T}$ and that they are equal if T is not onto.

Lemma 5.4. *Let $\lambda \in \partial\widehat{\sigma}^{\mathbb{K}}(T)$. Then $T - \lambda I_X$ and $(T - \lambda I_X)^*$ are infinitely singular,*

$$\partial\widehat{\sigma}^{\mathbb{K}}(T) \subset \sigma_\infty^{\mathbb{K}}(T) \cap \sigma_\infty^{*\mathbb{K}}(T).$$

Proof. We know that $T - \lambda I_X$ is not Fredholm since $\lambda \in \widehat{\sigma}^{\mathbb{K}}(T)$ but $T - \lambda I_X$ is arbitrarily close to Fredholm operators since $\lambda \in \partial\widehat{\sigma}^{\mathbb{K}}(T)$. It follows then from Lemma 4.5 that $T - \lambda I_X$ is not semi-Fredholm.

Corollary 5.2. *If $\mathbb{K} = \mathbb{C}$ and if $\dim X = \infty$, then for every $T \in \mathcal{L}(X)$ there exists $\lambda \in \mathbb{C}$ such that $T - \lambda I_X$ is infinitely singular (and also $(T - \lambda I_X)^*$).*

Proof. Since $\dim X = \infty$ the algebra $\mathcal{L}(X)/\mathcal{K}(X)$ is not $\{0\}$ hence the spectrum of the image $\widehat{T}$ in $\mathcal{C}(X)$ is not empty, and we simply have to pick any boundary point λ of this spectrum.

Exercise. If X is a real Banach space and $T \in \mathcal{L}(X)$, then either there exists $\lambda \in \mathbb{R}$ such that $T - \lambda I_X$ is infinitely singular, or there exists $p, q \in \mathbb{R}$ with $p^2 - 4q < 0$ such that $T^2 + pT + qI_X$ is infinitely singular.

Lemma 5.5. *Let $T \in \mathcal{L}(X)$ and let K be a compact subset of $\mathbb{K}$ such that $\sigma_\infty^{\mathbb{K}}(T) \cap \sigma_\infty^{*\mathbb{K}}(T) \subset K$. If Ω is a connected component of $\operatorname{int} \sigma^{\mathbb{K}}(T) \setminus K$, the boundary $\partial\Omega$ is contained in K.*

Proof. Suppose that $\lambda \in \partial\Omega$ but $\lambda \notin K$; then $\lambda \in \partial\sigma^{\mathbb{K}}(T)$. Let $U = T - \lambda I_X$; since $\lambda \notin K$, U or U^* is finitely singular, and $0 \in \partial\sigma^{\mathbb{K}}(U)$. By Proposition 5.1, 0 is isolated in $\sigma^{\mathbb{K}}(U)$, a contradiction to the fact that $\lambda \in \overline{\Omega}$.

Corollary 5.3. Every $\lambda \in \sigma^{\mathbb{K}}(T)$ belonging to the unbounded connected component of $\widehat{\rho}^{\mathbb{K}}(T)$ is an isolated eigenvalue of T with finite multiplicity (by this we mean that X splits as $X = E \oplus Y$, where E and Y are invariant under T, E is finite dimensional and $\sigma(T_{|E}) = \{\lambda\}$, and $T_{|Y}$ is an isomorphism from Y onto Y).

Proof. Let K be the complement in $\mathbb{K}$ of the unbounded connected component ω of $\widehat{\rho}^{\mathbb{K}}(T)$; then K is compact and contains $\sigma_\infty^{\mathbb{K}}(T) \cap \sigma_\infty^{*\mathbb{K}}(T)$. We want to prove that $\operatorname{int} \sigma^{\mathbb{K}}(T) \setminus K$ is empty. The open subset $\operatorname{int} \sigma^{\mathbb{K}}(T) \setminus K$ of ω is bounded, hence different from ω; if it is non empty it is not closed *in* ω; there must therefore exist some $\mu \in \partial\sigma^{\mathbb{K}}(T) \cap \omega$, contradicting Lemma 5.5). Assume that $\lambda \in \sigma^{\mathbb{K}}(T) \cap \omega$. By the preceding remark, λ does not belong to the interior of $\sigma^{\mathbb{K}}(T)$; also $U = T - \lambda I_X$ is finitely singular, and 0 belongs to the boundary of the spectrum of U. The result follows by Proposition 5.1.

Remark. We know by Proposition 5.1 that every non isolated point in $\partial\sigma(T)$ belongs to $\sigma_\infty(T)$.

Corollary 5.4. ($\mathbb{K} = \mathbb{C}$) If $\sigma_\infty(T) \cap \sigma_\infty^*(T)$ cannot contain the boundary of any bounded non empty open subset of $\mathbb{C}$, every $\lambda \in \sigma(T) \setminus (\sigma_\infty(T) \cap \sigma_\infty^*(T))$ is isolated and is an eigenvalue with finite multiplicity.

Proof. By Lemma 5.5 the spectrum has an empty interior. If $\lambda \in \sigma(T)$ but $\lambda \notin \sigma_\infty(T) \cap \sigma_\infty^*(T)$, the operator $U = T - \lambda I_X$ is semi-Fredholm and 0 belongs to the boundary of $\sigma(U)$. The result follows by Proposition 5.1.

This Corollary applies for example when $\widehat{\sigma}(T)$ is countable; also if A is hermitian in $\mathcal{L}(H)$ (complex case) and if K is a compact operator, then the essential spectrum of $A+K$ is real, hence does not contain the boundary of any bounded non empty open subset of the complex plane; every non real λ in the spectrum of $A + K$ is an isolated eigenvalue with finite multiplicity.

If $\widehat{\sigma}(T)$ does not contain the boundary of any bounded non empty open subset of $\mathbb{C}$ and if S is strictly singular, then…

6. Strictly singular operators. More on Fredholm operators

See Kato [Kt1], Pełczyński [Pe2]; also [LT1], section 2.c. for a concise presentation.

Definition 6.1. We say that $S \in \mathcal{L}(X, Y)$ is *strictly singular* if for every (infinite dimensional) subspace Z of X and every $\varepsilon > 0$, there exists $z \in Z$ such that $\|Sz\| < \varepsilon \|z\|$.

Let $\mathcal{S}(X, Y)$ denote the set of strictly singular operators from X to Y. When $Y = X$ we simply write $\mathcal{S}(X)$.

Exercise 6.1.

1. Let $S \in \mathcal{S}(X, Y)$. For every (infinite dimensional) subspace $Z \subset X$ and every $\varepsilon > 0$, there exists (an infinite dimensional) subspace $Z' \subset Z$ such that $\|S_{|Z'}\| < \varepsilon$ (compare to Proposition 3.2).
2. If $S_1, S_2 \in \mathcal{S}(X, Y)$, then $S_1 + S_2$ is strictly singular. Show that $\mathcal{S}(X, Y)$ is a closed vector subspace of $\mathcal{L}(X, Y)$.
3. Let $S \in \mathcal{S}(X, Y)$. For every $T \in \mathcal{L}(W, X)$ and $U \in \mathcal{L}(Y, Z)$, show that ST and US are strictly singular. When $Y = X$, $\mathcal{S}(X)$ is a closed two-sided ideal of $\mathcal{L}(X)$.
4. Show that $\mathcal{K}(X, Y) \subset \mathcal{S}(X, Y)$, and that they coincide when $X = Y = H$ is a Hilbert space, or when $X = Y = \ell_p$. Give an example of $S \in \mathcal{S}$ non compact. Give an example where $S \in \mathcal{S}(X, Y)$ but the adjoint S^* is not strictly singular.
5. Matrix of strictly singular operators. Let $T \in \mathcal{L}(X^n, Y^m)$. Then T can be represented by a $m \times n$ matrix $(T_{i,j})$ of operators from X to Y. Show that T is strictly singular iff each $T_{i,j}$ is strictly singular.
6. Complexification of a strictly singular operator. If S is a strictly singular operator between two real spaces X and Y, then $S_{\mathbb{C}}$ is strictly singular from $X_{\mathbb{C}}$ to $Y_{\mathbb{C}}$.
7. Show that $\mathcal{L}(\ell_1, \ell_2) = \mathcal{S}(\ell_1, \ell_2)$.

Remark. The essentially dual notion of *strictly cosingular* operators was defined by Pełczyński [Pe2] in the following way: an operator $T : X \to Y$ is strictly cosingular if, for every linear q from Y onto some Banach space Z, the map $q \circ T$ is not onto.

Lemma 6.1. *Assume that $T \in \mathcal{L}(X, Y)$ is finitely singular and that $S \in \mathcal{S}(X, Y)$. Then $T + S$ is finitely singular.*

Proof. There exists a finite codimensional subspace $X_0 \subset X$ and $c > 0$ such that $\|Tx\| \geq c\|x\|$ for every $x \in X_0$. Assuming $T + S$ infinitely singular, there would exist by Proposition 3.2 a subspace $Z \subset X$ (infinite dimensional) such that $\|(T+S)_{|Z}\| < c/2$. Then $Z' = Z \cap X_0$ is infinite dimensional, hence there exists $z \in Z' \subset X_0$ such that $\|Sz\| < (c/2)\|z\|$. But this implies $\|Tz\| < c\|z\|$, contradicting the choice of X_0 and c.

Remark 6.1. Let $U \in \mathcal{L}(X, Y)$. In order that the above proof works for $T + U$, it is enough that

$$s(U) = \sup_{\substack{Z \subset X \\ \dim Z = \infty}} \inf\{\|Uz\| : z \in Z, \|z\| = 1\}$$

is strictly less than $c(T)$. Note that U is strictly singular iff $s(U) = 0$.

Lemma 6.2. *Let* $\mathbb{K} = \mathbb{C}$. *For every* $U \in \mathcal{L}(X)$,

$$\widehat{r}(U) \leq s(U).$$

Proof. There exists $\lambda \in \partial\widehat{\sigma}(U)$ such that $|\lambda| = \widehat{r}(U)$ and $U - \lambda I_X$ infinitely singular by Lemma 5.4, hence there exists an infinite dimensional subspace Z on which $U \sim \lambda I_X$ by Proposition 3.2, so that $s(U) \geq |\lambda|$.

Corollary 6.1. *Let* $T, U \in \mathcal{L}(X, Y)$. *If* T *is Fredholm and if* $s(U) < c(T)$, *then* $T + U$ *is Fredholm, and* $\operatorname{ind}(T + U) = \operatorname{ind} T$.

Proof. For every $t \in [0, 1]$, we have $s(tU) < c(T)$, hence $T + tU$ is finitely singular for every $t \in [0, 1]$ by Remark 6.1. The result follows by Lemma 4.6.

Remark. The corresponding result holds also if T is finitely singular and not Fredholm.

Corollary 6.2. *Let* $S, T \in \mathcal{L}(X, Y)$. *If* T *is Fredholm and* S *strictly singular, then* $T + S$ *is Fredholm, and* $\operatorname{ind}(T + S) = \operatorname{ind} T$.

Corollary 6.3. T *is Fredholm iff it is invertible modulo strictly singular operators.*

Corollary 6.4. T *is Fredholm iff it is invertible modulo compact operators.*

Proposition 6.1. *Let* $S \in \mathcal{L}(X)$ *be strictly singular,* X *complex. Then every* $\lambda \neq 0$ *in the spectrum of* S *is isolated in* $\sigma(S)$ *and is an eigenvalue with finite multiplicity. It follows that the spectrum of* S *is finite or consists of a sequence converging to* 0.

(This is a particular case of a *Riesz operator*; for the proof below it is enough to know that $\widehat{\sigma}(S) = \{0\}$.) If X is real and $S \in \mathcal{S}(X)$ we obtain considering $S_{\mathbb{C}}$ a complex spectrum invariant under complex conjugation and consisting of a sequence converging to 0; the real spectrum is at most a sequence converging to 0.

Proof. If S is strictly singular it is clear that $\widehat{\sigma}(S) = \{0\}$ by Lemma 6.2. It follows then from Corollary 5.3 that every $\lambda \neq 0$ in $\sigma(S)$ is isolated and is an eigenvalue with finite multiplicity.

Exercise. Let $T \in \mathcal{L}(X)$, $S \in \mathcal{S}(X)$. If λ belongs to the unbounded component of $\widehat{\rho}^{\mathbb{K}}(T)$ and if $\lambda \in \sigma^{\mathbb{K}}(T + S)$, then λ is isolated in $\sigma^{\mathbb{K}}(T + S)$ and is an eigenvalue with finite multiplicity for $T + S$.

7. Ultraproducts

Ultraproducts appeared in model theory (Łoś) and as models for non-standard analysis. The notion of "restricted ultraproduct", also called ultraproduct of Banach spaces, is more suitable to classical analysis and was developed among others by Dacunha-Castelle and Krivine in [DK] (see also [Ja], approximately at the same period).

Let I be an infinite index set and let $\mathcal{U}$ be a non-trivial ultrafilter on I. Let $(X_i)_{i \in I}$ be a family of Banach spaces indexed by I. Let $L = \ell_\infty(I, (X_i))$ be the space of all bounded families $\widetilde{x} = (x_i)_{i \in I}$ such that $x_i \in X_i$ for every $i \in I$ and $\sup_i \|x_i\|_{X_i} < \infty$. The norm of $\widetilde{x} = (x_i)_{i \in I} \in L$ is given by $\|\widetilde{x}\| = \sup_{i \in I} \|x_i\|_{X_i}$. We define a seminorm on L by

$$p(\widetilde{x}) = \lim_{\mathcal{U}} \|x_i\|_{X_i} \ (\leq \|\widetilde{x}\|)$$

and we let $N = \{\widetilde{x} : p(\widetilde{x}) = 0\}$. Then N is a closed subspace of L and we define the *ultraproduct* $\widetilde{X} = \prod X_i/\mathcal{U}$ to be the quotient Banach space L/N. When all spaces X_i are equal to the same space X, we call $\widetilde{X}$ an *ultrapower* of X. We can embed X isometrically in the ultrapower $\widetilde{X}$ by mapping each $x \in X$ to the constant sequence $(x_i)_i$ where $x_i = x$ for every $i \in I$.

By a slight abuse, we shall consider $(x_i)_i$ as representing an element in $\widetilde{X}$, instead of the correct formulation which refers to the equivalence class modulo $\mathcal{U}$ (same tradition in measure theory when speaking about a "function" in L_1, instead of a class modulo negligible functions).

Exercise.

1. Show that $\widetilde{X} = X$ when $X_i = X$ is finite dimensional, and that $\widetilde{X} \neq X$ when X is infinite dimensional. In this case $\widetilde{X}$ is non separable.

2. Show that $\widetilde{X}$ is finite dimensional iff $d = \lim_{\mathcal{U}} \dim X_i$ is finite. In this case d is the dimension of $\widetilde{X}$.

The index set is usually the set $\mathbb{N}$ of integers, but more general sets are useful when dealing for example with ultrapowers $\widetilde{X}$ of spaces X with non-separable dual; in this case a natural index set is the set of finite subsets of the dual space X^* (or finite subsets of a dense subset in X^*: this is why the case of separable dual reduces to the index set $\mathbb{N}$); equivalently we can work with the set I of (closed) finite codimensional subspaces of X.

The weakly null part $\widetilde{X}_0$ of the ultrapower $\widetilde{X}$ of X consists of elements $\widetilde{x}$ that have a representative $(x_i)_{i \in I}$ such that $w\text{-}\lim_{\mathcal{U}} x_i = 0$ and plays a role in several questions.

Exercise. If X is reflexive, then

$$\widetilde{X} = X \oplus \widetilde{X}_0.$$

When H is a Hilbert space, the ultrapower $\widetilde{H}$ is also a Hilbert space: we may define a scalar product on $\widetilde{H}$ that extends the scalar product of H, and such that the corresponding norm is the norm of $\widetilde{H}$. Indeed, let for $\widetilde{x} = (x_i)$ and $\widetilde{y} = (y_i)$

$$\langle \widetilde{x}, \widetilde{y} \rangle = \lim_{i, \mathcal{U}} \langle x_i, y_i \rangle.$$

Then

$$\|\widetilde{x}\|^2 = \langle \widetilde{x}, \widetilde{x} \rangle.$$

It is also true, but more complicated, that for any given $p \in [1, +\infty)$, the class of L_p spaces is stable under ultraproduct (Dacunha-Castelle and Krivine [DK]).

Ultrapower of an operator

We fix an index set I and an ultrafilter $\mathcal{U}$ throughout this paragraph; all ultrapowers of possibly different spaces are taken with respect to I and $\mathcal{U}$. Given $T \in \mathcal{L}(X, Y)$ we get in the obvious way a bounded linear operator $\widetilde{T}$ from the ultrapower $\widetilde{X}$ to the corresponding ultrapower $\widetilde{Y}$: if $\widetilde{x} = (x_i)_i$ we simply let $\widetilde{T}\widetilde{x} = (Tx_i)_i$. It is clear that $\|\widetilde{T}\| = \|T\|$. If $T \in \mathcal{L}(X, Y)$ and $U \in \mathcal{L}(Y, Z)$, then $\widetilde{UT} = \widetilde{U}\widetilde{T}$. Also $\widetilde{I}_X$ is the identity of $\widetilde{X}$. It follows that $\widetilde{T}$ is invertible when T is invertible. In the case of a Hilbert space, it is easy to see that the ultrapower of the hilbertian adjoint T^* of T is the adjoint of $\widetilde{T}$.

When $Y = X$, we see that $T \to \widetilde{T}$ is a unital Banach algebra morphism from $\mathcal{L}(X)$ to $\mathcal{L}(\widetilde{X})$. Furthermore, it is a $*$-morphism when $X = H$ is a Hilbert space.

Let $\widetilde{X}_0$ be the weakly null part of the ultrapower that was defined before. Then, for every $T \in \mathcal{L}(X, Y)$, $\widetilde{T}\widetilde{X}_0 \subset \widetilde{Y}_0$. For a compact operator T, $\widetilde{T}\widetilde{X}_0 = \{0\}$. Conversely, when the index set is rich enough for coding every weakly null net, $\widetilde{T}_{|\widetilde{X}_0} = 0$ implies that T is compact.

Exercise. Fredholm and ultrapowers.

1. If T is finitely singular, then so is $\widetilde{T}$.
2. When T is Fredholm from X to Y, show that $\widetilde{T}$ is Fredholm from $\widetilde{X}$ to $\widetilde{Y}$, with the same index.
3. When $\lambda \in \partial\sigma^{\mathbb{K}}(T)$, there exists an eigenspace for $\widetilde{T}$ corresponding to λ. What is the spectrum of $\widetilde{T}$?

Finite representability

Definition 7.1. We say that Y is *finitely representable* into X if for every finite dimensional subspace F of Y and every $\varepsilon > 0$, there exists a linear map $A : F \to X$ such that

$$\forall y \in F,\ (1-\varepsilon)\|y\| \le \|Ay\| \le (1+\varepsilon)\|y\|.$$

Proposition 7.1. *If every X_i is finitely representable in X then the ultraproduct $\widetilde{X}$ of the family (X_i) is finitely representable in X.*

Proof. Let $F \subset \widetilde{X}$ be finite dimensional and let $(\widetilde{y}^{(\alpha)})_\alpha$ be an algebraic basis for F; let $(y^{(\alpha)})_{i\in I}$ be a representative of $\widetilde{y}^{(\alpha)}$. By assumption there exists for every $i \in I$ a linear map $A_i : F_i = [y_i^{(\alpha)}]_\alpha \to X$ such that

$$\forall y \in F_i,\ (1-\varepsilon)\|y\| \le \|A_i y\| \le (1+\varepsilon)\|y\|.$$

We obtain a linear map $A : F \to X$ by setting $A\widetilde{y}^{(\alpha)} = (A_i y_i^{(\alpha)})_i$. If $\widetilde{x} = \sum_\alpha a_\alpha \widetilde{y}^{(\alpha)} \in F$, then $x_i = \sum_\alpha a_\alpha y_i^{(\alpha)} \in F_i$ and

$$(1-\varepsilon)\|\widetilde{x}\| = (1-\varepsilon)\lim_{\mathcal{U}} \|x_i\| \le \lim_{\mathcal{U}} \|A_i x_i\| = \|A\widetilde{x}\| \le (1+\varepsilon)\|\widetilde{x}\|.$$

Spreading models ([BS], [BL])

Let $\mathcal{U}$ be a non trivial ultrafilter on $\mathbb{N}$; consider the successive ultrapowers of X defined in the following way. Let $\widetilde{X}_1$ be the usual ultrapower with index set $\mathbb{N}$. Let $\widetilde{X}_2$ be the vector space of classes of double sequences $\widetilde{x} = (x_{n_1,n_2})$ with the norm

$$\|\widetilde{x}\| = \lim_{n_1,\mathcal{U}} \lim_{n_2,\mathcal{U}} \|x_{n_1,n_2}\|.$$

Similarly $\widetilde{X}_3$ is defined from triple sequences, and so on... There is a natural isometric embedding from $\widetilde{X}_n$ into $\widetilde{X}_{n+1}$, which allows to consider $\widetilde{X}_n$ as a subspace of $\widetilde{X}_{n+1}$ and then to define the completion $\widetilde{X}_\infty$ of the union $\bigcup_n \widetilde{X}_n$ (this type of construction is very similar to the notion of *ultralimit* in model theory). Let us describe the embedding i_k from $\widetilde{X}_k$ into $\widetilde{X}_{k+1}$: to $\widetilde{x} = (x_{n_1,\ldots,n_k}) \in \widetilde{X}_k$ we associate $i_k(\widetilde{x}) = (y_{n_1,\ldots,n_{k+1}}) \in \widetilde{X}_{k+1}$ defined by $y_{n_1,\ldots,n_k,n_{k+1}} = x_{n_1,\ldots,n_k}$ for every $(n_1, \ldots, n_{k+1})$. For every operator $T \in \mathcal{L}(X,Y)$, there exists an operator $\widetilde{T}_\infty : \widetilde{X}_\infty \to \widetilde{Y}_\infty$ defined in the obvious way. This space $\widetilde{X}_\infty$ is finitely representable into X. We can define on $\widetilde{X}_\infty$ an isometric shift D in the following manner: if $\widetilde{x} = (x_{n_1,\ldots,n_k})$ belongs to $\widetilde{X}_k$, let

$$D\widetilde{x} = (y_{n_1,\ldots,n_{k+1}}) \in \widetilde{X}_{k+1}, \ \text{where } y_{n_1,n_2,\ldots,n_{k+1}} = x_{n_2,\ldots,n_{k+1}}.$$

The spreading model operation corresponds then to an iterated action of this shift operator on an element $\widetilde{x} \in \widetilde{X}_1$. This point of view was popularized by Krivine (the original approach of Brunel and Sucheston to spreading models uses a precise extraction of subsequence, with the help of Ramsey's theorem). The ultrapower point of view has the advantage of being "functorial": if Y is a second Banach space, and if we construct the corresponding space $\widetilde{Y}_\infty$, there exists a similar shift D' on $\widetilde{Y}_\infty$, and for every $T \in \mathcal{L}(X,Y)$ we have $D'\widetilde{T}_\infty = \widetilde{T}_\infty D$.

Let $(x_n)_{n\geq 1}$ be a sequence in X with no Cauchy subsequence. We consider in the successive ultrapowers the vectors $e_1 = (x_n)$, $e_2 = De_1$ (note that e_2 is *not* the image of e_1 under the canonical embedding of $\widetilde{X}_1$ into $\widetilde{X}_2$), and generally $e_k = D^{k-1}e_1$ for every $k \geq 1$. The norm in $\widetilde{X}_\infty$ of a linear combination of the vectors $e_1, \ldots, e_k$ is given by

$$\|\sum_{i=1}^k a_i e_i\| = \lim_{n_1,\mathcal{U}} \ldots \lim_{n_k,\mathcal{U}} \|a_1 x_{n_1} + a_2 x_{n_2} + \cdots + a_k x_{n_k}\|.$$

This norm on $[e_n]_{n\geq 1}$ is *invariant under spreading*, which means that

$$\|\sum_{i=1}^k a_i e_i\| = \|\sum_{i=1}^k a_i e_{m_i}\|$$

for every k, all scalars $(a_i)_{i=1}^k$ and all $m_1 < m_2 < \ldots < m_k$. Note that $\|e_1 - e_2\| > 0$, otherwise (x_n) would have a Cauchy subsequence.

Let $f_1 = e_1 - e_2$, $f_2 = e_3 - e_4, \ldots$ Then $\|f_n\| = \|f_1\| > 0$ and (f_n) is a monotone basic sequence (see below). The space generated by this sequence is contained in $\widetilde{X}_\infty$ and hence finitely representable in X. The norm is invariant under spreading. We call this space generated by (f_n) a *monotone spreading model* of X, generated by the sequence (x_n).

We prove that $(f_n)_n$ is a monotone basic sequence. By the spreading invariance property, we obtain

$$\|\sum_{i=1}^{k} a_i f_i\| = \|\sum_{i=1}^{k-1} a_i f_i + a_k(e_l - e_{l+1})\|$$

for every $l \geq 2k-1$. Taking averages from $l = 2k-1$ to $l = 2k+n-2$ we obtain $\|\sum_{i=1}^{k} a_i f_i\| \geq \|\sum_{i=1}^{k-1} a_i f_i + a_k(e_{2k-1} - e_{2k+n-1})/n\|$, hence letting $n \to \infty$

$$\|\sum_{i=1}^{k} a_i f_i\| \geq \|\sum_{i=1}^{k-1} a_i f_i\|.$$

Block finite representability

Let (x_n) be a sequence in a Banach space X, with no Cauchy subsequence. We say that a space Y with a basis (f_n) is *block finitely representable* in the span of (x_n) if for every finite sequence $y_1, \ldots, y_k$ of (successive) blocks in Y and every $\varepsilon > 0$ there exists a linear map $A : [y_1, \ldots, y_k] \to X$ such that $Ay_1, \ldots, Ay_k$ are successive linear combinations of the (x_n), and

$$\forall y \in [y_1, \ldots, y_k],\ (1-\varepsilon)\|y\| \leq \|Ay\| \leq (1+\varepsilon)\|y\|.$$

A (monotone) spreading model generated by a sequence (x_n) is block finitely representable into this sequence.

Ultrapowers of commuting or almost commuting operators

When $U, T \in \mathcal{L}(X)$ commute then $\widetilde{U}$ and $\widetilde{T}$ commute on $\widetilde{X}$. In some situations U and T do not exactly commute but the restrictions of $\widetilde{T}$ and $\widetilde{U}$ to the weakly null part $\widetilde{X}_0$ commute. For example, suppose that $TU - UT$ is compact. Then $\widetilde{T}\widetilde{U} - \widetilde{U}\widetilde{T}$ vanishes on the subspace $\widetilde{X}_0$ of $\widetilde{X}$. We shall give an easy application to the existence of common approximate eigenvectors.

Lemma 7.1. (Complex scalars) Let $T, U \in \mathcal{L}(X)$ with $TU = UT$. If T is not an into isomorphism, there exists $\lambda \in \mathbb{C}$ and a normalized sequence (x_n) (possibly constant) such that

$$Tx_n \to 0,\ (U - \lambda I_X)x_n \to 0.$$

Proof. We only have to find $\lambda \in \mathbb{C}$ such that for every $\varepsilon > 0$, there exists a norm one vector $x \in X$ with $\|Tx\| < \varepsilon$ and $\|Ux - \lambda x\| < \varepsilon$. Since T is not an into isomorphism, we can find a normalized sequence $(y_n) \subset X$ such that $Ty_n \to 0$ (this sequence (y_n) may be constant). If we consider $\widetilde{y} = (y_n)$ in the ultrapower $\widetilde{X}$, we get $\widetilde{T}\widetilde{y} = 0$. This shows that $Z = \ker \widetilde{T} \neq \{0\}$. Since $\widetilde{U}$ and $\widetilde{T}$ commute, we know that $\widetilde{U}Z \subset Z$. Let $V = \widetilde{U}_{|Z} \in \mathcal{L}(Z)$

and let $\lambda \in \partial\sigma(V)$. There exists $\widetilde{z}$ such that $\widetilde{z} \in Z$ and $V\widetilde{z} \sim \lambda\widetilde{z}$. Pulling back $\widetilde{z}$ to X gives the desired vector x.

Exercise. If X is a real Banach space and T, U are as in Lemma 7.1 we can find $r > 0$, $\theta \in \mathbb{R}$ and two sequences (x_n), (y_n) in X with $\|x_n\| + \|y_n\| = 1$, $Tx_n \to 0$, $Ty_n \to 0$, $Ux_n - r(\cos\theta\, x_n - \sin\theta\, y_n) \to 0$ and $Uy_n - r(\sin\theta\, x_n + \cos\theta\, y_n) \to 0$.

Corollary 7.1. ($\mathbb{K} = \mathbb{C}$) Let $T_1, \ldots, T_n \in \mathcal{L}(X)$ commute. There exist $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ such that for every $\varepsilon > 0$, there exists $x \in X$, $\|x\| = 1$ and

$$\forall i = 1, \ldots, n, \quad \|T_i x - \lambda_i x\| < \varepsilon.$$

Proof. The proof is by induction, using the argument of Lemma 7.1. When $n = 2$, we choose $\lambda_1 \in \partial\sigma(T_1)$; then $T = T_1 - \lambda_1 I_X$ is not an into isomorphism and we can find by Lemma 7.1 some λ_2 for which there exist common approximate vectors. Passing to an ultrapower we see that the eigenspaces for $\widetilde{T}_1$ and λ_1, and for $\widetilde{T}_2$ and λ_2 intersect, and their intersection is stable under $\widetilde{T}_3$ since the operators commute. It is therefore possible as before to find an approximate eigenvector for $\widetilde{T}_3$ in this intersection.

There is a variant of Lemma 7.1 where one assumes that T is infinitely singular and then the normalized sequence (x_n) in the result can be chosen basic. In order to construct this basic sequence, we only need to show that the vector x in the above proof can be chosen in any given finite codimensional subspace of X. Let I be the set of finite codimensional subspaces of X and let $\mathcal{U}$ be a ultrafilter on I containing the set $\{Z \in I : Z \subset Y\}$ for every $Y \in I$. Since T is infinitely singular, we may choose for every $i = Y \in I$ a norm one vector $y_i \in i$ such that $\|Ty_i\| < (\operatorname{codim} i)^{-1}$. The corresponding net $(y_i)_i$ belongs to the weakly null part $\widetilde{X}_0$ of the ultrapower, and $\widetilde{X}_0$ is stable under $\widetilde{T}$ and $\widetilde{U}$. It is easy then to adapt the above argument to the present case. The final $\widetilde{z}$ is now a weakly null net, so it can be pulled back in any finite codimensional subspace.

Exercise. Assume that $T_1, \ldots, T_k$ commute and that there exists a normalized (basic) sequence (x_n) such that $T_i x_n \to 0$ for every $i = 1, \ldots, k$. Let U commute with each T_i. Then there exists $\lambda \in \mathbb{C}$ and a normalized (basic) sequence (y_n) such that $T_i y_n \to 0$ for each $i = 1, \ldots, k$ and $(U - \lambda)y_n \to 0$.

The above variant of Lemma 7.1 remains true if T and U weakly commute.

Lemma 7.2. (Complex scalars) Let $T, U \in \mathcal{L}(X)$. Assume that T is infinitely singular and that $TU - UT$ is compact. We can then find $\lambda \in \mathbb{C}$ and a normalized basic sequence (x_n) in X such that $Tx_n \to 0$ and $(U - \lambda I_X)x_n \to 0$.

Wiener's algebra again

Let $f = \sum_{n\in\mathbb{Z}} a_n\, \mathrm{e}^{in\theta} \in W$ and $T = \sum_{n\in\mathbb{Z}} a_n R^n$ where R is the right shift on $\ell_1(\mathbb{Z})$. Then

$$\|f\|_W = \|T\|_{\mathcal{L}(\ell_1(\mathbb{Z}))}.$$

Note that T commutes with R. Conversely, to $T = \sum_{n\in\mathbb{Z}} a_n R^n$ with $\sum_{\mathbb{Z}} |a_n| < \infty$ we associate $f = \varphi_T = \sum_{n\in\mathbb{Z}} a_n\, \mathrm{e}^{in\theta} \in W$. If f does not vanish on $\mathbb{T}$, we show first that T is an into isomorphism. Otherwise there exists by Lemma 7.1 a $\lambda \in \mathbb{T}$ and a normalized sequence (x_n) such that $Tx_n \to 0$ and $Rx_n - \lambda x_n \to 0$; on the other hand, $Tx_n \sim f(\lambda)x_n$, hence $f(\lambda) = 0$, contradiction. It follows that $\partial\sigma(T) \subset f(\mathbb{T})$; indeed, if $\lambda \notin f(\mathbb{T})$, we see that $\varphi_{T-\lambda Id} = \varphi_T - \lambda$ does not vanish on $\mathbb{T}$, hence $T - \lambda Id$ is an into isomorphism and therefore $\lambda \notin \partial\sigma(T)$ by the boundary of spectrum Lemma.

Assuming that φ_T does not vanish on $\mathbb{T}$, we can find a trigonometric polynomial g such that $|g\varphi_T - 1| < 1/2$ on $\mathbb{T}$. Let U be the finite linear combination of powers of R such that $\varphi_U = g$. Then, since $\varphi_{TU} = \varphi_T\varphi_U = \varphi_T g$, the boundary of the spectrum of TU is contained in the disc $\Delta(1, 1/2)$, hence the spectrum itself is contained in the same disc and it follows that TU is invertible; similarly UT is invertible, therefore T is invertible. The inverse must commute to R, therefore $T^{-1} = \sum_{n\in\mathbb{Z}} b_n R^n$, where the sequence (b_n) is defined by $T^{-1}e_0 = \sum_{n\in\mathbb{Z}} b_n e_n$; finally $\sum_{n\in\mathbb{Z}} |b_n| < \infty$ since $T^{-1}e_0 \in \ell_1$. If we set $h = \sum_{n\in\mathbb{Z}} b_n \, \mathrm{e}^{in\theta}$, we see that $hf = 1$ on $\mathbb{T}$. Summing up, we found an alternate proof for the fact that a non-vanishing function in W is invertible in W (end of section 2).

Cuntz algebras

See Cuntz [C1].

Let $\mathcal{P}$ be the unital complex algebra generated by six elements (u_j) and (u_j^*), $j = 0, 1, 2$, satisfying the relations

$$u_i^* u_j = \delta_{i,j} 1_{\mathcal{P}}; \quad \sum_{j=0}^{2} u_j u_j^* = 1_{\mathcal{P}}.$$

Notice that $p_j = u_j u_j^*$ is an idempotent, and $p_i p_j = 0$ when $i \neq j$. We shall also consider the unital complex algebra $\mathcal{Q}$ generated by six elements (v_j) and (v_j^*), $j = 0, 1, 2$, satisfying the relations

$$v_i^* v_j = \delta_{i,j} 1_{\mathcal{Q}}.$$

Again $q_j = v_j v_j^*$ is an idempotent, $q_i q_j = 0$ when $i \neq j$, but $q = 1_{\mathcal{Q}} - \sum_{j=0}^{2} q_j$ is a non zero idempotent in $\mathcal{Q}$.

Exercise. Let (q) denote the ideal generated by q in $\mathcal{Q}$. Show that $\mathcal{P} \simeq \mathcal{Q}/(q)$.

Let $\mathcal{T}$ be the ternary tree $\bigcup_{n=0}^{\infty} \{0, 1, 2\}^n$. The root is the empty word, denoted $\emptyset$. If $s, t \in \mathcal{T}$, let (s, t) stand for the concatenation of s and t. For $t \in \mathcal{T}$, we define u_t and u_t^* in $\mathcal{P}$ (and in a similar way v_t and v_t^* in $\mathcal{Q}$) inductively by $u_{(t,i)} = u_i u_t$ and $(u_{(t,i)})^* = u_t^* u_i^*$. (We also let $u_\emptyset = u_\emptyset^* = 1_{\mathcal{P}}$.) Then because $u_i^* u_j = 0$ when $i \neq j$, every $w \in \mathcal{P}$ has a decomposition

$$w = \sum_{l=1}^{N} c_l \, u_{\alpha_l} u_{\beta_l}^* \ ,$$

where α_l and β_l are words in $\mathcal{T}$ and $c_l \in \mathbb{C}$. Notice that $u_\alpha^* u_\alpha = 1_{\mathcal{P}}$ and that $u_\alpha u_\alpha^*$ is an idempotent p_α for every word $\alpha \in \mathcal{T}$. Similarly every $w \in \mathcal{Q}$ has a decomposition

$$w = \sum_{l=1}^{N} c_l \, v_{\alpha_l} v_{\beta_l}^* \ ,$$

where α_l and β_l are words in $\mathcal{T}$ and $c_l \in \mathbb{C}$.

We shall describe a model for $\mathcal{Q}$ and two models for $\mathcal{P}$. Let Y_{00} be the vector space of finitely supported scalar sequences indexed by $\mathcal{T}$. Denote the canonical basis for Y_{00} by $(e_t)_{t\in\mathcal{T}}$ and denote the length of a word $t \in \mathcal{T}$ by $|t|$. We shall now describe some operators on Y_{00}. Let Id denote the identity operator on the space of sequences. Let V_i, for $i = 0, 1, 2$ be defined by their action on the basis as follows:

$$V_i e_t = e_{(t,i)}.$$

Thus V_i can be thought of as the map taking each vertex of $\mathcal{T}$ to the i^{th} vertex immediately below it. The adjoints V_i^* act in the following way: $V_i^* e_t = e_s$ if t is of the form $t = (s, i)$, and $V_i^* e_t = 0$ otherwise. The following facts are easy to check: $V_i^* V_j = \delta_{i,j} Id$; if Q denotes the natural rank one projection on the line $\mathbb{C}e_\emptyset$, then $\sum_{i=0}^{2} V_i V_i^* = Id - Q$. This is a model for $\mathcal{Q}$: we have a representation $\rho : \mathcal{Q} \to \mathcal{L}(Y_{00})$ defined by $\rho(v_i) = V_i$, $\rho(v_i^*) = V_i^*$ for $i = 0, 1, 2$. We shall see below that ρ is injective.

For constructing our first model for $\mathcal{P}$, consider the subset $\mathcal{T}_0$ of $\mathcal{T}$ consisting of all words $t \in \mathcal{T}$ that do not start with 0 (including the empty sequence). Let L_0 be the vector subspace of Y_{00} generated by the $(e_t)_{t\in\mathcal{T}_0}$. In order to define U_0 on L_0, we modify the definition of V_0 slightly, by letting $U_0 e_\emptyset$ equal $e_\emptyset$ instead of e_0. Operators U_1 and U_2 are defined exactly as V_1 and V_2 were. We still have that the $U_i {U_i}^*$ are projections and that ${U_i}^* U_j = \delta_{i,j} Id$, but this time $\sum_{i=0}^{2} U_i {U_i}^* = Id$. We have a model for $\mathcal{P}$, that we call $\mathcal{P}_0$. The mapping ρ_0 from $\mathcal{P}$ to $\mathcal{L}(L_0)$ which takes u_i to U_i and u_i^* to U_i^* is a representation of $\mathcal{P}$ into $\mathcal{L}(L_0)$. It is injective.... There is a simpler way to present this model: define on $c_{00}(\mathbb{N})$ the operators

$$U_i' e_n = e_{3n+i-2}, \ i = 0, 1, 2.$$

This is the same model, up to isomorphism; indeed, to each $s = (i_1, \ldots, i_n) \in \mathcal{T}_0$ we can associate the integer $n_s = 3^{n-1} i_1 + \cdots + 3 i_{n-1} + i_n + 1$, (with $n_\emptyset = 1$), and this defines a bijection between $\mathcal{T}_0$ and $\mathbb{N}$ such that $U_i' = \varphi U_i \varphi^{-1}$ for $i = 0, 1, 2$, where φ is the isomorphism from $\mathcal{L}(L_0)$ to $\mathcal{L}(c_{00})$ deduced from that bijection.

Our second model for $\mathcal{P}$ uses the index set $\mathcal{T}_\infty = \mathbb{Z} \times \mathcal{T}$. We consider the vector space L_∞ of finitely supported complex functions on $\mathcal{T}_\infty$, with its natural basis $e_{n,t}$, $n \in \mathbb{Z}$, $t \in \mathcal{T}$. Now define

$$U_0(e_{n,\emptyset}) = e_{n+1,\emptyset}; \ \ U_0(e_{n,t}) = e_{n,(t,0)} \text{ if } t \neq \emptyset,$$

and for $j = 1, 2$

$$U_j(e_{n,t}) = e_{n,(t,j)}.$$

Then we set $U_0^*(e_{n,\emptyset}) = e_{n-1,\emptyset}$ and for $j = 1, 2$, $U_j^*(e_{n,\emptyset}) = 0$; when $t = (s, k)$ is not the empty word ($k = 0, 1, 2$) we let $U_j^*(e_{n,t}) = 0$ when t does not end with j, that is $j \neq k$, and U_j^* removes that last j otherwise: $U_j^*(e_{n,(s,j)}) = e_{n,s}$. We write e_n instead of $e_{n,\emptyset}$; one can think of e_n as a vector e_t with an infinite t, having infinitely many 0s at the left of the nth place. We have a second model $\mathcal{P}_\infty$ for $\mathcal{P}$, with a representation ρ_∞ from $\mathcal{P}$ in $\mathcal{L}(L_\infty)$. This vector space L_∞ admits a natural graduation by

$$L_{\infty,n} = \operatorname{span}\{e_{m,t} : m + |t| = n\}, \quad n \in \mathbb{Z}.$$

It should be observed that each U_j sends $L_{\infty,n}$ to $L_{\infty,n+1}$ while each U_j^* sends $L_{\infty,n}$ to $L_{\infty,n-1}$.

Consider in $\mathcal{P}$ the subset $\mathcal{B}_k$ generated by products $u_\alpha u_\beta^*$ where $|\alpha| = |\beta| \leq k$. It is easy to see that $\mathcal{B}_k$ is a subalgebra of $\mathcal{P}$. Furthermore, every element in $\mathcal{B}_k$ is a linear combination of words $u_\alpha u_\beta^*$ with $|\alpha| = |\beta| = k$. This follows from the relation

$$u_\alpha u_\beta^* = \sum_{j=0}^{2} u_\alpha u_j u_j^* u_\beta^*$$

which shows that products of length r can be expressed as sum of products of length $r+1$. Let $f_{\alpha,\beta} = u_\alpha u_\beta^*$, for $|\alpha| = |\beta| = k$. It is easy to check that they generate an algebra isomorphic to the matrix algebra M_{3^k}, thus $\mathcal{B}_k$ is isomorphic to M_{3^k}.

Let $\mathcal{B}$ denote the subalgebra of $\mathcal{P}$ obtained as union of the increasing sequence $(\mathcal{B}_k)$.

Lemma 7.3. Let $L \neq \{0\}$ be a complex vector space and let (U_i), (U_i^*), $i = 0, 1, 2$ be operators on L such that

$$U_i^* U_j = \delta_{i,j} 1_{\mathcal{L}(L)}; \quad \sum_{j=0}^{2} U_j U_j^* = 1_{\mathcal{L}(L)}.$$

Then $\rho(u_i) = U_i$, $\rho(u_i^*) = U_i^*$ for $i = 0, 1, 2$ defines an injective representation of $\mathcal{P}$ into $\mathcal{L}(L)$. Similarly suppose (V_i), (V_i^*), $i = 0, 1, 2$ are operators on L such that

$$V_i^* V_j = \delta_{i,j} 1_{\mathcal{L}(L)}.$$

Then $\rho(u_i) = V_i$, $\rho(u_i^*) = V_i^*$ defines an injective representation of $\mathcal{Q}$ into $\mathcal{L}(L)$.

Proof. It is clear that ρ exists. Let $L_i = U_i(L)$; it is easy to check that L is the direct sum of L_0, L_1 and L_2; for every $t \in \mathcal{T}$ let $L_t = U_t(L)$; it is also clear that for every k, L is the direct sum of the L_t, $|t| = k$. This implies that the restriction of ρ to $\mathcal{B}_k$ is injective. ...

Using $\mathcal{B}$, it is possible to give a useful representation for every $w \in \mathcal{P}$. For every $n \geq 0$, let 0^n denote the word consisting of n zeros and let r_n be the idempotent $r_n = u_{0^n} u_{0^n}^* = u_0^n (u_0^*)^n$.

We shall also define the *weight* $q(w)$ of an element $w \in \mathcal{P}$ by letting $q(10 = 0$, $q(u_i) = 1$ and $q(u_i^*) = -1$, and defining the weigth of w as the sums of weights of the factors (note that the fundamental relation between the generators is compatible with this definition).

Proposition 7.2. *Every $w \in \mathcal{P}$ has a unique representation*

$$w = \sum_{n<0} (u_0^*)^{-n} b_n + \sum_{n \geq 0} b_n u_0^n,$$

where $b_n \in \mathcal{B}$ satisfy

$$b_n = b_n r_n \text{ for } n \geq 0, \quad b_n = r_{-n} b_n, n < 0.$$

Proof. It is possible to express w as $w = \sum_{n\in\mathbb{Z}} w_n$, where each w_n is of the form

$$w_n = \sum_{l=1}^{k} c_l\, u_{\alpha_l} u^*_{\beta_l},$$

with $|\alpha_l| - |\beta_l| = n$ for each $l = 1, \dots, k$. Then for $n \geq 0$ we have

$$w_n = (w_n (u_0^*)^n u_0^n (u_0^*)^n) u_0^n = b_n u_0^n$$

and for $n < 0$ we may write $w_n = (u_0^*)^{-n} b_n$; it is easy to check that $b_n \in \mathcal{B}$ and $b_n = b_n r_n$ for $n \geq 0$, $b_n = r_{-n} b_n$ for $n < 0$. This shows the existence. For the uniqueness, assume that for some $N \geq 0$, we have

$$0 = \sum_{n<0} (u_0^*)^{-n} b_n + \sum_{n\leq N} b_n u_0^n = w$$

and that this is a representation with the above properties; we want to show that $b_N = 0$ (the case where the largest index N in the summation is < 0 can be treated in a similar way and will be left to the reader). For proving $b_N = 0$ we use the model $\mathcal{P}_\infty$. Observe that for $v \in \mathcal{B}$, $V = \rho_\infty(v)$ leaves each subspace $L_{\infty,n}$ invariant. We have, letting $B_n = \rho_\infty(b_n)$

$$0 = \sum_{n<0} (U_0^*)^{-n} B_n + \sum_{n\leq N} B_n U_0^n = W = \rho_\infty(w).$$

Let $y \in L_{\infty,0}$. Then $P_N W y = B_N U_0^N y$, where P_N denotes the projection of L_∞ onto $L_{\infty,N}$; since $b_N \in \mathcal{B}$, it belongs to some $\mathcal{B}_M$ and since $B_N = B_N R_N$ we get, setting $F_{t,t'} = \rho_\infty(f_{t,t'})$

$$B_N = \sum_{t,t''} a_{t,t''} F_{t,(t'',0^N)},$$

where $|t| = M$, $|t''| = M - N$. Choosing $y = e_{N-M,t''}$ we get $B_N U_0^N y = \sum_t a_{t,t''} e_{N-M,t} = 0$ hence all $a_{t,t''}$ are 0.

Remark. We may obtain analogous results for $\mathcal{Q}$.....

It follows that the projection π_n in $\mathcal{P}$ on the set of elements of weight n is well defined. It also follows that for every $\lambda \in \mathbb{T}$ the transform defined by

$$\varphi_\lambda(u_j) = \lambda u_j; \quad \varphi_\lambda(u_j^*) = \overline{\lambda} u_j^*$$

extends to a morphism of $\mathcal{P}$. Indeed, if

$$0 = \sum_l c_l\, u_{\alpha_l} u^*_{\beta_l} = w$$

then $\pi_n(w) = 0$, but φ_λ multiplies by λ^n the set of elements of weight n hence

$$0 = \sum_l c_l \lambda^{|\alpha_l| - |\beta_l|} u_{\alpha_l} u^*_{\beta_l}$$

and φ_λ is well defined on $\mathcal{P}$. In the same way the $*$-transform is well defined on $\mathcal{P}$.

Hilbertian representations of $\mathcal{P}$ and $\mathcal{Q}$

Suppose that H is a Hilbert space. A Hilbertian representation of $\mathcal{P}$ is a representation $\rho : \mathcal{P} \to \mathcal{L}(H)$ such that $U_i^* = \rho(u_i^*)$ is the Hilbertian adjoint of $U_i = \rho(u_i)$. This implies immediately that U_i is an (into) isometry on H. We obtain an orthogonal decomposition of H into three subspaces H_i, with $H_i = U_i H$, $i = 0, 1, 2$, and each of these subspaces again decomposes into a sum of three... For every Hilbertian model of $\mathcal{P}$, we obtain a C^*-algebra norm on $\mathcal{P}$. We shall prove the result of Cuntz [C1] that the norm of $\rho(w)$ does not depend from the representation, or in other words that there exists a unique C^*-norm on $\mathcal{P}$. We first observe that this is the case for the subalgebra $\mathcal{B}$: for every k, it is easy to check that ρ is injective on $\mathcal{B}_k$; the image $\rho(\mathcal{B}_k)$ is a finite dimensional C^*-algebra, hence has a unique C^*-norm (namely, the norm of operators on $\ell_2^{3^k}$).

Suppose that ρ is a $*$-representation of $\mathcal{P}$ in some $\mathcal{L}(H)$, H a Hilbert space. Then, considering an ultrapower $\widetilde{H}$ of H we may define

$$\forall w \in \mathcal{P}, \quad \widetilde{\rho}(w) = \widetilde{\rho(w)};$$

this is a $*$-representation of $\mathcal{P}$ in $\mathcal{L}(\widetilde{H})$. We define a Hilbertian model from $\mathcal{P}_\infty$ in the obvious way: we associate to the vector space L_∞ a Hilbert space H_∞ admitting $(e_{n,t})_{n \in \mathbb{Z}, t \in \mathcal{T}}$ for hilbertian basis. Every U_i extends clearly to an isometry of H_∞ and U_i^* is the Hilbertian adjoint of U_i. We want to show that $\mathcal{P}_\infty$ with this ℓ_2-norm is minimal among the hilbertian models of $\mathcal{P}$.

Lemma 7.4. For every $*$-representation ρ of $\mathcal{P}$ into some $\mathcal{L}(H)$, there exists an invariant subspace $M \subset \widetilde{H}$ for $\widetilde{\rho}$ and an onto isometry $j : H_\infty \to M$ such that

$$\forall w \in \mathcal{P}, \quad \widetilde{\rho}_M(w) = j \rho_\infty(w) j^{-1},$$

where $\widetilde{\rho}_M$ denotes the restriction of $\widetilde{\rho}$ to M. Hence

$$\forall w \in \mathcal{P}, \quad \|\rho_\infty(w)\| = \|\widetilde{\rho}_M(w)\| \le \|\widetilde{\rho}(w)\| = \|\rho(w)\|.$$

Proof. Let $U_j = \rho(u_j)$, $j = 0, 1, 2$. Let $y \perp U_0(H)$, $\|y\| = 1$; it is easy to check that the sequence $(U_0^n y)_{n \ge 0}$ is orthonormal. We define in $\widetilde{H}$ the vectors

$$f_{n,t} = (U_t U_0^{k+n} y)_{k \ge 0} \in \widetilde{H}.$$

It is not hard to show that these vectors are normalized and pairwise orthogonal, so that $j(e_{n,t}) = f_{n,t}$ defines an isometry from H_∞ into $\widetilde{H}$. We set $M = j(H_\infty)$ and the conclusion follows easily.

In a similar way one can show that the model $\mathcal{P}_0$ is also minimal. Let

$$z_n = \frac{1}{\sqrt{n}} \Big(\sum_{k=0}^{n-1} U_0^k y \Big),$$

where y is chosen as before; then z_n is almost fixed under U_0, and $\widetilde{z} = (z_n)$ is fixed under $\widetilde{U}_0$; let

$$j(e_t) = \widetilde{U}_t \widetilde{z}, \ t \in \mathcal{T}_0.$$

Proposition 7.3. Suppose that ρ is a representation of $\mathcal{P}$ in a Banach algebra $\mathcal{A}$ such that $U_j = \rho(u_j)$ and $U_j^* = \rho(u_j^*)$ have norm one for $j = 0, 1, 2$ and such that φ_λ is isometric, i.e. $\|\rho(\varphi_\lambda(w))\| = \|\rho(w)\|$ for every $w \in \mathcal{P}$. Let $\Phi_\lambda(W) = \rho(\varphi_\lambda(w))$ where $W = \rho(w)$, $w \in \mathcal{P}$. Then $\lambda \to \varphi_\lambda(W)$ is well defined and continuous on $\mathbb{T}$ for every W in the closure of $\rho(\mathcal{P})$. We can write

$$\Phi_\lambda(W) \sim \sum_{-\infty}^{+\infty} \lambda^n W_n,$$

where W_n is uniquely defined by

$$W_n = \int_{\mathbb{T}} \lambda^{-n} \Phi_\lambda(W)\, d\mu(\lambda),$$

(μ is the invariant probability on $\mathbb{T}$); we have for $n \geq 0$

$$W_n = B_n U_0^n, \quad B_n \in \overline{\rho(\mathcal{B})},\ B_n = B_n R_n,\ R_n = \rho(r_n),$$

and for $n < 0$

$$W_n = (U_0^*)^{-n} B_n, \quad B_n \in \overline{\rho(\mathcal{B})},\ B_n = R_{-n} B_n.$$

Proof. It is clear that $\lambda \to \rho(\varphi_\lambda(w))$ is continuous from $\mathbb{T}$ to $\mathcal{A}$ for every $w \in \mathcal{P}$, and since φ_λ is isometric $\lambda \to \Phi_\lambda(W)$ is well defined and is the uniform limit of continuous functions on $\mathbb{T}$ for every W in the closure of $\rho(\mathcal{P})$. Let $w = \sum_n w_n \in \mathcal{P}$, where each w_n has weight $n \in \mathbb{Z}$; then $\Phi_\lambda(\rho(w)) = \sum_n \lambda^n \rho(w_n)$, and the integral formula above implies that $\|\rho(w_n)\| \leq \|\rho(w)\|$. Observe that, writing $w_n = b_n u_0^n$ as in Proposition 7.2???, we get $\|\rho(b_n)\| \leq \|\rho(w_n)\|$. When $\rho(w)$ converges in $\mathcal{A}$ to some W belonging to the closure of $\rho(\mathcal{P})$, we see thus that the corresponding $\rho(w_n)$ converges in $\mathcal{A}$ to some W_n, and that $\rho(b_n)$ converges to some $B_n \in \overline{\rho(\mathcal{B})}$. The equations for W_n and B_n follow by continuity.

Proposition 7.4. Suppose that Y_∞ is equipped with a norm such that $(e_{n,t})$ is 1-unconditional (this is in particular true for the Hilbert space H_∞). Then $D_\lambda e_{n,t} = \lambda^{n+|t|} e_{n,t}$ defines an isometry on Y_∞ and

$$\forall w \in \mathcal{P}, \quad D_\lambda \rho_\infty(w) D_{\lambda^{-1}} = \rho_\infty(\varphi_\lambda(w)).$$

Proof. Immediate.

It follows that $\|\rho_\infty(\varphi_\lambda(w))\|_{\mathcal{L}(H_\infty)} = \|\rho_\infty(w)\|_{\mathcal{L}(H_\infty)}$ for every $w \in \mathcal{P}$ and every $\lambda \in \mathbb{T}$. In particular, the Hilbertian model $\mathcal{P}_{\infty,2}$ satisfies the hypothesis of Proposition 7.3.

Theorem 7.1. *There exists a unique C^*-norm on $\mathcal{P}$.*

Proof. We already know a C^*-norm on $\mathcal{P}$, namely the norm given by $\rho_\infty : \mathcal{P} \to \mathcal{L}(H_\infty)$. We also know by Lemma 7.4 that this norm is smaller than any other C^*-norm on $\mathcal{P}$. Conversely, let $\rho : \mathcal{P} \to \mathcal{L}(H)$ be a Hilbertian representation of $\mathcal{P}$. We already know that $\|\rho(w)\| \geq \|\rho_\infty(w)\|$. Define

$$|w| = \sup_{\lambda \in \mathbb{T}} \|\rho(\varphi_\lambda(w))\| \geq \|\rho(w)\|.$$

This norm corresponds to the direct sum of the family of representations $(\rho \circ \varphi_\lambda)_{\lambda \in \mathbb{T}}$ and is therefore a C^*-norm on $\mathcal{P}$. Let $\mathcal{A}$ denote the completion of $\mathcal{P}$ under this norm. It is a C^*-algebra and we may define a $*$-representation ψ from $\mathcal{A}$ to the closure $\mathcal{A}_\infty$ of $\rho_\infty(\mathcal{P})$ in $\mathcal{L}(H_\infty)$. All we have to show (by Proposition 2.2) is that ψ is injective. It will follow that for every $w \in \mathcal{P}$, we have $|w| = \|\rho_\infty(w)\|$, so that finally since

$$\|\rho_\infty(w)\| \leq \|\rho(w)\| \leq |w|,$$

the three norms coincide.

We first observe that ψ is isometric on $\mathcal{B}$, since $\mathcal{B}$ has a unique C^*-algebra norm. Next, if $n \geq 0$ and $W_n = B_n U_0^n$, with $B_n \in \mathcal{B}$ and $B_n = B_n R_n$, we get $\|\psi(B_n)\| = \|B_n\|$, $\|R_n\| \leq 1$ since U_0 is an isometry, hence $\|W_n\| \leq \|B_n\|$ and $\|B_n\| = \|W_n (U_0^*)^n\| \leq \|W_n\|$. Finally

$$\|W_n\| \leq \|B_n\| = \|\psi(B_n)\| = \|\psi(B_n R_n)\| \leq \|\psi(W_n)\| \leq \|W_n\|$$

for every such W_n. A similar computation gives the case where $W_n = (U_0^*)^{-n} B_n$, for $n < 0$. Suppose now that $W \in \mathcal{A}$ and that $\psi(W) = 0$. We have that φ_λ is isometric in $\mathcal{A}$ by construction and also isometric in $\mathcal{A}_\infty$ (because the hypothesis of Proposition ?? is satisfied), so that we may apply Proposition 7.3 and obtain

$$\Phi_\lambda(W) \sim \sum_{n \in \mathbb{Z}} \lambda^n W_n;$$

taking the image under ψ gives

$$\Phi_\lambda(\psi(W)) \sim \sum_{n \in \mathbb{Z}} \lambda^n \psi(W_n) = 0,$$

hence $\psi(W_n) = 0$ thus $W_n = 0$ for every n and $W = 0$.

The unique C^*-algebra constructed from $\mathcal{P}$ is called $\mathcal{O}_3$. The above result says that any unital C^*-algebra generated by three elements U_j, $j = 0, 1, 2$ such that $U_j^* U_j = 1$ and $\sum U_j U_j^* = 1$ is isometric to $\mathcal{O}_3$ (in the C^*-case, it is easy to see that the property $U_j^* U_i = 0$ when $i \neq j$ follows from the two properties above).

Simplicity: let I be a proper two-sided closed ideal in $\mathcal{O}_3$; the quotient algebra $\mathcal{O}_3/I$ is a C^*-algebra generated by three elements $U'_j = \pi(U_j)$, such that $U'^*_j U'_i = \delta_{i,j} 1$, $\sum_j U'_j U'^*_j = 1$, therefore this quotient map is isometric and $I = 0$. Thus $\mathcal{O}_3$ is simple.

Extensions by compacts. Let $\mathcal{E}$ be a Hilbertian model for $\mathcal{Q}$. We know ??? that we have a map from $\mathcal{E}$ to $\mathcal{O}_3$: it is a quotient map.

We have described $\mathcal{O}_3$; all the proofs generalize easily to the C^*-algebra $\mathcal{O}_n$ generated by n partial isometries such that $\sum_{j=0}^{n-1} = 1$.

Embedding $\mathcal{Q}$ in $\mathcal{O}_4$;

8. Krivine's theorem

Theorem 8.1. ([K], see also [Le], [MiS]) Let X be a Banach space and let (x_n) be a sequence in X with no Cauchy subsequence. There exists $p \in [1, \infty]$ such that ℓ_p (or c_0 if $p = \infty$) is block finitely representable in the span of the given sequence. In other words, there exists $p \in [1, \infty]$ such that for every k and $\varepsilon > 0$, we can find successive blocks $y_1, \ldots, y_k$ of the sequence (x_n) such that for all scalars $(a_i)_{i=1}^k$

$$(1-\varepsilon)(\sum_{i=1}^k |a_i|^p)^{1/p} \le \|\sum_{i=1}^k a_i y_i\| \le (1+\varepsilon)(\sum_{i=1}^k |a_i|^p)^{1/p}.$$

We proceed by successive reductions of the problem, each time constructing a space with basis, block finitely representable in the preceding, thus block finitely representable in the given sequence. We assume that the scalars are real. The first reduction is to a space with a monotone basis and a norm invariant by spreading. This is given by any monotone spreading model generated by the sequence (x_n), as explained before in section 7.

Building unconditionality

We shall use an operator trick with the right shift defined on our spreading invariant space. After the first reduction we have a Banach space Y with a monotone basis $(f_n)_{n=1}^\infty$ and a norm invariant under spreading. In particular the right shift R on Y defined by $Rf_n = f_{n+1}$ is an isometry on Y. It follows that the real spectrum of R is contained in $[-1, +1]$. Also, it is easy to check that $R + I_Y$ is not onto (check that f_1 is not in the range) hence -1 belongs to the boundary of the real spectrum of R. It follows by Lemma 3.3 that one can find for every $\varepsilon > 0$ a vector $y \in Y$ such that $\|y\| = 1$ and $\|y + Ry\| < \varepsilon$. One can assume that y has finite support, $y = \sum_{i=1}^{N-1} a_i f_i$. Consider $y_0 = y$, $y_1 = R^N y$, $y_2 = R^{2N} y$, $\ldots, y_k = R^{kN} y$, and so on... It is easy to check that this sequence (y_k) is invariant under spreading and that changing one sign in a linear combination gives

$$\|\sum_{i \ne i_0} a_i y_i - a_{i_0} y_{i_0}\| \le \|\sum_{i \ne i_0} a_i y_i + a_{i_0} R y_{i_0}\| + \varepsilon |a_{i_0}| = \|\sum_i a_i y_i\| + \varepsilon |a_{i_0}|.$$

For every given integer $n > 0$ and $\varepsilon = 1/n$ we can find such a vector $y^{(n)}$ with $\|y^{(n)} + Ry^{(n)}\| < 1/n$ and we form the sequence $y_1^{(n)}, \ldots, y_k^{(n)}, \ldots$ as above. This sequence is spreading invariant. Then in the ultrapower $\widetilde{Y}$ we obtain for every $k \ge 1$ a vector $\widetilde{e}_k = (y_k^{(n)})_n$ with the property that $\|\widetilde{R}\,\widetilde{e}_k + \widetilde{e}_k\| = 0$. This sequence $(\widetilde{e}_k)$ is invariant under spreading and also 1-unconditional because we obtain in the limit

$$\|\sum_{i \ne i_0} a_i \widetilde{e}_i - a_{i_0} \widetilde{e}_{i_0}\| \le \|\sum_i a_i \widetilde{e}_i\|$$

for every i_0 and all scalars $(a_i)_i$.

Exercise. Construct the unconditionality in the case of complex scalars.

Lemberg's method: a space on $\mathbb{Q}_+$

At this point we have a space with a 1-unconditional basis $(y_n)_{n=1}^{\infty}$ invariant under spreading. We may define in a further ultrapower the vectors

$$f_q = (y_{1+[nq]})_n,$$

for every non-negative rational q. Let Ξ be the closed subspace generated by $(f_q)_{q\geq 0}$ and let Ξ_0 be the closed subspace of Ξ generated by $(f_q)_{0\leq q<1}$. The family (f_q) is still invariant under spreading in the following sense: if $q_1 < \ldots < q_n$ and $r_1 < \ldots < r_n$, then

$$\| \sum_i a_i f_{q_i} \| = \| \sum_i a_i f_{r_i} \|$$

for all scalars $(a_i)_{i=1}^n$ (and it is equal to $\| \sum_i |a_i| \, f_{q_i} \|$ because the family is 1-unconditional). We can consider elements of Ξ as (real) functions defined on $\mathbb{Q}_+$ (for example, f_q is the function equal to 0 at every $s \in \mathbb{Q}_+$, except $f_q(q) = 1$). We define operators D_n on Ξ_0 by

$$\forall t \in \mathbb{Q}_+, \ (D_n f)(t) = f(nt \bmod 1).$$

The operators D_n commute. It is easy here to complexify the space Ξ by simply defining $\Xi_{\mathbb{C}}$ to be the space of complex functions f on $\mathbb{Q}_+$ such that $|f|$ belongs to Ξ, and with the norm $\|f\| = \| \, |f| \, \|$.

Common approximate eigenvectors

Since the operators (D_n) commute, it is possible by Corollary 7.1?? to find, for every integer $N \geq 2$, scalars $\lambda_2, \ldots, \lambda_N$ and a common approximate eigenvector $g \in \Xi_0$ such that $\|g\| = 1$ and $D_i g \sim \lambda_i g$ for all D_i, $i = 2, \ldots, N$. We can replace each λ_i by $|\lambda_i|$ and g by $|g|$ because $|D_i g| = D_i |g|$; we assume therefore that $\lambda_i \geq 0$, $i = 2, \ldots, N$, and $g \geq 0$ in what follows. One shows that $\ln \lambda_i / \ln i$ is constant; this is not totally obvious: let R be the right shift by 1 on Ξ (defined by $(Rf)(t) = f(t-1)$); if $2^m < 3^n$, we see that

$$\sum_{j=0}^{2^m-1} R^j g \leq \sum_{j=0}^{3^n-1} R^j g$$

in the Banach lattice $\Xi_{\mathbb{C}}$, thus

$$\|(D_2)^m g\| = \Big\| \sum_{j=0}^{2^n-1} R^j g \Big\| \leq \Big\| \sum_{j=0}^{3^n-1} R^j g \Big\| = \|(D_3)^n g\|,$$

therefore $\lambda_2^m \leq \lambda_3^n$, and this yields that $\ln \lambda_2 / \ln \lambda_3 \leq \ln 2 / \ln 3$; the argument can be reversed to get $\ln \lambda_2 / \ln 2 = \ln \lambda_3 / \ln 3$, hence there exists $p \in [1, +\infty]$ $(1/p = \ln \lambda_i / \ln i)$ such that $\lambda_i = i^{1/p}$ for $i = 1, \ldots, N$.

Construction of ℓ_p^m

Assume $p < \infty$. We choose N so large that the set of all vectors in $\mathbb{R}^m$ with coordinates of the form $(k/N)^{1/p}$, k integer with $0 \le k \le N$, gives a good approximation for the positive part of the unit ball of ℓ_p^m (in more precise terms: an ε-net for some small $\varepsilon > 0$). Let $y_1, \dots, y_m$ be defined by $y_i = R^i y$ for $i = 1, \dots, m$, where $y \in \Xi_0$ satisfies $\|y\| = 1$, $D_i y \sim i^{1/p} y$ for $i = 1, \dots, Nm$. We obtain

$$\|(k_1/N)^{1/p} y_1 + \cdots + (k_m/N)^{1/p} y_m\| \simeq$$

$$N^{-1/p} \|D_{k_1} y + R D_{k_2} y + \cdots + R^{m-1} D_{k_m} y\| =$$

$$= N^{-1/p} \|D_{k_1+k_2+\cdots+k_m} y\| \simeq \big(\frac{k_1 + k_2 + \cdots + k_m}{N}\big)^{1/p}.$$

If $p = \infty$, we have $D_2 y \sim y$ and then $D_{2^n} y \sim y$. If $y_i = R^i y$, $i = 1, \dots, 2^n$, we get that

$$\|\sum_i \pm y_i\| \simeq \|y\| = 1$$

hence $(y_1, \dots, y_{2^n})$ is well equivalent to the usual vector basis for $\ell_\infty^{2^n}$.

9. K-theory of Banach algebras

See [Bl], [Ta], [WO], [C2], [Sk]. We work in this section with complex Banach spaces and complex Banach algebras. Let A be a unital Banach algebra over $\mathbb{C}$. We denote by $M_n(A)$ the unital algebra of $n \times n$ matrices with entries in A. It can be identified with $M_n \otimes A$, where $M_n = M_n(\mathbb{C})$. It is easy to see that $M_n(A)$ is a Banach algebra, but we will not insist on defining a Banach algebra norm on it. We denote by 1_n and 0_n respectively the unit matrix and the zero matrix in M_n, and by $1_{n,A} = 1_n \otimes 1_A$ and $0_{n,A} = 0_n \otimes 0_A$ the unit matrix and the zero matrix in $M_n(A)$. Given $a \in M_n(A)$ and $b \in M_p(A)$, we denote by $a \oplus b$ the matrix in $M_{n+p}(A)$ equal to

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

When X is a Banach space and $A = \mathcal{L}(X)$, the algebra $M_n(A)$ is naturally identified to $\mathcal{L}(X^n)$. Let $GL_n = GL_n(\mathbb{C})$ denote the group of complex $n \times n$ invertible matrices, and $GL_n(A)$ the (topological) group of $n \times n$ invertible matrices with entries in A. By $GL(A) = GL_1(A)$ we denote the group of invertible elements in A, and by $GL^{(0)}(A)$ the connected component of the identity 1_A in this group.

Exercise 9.1. Show that $GL_n(\mathbb{C})$ is connected (let M be an invertible matrix; since $\sigma(M)$ is finite and does not contain 0, one can find $\mu \ne 0$ such that the half-line $\mathbb{R}_+ \mu$ does not intersect $\sigma(M) \cup \{1\}$; consider then $M_t = (1 - t\mu)^{-1}(M - t\mu Id)$, t varying from 0 to $+\infty$).

The following fact will be very useful to the discussion.

Proposition 9.1. *For every unital Banach algebra B, the set of all finite products*

$$\mathrm{e}^{b_1}\, \mathrm{e}^{b_2} \dots \mathrm{e}^{b_n},$$

for $b_i \in B$ and $n \in \mathbb{N}$, is equal to the connected component $GL^{(0)}(B)$ of 1_B in $GL(B)$.

Proof. Our first remark is that there is an obvious continuous path from e^b to 1_B, namely $t \to \mathrm{e}^{tb}$, t varying from 1 to 0, hence all finite products of exponentials belong to $GL^{(0)}(B)$. For proving the converse, we may assume that the norm on B is a Banach algebra norm. We observe that the Taylor series of $\ln(1_B + x)$ converges when $\|x\| < 1$. It follows that each $a \in B$ such that $\|1_B - a\| < 1$ is an exponential $a = \mathrm{e}^b$, where $b = \ln(1_B + (a - 1_B))$. If u is invertible and if $\|v - u\| < \|u^{-1}\|^{-1}$, this implies that $v = \mathrm{e}^b\, u$ for some $b \in B$. With these remarks it is easy to see that the set of finite products of exponentials is open and closed in $GL(B)$ and since it contains 1_B, it is equal to $GL^{(0)}(B)$.

Corollary 9.1. *Let B be a unital Banach algebra and let J be a closed two-sided ideal in B. Then every invertible element in $GL^{(0)}(B/J)$ can be lifted to an invertible element in $GL^{(0)}(B)$.*

Proof. Simply write our invertible element in $GL^{(0)}(B/J)$ as product of exponentials $\mathrm{e}^{b_1} \dots \mathrm{e}^{b_n}$, with $b_i \in B/J$, and lift each b_i arbitrarily in B.

Similarity, equivalence and homotopy

Let p and q be two *idempotents* in A, i.e. $p^2 = p$, $q^2 = q$. We say that p and q are *equivalent* in A if there exist $x, y \in A$ such that $p = yx$, $q = xy$. We say that p and q are *similar* in A if there exists an invertible u in A such that $q = upu^{-1}$. It is clear that similar implies equivalent. We say that p and q are *homotopic* in A if there exists a continuous path $t \to p_t$ from $[0, 1]$ into A such that $p_0 = p$, $p_1 = q$ and $p_t^2 = p_t$ for all $t \in [0, 1]$.

We shall also say that two invertible elements $a, b \in GL(A)$ are homotopic in $GL(A)$ if there exists a continuous path in $GL(A)$ joining a and b.

Let B be a unital Banach algebra. Consider the following path in $GL_2(B)$:

$$r_{\theta,B} = \begin{pmatrix} (\cos\theta)1_B & -(\sin\theta)1_B \\ (\sin\theta)1_B & (\cos\theta)1_B \end{pmatrix}.$$

Using $(r_{\theta,B})$ for θ varying from $\theta = 0$ to $\theta = \pi/2$ we build a continuous path $r_{\theta,B}\,(a \oplus b)\, r_{-\theta,B}$ in $M_2(B)$ between $a \oplus b$ and $b \oplus a$, for any $a, b \in B$. Observe that when a and b are invertible in B this path is contained in $GL_2(B)$. When a, b are idempotents, it is a path of idempotents in $M_2(B)$. We get in this way an homotopy in $GL_2(B)$ between $a \oplus 1_B$ and $1_B \oplus a$. Multiplying it by $1_B \oplus a^{-1}$, we get an homotopy between $a \oplus a^{-1}$ and $1_{2,B}$. In particular $a \oplus a^{-1} \in GL_2^{(0)}(B)$. This is an important fact that will be used several times later. If we apply this to $B = M_n(A)$, and if $a \in GL_n(A)$, we get an homotopy in $GL_{2n}(A)$ between $a \oplus 1_{n,A}$ and $1_{n,A} \oplus a$, and an homotopy between $a \oplus a^{-1}$ and $1_{2n,A}$. In particular $a \oplus a^{-1} \in GL_{2n}^{(0)}(A)$. We have thus obtained

Lemma 9.1. *If p and q are idempotents in $M_n(A)$, then $p \oplus q$ and $q \oplus p$ are homotopic in $M_{2n}(A)$. If $a, b \in GL_n(A)$, then $a \oplus b$ and $b \oplus a$ are homotopic in $GL_{2n}(A)$, and $a \oplus a^{-1}$ belongs to $GL_{2n}^{(0)}(A)$.*

Exercises and examples 9.2.

1. Show that equivalence is an equivalence relation (if $q = vu$ and $r = uv$, consider uqx and yqv).

2. In $M_n(\mathbb{C})$, two idempotents are equivalent iff they have the same rank. They are then similar and homotopic (because $GL_n(\mathbb{C})$ is arcwise connected).

3. Let X be a Banach space and let p and q be two projections in $\mathcal{L}(X)$. Show that p and q are equivalent in $\mathcal{L}(X)$ iff the ranges pX and qX are isomorphic Banach spaces. Consequently, any rank one projections p and q are equivalent (and also similar and homotopic) in $\mathcal{L}(X)$.

For example, assume that X is a Banach space such that $X \simeq X^2$, one has that $I_X \oplus I_X$ and $I_X \oplus 0$ are equivalent in $\mathcal{L}(X^2)$. Indeed, let $U : X \to X \oplus X$ be an onto isomorphism and let $V : X \oplus X \to X$ be its inverse. In $M_2(\mathcal{L}(X))$ we have the equations

$$e = \begin{pmatrix} I_X & 0 \\ 0 & I_X \end{pmatrix} = \begin{pmatrix} U & 0 \end{pmatrix} \begin{pmatrix} V \\ 0 \end{pmatrix}; \ f = \begin{pmatrix} I_X & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} V \\ 0 \end{pmatrix} \begin{pmatrix} U & 0 \end{pmatrix}.$$

This means that the two idempotents e and f in $\mathcal{L}(X^2) = M_2(\mathcal{L}(X))$ are equivalent.

4. If p is an idempotent in B, then there is an homotopy in $M_2(B)$ between $p \oplus (1_B - p)$ and $1_B \oplus 0_B$: consider the path of idempotents in $M_2(B)$

$$\begin{pmatrix} p + (1_B - p)\sin^2\theta & (1_B - p)\sin\theta\cos\theta \\ (1_B - p)\sin\theta\cos\theta & (1_B - p)\cos^2\theta \end{pmatrix},$$

where θ varies from 0 to $\pi/2$. More generally, if p and q are two idempotents in B such that $pq = qp = 0$, then there exists an homotopy in $M_2(B)$ between $p \oplus q$ and $(p+q) \oplus 0_B$. Applying this to $B = M_n(A)$, if p and q are two idempotents in $M_n(A)$ such that $pq = qp = 0$, then there exists an homotopy in $M_{2n}(A)$ between $p \oplus q$ and $(p+q) \oplus 0_{n,A}$.

We shall now investigate the relations between the three notions of equivalence, similarity and homotopy.

Proposition 9.2. *Let B be a unital complex Banach algebra. If two idempotents p and q are similar in B, they are equivalent in B; if p and q are homotopic in B, they are similar in B.*

Proof. The first assertion is obvious; for the second we need the following lemma

Lemma 9.2. *For two idempotents p and q in a unital Banach algebra B (with a Banach algebra norm), $\|p - q\| < (\|p\| + \|q\|)^{-1}$ implies that p and q are similar.*

Proof. Let $u = qp + (1_B - q)(1_B - p)$. Then $qu = qp = up$, and $u = 1_B - p - q + 2qp = 1_B - q(q-p) + (q-p)p$ is invertible when $\|q - p\|(\|p\| + \|q\|) < 1$.

Suppose that p_t is a continuous path of idempotent elements in B, $t \in [0,1]$. Then $\|p_t\|$ is bounded by some M, and the condition $\|p_t - p_s\| < 1/(2M)$ implies that p_t and p_s are similar by Lemma 9.2. By uniform continuity we can find $\varepsilon > 0$ such that $\|p_t - p_s\| < 1/(2M)$ whenever $|t - s| < \varepsilon$. We can then pass from p_0 to p_1 by a finite number of similarities (actually, we may find a continuous path of invertible elements (u_t) such that $p_t = u_t\, p_0\, u_t^{-1}$ for every $t \in [0,1]$).

Proposition 9.3. *Let B be a unital complex Banach algebra. If two idempotents p and q are equivalent in B, then $p \oplus 0$ and $q \oplus 0$ are similar in $M_2(B)$; if p and q are similar in B, they are homotopic in $M_2(B)$.*

Proof. Suppose first that p and q are equivalent, with $p = xy$, $q = yx$. We may assume that $x = pxq$ and $y = qyp$. Indeed, we have $p = xyxyxyxyxy = (xy)x(yx)\,(yx)y(xy)$, which gives that $p = (pxq)(qyp)$ and similarly for q. We can write

$$\begin{pmatrix} 0 & 0 \\ 0 & p \end{pmatrix} = \begin{pmatrix} 1-q & y \\ x & 1-p \end{pmatrix} \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1-q & y \\ x & 1-p \end{pmatrix},$$

and we know that $p \oplus 0$ and $0 \oplus p$ are similar (actually homotopic).

Suppose now that p and q are similar in B; we can write $q = upu^{-1}$, with u invertible in B. Let v_t denote a path in $GL_2(B)$ from $1_B \oplus 1_B$ to $u \oplus u^{-1}$ (Lemma 9.1). We get an homotopy between $p \oplus 0_B$ and $q \oplus 0_B$ in $M_2(B)$ given by $r_t = v_t\,(p \oplus 0_B)\,v_t^{-1}$.

Examples 9.1.

1. Triangular matrices. Let $T_2(A)$ denote the algebra of 2×2 upper triangular matrices with entries in A. Every triangular idempotent is homotopic to its diagonal, using the path

$$\begin{pmatrix} \lambda^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b/\lambda \\ 0 & d \end{pmatrix},$$

λ varying from 1 to $+\infty$. A similar reasoning applies to the algebra $T(Y, X)$ of operators on a Banach space X that leave a complemented subspace $Y \subset X$ invariant; let Z be the kernel of a projection from X onto Y. Then every idempotent in this algebra $T(Y, X)$ is homotopic to a projection sending Y to Y and Z to Z.

2. Vector bundles. Let K be a compact topological space. Consider the Banach algebra $C(K)$ of continuous complex functions on K. An idempotent in $M_n(A)$ identifies with a continuous function p from K to M_n, such that $p(x)$ is a projection for every $x \in K$. This data defines a (complex) vector bundle over K; for every $x \in K$, the fiber at x is the complex vector space $F_x = p(x)(\mathbb{C}^n) \subset \mathbb{C}^n$; the dimension of the fiber at $x \in K$ is the rank of $p(x)$. Assume that $K = [0, 1]$. Setting $p_s(t) = p(st)$, we get an homotopy between p and the constant function $p(0) \in M_n$. The same reasoning applies to any contractible compact space.

Example of the Hopf fibration. The space $\mathbb{P}_1(\mathbb{C})$ admits a canonical (complex) line bundle. Recall that $\mathbb{P}_1(\mathbb{C})$ is the set of complex lines in $\mathbb{C}^2$. To every line $t \in \mathbb{P}_1(\mathbb{C})$ we associate the one-dimensional vector space $t \subset \mathbb{C}^2$. Every line t can be parametrized by a non zero point $\zeta = (z_1, z_2) \in t$. To $t \in \mathbb{P}_1(\mathbb{C})$ containing (z_1, z_2) we associate the orthogonal projection on the line t,

$$p(t) = \frac{1}{|\zeta|^2} \begin{pmatrix} z_1\overline{z}_1 & z_1\overline{z}_2 \\ z_2\overline{z}_1 & z_2\overline{z}_2 \end{pmatrix};$$

if we represent t by $(z, 1)$ (this is possible except for the line passing trough $(1, 0)$) we get a continuous idempotent defined for $z = \rho\,\mathrm{e}^{i\theta} \in \mathbb{C}$ by

$$p(z) = \frac{1}{\rho^2 + 1} \begin{pmatrix} \rho^2 & \rho\,\mathrm{e}^{i\theta} \\ \rho\,\mathrm{e}^{-i\theta} & 1 \end{pmatrix},$$

and this converges when z tends to infinity to

$$p(\infty) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

that corresponds to the line t passing trough $(1, 0)$. It is clear that $\mathbb{P}_1(\mathbb{C})$ is homeomorphic to the one-point compactification of $\mathbb{C}$, or also to the sphere S^2. Identifying S^2 with the closed unit disc D in $\mathbb{C}$, with all points on the unit circle identified to a single point, we get from the preceding p, setting $r = \tan(\pi\rho/2)$, for $z = r\,\mathrm{e}^{i\theta} \in D$

$$q(z) = \frac{1}{2}\begin{pmatrix} 1 - \cos\pi r & \mathrm{e}^{i\theta}\sin\pi r \\ \mathrm{e}^{-i\theta}\sin\pi r & 1 + \cos\pi r \end{pmatrix}$$

(note that $q(\mathrm{e}^{i\theta}) = 1 \oplus 0$ for every θ, so that q is constant on the unit circle $\mathbb{T}$ and can therefore be considered as a function on S^2). This idempotent q is not equivalent to a constant function on S^2 (more generally, for every integer $k \geq 0$, $q \oplus 0_k$ is not equivalent in $M_{k+2}(C(S^2))$ to a constant idempotent; this is a relatively difficult exercise, where the decisive argument is Brouwer's fixed point theorem or the notion of degree).

3. Cuntz algebras. For every $n \geq 2$ let $\mathcal{O}_n$ be the C^*-algebra generated in $\mathcal{L}(\ell_2)$ by n into isometries $U_1, \ldots, U_n$ (so $U_i^* U_i = I$ for every i) such that $\sum_{i=1}^n U_i U_i^* = I$ (it follows that $U_j^* U_i = 0$ when $i \neq j$). It is proved in [C1] that $\mathcal{O}_n$ does not depend upon the particular choice of (U_i) (see section 7 for $\mathcal{O}_3$). For every $i = 1, \ldots, n$, we have a projection $p_i = U_i U_i^*$ equivalent to I since $p_i = U_i U_i^*$ and $I = U_i^* U_i$.

The algebra $\mathcal{E}_n$ is generated in $\mathcal{L}(\ell_2)$ by n into isometries $V_1, \ldots, V_n$ such that

$$\sum_{i=1}^n V_i V_i^* < I$$

(it also follows that $V_j^* V_i = 0$ when $i \neq j$). The projections $q_i = V_i V_i^*$, $i = 1, \ldots, n$ are again equivalent to I.

The semi-group of classes of idempotents

We introduce now the algebra $M_\infty(A)$ equal to the union of the $M_n(A)$, the embedding of $M_n(A)$ into $M_{n+p}(A)$ being given by $a \to a \oplus 0_{p,A}$; we say that two idempotents a and b are equivalent, similar or homotopic in $M_\infty(A)$ iff there is some n such that $a, b \in M_n(A)$ and a and b are equivalent, similar or homotopic in $M_n(A)$; in $M_\infty(A)$ the three notions of comparison for idempotents coincide. Let $\mathrm{Pr}(A)$ denote the set of equivalence classes of idempotents in $M_\infty(A)$. Let $\{p\}$ denote the equivalence class of an idempotent p of $M_\infty(A)$ in $\mathrm{Pr}(A)$.

Additive structure on $\mathrm{Pr}(A)$

If $\{p\}$ and $\{q\}$ are two classes in $\mathrm{Pr}(A)$, we define their sum by $\{p\} + \{q\} = \{p \oplus q\}$.

Exercise. Show that this operation is well defined, associative and that $\{0\}$ is a neutral element. In other words, $\mathrm{Pr}(A)$ is a monoid.

This addition is commutative; indeed, we know by Lemma 9.1 that $p \oplus q$ and $q \oplus p$ are homotopic.

Examples 9.2.

1. In $M_n(\mathbb{C})$, we know that two idempotents are equivalent iff they have same rank. This shows that $\mathrm{Pr}(\mathbb{C})$ can be identified to $\mathbb{Z}_+$. Furthermore, the addition corresponds to the addition of ranks. This shows that $\mathrm{Pr}(\mathbb{C})$ is simply $\mathbb{Z}_+$ (as monoid), where $1 \in \mathbb{Z}$ is identified to the class of rank one projections. If we consider $\mathrm{Pr}(M_n)$, it is clear that $M_\infty(M_n) \simeq M_\infty(\mathbb{C})$ and hence $\mathrm{Pr}(M_n) \simeq \mathbb{Z}_+$, where again $1 \in \mathbb{Z}_+$ corresponds to the class of rank one projections.

2. $\{1_{n,A}\} = n\,\{1_A\}$.

3. If p is an idempotent in $M_n(A)$, then $\{p\} + \{1_{n,A} - p\} = \{1_{n,A}\}$. More generally, if p and q are two idempotents in $M_n(A)$ such that $pq = qp = 0$, then $\{p\} + \{q\} = \{p+q\}$. This follows directly from Example 9.2, 4 above.

4. Let A, B be two unital Banach algebras. It is clear that $\mathrm{Pr}(A \times B) \simeq \mathrm{Pr}(A) \times \mathrm{Pr}(B)$. Let $T_2(A)$ be the algebra of 2×2 upper triangular matrices with entries in A. A matrix in $M_n(T_2(A))$ can be considered as an element of $T_2(M_n(A))$, after some reindexing. Applying the deformation from Example 9.1, 1 with A replaced by $M_n(A)$, we see that $\mathrm{Pr}(T_2(A)) \simeq (\mathrm{Pr}(A))^2$. For the algebra $T(Y,X)$ of operators on a Banach space X that leave a complemented subspace $Y \subset X$ invariant, we see that $\mathrm{Pr}(T(Y,X)) \simeq \mathrm{Pr}(\mathcal{L}(Y)) \times \mathrm{Pr}(\mathcal{L}(Z))$, where Z is the kernel of a projection from X onto Y.

5. Vector bundles. We said that an idempotent in $M_n(A)$, $A = C(K)$ identifies to a continuous map from K to the space of idempotents in M_n. If K splits into two closed and open subsets K_1 and K_2, then $\mathrm{Pr}(C(K))$ is isomorphic to $\mathrm{Pr}(C(K_1)) \times \mathrm{Pr}(C(K_2))$. If K is connected, the rank of $p(t)$ is constant when t varies in K, but this rank is not enough to characterize the class of p in $\mathrm{Pr}(C(K))$. This rank only gives the dimension (or rank) of the associated vector bundle on K. When K is contractible, every idempotent $p \in M_\infty(C(K))$ is equivalent to a constant function $p(x_0) \in M_n$, hence $\mathrm{Pr}(C(K)) \simeq \mathbb{Z}_+$ in this case. In any case $\mathbb{Z}_+$ is always a submonoid of $\mathrm{Pr}(C(K))$, corresponding to the classes of constant functions p (or of trivial bundles).

The example of the Hopf fibration yields an idempotent $p \in M_2(C(S^2))$ which is not equivalent to a constant function on S^2; the monoid $\mathrm{Pr}(C(S^2))$ contains at least $\mathbb{Z}_+^2$.

6. When $A = \mathcal{L}(X)$, the class $\{I_X\}$ in the monoid $\mathrm{Pr}(\mathcal{L}(X))$ is the class of complemented subspaces of X^n isomorphic to X. If X is a Banach space such that $X \simeq X^2$, we have seen in Example 9.2, 3 that $\{I_X \oplus I_X\} = \{I_X\}$ in $M_\infty(\mathcal{L}(X))$. One has therefore $2\{I_X\} = \{I_X\} + \{I_X\} = \{I_X\}$. Indeed, we know that $I_X \oplus I_X$ and $I_X \oplus 0$ are equivalent, and this implies by definition of the addition that $\{I_X\} + \{I_X\} = \{I_X\}$.

Several examples show that the monoid $\mathrm{Pr}(A)$ may fail the cancellation property (but it is true in $\mathrm{Pr}(\mathbb{C}) \simeq \mathbb{Z}_+$): for $A = \mathcal{L}(L^1)$ for example, we have

$$\ell_1 \oplus L_1 \simeq L_1 \oplus L_1 \simeq L_1 \simeq 0 \oplus L_1,$$

hence identifying projections and ranges (see Exercise 9.2,3),

$$\{\ell_1\} + \{L_1\} = \{L_1\} + \{L_1\} = \{L_1\}.$$

When the Banach space X is isomorphic to its hyperplanes, we have in $A = \mathcal{L}(X)$, $\{I_X\} = \{I_X - p\}$ when p is a rank one projector, hence

$$\{I_X\} + \{p\} = \{I_X\}.$$

7. Cuntz algebras. In $\mathcal{O}_n$ we have n projections $p_i = U_i U_i^*$, each equivalent to I, and $p_i p_j = 0$ when $i \neq j$. Then the relation $\sum_{i=1}^n U_i U_i^* = I$ implies that $n\{1_{\mathcal{O}_n}\} = \{1_{\mathcal{O}_n}\}$, using case 3 above.

In $\mathcal{E}_n$, $Q = I - \sum_{i=1}^n V_i V_i^*$ is a non zero projection and we get $n\{1_{\mathcal{E}_n}\} + \{Q\} = \{1_{\mathcal{E}_n}\}$.

Group associated to an additive monoid M. Additive group $K_0(A)$

On the set of couples $(m, n) \in M^2$, we define the equivalence relation $(m, n) \sim (m', n')$ if there exists $r \in M$ such that $m + n' + r = m' + n + r$. The quotient of M by this relation is an additive group G. If ϕ denotes the map from M to G that sends $m \in M$ to the class $[(m, 0)]$ of $(m, 0)$ in G, there exists for every monoid morphism f from M to a group H a unique group morphism $\overline{f} : G \to H$ such that $f = \overline{f} \circ \phi$. Every element $\alpha \in G$ can be written $\alpha = \phi(m) - \phi(n)$ for some $m, n \in M$.

Definition 9.1. If A is a unital Banach algebra, we denote by $K_0(A)$ the group associated to the monoid $\mathrm{Pr}(A)$. We denote by $[p]$ the image $\phi(\{p\})$ by $\phi : \mathrm{Pr}(A) \to K_0(A)$ of the class $\{p\}$ in $\mathrm{Pr}(A)$ of an idempotent $p \in M_\infty(A)$. Every element of $K_0(A)$ can be written $[p] - [q]$ for some idempotents p, q in $M_\infty(A)$.

Every $\alpha = [p] - [q] \in K_0(A)$ can be written $[p'] - [1_{n,A}]$ for some n and some idempotent p' in $M_{2n}(A)$. This follows from Example 9.3, 3 below.

We have $[p] = [q]$ if and only if there exists r such that $\{p\} + \{r\} = \{q\} + \{r\}$. Since r is an idempotent in some $M_n(A)$, there exists $s \in M_n(A)$ so that $\{r\} + \{s\} = \{1_{n,A}\}$ (we may simply choose $s = 1_{n,A} - r$), and finally $[p] = [q]$ if and only if there exists n such that $\{p\} + \{1_{n,A}\} = \{q\} + \{1_{n,A}\}$.

Examples 9.3.

1. For $\mathbb{C}$, we obtain of course $K_0(\mathbb{C}) \simeq \mathbb{Z}$. Recall that $1 \in \mathbb{Z}$ corresponds to the class of rank one projections. We also have $K_0(M_n) \simeq \mathbb{Z}$ for every integer $n \geq 1$, where again $1 \in \mathbb{Z}$ corresponds to the class of rank one projections.

2. $[1_{n,A}] = n\,[1_A]$ (since we had $\{1_{n,A}\} = n\,\{1_A\}$).

3. If p is an idempotent in $M_n(A)$, $[p] + [1_{n,A} - p] = [1_{n,A}]$. This is because $\{p\} + \{1_{n,A} - p\} = \{1_{n,A}\}$. More generally, if p and q are two idempotents in $M_n(A)$ such that $pq = qp = 0$, then $[p] + [q] = [p + q]$.

4. Let p be a finite rank projection in some Banach space X isomorphic to its hyperplanes. Then $[p] = 0$. Indeed, we saw in Example 9.2, 6 that $\{I_X\} + \{q\} = \{I_X\}$ if q has rank one, thus $[q] = 0$ and when p has rank n, we get $[p] = n[q] = 0$.

5. Suppose $A = \mathcal{L}(X)$, where X is a Banach space such that $X \simeq X^2$, we have $[I_X] + [I_X] = [1_{2,A}] = [I_X]$, thus $[I_X] = 0$. Since $\{I_X\}$ is the class in $\mathrm{Pr}(\mathcal{L}(X))$ of complemented subspaces of X^n isomorphic to X, the class of $[I_X]$ is 0 in $K_0(\mathcal{L}(X))$ if and only if there exists an integer n such that $X^n \simeq X^{n+1}$. It is relatively usual for many classical spaces that $X \simeq X^2$. I know of no example where $X \not\simeq X^2$ but $X^2 \simeq X^3$.

Suppose that $X \simeq \ell_p(X)$ for some $1 \leq p < \infty$. Then every complemented subspace Y of X containing a complemented copy of X is isomorphic to X. This is Pełczyński's decomposition method. If $X = Y \oplus Z$ and $Y = X \oplus U$ we have

$$X \oplus Y \simeq \ell_p(X) \oplus X \oplus U \simeq \ell_p(X) \oplus U \simeq Y$$

and

$$X \oplus Y \simeq \ell_p(Y \oplus Z) \oplus Y \simeq \ell_p(Y \oplus Z) \simeq X.$$

This implies that X is isomorphic to its square and isomorphic to its hyperplanes.

6. $K_0(\mathcal{L}(X))$ for a primary Banach space isomorphic to its square. The space X is said *primary* when for every decomposition $X = Y \oplus Z$, one has $Y \simeq X$ or $Z \simeq X$. We know for example after Enflo [E] that $L_p[0,1]$ is primary when $1 \le p < \infty$ ([AEO], [M1]). The space ℓ_p is more than primary, it is *prime*; recall that a Banach space is said to be *prime* if it is isomorphic to every infinite-dimensional complemented subspace of itself. The only known examples before [GM2] were c_0 and ℓ_p $(1 \le p \le \infty)$. These were shown to be prime by Pełczyński [P], apart from ℓ_∞ which is due to Lindenstrauss [L1]. If we assume that X^2 is primary, it follows that $X \simeq X^2$, and $X^n \simeq X$ is primary for every n. Let $A = \mathcal{L}(X)$. For every idempotent $p \in M_n(\mathcal{L}(X))$, either $[p] = [1_{n,A}] = [I_X]$ or $[1_{n,A} - p] = [I_X]$. We also know that $[I_X] = 0$ because $X \simeq X^2$. In either case $[p] = 0$, hence $K_0(\mathcal{L}(X)) = \{0\}$. In particular for a Hilbert space H we obtain $K_0(\mathcal{L}(H)) = \{0\}$. This shows that in most classical cases, the K_0-theory of $\mathcal{L}(X)$ is trivial (and thus mostly useless). It will not be so for the exotic spaces of sections 10, 11and 12.

7. It is clear that $K_0(A \times B) \simeq K_0(A) \times K_0(B)$. Since $\mathrm{Pr}(T_2(A)) \simeq (\mathrm{Pr}(A))^2$, it follows that $K_0(T_2(A)) \simeq K_0(A)^2$. For the algebra $T(Y,X)$ introduced previously, we see that $K_0(T(Y,X)) \simeq K_0(\mathcal{L}(Y)) \times K_0(\mathcal{L}(Z))$, where Z is the kernel of a projection from X onto Y. On the contrary, for $M_2(A)$ as well as for $M_n(A)$, we obtain $K_0(M_n(A)) \simeq K_0(A)$.

8. Let $A = C(K)$, where K is compact and connected. We see that $K_0(C(K))$ contains a canonical copy of $\mathbb{Z}$, corresponding to idempotents in $M_n(C(K))$ given by constant functions from K to M_n. When K is contractible, we know that every idempotent in $M_n(C(K))$ is homotopic to a constant function, hence we get $K_0(C(K)) \simeq \mathbb{Z}$.

9. Simpler presentation of the case of a Banach space X such that $X \oplus X$ is isomorphic to a complemented subspace of X. In this case we can avoid the use of $M_\infty(\mathcal{L}(X))$ and the symmetrisation from Pr to K_0 in the following way: let $X \simeq X \oplus X \oplus Y$. To every $T \in \mathcal{L}(X)$ we associate, in block notation with respect to the decomposition $X = X \oplus X \oplus Y$

$$i_1(T) = \begin{pmatrix} T & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad i_2(T) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & T & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $i_1(T)$ and $i_2(T)$ are equivalent to T; when T and U are equivalent, then $i_1(T)$ and $i_2(U)$ are homotopic in $\mathcal{L}(X)$ and given $T_1, \dots, T_n$ we can find using products of i_1 and i_2 operators $U_1, \dots, U_n$ such that $U_i \sim T_i$ and such that the U_i appear as disjoint diagonal blocks in a larger decomposition of X. We may then define the sum $\{T_1\} + \cdots + \{T_n\}$ as $\{U_1 + \cdots + U_n\}$.

10. Cuntz algebras. In $\mathcal{O}_n$ the relation $n\{1_{\mathcal{O}_n}\} = \{1_{\mathcal{O}_n}\}$ implies that $(n-1)[1_{\mathcal{O}_n}] = 0$. Cuntz proved in [C2] that $K_0(\mathcal{O}_n) = \mathbb{Z}/(n-1)\mathbb{Z}$. In $\mathcal{E}_n$ we obtain $(n-1)[1_{\mathcal{E}_n}] + [Q] = 0$.

K_0 functor

Let φ be a morphism of unital Banach algebras from A to B. Letting φ act on each entry of a matrix in $M_n(A)$, we get a unital algebra morphism $\varphi^{(n)}$ from $M_n(A)$ to

$M_n(B)$ for every $n \geq 1$, sending idempotents in $M_n(A)$ to idempotents in $M_n(B)$. Clearly, homotopic idempotents have homotopic images. This gives a morphism of monoids from $\mathrm{Pr}(A)$ to $\mathrm{Pr}(B)$, then a group morphism φ_* from $K_0(A)$ to $K_0(B)$. One can check that $A \to K_0(A), \varphi \to \varphi_*$ defines a functor from the category of unital Banach algebras to the category of additive groups.

An example. For every $n \geq 0$ let φ_n be the algebra morphism from M_{3^n} to $M_{3^{n+1}}$ defined by $\varphi_n(a) = a \oplus a \oplus a$. For every n, we have that $K_0(M_{3^n}) = \mathbb{Z}$, and the map φ_n sends rank one projections to rank three projections, hence $\varphi_{n,*}(1) = 3$. If we consider the Banach algebra A obtained as inductive limit of the sequence (M_{3^n}) with the (φ_n) as successive embeddings, we obtain a chain of maps

$$\mathbb{Z} \to \cdots \to \mathbb{Z} \to \mathbb{Z} \to \cdots$$

where each arrow is the map $\varphi_*(k) = 3k$; this implies (with some work) that

$$K_0(A) \simeq \{\frac{k}{3^n} \in \mathbb{Q} : k \in \mathbb{Z}, n \geq 0\}.$$

We may associate to $1 \in \mathbb{Q}$ the class $[1_A]$. This algebra A is closely related to the algebra $\mathcal{B}$ appearing in section 7 with the discussion of $\mathcal{P}$; actually A is the completion of $\mathcal{B}$ under its (unique) C^*-norm.

K_0 for non unital algebras

Let A be a Banach algebra without unit. We consider the unital algebra $A^+ = A \oplus \mathbb{C}$ from section 2. Then A is a closed two-sided ideal of A^+ and A^+/A is canonically isomorphic to $\mathbb{C}$. Let π be the projection from A^+ onto $\mathbb{C}$, and let $i : \mathbb{C} \to A^+$ be given by $i(\lambda) = \lambda 1_+$. We have $\pi \circ i = Id_{\mathbb{C}}$. By the functorial character we get $\pi_* : K_0(A^+) \to K_0(\mathbb{C})$, $i_* : K_0(\mathbb{C}) \to K_0(A^+)$, and $\pi_* \circ i_* = Id$. Hence $K_0(\mathbb{C}) \simeq \mathbb{Z}$ appears as factor in $K_0(A^+)$; we define $K_0(A)$ as the kernel of π_*:

$$K_0(A) = \ker \pi_* \subset K_0(A^+).$$

Let $\alpha \in K_0(A)$. We know that we can write $\alpha = [p] - [1_{n,A^+}] \in K_0(A^+)$, where p is an idempotent in $M_{2n}(A^+)$; by definition of $K_0(A)$ we have $\pi_*(\alpha) = 0$. This means that $\pi(p)$ and $1_n \oplus 0_n$ are equivalent idempotents in $M_{2n}(\mathbb{C})$. Hence there exists $u \in GL_{2n}(\mathbb{C})$ such that $u\pi(p)u^{-1} = 1_n \oplus 0_n$. Then, setting $\widetilde{u} = u \otimes 1_{A^+}$, we see that $r = \widetilde{u}\, p\, \widetilde{u}^{-1}$ is an idempotent in $M_{2n}(A^+)$, equivalent to p, and $r = 1_{n,A^+} \oplus 0_{n,A} + a$, with $a \in M_{2n}(A)$. Finally, replacing p by r we obtain:

Every α in $K_0(A)$ can be expressed as $\alpha = [r] - [1_{n,A^+}]$, where r is an idempotent in $M_{2n}(A^+)$ of the form $r = 1_{n,A^+} \oplus 0_{n,A} + a$, with $a \in M_{2n}(A)$.

Remark 9.1. The above construction is certainly necessary for Banach algebras without unit that have no idempotent, except 0 (for example, the algebra $A = C_0(K)$ of continuous functions on a compact connected space K, vanishing at some point $x_0 \in K$). Some algebras without unit, like $\mathcal{K}(X)$, already have idempotents (finite rank projections). If p

is an idempotent in A, its class in the above construction is given by $[1_{A^+}] - [1_{A^+} - p]$ or $[1_{A^+} \oplus p] - [1_{A^+}]$. One can check that when A is already unital, the above construction defines the same group $K_0(A)$. Indeed we have the exact sequence $A \xrightarrow{i} A^+ \xrightarrow{\pi} \mathbb{C}$, giving

$$K_0(A) \xrightarrow{i_*} K_0(A^+) \xrightarrow{\pi_*} K_0(\mathbb{C}),$$

where $K_0(A)$ here is defined according to our first definition for unital Banach algebras. By functoriality it is clear that $i_*(K_0(A))$ is contained in the kernel of π_*. Conversely, let $\alpha \in K_0(A^+)$ belong to $\ker \pi_*$. We write $\alpha = [p] - [1_{n,A^+}]$, where $p = 1_{n,A^+} \oplus 0_{n,A} + a$ is an idempotent in $M_{2n}(A^+)$, with $a \in M_{2n}(A)$. We see that $q = 1_{A^+} - 1_A$ is an idempotent in A^+ such that $qA = Aq = 0$. We may write p as $p = p' + q'$, where $p' = 1_{n,A} \oplus 0_{n,A} + a$ is an idempotent in $M_{2n}(A)$, and q' is an idempotent in $M_{2n}(A^+)$ (q' is the direct sum of n copies of q) such that $q'p' = p'q' = 0$, hence $[p] = [p'] + [q']$; similarly $[1_{n,A^+}] = [1_{n,A}] + [q']$ and finally $[p] - [1_{n,A^+}] = [p'] - [1_{n,A}]$ belongs to the image of $K_0(A)$ in $K_0(A^+)$.

The K_0 functor extends now to the category of Banach algebras, not necessary unital.

Examples.

1. Let K be a compact connected topological space. Let $C_0(K) = C_{x_0}(K)$ denote the closed ideal of $C(K)$ consisting of continuous functions on K vanishing at some point $x_0 \in K$; we see that $C(K) \simeq C_0(K)^+$, hence

$$K_0(C(K)) \simeq K_0(C_0(K)) \oplus \mathbb{Z}.$$

When $A = C([0,1])$, we have seen that $K_0(C([0,1])) \simeq \mathbb{Z}$ (true for any contractible space). For a contractible compact space K we get $K_0(C_0(K)) = \{0\}$. For $C_0(\mathbb{T})$, we also have $K_0(C(\mathbb{T})) = \mathbb{Z}$, thus $K_0(C_0(\mathbb{T})) = \{0\}$.

2. We have seen that $\mathbb{P}_1(\mathbb{C}) \simeq S^2$ admits a canonical (complex) line bundle (Hopf fibration). This bundle gives a non zero class in $K_0(C_0(S^2))$: if we identify $C_0(S^2)$ to the space of continuous functions on $\mathbb{C}$ vanishing at infinity, the idempotent described in Example 9.1, 2 gives a non zero element $[p] - [1 \oplus 0]$ in $K_0(C_0(S^2))$.

Example of $K_0(\mathcal{S})$

Here X is a Banach space, and we study the ideal $\mathcal{S}(X)$ of strictly singular operators.

Lemma. Let $X = U \oplus V$, $\dim U = \dim V = +\infty$, and let π_U be the projection of $U \oplus V$ onto U. Let $\mathcal{A}$ denote the unital subalgebra of $\mathcal{L}(U \oplus V)$ generated by π_U and $\mathcal{S}(U \oplus V)$. Let p be a projection in $\mathcal{L}(X)$, with the form $p = \pi_U + S$, where $S \in \mathcal{S}(U \oplus V)$. Then p is equivalent (in $\mathcal{A}$), either to a projection on Y, finite codimensional subspace of U, or to a projection on $U \oplus E$, where E is a finite dimensional subspace of V. In other words, p is equivalent to a projection $(\pi_U - r_1) \oplus r_2$, where r_1 and r_2 are finite rank projections in $\mathcal{L}(U)$ and $\mathcal{L}(V)$ respectively. The quantity $\operatorname{rank}(r_2) - \operatorname{rank}(r_1)$ is an invariant of the similarity class of p in $\mathcal{A}$.

Proof. Let $q = \pi_U \, p \, \pi_U = I_U + S'$, $S' \in \mathcal{S}(U)$, considered as operator on U. By Proposition 6.1, we know that q has a finite codimensional invariant subspace Z on which q gives an isomorphism $a \in \mathcal{L}(Z)$. Considering a new decomposition $Z \oplus W$ of $U \oplus V$ (notice that finite rank projections belong to $\mathcal{A}$, so the projections on Z and W belong to $\mathcal{A}$), we get in block notation in $Z \oplus W$

$$p = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = p^2,$$

with a invertible, b, c, d strictly singular; then p is similar in $\mathcal{A}$ to

$$\begin{pmatrix} a^{-1} & 0 \\ -ca^{-1} & I_W \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & 0 \\ c & I_W \end{pmatrix} = \begin{pmatrix} I_Z & b' \\ 0 & d' \end{pmatrix},$$

(we know that $a^{-1} \in \mathcal{A}$ by Lemma 2.2 and Proposition 6.1) with b', d' strictly singular. It follows from the fact that the above matrix is an idempotent that $b'd' = 0$, $d'^2 = d'$; next

$$\begin{pmatrix} I_Z & b' \\ 0 & I_W \end{pmatrix} \begin{pmatrix} I_Z & b' \\ 0 & d' \end{pmatrix} \begin{pmatrix} I_Z & -b' \\ 0 & I_W \end{pmatrix} = \begin{pmatrix} I_Z & 0 \\ 0 & d' \end{pmatrix}$$

is an idempotent equivalent in $\mathcal{A}$ to an idempotent of the desired form; indeed, d' is a strictly singular projection, thus has finite rank. It only remains to move the finite dimensional image F of d' in the correct position; if k denotes the codimension of Z in U and if $\dim F \le k$, we can move F to G inside U, in such a way that $Z \oplus G$ is a direct sum; if $\dim F > k$, we will move a k-dimensional subspace of F to some G inside U in such a way that $U = Z \oplus G$, and the remaining part of F into V.

Let us give now a partial proof for the claim in the last line of the Lemma. Let r be a finite rank non zero projection in $\mathcal{L}(V)$, and let $F = rV \neq \{0\}$; we will show that $q = \pi_U \oplus r$ is not similar to π_U in $\mathcal{A}$. If it was similar, we could find an invertible element u in $\mathcal{A}$ such that $\pi_U u = uq$. Then quq gives an isomorphism from $U \oplus F$ onto U, thus a Fredholm operator in $\mathcal{L}(U \oplus F)$ with non zero index. But quq has a block decomposition in $U \oplus F$

$$\begin{pmatrix} \lambda I_U + s & br \\ rc & rdr \end{pmatrix},$$

with s strictly singular and $\lambda \neq 0$ (because u is invertible). This operator is a strictly singular perturbation of $\lambda \pi_U$, and should therefore have zero index, a contradiction.

We arrive to the computation of $K_0(\mathcal{S}(X))$. Here $A = \mathcal{S}(X)$ and we assume of course that $\dim X = +\infty$; we can then identify A^+ to the subalgebra of $B = \mathcal{L}(X)$ consisting of all operators $\lambda Id + S$, $S \in A$ and $\lambda \in \mathbb{C}$. Let now $\alpha \in K_0(\mathcal{S}(X))$. There exists a projection p on X^{2n} of the form $1_{n,B} + S$, with $S \in \mathcal{S}(X^{2n})$, such that $\alpha = [p] - [1_{n,B}]$. By the Lemma, we know that p is equivalent either to a projection on Y, finite codimensional subspace of X^n, or to a projection on $X^n \oplus F$, $\dim F < \infty$. Furthermore we said that the integer k, equal to $-\operatorname{codim} Y$ or to $\dim F$ characterizes α, hence $K_0(\mathcal{S}(X))$ is isomorphic to $\mathbb{Z}$. We may choose for generator of $K_0(\mathcal{S}(X))$ the class of rank one projections, (see remark 9.1).

The above proof also applies to $\mathcal{K}(X)$. We get that $K_0(\mathcal{K}(X)) \simeq \mathbb{Z}$. In the case of the Hilbert space, a more natural approach uses the fact that $\mathcal{K}(\ell_2)$ is the inductive limit of the algebras (M_n).

Short exact sequence in K-theory

Suppose given a short exact sequence

$$0 \to I \xrightarrow{i} A \xrightarrow{\pi} A/I \to 0,$$

where I is a closed two-sided ideal in A, i the inclusion map and π the canonical quotient map. We get

$$K_0(I) \xrightarrow{i_*} K_0(A) \xrightarrow{\pi_*} K_0(A/I)$$

exact at $K_0(A)$ (we just remark that $\pi_* \circ i_* = 0$ by functoriality; the proof that $\ker \pi_*$ is precisely equal to the range of i_* uses Corollary 9.1 and Lemma 9.1).

Exercise. Prove the above: let $\alpha = [p] - [1_{n,A}]$, with p idempotent in $M_{2n}(A)$, such that $\pi_*(\alpha) = 0$. We know that for some m, the idempotent $\pi(p) \oplus 1_{m,A/I}$ is similar to $1_{n+m,A/I}$ in $M_{n+m}(A/I)$; apply now the last part of Lemma 9.1 to $M_{2n+2m}(A/I)$ and next apply Corollary 9.1.

Suspension and the group $K_1(A)$

Let A be a Banach algebra. Denote by SA the (non unital) algebra of continuous maps $f : \mathbb{T} \to A$ such that $f(1) = 0_A$, the product being the pointwise product in A. We will set $K_1(A) = K_0(SA)$, but some comments are necessary. For defining $K_0(SA)$ we consider idempotents in $M_n((SA)^+)$. Such an idempotent p is a $n \times n$ matrix with entries in $(SA)^+$. An element of $(SA)^+$ identifies to a continuous function f from $\mathbb{T}$ to A^+ such that $f(1) = \lambda 1_+$. The matrix p identifies to a continuous function $p(t)$ with values in idempotents of $M_n(A^+)$ such that $p(1)$ is a "scalar" matrix i.e. a matrix $q \otimes 1_{n,A^+}$, $q \in M_n(\mathbb{C})$. Up to equivalence we can assume that $p(1) = 1_{k,A^+} \oplus 0_{n-k,A^+}$. By definition we get an homotopy from $p(1)$ to $p(1)$ obtained by travelling around the circle. This gives as explained after Lemma 9.2 a similarity $up(1) = p(1)u$. But the special form of $p(1)$ implies that $u \in GL_n(A^+)$ leaves both factors $(A^+)^k$ and $(A^+)^{n-k}$ invariant, so $u = v \oplus w$, with $v \in GL_k(A^+)$ and $w \in GL_{n-k}(A^+)$. We have therefore associated to any idempotent p in $M_\infty((SA)^+)$ an invertible $v \in GL_k(A^+)$, for some k.

Conversely, let $v \in GL_k(A^+)$; we may associate to v some $w \in GL_{n-k}(A^+)$ such that $v \oplus w \in GL_n^{(0)}(A^+)$ (for example $n = 2k$, $w = v^{-1}$). We can find a continuous path u_t in $GL_n(A^+)$ such that $u_0 = v \oplus w$ and $u_1 = 1_{n,A}$, and the path of idempotents $p(t) = u_t\,(1_{k,A^+} \oplus 0_{n-k,A^+})\,u_t^{-1}$; we check that $p(1) = p(0)$ (because $u(0)$ and $u(1)$ commute to $p(0)$) and $p(0)$ is scalar), thus p identifies to an idempotent in $M_n((SA)^+)$. If v' belongs to the same connected component of $GL_k(A^+)$, we may find a continuous path (v_t) from v to v' in $GL_k(A^+)$, that gives us an invertible $\widetilde{v}$ in $GL_k(C([0,1], A^+))$. Applying the above construction to the algebra $C([0,1], A^+)$, we associate to $\widetilde{v}$ an idempotent $\widetilde{p}$ of $M_n(C([0,1], A^+))$, i.e. a continuous function from $[0,1]$ to the idempotents of $M_n(A^+)$. The values at 0 and 1 of that function $\widetilde{p}$ are the idempotents p et p' associated to v et v' by the above reasoning, so p and p' are homotopic. Finally, p and p' are homotopic when v and v' belong to the same component. This explains why $K_0(SA)$ is equivalent to the study of connected components of $GL_n(A^+)$ (for varying n).

The usual definition of $K_1(A)$ uses the family of groups $GL_n(A)$. Let A be a unital Banach algebra. Let $GL_\infty(A)$ denote the group equal to the union of the $GL_n(A)$, $n \in \mathbb{N}$, where the injection from $GL_n(A)$ into $GL_{n+p}(A)$ is now of course given by $u \to u \oplus 1_{p,A}$. The group $K_1(A)$ is the set of connected components of $GL_\infty(A)$, that is the set of equivalence classes of $u \in GL_\infty(A)$ for the homotopy relation in $GL_\infty(A)$: we say that $u, v \in GL_\infty(A)$ are homotopic in $GL_\infty(A)$ if there exists an integer n such that $u, v \in GL_n(A)$ and u, v are homotopic in $GL_n(A)$. The product of $[u]$ and $[v]$ is the component of the product uv. This product is commutative because $[uv] = [uv \oplus 1] = [u \oplus v] = [v \oplus u] = [vu]$ by Lemma 9.1. We shall actually use the additive notation in $K_1(A)$.

For an algebra without unit we set $K_1(A) = K_1(A^+)$. We obtain a second functor from the category of Banach algebras to the category of groups, the K_1 functor. Indeed, every homomorphism from A to B induces a map from $GL_\infty(A)$ to $GL_\infty(B)$ which gives a group homomorphism from $K_1(A)$ to $K_1(B)$.

Examples 9.4.

1. Since $GL_n(\mathbb{C})$ is connected we get $K_1(\mathbb{C}) = \{0\}$. Compare to $K_0(C_0(\mathbb{T})) = K_0(S\mathbb{C})$.

2. Products and triangular matrices; it is clear that $K_1(A \times B) = K_1(A) \times K_1(B)$. It is also clear that a triangular matrix is invertible iff its diagonal elements are invertible. We already explained that an element in $M_n(T_2(A))$ can be seen as an element of $T_2(M_n(A))$; if it is invertible we may deform it inside $GL_n(T_2(A))$ to the diagonal form as explained before, but also trivially by letting the non diagonal entries go to 0. It follows that $K_1(T_2(A)) \simeq (K_1(A))^2$. For $T(Y, X)$, we see that $K_1(T(Y, X)) \simeq K_1(\mathcal{L}(Y)) \times K_1(\mathcal{L}(Z))$, where Z denotes the kernel of the projection from X to Y.

3. Show that $K_1(\mathcal{S}(X)) = \{0\}$, $K_1(\mathcal{K}(X)) = \{0\}$.

Hint. Use Proposition 6.1 and a reasoning similar to the one used for proving the connectedness of $GL_n(\mathbb{C})$ in Exercise 9.1.

4. It is known that the linear group of a Hilbert space is connected [CL], hence $K_1(\mathcal{L}(H)) = \{0\}$. This proof uses the functional calculus for isometries on a Hilbert space. We can give a more Banach space theoretic proof (which is essentially Kuiper's lemma from [Ku]), that also works for ℓ_p, when $1 \le p \le \infty$ (and for c_0). Let T be an invertible operator in $\mathcal{L}(\ell_p)$. We shall prove that $T \oplus I_{\ell_p}$ is homotopic to $I_{\ell_p} \oplus I_{\ell_p}$ in $\mathcal{L}(\ell_p \oplus \ell_p) = M_2(\mathcal{L}(\ell_p))$. Let us represent the second factor ℓ_p in the sum as $X = \ell_p(\ell_p \oplus \ell_p)$. In each component $\ell_p \oplus \ell_p$ of X, we may find an homotopy from $I_{\ell_p} \oplus I_{\ell_p}$ to $T \oplus T^{-1}$. This gives an homotopy from I_X, written symbolically as $\ell_p(I_{\ell_p} \oplus I_{\ell_p})$ to $\ell_p(T \oplus T^{-1})$. Then $T \oplus I_X$ is homotopic to $T \oplus \ell_p(T \oplus T^{-1})$; using a different grouping of the T and T^{-1} we may deform this last operator back to $I_{\ell_p} \oplus I_X$. Finally $K_1(\mathcal{L}(\ell_p)) = \{0\}$. The same proof works for any Banach space X such that $X \simeq \ell_p(X)$.

Actually, more is true: Neubauer proved that $GL(\mathcal{L}(\ell_p))$ is contractible; see also Mityagin [Mt] for more examples. To the contrary, the linear group of $\ell_p \oplus \ell_q$, $1 \le p < q < \infty$, is not connected (Douady [Do]). We shall compute later $K_1(\mathcal{L}(\ell_p, \ell_q))$.

5. Let $\alpha \in K_1(C([0,1]))$; it corresponds to an invertible $v \in GL_n(C([0,1]))$, that is to say a continuous map from $[0,1]$ to GL_n. Letting $v_s(t) = v(st)$ we define an homotopy

between v and the constant function $v(0) \in GL_n(\mathbb{C})$. Since GL_n is connected we get $K_1(C[0,1])) = \{0\}$. This argument generalizes to any contractible compact space K.

The situation is different for $C(S^1)$. In this case the determinant of $v(t)$ can make a non trivial loop around 0 in $\mathbb{C}$, therefore $K_1(C(S^1)) \neq \{0\}$; we get actually $K_1(C(S^1)) \simeq \mathbb{Z}$. This will correspond to a special case of Bott's theorem.

The index map

Let I be a closed two sided proper ideal of A. We are going to define a map ∂ from $K_1(A/I)$ to $K_0(I)$ that plays the role of the connecting map in homological theories. Recall that for every unital Banach algebra B, every invertible element in $GL^{(0)}(B/J)$ can be lifted to an invertible element in $GL^{(0)}(B)$.

In order to simplify the discussion of the index map let us assume that A is unital; let I be a proper closed two-sided ideal in A, let $i : I \to A$ be the inclusion map and $\pi : A \to A/I$ the quotient map; let us identify I^+ with the subalgebra of A consisting of all elements $\lambda 1_A + x$, $\lambda \in \mathbb{C}$ and $x \in I$. Let u be invertible in $GL_n(A/I)$. Notice that $M_n(A/I) \simeq M_n(A)/M_n(I)$, so that we can apply the above remarks to $B = M_n(A)$ and to the ideal $J = M_n(I)$ in B. We have seen that $u \oplus u^{-1}$ belongs to $GL_{2n}^{(0)}(A/I)$. We know that every element in $GL_{2n}^{(0)}(A/I)$ can be lifted to $GL_{2n}^{(0)}(A)$. Let $v \in GL_{2n}(A)$ be any lifting of $u \oplus u^{-1}$. Consider $p = v\,(1_{n,A} \oplus 0_{n,A})\,v^{-1}$. This is an idempotent and $\pi(p) = 1_{n,A/I} \oplus 0_{n,A/I}$. The matrix p is thus in $M_{2n}(I^+)$. If we set

$$\partial[u] = [p]_{I^+} - [1_{n,A}]_{I^+}$$

we get an element of $K_0(I^+)$ such that $q_*(\partial[u]) = 0$, where $q : I^+ \to I^+/I \simeq \mathbb{C}$ is the canonical quotient map; therefore we get an element of $K_0(I)$ (of course one has to show that this class only depends upon $[u]$; the notation $[p]_{I^+}$ means that this is a class computed in $K_0(I^+)$; notice that as idempotent in $M_{2n}(A)$, p is similar to $1_{n,A} \oplus 0_{n,A}$, so that

$$[p]_A - [1_{n,A}]_A = 0 \in K_0(A);$$

in other words $i_*\partial[u] = 0$). It is easy to check that ∂ is a group morphism.

Example. The right shift on $\ell_2(\mathbb{N})$.

Let R be the right shift on $\ell_2(\mathbb{N})$, and let L denote the left shift; we see that $LR = I = I_{\ell_2}$ and $RL = I - e_1 \otimes e_1$. The image $\widehat{R}$ of R in the Calkin algebra $\mathcal{C} = \mathcal{L}(\ell_2)/\mathcal{K}(\ell_2)$ is thus invertible, and the inverse is the image of L. Apply the preceding discussion to $u = \widehat{R} \in \mathcal{C}$. Let P be the rank one projection $P = e_1 \otimes e_1$. We obtain an explicit lifting of $u \oplus u^{-1}$ given by

$$v = \begin{pmatrix} R & P \\ 0 & L \end{pmatrix}, \text{ and } v^{-1} = \begin{pmatrix} L & 0 \\ P & R \end{pmatrix},$$

(Notice that $LP = PS = 0$). The idempotent p from the general discussion is now

$$p = v\,(1 \oplus 0)\,v^{-1} = \begin{pmatrix} RL & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I - P & 0 \\ 0 & 0 \end{pmatrix}.$$

According to the discussion about $K_0(\mathcal{S})$, the element $\partial\widehat{R} = [p] - [I]$ is identified to the opposite of the codimension of the image of p in the first factor ℓ_2, which gives here $\partial\widehat{R} = -1$. This value is equal to the index of R. This is not an accident:

Exercise. Generalize the above discussion to the quotient algebra $\mathcal{L}(X)/\mathcal{S}(X)$ and to an arbitrary Fredholm operator T on any Banach space X.

Exactness

The construction of the index map extends to the non unital case. Suppose given a short exact sequence

$$0 \to I \xrightarrow{i} A \xrightarrow{\pi} A/I \to 0.$$

We deduce a sequence

$$K_1(A) \xrightarrow{\pi_*} K_1(A/I) \xrightarrow{\partial} K_0(I) \xrightarrow{i_*} K_0(A) \xrightarrow{\pi_*} K_0(A/I)$$

exact at $K_1(A/I)$ and at $K_0(I)$. We shall only explain that $i_* \circ \partial$ and $\partial \circ \pi_*$ vanish. We already observed that $i_*\partial = 0$; if $w \in GL_n(A)$ and $u = \pi(w)$, we may choose as lifting $v \in GL_{2n}(A)$ of $u \oplus u^{-1}$ simply $v = w \oplus w^{-1}$, and with this choice it is clear that $\partial[u] = 0$; this reasoning shows that $\partial \circ \pi_* = 0$.

Triviality of ∂; when for each integer n, every invertible $u \in GL_n(A/I)$ can be lifted to an invertible $v \in GL_n(A)$, we see by definition of $\partial[u]$ that $\partial[u] = 0$. This is in particular true if the quotient map $\pi : A \to A/I$ is splitted by an algebra morphism σ such that $\pi \circ \sigma = Id$. In this case $\partial = 0$.

The above exact sequence is related to the discussion about $K_1(A) \simeq K_0(SA)$. Suppose that A is unital, and represent SA as the algebra of continuous functions from $[0,1]$ to A such that $f(0) = f(1) = 0_A$. Let $\widetilde{S}A$ be the algebra of continuous functions from $[0,1]$ to A such that $f(0) = 0_A$. It is clear that $\widetilde{S}A/SA \simeq A$ in a canonical way. The above exact sequence gives

$$K_1(\widetilde{S}A) \to K_1(A) \xrightarrow{\partial} K_0(SA) \to K_0(\widetilde{S}A);$$

using the homotopy $f_t(s) = f(ts)$ which moves every element in $M_n(\widetilde{S}A)$ to the zero matrix when t decreases from 1 to 0, it is easy to see that K_1 and K_0 vanish for $\widetilde{S}A$, so that ∂ gives our isomorphism between $K_1(A)$ and $K_0(SA)$.

$K_2(A)$ and Bott's periodicity theorem

We set now $K_2(A) = K_1(SA)$; then Bott's periodicity theorem states that $K_2(A) \simeq K_0(A)$. We define $K_2(A) = K_1(SA) = K_1((SA)^+)$; assume that A is unital for simplicity. We describe again SA as the algebra of continuous functions from the circle $\mathbb{T}$ to A such that $f(1) = 0_A$, and $(SA)^+$ as the algebra of continuous functions from $\mathbb{T}$ to A such that $f(1) = \lambda 1_A$ for some $\lambda \in \mathbb{C}$. We say that $a \in M_n(A)$ is "scalar" if $a = \Lambda \otimes 1_A$, where $\Lambda \in M_n$. The group $K_1((SA)^+)$ is defined using invertible elements in $M_n((SA)^+)$. But giving such an invertible is the same as giving a continuous path u from $\mathbb{T}$ into $GL_n(A)$, such that $u(1)$ is a "scalar" matrix. We can find an equivalent element v such that $v(1) = 1_{n,A}$ in the following way: let $u(1) = \Lambda \otimes 1_A$ and let (μ_t) be a continuous path in GL_n from 1_n to Λ^{-1}; then $u_t = (\mu_t \otimes 1_A)u$ is a path from u to v in $GL_n((SA)^+)$, and v satisfies $v(1) = 1_{n,A}$.

We define a map $j : K_0(A) \to K_2(A)$ which is easy to describe; the difficult part will be to show that j is onto; to each idempotent p in $M_n(A)$ we associate the "loop" $f(z) = zp + (1_{n,A} - p)$ in $GL_n(A)$, $z \in \mathbb{T}$. Then $f(1) = 1_{n,A}$, f belongs to $GL_n((SA)^+)$, and we set $j(p) = [f]$ in $K_2(A)$. We can see easily that the image of $p \oplus q$ in $K_2(A)$ is the sum of images, hence our map j is a monoid morphism from $\mathrm{Pr}(A)$ to $K_2(A)$, hence gives a group morphism still denoted j from $K_0(A)$ to $K_2(A)$. As we said, the difficult part in the proof of Bott's theorem is the fact that j is onto from $K_0(A)$ to $K_2(A)$; the fact that j is injective from $K_0(A)$ into $K_2(A)$ is not obvious but will follow from the fact that j is onto for every Banach algebra. Let us indicate the main steps of the proof that j is onto. We first observe that the path $z^k 1_{n,A}$ of invertible defines the same class as $z1_{kn,A}$ in $K_2(A)$. Multiplying a path of invertible by $z^k 1_{n,A}$ amounts thus to add the loop $j(1_{kn,A})$ associated to the idempotent $1_{kn,A}$.

Let $\alpha \in K_2(A)$. We can find a continuous map w from $\mathbb{T}$ to $GL_n(A)$ such that $[w] = \alpha$, and we may assume that $w(1) = 1_{n,A}$. We want to find an element β in $K_0(A)$ such that $j(\beta) = [w] = \alpha$.

1. We begin by approximating our continuous map w from $\mathbb{T}$ to $GL_n(A)$ by a map $z \to w'(z)$ from $\mathbb{T}$ to $GL_n(A)$ which is a trigonometric polynomial $w'(z) = \sum_{k=-N}^{N} v_k z^k$, with each $v_k \in M_n(A)$, and such that $w'(1) = 1_{n,A}$; the standard way is to use convolution with a Fejer kernel; we have $[w'] = [w]$ if the approximation is good enough; multiplying by z^N we get a polynomial $u(z) = z^N w'(z)$ in $z \in \mathbb{T}$ with coefficients in $M_n(A)$. With this multiplication we have added to $[w]$ the class of the loop associated to $1_{nN,A}$, that we should substract at the end. We are going to find an idempotent p in $M_\infty(A)$ such that $j(\{p\}) = [u]$. It will follow that $j([p] - [1_{nN,A}]) = [w] = \alpha$, thus proving that j is onto.

2. Passing to a larger dimension K we may find an equivalent path u_1 that is linear in z, namely $u_1(z) = a + bz$, $a, b \in M_K(A)$ and $u_1(1) = 1_{K,A}$. We may describe this (in a way that is not the cheapest on dimensions) in the following way: for every polynomial $P(z)$ with coefficients in a unital Banach algebra B and such that $\deg P \le m$, we may write

$$P(z) = (z-1)^2 Q(z) + R(z),$$

where $\deg R \le 1$ and $\deg Q \le m-2$. Let for every $\lambda \in [0,1]$

$$\varphi_\lambda(z) = \begin{pmatrix} 1_B & \lambda(z-1)1_B \\ 0 & 1_B \end{pmatrix} \begin{pmatrix} P(z) & 0 \\ 0 & 1_B \end{pmatrix} \begin{pmatrix} 1_B & 0 \\ -\lambda Q(z) & 1_B \end{pmatrix} =$$

$$\begin{pmatrix} P(z) - \lambda^2 (z-1)^2 Q(z) & \lambda(z-1)1_B \\ -\lambda(z-1)Q(z) & 1_B \end{pmatrix}.$$

For every λ, we get that $z \to \varphi_\lambda(z)$ is a path of invertible and $\varphi_\lambda(1) = 1_{2,B}$, $\varphi_0 = P(z) \oplus 1_B$. When $\lambda = 1$, the result is a path of invertible elements with degree $\le m-1$ in z,

$$\varphi_1 z) = \begin{pmatrix} R(z) & (z-1)1_B \\ -(z-1)Q(z) & 1_B \end{pmatrix}.$$

When λ varies from 0 to 1 we get an homotopy in $GL_2((SB)^+)$ between $P(z) \oplus 1_B$ and this path $\varphi_1(z)$ of degree $\leq m-1$ in z.

3. Using spectral theory one finally shows that a linear path of invertible elements is equivalent to a loop $z \to zp + (1-p)$ where p is an idempotent. We shall prove a crucial Lemma. We recall the notion of spectral projection. Let B be a unital Banach algebra. Assume that the spectrum of $b \in B$ does not meet the imaginary axis in $\mathbb{C}$. It is then possible to find a circle γ centered at some real point M, with $M > 0$ large, and with radius $M - \varepsilon$, $\varepsilon > 0$ small, such that every $\lambda \in \sigma(b)$ with $\operatorname{Re}\lambda > 0$ will be contained in the interior of γ. Let

$$p = \frac{1}{2\pi i} \int_\gamma (z-b)^{-1}\, dz.$$

Then p is a spectral projection commuting with b.

Lemma 9.3. *Let B be a unital Banach algebra. Assume that $b \in B$ is such that $b - it1_B$ is invertible for every $t \in \mathbb{R}$, and let p be the above spectral projection corresponding to the half complex plane $C_+ = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. Then $b + sp$ is invertible for every real number $s \geq 0$. In other words: if $\sigma(b)$ does not meet the imaginary axis $i\mathbb{R}$, the same is true for $\sigma(b+sp)$ for every real $s \geq 0$.*

Proof. Let $B_p = pBp$. This is a Banach algebra with unit p. We know that $bp = pb = pbp$ and the spectrum of $b_p = pbp$ in B_p is contained in $\mathbb{C}_+$. It follows that $b_p + sp$ is invertible in B_p for every real $s \geq 0$, hence there exists $u \in B$ such that

$$(pbp + sp)pup = pup(pbp + sp) = p.$$

Let $q = 1_B - p$. The spectrum of qbq in B_q is contained in $\mathbb{C}_-$, hence b_q is invertible in B_q and there exists $v \in B$ such that

$$(qbq)qvq = qvq(qbq) = q.$$

Now

$$(b+sp)(pup+qvq) = (b+sp)(p+q)(pup+qvq) =$$

$$(bp+sp)(pup) + bq(qvq) = (pbp+sp)(pup) + (qbq)(qvq) = p + q = 1_B$$

and similarly for the other direction, $(pup+qvq)(b+sp) = 1_B$.

For the sake of completeness, let us give a sketch of a proof that the spectrum of b_p is contained in $\mathbb{C}_+$. First of all, it is clear that the spectrum of b_p in B_p is contained in the spectrum of b in B (if $(b-\lambda 1_B)u = u(b-\lambda 1_B) = 1_B$, then u commutes with p and $(pbp - \lambda p)(pup) = (pup)(pbp - \lambda p) = p$). If the spectrum of b_p contains elements $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \leq 0$, then the boundary of the spectrum will also contain such λ. Then there exists by Remark ?? a norm one $d = pcp \in B_p$ such that $(b_p - \lambda p)d \sim 0$. But in B_p,

$$p = p^2 = \frac{1}{2\pi i} \int_\gamma (zp - b_p)^{-1}\, dz.$$

We have $b_p d \sim \lambda d$, thus $(zp - b_p)d \sim (z-\lambda)d$ and $(zp-b_p)^{-1}d \sim (z-\lambda)^{-1}d$, therefore

$$d = pd \sim \frac{1}{2\pi i}\Big(\int_\gamma (z-\lambda)^{-1}\,dz\Big)d = 0.$$

This contradicts $\|d\| = 1$.

Corollary. If $b \in B$ is as in the Lemma, there exists an homotopy (b_s) in $GL(B)$ from b to p such that for every s, $b_s + it1_B$ is invertible for every $t \in \mathbb{R}$.

Proof. For s real varying from 1 to $+\infty$ we set $b_s = (1+s)^{-1}(b+sp)$.

Let us come back to the proof of the third step. We have a linear path $u_1(z) = a + zb$ in $GL_K(A)$, with $u_1(1) = 1_{K,A}$, thus $a+b = 1_{K,A}$. We see that $1_{K,A} + (z-1)b$ is invertible for every $z \in \mathbb{T}$; this yields that the spectrum of b in $M_K(A)$ does not meet the line $\operatorname{Re}\lambda = 1/2$. Applying the preceding Corollary with $B = M_K(A)$ to $b - (1/2)1_{K,A}$, we get an homotopy (b_s) from b to an idempotent p, such that $b_s + \lambda 1_{K,A}$ is invertible for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda = 1/2$. It follows that for every s, the path $1_{K,A} + (z-1)b_s$, for $z \in \mathbb{T}$, is a path of invertible. We have thus found a deformation in $GL_K((SA)^+)$ from the original path $1_{K,A} + (z-1)b$ to a loop $1_{K,A} + (z-1)p$ associated to an idempotent p, as was to be proved.

This proof of the third step is nicer with the full strengh of the functional calculus. Let us find two circles γ_1 and γ_2 containing the spectrum of b, where γ_1 is contained in $\{\operatorname{Re}\zeta < 1/2\}$ and γ_2 in $\{\operatorname{Re}\zeta > 1/2\}$. We have

$$b = \frac{1}{2i\pi}\int_{\gamma_1}\frac{z}{z-b}\,dz + \frac{1}{2i\pi}\int_{\gamma_2}\frac{z}{z-b}\,dz.$$

What we do is to find a deformation $\varphi_1(z,t)$, $t \in [0,1]$, of the identity function $z \to z$ to $z \to 0$ at the left of the line $\operatorname{Re}\zeta = 1/2$ and a deformation $\varphi_2(z,t)$ from $z \to z$ to $z \to 1$ at the right of the same line in such a way that $\varphi_1(z,t)$ and $\varphi_2(z,t)$ never meet the line $\operatorname{Re}\zeta = 1/2$. There is an easy choice: $\varphi_1(z,t) = (1-t)z$ if $z \in \gamma_1$ and $\varphi_2(z,t) = (1-t)z + t$ for $z \in \gamma_2$. We obtain a continuous path

$$b_t = \frac{1}{2i\pi}\int_{\gamma_1}\frac{\varphi_1(z,t)}{z-b}\,dz + \frac{1}{2i\pi}\int_{\gamma_2}\frac{\varphi_2(z,t)}{z-b}\,dz$$

of elements such that $1_{K,A} + (z-1)b_t$ is invertible for every $z \in \mathbb{T}$, with $b_0 = b$ and $b_1 = p$.

Finally, let us explain why $j : K_0(A) \to K_2(A)$ is injective; suppose that $\alpha = [p] - [q]$ and $j(\alpha) = 0$; this means that the two loops $zp + (1-p)$ and $zq + (1-q)$, $z \in \mathbb{T}$, are homotopic in $GL_\infty((SA)^+)$. We can therefore construct a continuous map $\varphi(t,z)$, $t \in [0,1]$, such that $\varphi(0,z) = zp + (1-p)$ and $\varphi 9!, z) = zq + (1-q)$; we may consider this as a map from $\mathbb{T}$ to the space of continuous maps from $[0,1]$ to $GL(A)$. More precisely, the values at 0 and 1 belong respectively to the unital subalgebras C_p and C_q generated by p and q (thus C_p is the subalgebra of elements $\lambda 1_A + \mu p$, and similarly for C_q). Let B be the algebra of continuous functions from $[0,1]$ to A such that $f(0) \in C_p$ and $f(1) \in C_q$; assume for simplicity that p and q are different from 1_A; the two preceding subalgebras are then isomorphic to $\mathbb{C}^2$. Since the map j is onto for the algebra B, we may deform φ to a loop $zP + (1 - P)$, where P is an idempotent in $M_\infty(B)$, i.e. a continuous map from $[0, 1]$ to $M_\infty(A)$ such that $zP(0) + (1 - P(0))$ is homotopic to $zp + (1 - p)$ and similarly for $t = 1$. Since the subalgebra is isomorphic to $\mathbb{C}^2$, this implies that the ranges of $P(0)$ and p have equal dimensions (use determinant??), and are therefore equivalent as projectors; similarly for $P(1)$ and q; finally, $P(t)$ is a continuous path of idempotents from $[p]$ to $[q]$, and therefore $\alpha = [p] - [q] = 0$.

Examples 9.5.

1. $K_1(C(S^1))$ corresponds to $K_2(\mathbb{C})$. Indeed, $K_2(\mathbb{C})$ is $K_1((S\mathbb{C})^+)$, and $(S\mathbb{C})^+$ identifies with $C(S^1)$. It follows from Bott's theorem that $K_1(C(S^1)) = K_2(\mathbb{C}) \simeq K_0(\mathbb{C}) \simeq \mathbb{Z}$. We get a generator of $K_1(C(S^1))$ by considering a continuous map $v(t)$ from $\mathbb{T}$ to GL_n such that the determinant of $v(t)$ makes a loop around 0 in $\mathbb{C}$ with index 1.

2. $C(S^2)$. The suspension $S\mathbb{C}$ identifies to the space of continuous functions on $[0, 1]$, vanishing at 0 and 1. Then $SS\mathbb{C}$ identifies to continuous functions on the square $[0, 1]^2$, vanishing on the boundary. But this algebra can also be identified to continuous functions on S^2, vanishing at a given point (here, the point obtained by identifying all points on the boundary to a single point). Therefore, $K_0(C_0(S^2)) \simeq K_2(\mathbb{C}) \simeq K_0(\mathbb{C}) \simeq \mathbb{Z}$. This is a fundamental example, and it is possible to deduce the general case from it by a tensor product technique.

With this Bott's isomorphism we get a new connecting map ∂_0 from $K_0(A/I) \simeq K_2(A/I) = K_1(S(A/I))$ to $K_1(I) = K_0(SI)$, (observe that $S(A/I) \simeq SA/SI$) and a new exact sequence. Call now ∂_1 the connecting map that was defined earlier. Suppose given a short exact sequence

$$0 \to I \xrightarrow{i} A \xrightarrow{\pi} A/I \to 0.$$

We obtain a cyclic exact sequence with period 6

$$K_1(A/I) \xrightarrow{\partial_1} K_0(I) \xrightarrow{i_*} K_0(A) \xrightarrow{\pi_*} K_0(A/I) \xrightarrow{\partial_0} K_1(I) \xrightarrow{i_*} K_1(A) \xrightarrow{\pi_*} K_1(A/I) \xrightarrow{\partial_1} \dots.$$

Some examples with the cyclic exact sequence

1. Let $A = \mathcal{L}(\ell_p)$ and $I = \mathcal{K}(\ell_p)$, $A/I = \mathcal{C}_p = \mathcal{C}(\ell_p)$. We know that $K_1(I) = \{0\}$, $K_0(I) \simeq \mathbb{Z}$, $K_1(A) = \{0\}$ (see Example 9.4, 4) et $K_0(A) = \{0\}$ because ℓ_p is prime. We get

$$0 \xrightarrow{\pi_*} K_1(A/I) \xrightarrow{\partial_1} \mathbb{Z} \xrightarrow{i_*} 0 \xrightarrow{\pi_*} K_0(A/I) \xrightarrow{\partial_0} 0,$$

thus $K_1(\mathcal{C}_p) \simeq \mathbb{Z}$, $K_0(\mathcal{C}_p) = \{0\}$.

When $X \simeq \ell_p(X)$, we have $K_1(\mathcal{L}(X)) = \{0\}$ by Example 9.4, 4, and we know that $i_*(1) = 0$ because X is isomorphic to its hyperplanes (Example 9.2, 6); this gives

$$K_0(\mathcal{L}(X)) \simeq K_0(\mathcal{C}(X)); \quad K_1(\mathcal{L}(X)) \simeq \mathbb{Z}.$$

2. Let now $A = \mathcal{L}(\ell_p \oplus \ell_q)$. Assume $1 \le p < q < \infty$. Then every operator from ℓ_q to ℓ_p is compact (Pitt's Theorem, see [LT1], Proposition 2.c.3). This implies that the Calkin

algebra $\mathcal{C}_{p,q} = \mathcal{C}(\ell_p \oplus \ell_q)$ is triangular, hence $K_0(\mathcal{C}_{p,q}) \simeq K_0(\mathcal{C}_p) \times K_0(\mathcal{C}_q) = \{0\}$. Also, $K_1(\mathcal{C}_{p,q}) \simeq K_1(\mathcal{C}_p) \times K_1(\mathcal{C}_q) = \mathbb{Z} \times \mathbb{Z}$. We get using $\mathbb{Z} \simeq K_0(\mathcal{K}_{p,q})$

$$0 \xrightarrow{i_*} K_1(A) \xrightarrow{\pi_*} K_1(\mathcal{C}_{p,q}) \xrightarrow{\partial_1} \mathbb{Z} \xrightarrow{i_*} K_0(A) \xrightarrow{\pi_*} K_0(\mathcal{C}_{p,q}),$$

this gives

$$0 \xrightarrow{i_*} K_1(A) \xrightarrow{\pi_*} \mathbb{Z} \times \mathbb{Z} \xrightarrow{\partial_1} \mathbb{Z} \xrightarrow{i_*} K_0(A) \xrightarrow{\pi_*} 0.$$

The map ∂_1 from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z}$ is simply the sum $\partial_1(n, m) = n + m$. Finally $K_1(A)$ is isomorphic to the kernel of this sum map, thus isomorphic to $\mathbb{Z}$. Since this ∂_1 is onto, we get that $i_* = 0$, but i_* is also onto and thus $K_0(A) = \{0\}$ (for this last statement one can use directly Edelstein-Wojtaszczyk [EW] who say that every projection on $\ell_p \oplus \ell_q$ is equivalent to a direct sum of projections in ℓ_p and ℓ_q).

3. Cuntz algebras. We want to show that $[1_{\mathcal{O}_n}] \neq 0$ for the Cuntz algebra $\mathcal{O}_n$ when $n \geq 3$. To this end Cuntz introduces the auxiliary algebra $\mathcal{E}_n$ generated by n into isometries $V_1, \ldots, V_n$ such that $\sum_{i=1}^n V_i V_i^* < 1$. The projection $Q = 1 - \sum V_i V_i^*$ generates an ideal I isomorphic to $\mathcal{K}$, with Q playing the role of a rank one projection in $\mathcal{K}$, and $\mathcal{O}_n \simeq \mathcal{E}_n / I$ by the uniqueness of $\mathcal{O}_n$; we obtain

$$K_1(\mathcal{O}_n) \xrightarrow{\partial_1} K_0(I) \simeq \mathbb{Z} \xrightarrow{i_*} K_0(\mathcal{E}_n) \xrightarrow{\pi_*} K_0(\mathcal{O}_n) \xrightarrow{\partial_0} K_1(I) = 0.$$

One can show that $\partial_1 = 0$ (for example by the reasoning of Corollary 12.1). Since $i_*(1)$ is the class of rank one projections in $\mathcal{K}$ we have $i_*(1) = [Q]$, and since $\partial_1 = 0$ we know that i_* is injective, thus $[Q]$ generates a subgroup of $K_0(\mathcal{E}_n)$ isomorphic to $\mathbb{Z}$. If $[1_{\mathcal{O}_n}] = 0$, we must have $[1_{\mathcal{E}_n}] = m[Q]$ for some $m \in \mathbb{Z}$ by exactness. On the other hand (Example 9.3, 10),

$$n[1_{\mathcal{E}_n}] + [Q] = [1_{\mathcal{E}_n}],$$

hence $(m(n-1)+1)[Q] = 0$, which is impossible since $n - 1 \geq 2$ and since $[Q]$ generates a group isomorphic to $\mathbb{Z}$. Cuntz proved in [C2] that $K_0(\mathcal{O}_n) = \mathbb{Z}/(n-1)\mathbb{Z}$. This is significantly more difficult than the above remark (see also Pimsner-Voiculescu).

10. Exotic Banach spaces

H.I. spaces

Definition 10.1. Let X be an infinite dimensional Banach space, real or complex. We say that X is *Indecomposable* if X cannot be written as the topological direct sum of two infinite dimensional closed subspaces Y_1 and Y_2. We say that X is *Hereditarily Indecomposable* (in short, H.I.) if every closed infinite dimensional subspace Y of X is indecomposable, that is if no subspace Y of X can be written as the topological direct sum of two infinite dimensional closed subspaces Y_1 and Y_2 of X.

Obviously, if X is H.I. then every (infinite dimensional) subspace $Y \subset X$ is H.I.

Exercise.

1. A Banach space X is H.I. iff for all infinite dimensional subspaces Y and Z of X, we have

$$\inf\{\|y - z\| : y \in Y, z \in Z, \|y\| = \|z\| = 1\} = 0.$$

In other words, X is H.I. when the "angle" between any two (infinite dimensional) subspaces of X is equal to 0.

2. X is H.I. iff for every (infinite dimensional) subspace $Y \subset X$, the quotient map $\pi_Y : X \to X/Y$ is strictly singular.

Fact [GM1]. There exist H.I. Banach spaces.

Actually the first example in [GM1] of a H.I. space was also the first example of an indecomposable space. The existence of an indecomposable space answered a question of Lindenstrauss [L2]. This example in [GM1] is a special case of the construction presented in section 11.

When X is a H.I. Banach space, then X contains no (infinite) Unconditional Basic Sequence (UBS in short). This is clear, because a space with unconditional basis is easily decomposable, for example into the two subspaces generated by basis vectors with odd indices and even indices. Tim Gowers and the author of these Notes were actually looking for an example of a space without UBS. The example turned out to have the stronger property of being H.I. (this was observed by W.B. Johnson). This was not totally accidental (although not deliberate), as Gowers' dichotomy theorem will explain (see Theorem 10.2 below).

Since every Banach space contains a subspace with basis, it is formal from the existence of any H.I. space that there exist H.I. spaces with monotone basis; actually the example in [GM1] is also reflexive; much more difficult is another example due to Gowers [G2] of a H.I. space without any reflexive subspace. Being H.I. this last example does not contain c_0 or ℓ_1, solving another longstanding conjecture in Banach space theory (see [LT, ???]).

Ferenczi [F1] has constructed an example of uniformly convex H.I. space. The proof adds to the ideas of the construction of [GM1] the notion of complex interpolation for families of Banach spaces developed by Coifman, Cwikel, Rochberg, Sagher and Weiss in [CW]. Argyros-Delyanni [AD] constructed asymptotically ℓ_1 H.I. spaces, using a technique closer to the original Tsirelson example [T] instead of working with a modification of Schlumprecht's example as it is done in [GM1] or [F1].

Kalton [K] has constructed an example of a quasi-Banach space X with the very strange property that there is a vector $x \neq 0$ such that every closed infinite dimensional subspace of X contains x. It follows that this quasi-Banach space does not contain any infinite basic sequence. This example is related to an example of Gowers [G1] of a space with unconditional basis not isomorphic to its hyperplanes; Kalton's construction uses the technique of *twisted sums* together with the properties of the space in [G1].

"Germs" of H.I. spaces; in some sense all subspaces of a H.I. space intersect; we may define a net of subspaces that captures a good part of the structure of a H.I. space; the order of this net is not the inclusion, as it is not true that any two infinite dimensional subspaces have an infinite dimensional intersection, but almost.... We say that $Y \leq Z$ if there exists a compact operator $K : Y \to X$ such that $i_{Y,X} + K)(Y) \subset Z$. Given $Z_1, Z_2 \subset X$ there exists Y such that $Y \leq Z_1$ and $Y \leq Z_2$. We could call "germ" of H.I. space an equivalence class of such nets, in a way to be made precise. An interesting class of examples is the family of spaces containing no UBS but having only a finite set of germs of H.I. spaces. The example of Gowers [G2] of a non reflexive H.I. space is mainly an example of non reflexive germ. See also Remark 10.1 below.

Spectral theory and consequences

Theorem 10.1. *Let X be a complex H.I. space. Then every $T \in \mathcal{L}(X)$ can be written as $T = \lambda I_X + S$, where $\lambda \in \mathbb{C}$ and S is strictly singular. Thus every $T \in \mathcal{L}(X)$ is either strictly singular or Fredholm with index 0. Furthermore, the spectrum of T is either finite, or consists of a sequence (λ_n) converging to λ. In this second case, each $\lambda_n \neq \lambda$ is an eigenvalue of T with finite multiplicity.*

Proof. Let $T \in \mathcal{L}(X)$. We know by Lemma 5.4 or Corollary 5.2 that there exists $\lambda \in \mathbb{C}$ such that $T - \lambda I_X$ is infinitely singular. Let $U = T - \lambda I_X$. For every $\varepsilon > 0$ there exists by Proposition 3.2 an infinite dimensional subspace $Y_\varepsilon \subset X$ such that $\|U_{|Y_\varepsilon}\| < \varepsilon$. Now let Z be any infinite dimensional subspace of X. Since X is H.I., we can find a vector $z \in Z$ such that $\|z\| = 1$ and $\operatorname{dist}(z, S_{Y_\varepsilon}) < \varepsilon$. It follows that $\|Uz\| < (1 + \|U\|)\varepsilon$, showing that $U = T - \lambda I_X$ is strictly singular. The rest is given by Proposition 6.1.

Remark 10.1. Ferenczi [F2] has shown that, given a complex H.I. space X and a bounded linear operator T from a subspace Y of X to X, one can write $T = \lambda i_{Y,X} + S$, where $\lambda \in \mathbb{C}$, S is strictly singular and $i_{Y,X}$ denotes the inclusion map from Y to X. This property was shown to be true for the specific H.I. example in [GM1]. Conversely, it is easy to see that any Banach space X with the property that for every subspace Y, every $T \in \mathcal{L}(Y, X)$ can be written as $\lambda i_{Y,X} + S$ is a H.I. space, so that the above result is a characterization of complex H.I. spaces. Ferenczi's proof consists essentially to show that the space of "germs" of operators on X is a Banach field, hence isomorphic to $\mathbb{C}$. In the case of a real H.I. Banach space X, his proof shows that the quotient $\mathcal{L}(X)/\mathcal{S}(X)$ is a division ring, hence isomorphic to $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. A germ of operator is an equivalence class for the relation where $T_1 \in \mathcal{L}(Z_1, X)$ and $T_2 \in \mathcal{L}(Z_2, X)$ are equivalent if there exists $Y \leq Z_1, Z_2$ such that $T_1 \circ (i_{Y,X} + K_1) - T_2 \circ (i_{Y,X} + K_2)$ is compact on Y, and K_1, K_2 are the compact operators from the definition of the order.

Exercise 10.1.

1. Operators on real H.I. spaces. Let X be a real H.I. space and let $T \in \mathcal{L}(X)$. Either there exists $\lambda \in \mathbb{R}$ such that $T - \lambda I_X$ is strictly singular, or there exists $\lambda \in \mathbb{C} \setminus \mathbb{R}$ such that $T^2 - 2 \operatorname{Re} \lambda\, T + \lambda \bar{\lambda} I_X$ is strictly singular. Check that T is either strictly singular or Fredholm with index 0. The spectrum of the complexified operator $T_{\mathbb{C}}$ is invariant under complex conjugation, and the part of the spectrum contained in the upper half plane is finite or consists of a convergent sequence with its limit.

2. Operators on X^n. If X is a complex H.I. space and if $T \in \mathcal{L}(X^n)$, there exists a matrix $\Lambda \in M_n(\mathbb{C})$ such that $T = \Lambda \otimes I_X + S$, with S strictly singular on X^n.

3. If X is H.I. then $X^n \not\simeq X^m$ when $m \neq n$.

4. If X is a complex H.I. space, then $K_1(\mathcal{L}(X)) = \{0\}$. Also, $K_0(\mathcal{L}(X)) \neq \{0\}$.

The hyperplane problem

Corollary 10.1. *Let X be a H.I. space, real or complex. Then X is not isomorphic to any proper subspace. In particular, X is not isomorphic to its hyperplanes.*

The first example of a space not isomorphic to its hyperplanes appeared in [G1].

Proof. Let T be an isomorphism from X into itself; then T is not strictly singular, hence it must be Fredholm with index 0 by Theorem 10.1 or Exercise 10.1 and thus $TX = X$.

Exercise.

1. If X is H.I. and $Z \subset Y \subset X$, $Z \neq Y$, then Z and Y are not isomorphic.

2. Show that an H.I. space X is not isomorphic to any quotient X/Y.

Hint. Use the fact that the spectrum of every operator on X is countable. Let $T : X/Y \to X$ be an isomorphism, consider $T \circ \pi_Y$, then its adjoint and the spectrum of the adjoint.

3. Homotopy of subspaces. Two subspaces Y and Z of a complex H.I. space X are isomorphic if and only if there exists an homotopy in $\mathcal{L}(Y, X)$ from $i_{Y,X}$ to T consisting of into isomorphisms from Y to X, where $i_{Y,X}$ denotes the injection from Y to X and T is an into isomorphism from Y to X such that $TY = Z$.

Ferenczi showed that the dual of the example in [GM1] is also H.I. and even that every quotient of this space is still H.I. This question is not at all clarified in general. What is clear is that the dual of a reflexive indecomposable space (not necessarily *hereditarily* indecomposable) is indecomposable; therefore, if every quotient of a reflexive space is H.I., then every subspace of a quotient is indecomposable and this property clearly passes to the dual. However Ferenczi gave an example of a H.I. space such that the dual is not H.I.

Gowers' dichotomy theorem and homogeneous Banach spaces

Recall that a Banach space X is said *homogeneous* if X is isomorphic to all its infinite dimensional closed subspaces. What can we say about a homogeneous Banach space? Since every Banach space contains a subspace with a basis, X must have a basis. Furthermore, every subspace will also have a basis. It follows from the work of Enflo on the approximation property, extended by Szankowski, that every Banach space with this property must have type $2 - \varepsilon$ and cotype $2 + \varepsilon$ for every $\varepsilon > 0$ ([LT2], Theorem 1.g.6). These results have been obtained in the '70s; more recently, Komorowski and Tomczak proved in [KT] the following result:

Theorem. Let X be a Banach space with finite cotype and not containing any subspace isomorphic to ℓ_2. Then there exists a subspace Y of X without unconditional basis.

The proof of [KT] is rather difficult and complicated, and uses techniques different from those of these Notes.

Corollary. [KT] If X is a homogeneous Banach space not isomorphic to ℓ_2, then X does not contain any UBS.

Proof. We said that X must have finite cotype if it is homogeneous. If X is not isomorphic to ℓ_2, we know by the Theorem above that X contains a subspace Y without unconditional basis. Since X is homogeneous, it follows that X has no subspace with unconditional basis.

It was then very tempting to try to relate the fact that a space does not contain any UBS to the H.I. property. This was done by T. Gowers in a beautiful "dichotomy Theorem"; Gowers obtains more general combinatorial statements (in [G4] and [G5]) that we shall not give here, and which are somewhat analogous to the infinite versions of

Ramsey's Theorem; he then deduce the result about UBS and H.I. from them. We shall only prove the particular case which is needed here.

Theorem 10.2. (Gowers' dichotomy theorem, [G4], [G5]). *Let X be an arbitrary (infinite dimensional) Banach space. Either X contains an infinite unconditional basic sequence, or X contains a H.I. subspace.*

This result gives a very good reason for introducing this notion of H.I. spaces. If one is interested to know whether a general Banach space contains a subspace with unconditional basis, one has to encounter H.I. spaces some day.

Proof of Theorem 10.2. For the proof we need a more quantitative result; let $\varepsilon > 0$; we shall say that a Banach space X is HI(ε) when for all subspaces Y and Z of X, there exist two vectors $y \in Y$, $z \in Z$ such that

$$\|y - z\| < \varepsilon \|y + z\|.$$

Setting $\lambda = \|(y+z)/2\| > 0$ and $y' = y/\lambda$, $z' = z/\lambda$ we see that $1 - \varepsilon < \|y'\|, \|z'\| < 1 + \varepsilon$ and $\|y' - z'\| < 2\varepsilon$. It is thus clear that X is H.I. if and only if it is HI(ε) for every $\varepsilon > 0$.

Lemma 10.1. *Let X be a Banach space. For every $\varepsilon > 0$, either X contains an UBS with constant $2/\varepsilon$, or X contains an HI(2ε) subspace Z.*

Proof (from [M2]). The approach is combinatorial. We shall need to discretize the problem to make the situation countable, and even finite later on. We shall restrict now to the real case. Let us choose in X a normalized basic sequence $(x_n)_{n \ge 1}$ with constant 2 (say) and denote by X_0 the $\mathbb{Q}$-vector subspace generated by this sequence. This space X_0 is countable and infinite dimensional (over $\mathbb{Q}$); furthermore, for every infinite dimensional $\mathbb{Q}$-vector subspace Y of X_0, the closure $\overline{Y}$ in X is an infinite dimensional Banach space over $\mathbb{R}$ or $\mathbb{C}$.

From now on, in the proof of the Lemma, the notation Y, Z, or U, V, W will be used for infinite dimensional $\mathbb{Q}$-vector subspaces of X_0. Let us consider the set

$$A = \{(x, y) \in X_0 \times X_0; \|x - y\| < \varepsilon \|x + y\|\}.$$

This set A is countable and symmetric. We introduce a convenient terminology, inspired by [GP]. Let (x, y) be a couple of vectors in X_0 and let Z be an infinite dimensional subspace of X_0. We say that (x, y) *accepts* Z if for all subspaces U, V of Z there exists $(u, v) \in U \times V$ such that $(x + u, y + v) \in A$. Since A is symmetric, acceptation is also symmetric. Observe also that if $(x, y) \in A$, then (x, y) accepts every subspace Z: just take $u = 0$ and $v = 0$.

We say that (x, y) *rejects* Z if no subspace $Z' \subset Z$ is accepted by (x, y). Rejection is also symmetric, and saying that (x, y) rejects some subspace Z implies that $(x, y) \notin A$. Observe that when (x, y) accepts or rejects a subspace Z, it remains true for every subspace Z' of Z, and it is also true for "supspaces" of Z of the form $Z + F$, when F is finite dimensional; combining these two observations, we see that when (x, y) accepts or rejects Z, the same is true for every Z' such that $Z' \subset Z + F$, when F is finite dimensional. This simple remark is the basis for our first step:

Claim: there exists an infinite dimensional subspace Z_0 of X_0 such that for every couple $(x, y) \in X_0 \times X_0$, either (x, y) accepts Z_0 or (x, y) rejects Z_0.

We use for this a very usual diagonal argument; since X_0 is countable, we may form the list $(x_n, y_n)_{n \geq 1}$ of all elements of $X_0 \times X_0$. We then construct a decreasing sequence $(X_n)_n$ of subspaces in the following way: if (x_{n+1}, y_{n+1}) rejects X_n, we simply let $X_{n+1} = X_n$. Otherwise, there exists a subspace X_{n+1} of X_n such that (x_{n+1}, y_{n+1}) accepts X_{n+1}. We consider then a diagonal subspace Z_0 built by taking one vector in each X_n, in such a way that Z_0 is infinite dimensional.

From now on the whole construction will be performed inside our "stabilizing" subspace Z_0. For every couple (x, y) in $Z_0 \times Z_0$, (x, y) accepts ou rejects, and we don't need anymore to specify "accepts or rejects a subspace $Z' \subset Z_0$".

There are two possibilities: either the couple $(0, 0)$ accepts, or it rejects. If $(0, 0)$ accepts, we see that the Banach space $Z = \overline{Z_0}$ is HI$(\varepsilon + \varepsilon')$ for every $\varepsilon' > 0$, in particular HI(2ε). Indeed, if U and V are two subspaces of Z, we may approximate them by two $\mathbb{Q}$-subspaces U' and V' of Z_0. Since $(0, 0)$ accepts, there exist u' and v' in U' and V' such that $(u', v') \in A$ which gives by approximation two vectors $u \in U$ and $v \in V$ such that $\|u - v\| < (\varepsilon + \varepsilon')\|u + v\|$.

Suppose now that $(0, 0)$ rejects; we will find in Z_0 an unconditional sequence $(e_k)_{k \geq 1}$ with constant $2/\varepsilon$, namely such that

$$\|\sum_k b_k e_k\| \leq \frac{2}{\varepsilon} \|\sum_k \eta_k b_k e_k\|$$

for all scalars $(b_k)_k$ and every choice of signs $(\eta_k)_k$ (signs appear usually on the other side of the inequality, but it is clearly equivalent to put them on the right). In order to deal with this in a combinatorial manner, we discretize our problem as follows: it is easy to see that we only need to make sure that $1 < \|e_k\| < 2$ for every integer $k \geq 1$ and that

$$(*) \qquad \|\sum_{k=1}^{K} a_k e_k\| \leq \frac{1}{\varepsilon} \|\sum_{k=1}^{K} \eta_k a_k e_k\|$$

for every integer $K \geq 1$, all choices of signs $(\eta_k)_{k=1}^K$ and all scalars $(a_k)_{k=1}^K$ taking the values $a_k = j/(N2^k)$, $j = -N2^k, \ldots, N2^k$, where N is an integer larger than $16/\varepsilon$. We call *reasonable* such a choice of scalars $(a_k)_{k=1}^K$, and *reasonable combination* (of length K) a linear combination of the form $\sum_{k=1}^K a_k e_k$. Relation $(*)$ means that $\|x - y\| \geq \varepsilon\|x + y\|$ whenever $x = \sum_{k \in I} a_k e_k$ and $y = \sum_{k \in J} a_k e_k$, where the coefficients are reasonable and (I, J) is the partition of $\{1, \ldots, K\}$ corresponding to the signs $(\eta_k)_{k=1}^K$. We call such a couple (x, y) a *partition of a reasonable combination*. In other words, we want to make sure that $(x, y) \notin A$ whenever (x, y) is a partition of a reasonable combination $\sum_{k=1}^K a_k e_k$ with arbitrary length K. As we observed, it is enough to know that every partition of a reasonable combination rejects.

Sublemma: if (x, y) rejects, then for every infinite dimensional subspace W of Z_0 there exists a further subspace $W' \subset W$ such that for every $w' \in W'$, the couple $(x + w', y)$ rejects.

(Otherwise, for every subspace $U \subset W$, there would exist $u_0 \in U$ such that $(x + u_0, y)$ accepts; then, for every subspace $V \subset W$ we could find a couple (u_1, v) in $U \times V$ such that $(x + u_0 + u_1, y + v) \in A$, which implies that (x, y) accepts, contrary to the initial hypothesis.)

Let us finish the proof of Lemma 10.1. Assuming that $(0, 0)$ rejects, we build by induction a sequence $(e_k)_{k=1}^{\infty}$, such that for every integer $n \geq 1$, every partition (x, y) of a reasonable combination with length n rejects. If $e_1, \ldots, e_n$ are already constructed, consider the finite list of all partitions (x_i, y_i) of reasonable combinations of length n. By our induction hypothesis every such couple (x_i, y_i) rejects; applying successively the sublemma to each (x_i, y_i) from the list, we obtain a subspace W such that for every $w \in W$ and every i, $(x_i + w, y_i)$ rejects; observe that (y_i, x_i) also belongs to the list, hence $(y_i + w, x_i)$ rejects, and since A is symmetric $(x_i, y_i + w)$ also rejects. We choose now a vector e_{n+1} in W, such that $1 < \|e_{n+1}\| < 2$. We check that the induction hypothesis is verified for $n + 1$. Indeed, every partition (x', y') of a reasonable combination of length $(n + 1)$ is either of the form $(x + ae_{n+1}, y)$ or of the form $(x, y + ae_{n+1})$, where (x, y) is a partition of a reasonable combination of length n. It follows from the choice of W and e_{n+1} that (x', y') rejects.

We can now finish the proof of Theorem 10.2. Assume that X does not contain any UBS. Let Y be a Banach subspace of X. Since Y does not contain an UBS, it follows from Lemma 10.1 that for every $\varepsilon > 0$, Y contains a subspace Z which is HI(ε).

Taking successively $\varepsilon = 2^{-n}$, we can construct a decreasing sequence (Z_n) of subspaces corresponding to $\varepsilon_n = 2^{-n}$. Let Z be a subspace obtained by a diagonal procedure from the sequence (Z_n), which means that for every n this space Z is contained in Z_n up to finitely many dimensions. Let $\varepsilon > 0$, and let U and V be infinite dimensional subspaces of Z. Let n be such that $2^{-n} < \varepsilon$. We can find infinite dimensional subspaces U' of U and V' of V such that $U' \subset Z_n$, $V' \subset Z_n$. By the construction of Z_n there exists a couple (u, v) such that $u \in U'$, $v \in V'$ and $\|u - v\| < \varepsilon \|u + v\|$. Therefore Z is HI(ε) for every $\varepsilon > 0$, so Z is H.I.

Exercise. Finite field.

Theorem. Every homogeneous Banach space is isomorphic to ℓ_2.

Proof. Let X be a homogeneous Banach space, not isomorphic to ℓ_2. We know that X does not contain any UBS by the Corollary of [KT]. By Gowers' result, X must contain a H.I. subspace, hence X itself is H.I. But an H.I. space is obviously not homogeneous, and in a very strong way, since we have seen that it is not isomorphic to any proper subspace by Corollary 10.1.

11. A class of examples of exotic spaces

The contents of this section come from [GM2].

Let c_{00} be the vector space of all scalar sequences of finite support. Let $(\mathbf{e}_n)_{n=1}^{\infty}$ be the standard basis of c_{00}. Given a vector $\mathbf{a} = \sum_{n=1}^{\infty} a_n \mathbf{e}_n$ its *support*, denoted $\mathrm{supp}(\mathbf{a})$, is the set of n such that $a_n \neq 0$. Given two subsets $E, F \subset \mathbb{N}$, we say that $E < F$ if $\max E < \min F$. If $x, y \in c_{00}$, we say that $x < y$ if $\mathrm{supp}(x) < \mathrm{supp}(y)$. We also write $n < x$ when $n \in \mathbb{N}$ and $n < \min \mathrm{supp}(x)$. If $x_1 < \ldots < x_n$, then we say that the vectors $x_1, \ldots, x_n$ are *successive*. An infinite sequence of successive non-zero vectors is also called a *block basis* and a subspace generated by a block basis is a *block subspace*. Given a subset $E \subset \mathbb{N}$ and a vector $\mathbf{a}$ as above, we write $E\mathbf{a}$ for the vector $\sum_{n\in E} a_n \mathbf{e}_n$. An *interval* of integers is a set of the form $\{n, n+1, \ldots, m\}$ and the *range* of a vector x, written $\mathrm{ran}(x)$, is the smallest interval containing $\mathrm{supp}(x)$.

Let $\mathcal{X}$ stand for the set of Banach spaces obtained as the completion of c_{00} for a norm $\|.\|$ such that the sequence $(\mathbf{e}_n)_{n=1}^{\infty}$ is a normalized bimonotone basis. A first extremely important example in this class is the space T constructed by Tsirelson [T] (see also [FJ]). Let B_T^* be the smallest convex subset of $B(c_0) \cap c_{00}$ containing $\pm \mathbf{e}_n$ for each $n \geq 1$ and such that

$$(x_1^* + \cdots + x_n^*) \in 2B_T^*$$

whenever $x_1^*, \ldots, x_n^*$ are successive in B_T^* and $n < x_1^*$. The norm is then defined on c_{00} by

$$\|x\|_T = \sup\{|x^*(x)| : x^* \in B_T^*\}.$$

A second extremely important example is the space S constructed by Schlumprecht [S1], [S2], which is a very useful variation of the construction of T. Let $f(t) = \log_2(t+1)$ for $t \geq 0$. The relevant properties of this function will be listed below. Let B_S^* be the smallest convex subset of $B(c_0) \cap c_{00}$ containing $\pm \mathbf{e}_n$ for each n and such that

$$(x_1^* + \cdots + x_n^*)/f(n) \in B_S^*$$

whenever $x_1^*, \ldots, x_n^*$ are successive in B_S^* and $n \geq 2$. The norm is then defined on c_{00} by

$$\|x\|_S = \sup\{|x^*(x)| : x^* \in B_S^*\}.$$

The basic idea for the construction of our class of examples uses the technology of lower f-estimate introduced by Schlumprecht in [S1], [S2]. Given $X \in \mathcal{X}$, we shall say that X *satisfies a lower f-estimate* if, given any vector $x \in X$ and any sequence of intervals $E_1 < \ldots < E_n$, we have $\|x\| \geq f(n)^{-1} \sum_{i=1}^{n} \|E_i x\|$. In the dual formulation, this property means that whenever $x_1^*, \ldots, x_n^*$ are successive functionals with norm ≤ 1 in X^*, we have

$$\|(x_1^* + \cdots + x_n^*)/f(n)\|_{X^*} \leq 1.$$

The norm of S appears then as the smallest norm for a space in $\mathcal{X}$ satisfying a lower-f estimate.

Schlumprecht introduced the important notion of Rapidly Increasing Sequences, in short RIS. Let us mention first that every subspace Y of S generated by a block basis contains for every $n \geq 1$ a sequence $y_1 < \ldots < y_n$ of normalized vectors which is almost isometrically equivalent to the unit vector basis of ℓ_1^n (see Lemma 11.1 below). Roughly speaking, a RIS is a normalized sequence $x_1 < \ldots < x_k$ where each x_i is the average of an $\ell_1^{n_i}$ sequence, with $n_1 < n_2 < \ldots < n_k$ growing extremely rapidly (a precise definition will be given later). Schlumprecht proved that the norm of the sum of RIS sequences has an almost minimal behaviour; indeed, Schlumprecht's space satisfies a lower f-estimate, hence $\| \sum_{i=1}^n x_i \| \geq n/f(n)$ for every sequence of successive norm one vectors. Schlumprecht's Lemma states that for an RIS, we almost get an equality, $\| \sum_{i=1}^n x_i \| \leq (1+\varepsilon)n/f(n)$. We obtain in this way one of the most important features of Schlumprecht's example: on one hand, we can find ℓ_1^n in every subspace; on the other hand, we can always combine very different $\ell_1^{n_i}$ in a RIS and get a behaviour arbitrarily far from the ℓ_1 behaviour. But these new vectors can again be combined to give further ℓ_1^n, and so on...

For our construction, we need to work with more than one function f. To this end we introduce the family $\mathcal{F}$ of functions $g : [1, \infty) \to [1, \infty)$ satisfying the following conditions:

(i) $g(1) = 1$ and $g(t) < t$ for every $t > 1$;
(ii) g is strictly increasing and tends to infinity;
(iii) $\lim_{t\to\infty} t^{-q} g(t) = 0$ for every $q > 0$;
(iv) the function $t/g(t)$ is concave and non-decreasing;
(v) $g(st) \leq g(s)g(t)$ for every $s, t \geq 1$;

It is easy to check that $f(t) = \log_2(t+1)$ satisfies these conditions, as does the function $\sqrt{f(t)}$. Note also that some of the conditions above are redundant. In particular, it follows from the other conditions that $g(x)$ and $x/g(x)$ are strictly increasing.

Let $X \in \mathcal{X}$ and $y \in X$. For every $n \geq 1$, let

$$\|y\|_{(n)} = \sup \sum_{i=1}^n \|E_i y\|$$

where the supremum is extended to all families $E_1 < \ldots < E_n$ of successive intervals. This quantity is clearly increasing with n, and $\|y\| = \|y\|_{(1)}$ since $(\mathbf{e}_n)$ is a bimonotone basis for the space X. Observe that the basis (e_i) satisfies $\|e_i\|_{(n)} = 1$ for every $n \geq 1$.

Lemma 11.1. *Let $X \in \mathcal{X}$ satisfy a lower f-estimate. Given $n \geq 1$ and $\varepsilon > 0$, there exists an integer $N(n, \varepsilon)$ such that for every sequence $x_1, \ldots, x_N$ of successive norm one vectors with $N \geq N(n, \varepsilon)$, we may find x of the form $x = \lambda \sum_{i\in A} x_i$, where A is some subinterval of $\{1, \ldots, N\}$ such that $\|x\| = 1$ and $\|x\|_{(n)} \leq 1 + \varepsilon$.*

The proof of this Lemma uses a variant of a well known blocking procedure for constructing ℓ_1^n, originating in James [J3]; let us also mention Giesy [Gi], Pisier [P1]; and much more elaborated results of Elton (E], (Pajor [Pa] in the complex case).

Corollary 11.1. *Let $X \in \mathcal{X}$, satisfying a lower f-estimate. Then for every $n \in \mathbb{N}$ and $\varepsilon > 0$, every subspace Y of X contains a vector y such that $\|y\| = 1$ and $\|y\|_{(n)} \leq 1 + \varepsilon$.*

Proof. By the standard gliding hump procedure, we may find for every N a normalized sequence $y_1, \ldots, y_N$ of vectors in Y and successive vectors $x_1 < \ldots < x_N$ in X such that $\|y_i - x_i\| < \varepsilon/nN$. The result follows from Lemma 11.1 and an easy approximation argument.

Given a subspace $Y \subset X$, we will be interested in a seminorm $|\!|\!|.|\!|\!|$ defined on $\mathcal{L}(Y, X)$ as follows. We say that a sequence (x_n) is a sequence of *almost successive* vectors if there exists a sequence (y_n) of successive vectors such that $\lim_n \|x_n - y_n\| = 0$. Let $\mathcal{M}_Y$ be the set of sequences $(x_n)_{n=1}^{\infty}$ of almost successive vectors in Y such that $\limsup_n \|x_n\|_{(n)} \leq 1$. Now, given $T \in \mathcal{L}(Y, X)$ let

$$|\!|\!|T|\!|\!| = \sup_{\mathbf{x} \in \mathcal{M}(Y)} \limsup_n \|Tx_n\| .$$

Suppose that X is reflexive and that (x_n) is a weakly null sequence in a subspace Y such that $\limsup_n \|x_n\|_{(m)} \leq 1$ for every integer m. Let $T \in \mathcal{L}(Y, X)$ and $t = \limsup \|Tx_n\|$. We can find a subsequence (x'_n) such that $t = \lim_n \|Tx'_n\|$. Since (x'_n) is weakly null, we can extract a further subsequence (x''_n) which is almost successive, and we may also arrange that $\|x''_n\|_{(n)} \leq 1 + 2^{-n}$. Now (x''_n) belongs to $\mathcal{M}_Y$, therefore $t = \lim \|Tx''_n\| \leq |\!|\!|T|\!|\!|$. Let us say the same thing in a slightly different way: let P_m denote the projection on the interval $\{1, \ldots, m\}$; for every $T \in \mathcal{L}(Y, X)$ and for every $\varepsilon > 0$, there exists integers $m, n \geq 1$ such that, for every $y \in Y$, the condition $\|P_n y\| \leq 1/n$ implies that $\|Ty\| \leq (|\!|\!|T|\!|\!| + \varepsilon)\|y\|_{(m)}$.

Lemma 11.2. *Suppose that $X \in \mathcal{X}$ satisfies a lower f-estimate; then for every subspace Y of X and $T \in \mathcal{L}(Y, X)$*

$|\!|\!|T|\!|\!| = 0 \Rightarrow T$ is strictly singular;

if X is reflexive and T compact, then $|\!|\!|T|\!|\!| = 0$;

if for every z in some infinite dimensional subspace Z of Y, we have $\|Tz\| \geq \|z\|$, then $|\!|\!|T|\!|\!| \geq 1$.

Proof. By the preceding Lemma, every subspace Y contains for every $\varepsilon > 0$ normalized sequences in $\mathcal{M}_Y$. Hence every (infinite dimensional) subspace of Y contains a norm one vector x such that $\|Tx\| \leq (1 + \varepsilon)\, |\!|\!|T|\!|\!|$. As a consequence, for every $U \in \mathcal{L}(Y, X)$, we see that $s(U) \leq |\!|\!|U|\!|\!|$ (where $s(U)$ was defined in section 6); in particular, if $|\!|\!|T|\!|\!| = 0$, then T is strictly singular. Suppose that X is reflexive; then every normalized sequence of almost successive vectors is weakly null, therefore $\lim \|Tx_n\| = 0$ if T is compact, hence $|\!|\!|T|\!|\!| = 0$. Lastly, suppose that $\|Tz\| \geq \|z\|$ for every z in some infinite dimensional subspace Z of Y. We know from Corollary 11.1 that Z contains a normalized sequence (z_n) of almost successive vectors with $\lim \|z_n\|_{(n)} = 1$. By definition,

$$|\!|\!|T|\!|\!| \geq \lim_n \|Tz_n\| \geq 1.$$

Remark. if $U \in \mathcal{L}(X)$, then $\widehat{r}(U) \leq s(U) \leq |\!|\!|U|\!|\!|$. If $T - \lambda I_X$ is infinitely singular, we see that $|\lambda| \leq |\!|\!|T|\!|\!|$.

The second ingredient inspired by Schlumprecht is that of Rapidly Increasing Sequences, in short RIS. Let $X \in \mathcal{X}$. For $0 < \varepsilon \leq 1$, we say that a sequence $x_1, \ldots, x_N$ of successive vectors in X *satisfies the RIS(ε) condition* if there is a sequence

$$2^{N^2/\varepsilon^2} < n_1 < \cdots < n_N$$

of integers such that $\|x_i\|_{(n_i)} \le 1$ for each $i = 1, \dots, N$ and

$$\varepsilon\sqrt{f(n_i)} > |\operatorname{ran}(\sum_{j=1}^{i-1} x_j)|$$

for every $i = 2\dots, N$.

Given $g \in \mathcal{F}$, $M \in \mathbb{N}$ and $X \in \mathcal{X}$, an *(M, g)-form on X* is defined to be a functional x^* of norm at most one which can be written as $\sum_{j=1}^{M} x_j^*$ for a sequence $x_1^* < \dots < x_M^*$ of successive functionals all of which have norm at most $g(M)^{-1}$. Observe that if x^* is an (M, g)-form then $\|x^*\|_\infty \le 1/g(M)$ and $|x^*(x)| \le g(M)^{-1}\|x\|_{(M)}$ for any x.

Lemma 11.3. *Let $X \in \mathcal{X}$. Suppose that $(x_1, \dots, x_N)$ satisfies $RIS(\varepsilon)$ in X. If $g \in \mathcal{F}$, $\sqrt{f} \le g$ and if x^* is a (k, g)-form on X, we have*

$$|x^*(x_1 + \dots + x_N)| \le \max_{j=1,\dots,N} \|x_j\| + \varepsilon + \frac{N}{g(k)}.$$

In particular, $|x^*(x_1 + \dots + x_N)| \le \max \|x_j\| + 2\varepsilon$ when $k \ge 2^{N^2/\varepsilon^2}$.

Proof. Let $n_1 < n_2 < \dots < n_N$ be the sequence of integers associated to the RIS property. Let $i \in \{1, \dots, N\}$ be such that $n_i < k \le n_{i+1}$. Observe that the RIS condition implies $\|x_j\|_\infty \le 1$ for each $j = 1, \dots, N$. The result follows from three easy inequalities,

$$\big|x^*(\sum_{j=1}^{i-1} x_j)\big| \le \|x^*\|_\infty \,\big|\operatorname{ran}(\sum_{j=1}^{i-1} x_j)\big| \le \frac{1}{g(k)}\,\varepsilon\sqrt{f(n_i)} \le \varepsilon,$$

$$|x^*(x_i)| \le \|x_i\| \le \max_{j=1,\dots,N} \|x_j\|,$$

and for $j \ge n_{i+1}$,

$$|x^*(x_j)| \le \frac{1}{g(k)}\|x_j\|_{(k)} \le \frac{1}{g(k)}\|x_j\|_{(n_j)} \le \frac{1}{g(k)}.$$

When $k \ge 2^{N^2/\varepsilon^2}$, we get $g(k) \ge \sqrt{f(k)} \ge N/\varepsilon$.

The next Lemma is a variation of a main Lemma due to Schlumprecht. We already mentioned that in the case of the space constructed by Schlumprecht, this Lemma says that the norm of the sum of RIS sequences has an almost minimal behaviour. Our situation is technically more complicated; the space we want to construct will satisfy a lower f-estimate, but in some parts of our space the behaviour of RIS will be larger than $n/f(n)$, namely it could sometimes be as big as $n/\sqrt{f(n)}$. We need a more general statement that allows to play between the two possibilities. To this end we introduced the family $\mathcal{F}$ of functions. The next Lemma is similar to Lemma 3 from [GM2] or Lemma 7 from [GM1].

Lemma 11.4. *Let $X \in \mathcal{X}$, $g \in \mathcal{F}$, $\sqrt{f} \le g$, and let $x_1 < \dots < x_n$ in X satisfy $\|x_i\|_{(p^n)} \le 1$ for every $i = 1, \dots, n$ and some integer $p \ge 2$. Let $x = \sum_{i=1}^{n} x_i$ and suppose that*

$$\|Ex\| \le 1 \vee \sup\big\{|x^*(Ex)| : x^* \text{ is a } (k, g)\text{-form}, 2 \le k \le p\big\}$$

for every interval E. Then

$$\|x\| \leq ng(n)^{-1}.$$

Proof. Let $G(t) = t/g(t)$ when $t \geq 1$ and $G(t) = t$ when $0 \leq t \leq 1$. This function G is concave and increasing on $[0, +\infty)$. For every interval E and every integer $l \geq 0$, let

$$\sigma_l(E) = \sum_{i=1}^{n} \|Ex_i\|_{(p^l)}.$$

We shall prove by induction on l, $1 \leq l \leq n$ that

$$(*) \qquad \|Ex\| \leq G(\sigma_{\kappa(E)}(E)),$$

where $\kappa(E)$ is the number of $i \in \{1, \ldots, n\}$ such that $Ex_i \neq 0$ (if $\kappa(E) = 0$, then $Ex = 0$ and this case is obvious). Once this is done, we obtain the result for $l = n$, $E = \mathrm{ran}(x)$,

$$\|x\| \leq G(\sigma_n(\mathrm{ran}(x))) = G\Big(\sum_{i=1}^{n} \|x_i\|_{(p^n)}\Big) \leq G(n) = \frac{n}{g(n)}.$$

Observe first that when $\|Ex\| \leq 1$, we have $\|Ex\| = G(\|Ex\|) \leq G(\sum_{i=1}^{n} \|Ex_i\|) \leq G(\sigma_l(E))$. This shows that $(*)$ is true when $\kappa(E) = 1$. Assume $(*)$ true when $\kappa(E) \leq l < n$, and suppose there exists an interval E such that $\kappa(E) = l + 1$ and $\|Ex\| > G(\sigma_{l+1}(E))$; since $(*)$ is not true for E we know that $l \geq 1$ and $\|Ex\| > 1$. From our assumption there exists a (k, g)-form $x^* = (\sum_{j=1}^{k} A_j x_j^*)/g(k)$, $2 \leq k \leq p$, $\|x_j^*\| \leq 1$ and $A_1 < \ldots < A_k$, such that

$$G(\sigma_{l+1}(E)) < |x^*(Ex)|.$$

Assume first that $\kappa(A_j E) \leq l$ for every $j = 1, \ldots, k$. We have $\|A_j Ex\| \leq G(\sigma_l(A_j E))$ by the induction hypothesis, and using the concavity of G we obtain

$$|x^*(x)| \leq \frac{k}{g(k)} \sum_{j=1}^{k} \frac{1}{k} G(\sigma_l(A_j E)) \leq \frac{k}{g(k)} G\Big(\frac{1}{k} \sum_{j=1}^{k} \sigma_l(A_j E)\Big)$$

$$= \frac{k}{g(k)} G\Big(\frac{1}{k} \sum_{i=1}^{n} \sum_{j=1}^{k} \|A_j Ex\|_{(p^l)}\Big) \leq \frac{k}{g(k)} G\Big(\frac{1}{k} \sum_{i=1}^{n} \|Ex_i\|_{(p^{l+1})}\Big) = \frac{k}{g(k)} G\Big(\frac{\sigma_{l+1}(E)}{k}\Big).$$

If $\sigma_{l+1}(E) \leq k$, this last expression is $\sigma_{l+1}(E)/g(k) \leq \sigma_{l+1}(E)/g(\sigma_{l+1}(E)) = G(\sigma_{l+1}(E))$, otherwise it is equal to

$$\frac{\sigma_{l+1}(E)}{g(k)g(\sigma_{l+1}(E)/k)} \leq \frac{\sigma_{l+1}(E)}{g(\sigma_{l+1}(E))} = G(\sigma_{l+1}(E)),$$

so that we have reached a contradiction.

In the remaining case there exists $j_0 \in \{1, \ldots, k\}$ such that $A_{j_0} E x_i \neq 0$ for every i such that $E x_i \neq 0$. Assume for example $j_0 < k$ (otherwise $1 < j_0$ deserves a similar treatment). Let m be the last integer i such that $E x_i \neq 0$. Let $B_{j_0} = A_{j_0} \setminus \operatorname{ran}(E x_m)$, $B'_{j_0+1} = A_{j_0} \cap \operatorname{ran}(E x_m)$, $B''_{j_0+1} = A_{j_0+1}$, $B_{j_0+1} = B'_{j_0+1} \cup B''_{j_0+1}$ and $B_j = A_j$ otherwise. We see that

$$\|A_{j_0} E x\| + \|A_{j_0+1} E x\| \leq \|B_{j_0} E x\| + \|B'_{j_0+1} E x_m\| + \|B''_{j_0+1} E x_m\| \leq$$

$$\leq \|B_{j_0} E x\| + \|B_{j_0+1} E x_m\|_{(2)}.$$

Every B_j satisfies $\kappa(B_j E) \leq l$, so that the induction hypothesis applies and since $p^l \geq 2$ we obtain

$$\sum_{j=1}^{k} \|A_j E x\| \leq \sum_{j \neq j_0} \|B_j E x\| + \|B_{j_0+1} E x_m\|_{(2)} \leq \sum_{j \neq j_0} G(\sigma_l(B_j E)) + G(\sigma_l(B_{j_0+1} E)),$$

and the conclusion follows as before.

The next simple Lemma is useful in conjunction with the preceding.

Lemma 11.5. *Let $X \in \mathcal{X}$, and let $x_1 < \ldots < x_l$ in X be such that*

$$\|x_i\| \leq 1; \quad \|\sum_{i \in A} x_i\| \leq \frac{|A|}{f(|A|)}$$

for every interval $A \subset \{1, \ldots, l\}$ such that $m \leq |A| \leq l$. Then for every integer $n \geq 1$

$$\frac{f(l)}{l} \|\sum_{i=1}^{l} x_i\|_{(n)} \leq \frac{f(l)}{f(m)} + \frac{2nmf(l)}{l}.$$

Proof. Let $x = \sum_{i=1}^{l} x_i$ and let (E_j) be a sequence of n successive intervals. By adding at most n cuts, we may assume that we have a family (E_j) of at most $2n$ intervals and that for every $j = 1, \ldots, 2n$ there exists an interval $A_j \subset \{1, \ldots, l\}$ such that $E_j x = \sum_{i \in A_j} x_i$. Let

$$J = \{j : |A_j| \geq m\}.$$

We get

$$\sum_{j=1}^{2n} \|E_j x\| = \sum_{j \notin J} \|E_j x\| + \sum_{j \in J} \|E_j x\| \leq 2nm + \sum_{j \in J} \frac{|A_j|}{f(|A_j|)},$$

and the result follows.

Proper semi-groups of spreads.

Given two infinite subsets $A = \{a_1, a_2, \ldots\}$ and $B = \{b_1, b_2, \ldots\}$ of $\mathbb{N}$, define the *spread from A to B* to be the map $S_{A,B}$ on c_{00} that sends $\mathbf{e}_n$ to zero if $n \notin A$, and sends $\mathbf{e}_{a_k}$ to $\mathbf{e}_{b_k}$ for every $k \in \mathbb{N}$. $S_{A,A}$ is just the projection on to A. Note that $S_{B,C}S_{A,B} = S_{A,C}$. Note also that $S_{B,A}$ is (formally) the adjoint of $S_{A,B}$. Given any set $\mathbf{S}$ of spreads, we shall say that it is a *proper set* if it is closed under composition (note that this applies to all compositions and not just those of the form $S_{B,C}S_{A,B}$) and taking adjoints; we also make in [GM2] a technical assumption which means roughly that our semi-group is rather small: For every $(i,j) \neq (k,l)$, there are only finitely many spreads $U \in \mathbf{S}$ for which $e_i^*(Ue_j) \neq 0$ and $e_k^*(Ue_l) \neq 0$. Note that a proper semi-group is countable.

A good example of such a set is the collection of all spreads $S_{A,B}$ where $A = \{m, m+1, m+2, \ldots\}$ and $B = \{n, n+1, n+2, \ldots\}$ for some $m, n \in \mathbb{N}$. This is the proper set generated by the shift operator. The next theorem is the main result of [GM2].

Theorem 11.1. *Given a proper set $\mathbf{S}$ of spreads, there exists a Banach space $X = X(\mathbf{S}) \in \mathcal{X}$ such that:*

1. The space X is reflexive, satisfies a lower-f estimate (and the natural basis $(\mathbf{e}_n)$ of c_{00} is a bimonotone basis for X by definition of the class $\mathcal{X}$).

2. $\|U\| \leq 1$ for every $U \in \mathbf{S}$. It follows that $\|S_{A,B}x\| = \|x\|$ when $S_{A,B} \in \mathbf{S}$ and $\operatorname{supp} x \subset A$.

3. $⦀TU⦀ \leq ⦀T⦀\ ⦀U⦀$ for all $T, U \in \mathcal{L}(X)$.

4. Let $\mathcal{A}$ be the algebra generated by $\mathbf{S}$. For every subspace Y of X, every $\varepsilon > 0$ and every $T \in \mathcal{L}(Y,X)$, there exists $U \in \mathcal{A}$ such that $⦀T - U \circ i_{Y,X}⦀ < \varepsilon$, where $i_{Y,X}$ denotes the injection from Y into X.

Recall some facts from the general discussion: property 1 implies that every subspace $Y \subset X$ contains normalized sequences in $\mathcal{M}$. Since X is reflexive with a basis, normalized sequences of (almost) successive elements are weakly null, in particular sequences in $\mathcal{M}$ are weakly null. It follows that $⦀K⦀ = 0$ when K is compact from $Y \subset X$ to X. If $T \in \mathcal{L}(Y,X)$ and $⦀T⦀ = 0$, then T is strictly singular. We have seen that $\widehat{r}(U) \leq s(U) \leq ⦀U⦀$;

Corollary 11.2. *All spaces $X = X(\mathbf{S})$ from Theorem 11.1 have the property that they do not contain any UBS.*

Proof. This is because property 4 immediately implies that for every subspace Y of X, $\mathcal{L}(Y)$ is separable for $⦀.⦀$; on the other hand, if X contains an infinite dimensional subspace Y with unconditional basis (y_n), then we can find an uncountable set $\mathcal{P}$ of projections in $\mathcal{L}(Y)$ such that $⦀P - Q⦀ \geq 1$ when $P \neq Q$ in $\mathcal{P}$; for every infinite set $L \subset \mathbb{N}$, let Y_L denote the span $[y_n]_{n \in L}$ of the corresponding subsequence, and let P_L denote the corresponding projection from Y onto Y_L. If L and M are two subsets of $\mathbb{N}$ with infinite difference D, we obtain that $\|(P_L - P_M)z\| \geq \|z\|$ for every z in Y_D, hence $⦀P_L - P_M⦀ \geq 1$ by Lemma 11.2. Finally, it is well known that we may find uncountably many infinite subsets of $\mathbb{N}$ with pairwise infinite differences.

These examples are not necessarily H.I. spaces however. They give a good illustration for the dichotomy result of Tim Gowers (Theorem 10.2).

It follows from Gowers' dichotomy theorem that every Banach space X not containing an UBS has the property that every subspace Y of X contains an H.I. subspace. By the Corollary above this is the case for all the examples given by Theorem 11.1. It is possible actually to see this directly by a reasoning close to the proof of the Corollary. Suppose that $X = X(\mathbf{S})$ for some proper set of spreads, and that $Y \subset X$ contains no H.I. subspace. We can build by a standard ordinal construction a family $(Y_\alpha)_{\alpha<\omega_1}$ of subspaces of Y in the following way: since Y_α is not H.I. we can find a direct sum $U_\alpha \oplus V_\alpha$ in Y_α. Let T_α be Id on U_α and 0 on V_α. We choose then $Y_{\alpha+1} = U_\alpha$; for a limit ordinal, we construct Y_β by a diagonal procedure in such a way that Y_β is almost contained in the preceding spaces, up to finite dimension. If $\alpha < \beta$, $T_\beta - T_\alpha = Id$ on a finite codimensional space of V_β, hence $\|T_\alpha - T_\beta\| \geq 1$ by ???. On the other hand there should exist by Theorem 11.1 for every ordinal $\alpha < \omega_1$ an $A_\alpha \in \mathcal{A}$ such that $\|U_\alpha - T_\alpha\| \leq 1/4$, and this contradicts the separability of $\mathcal{A}$.

We start now the construction of the spaces $X(\mathbf{S})$ and the proof of Theorem 11.1. We introduce a lacunary subset J of $\mathbb{N}$, which we split into two disjoint parts K and L. Let $J \subset \mathbb{N}$ be a set such that, if $m < n$ and $m, n \in J$, then $\log\log\log n \geq 2m$. Let us write J in increasing order as $\{j_1, j_2, \ldots\}$. We also need $f(j_1) > 256$. Now let $K, L \subset J$ be the sets $\{j_1, j_3, j_5, \ldots\}$ and $\{j_2, j_4, j_6, \ldots\}$.

We mentioned before the statement of Theorem 11.1 two important ingredients of the construction. The third and last important ingredient is the notion of *special sequence*. Let us recall the definition from [GM1] of the *special functionals* on a space $X \in \mathcal{X}$. Let $\mathbf{Q} \subset c_{00}$ be the (countable) set of sequences with rational coordinates and maximum at most 1 in modulus. Let σ be an injection from the collection of finite sequences of successive elements of $\mathbf{Q}$ to the set L introduced above. Given $X \in \mathcal{X}$ such that X satisfies a lower f-estimate and given an integer $m \in \mathbb{N}$, let $A_m^*(X)$ be the set of (m, f)-forms on X, i.e. the set of all functionals x^* of the form $x^* = f(m)^{-1} \sum_{i=1}^m x_i^*$, where $x_1^* < \ldots < x_m^*$ and $\|x_i^*\| \leq 1$ for each $i = 1, \ldots, m$. Note that these functionals have norm at most 1 by the lower f-estimate. If $k \in \mathbb{N}$, let Γ_k^X be the set of sequences $y_1^* < \ldots < y_k^*$ such that $y_i^* \in \mathbf{Q}$ for each i, $y_1^* \in A_{j_{2k}}^*(X)$ and $y_{i+1}^* \in A_{\sigma(y_1^*, \ldots, y_i^*)}^*(X)$ for each $1 \leq i \leq k-1$. We call these *special sequences*. Let $B_k^*(X)$ be the set of all functionals y^* of the form

$$y^* = \frac{1}{\sqrt{f(k)}} \sum_{j=1}^k g_j$$

such that $(g_1, \ldots, g_k) \in \Gamma_k^X$ is a special sequence. These, when $k \in K$, are the *special functionals* (on X of size k). Note that if $g \in \mathcal{F}$ and $g(k) = f(k)^{1/2}$, then a special functional y^* of size k is also a (k, g)-form, and the same is true for every EUy^*, for every interval E and every $U \in \mathbf{S}$. The idea behind this notion of special fonctionals is that their normalization is very different from the usual normalization of functionals obtained by the "Schlumprecht operation" $(x_1^* + \cdots + x_n^*)/f(n)$, so they produce "spikes" in the unit ball of X^*, but they are extremely rare and easily identified: a relatively weak information about a part of a special functional $f(k)^{-1/2} \sum_{j=1}^k g_j$, namely knowing simply the integer $l \in L$ such that $g_j \in A_l^*$, allows us to trace back all the past of its construction, since there is at most one sequence $(g_1, \ldots, g_{j-1})$ such that $l = \sigma(g_1, \ldots, g_{j-1})$.

Now, given a proper set $\mathbf{S}$ of spreads, we define the space $X(\mathbf{S})$, inductively. It is the completion of c_{00} in the smallest norm satisfying the following equation.

$$\begin{aligned}\|x\| = \|x\|_{c_0} &\vee \sup\Big\{f(n)^{-1}\sum_{i=1}^{n}\|E_i x\| : 2 \le n \in \mathbb{N}, E_1 < \ldots < E_n \text{ intervals}\Big\} \\ &\vee \sup\Big\{|x^*(Ex)| : k \in K,\, x^* \in B_k^*(X),\, E \subset \mathbb{N} \text{ an interval}\Big\} \\ &\vee \sup\{\|Ux\| : U \in \mathbf{S}\}\end{aligned}$$

In the case $\mathbf{S} = \{Id_{c_{00}}\}$ the fourth term drops out and the definition reduces to that of the space constructed in [GM1]. The fourth term is there to force $X(\mathbf{S})$ to have property (2) claimed in Theorem 11.1. The second term ensures that X satisfies a lower f-estimate. It is also not hard to show that $X(\mathbf{S})$ is reflexive. (A proof can be found in [GM1], end of section 3, which works in this more general context.) It is also useful to understand the construction of X in a way similar to what we said about T and S: we construct a dense subset of the unit ball B_{X^*} in a sequence of steps, producing an increasing sequence (B_n) of convex subsets of B_{c_0}. We start with $B_0 = B_{\ell_1} \cap c_{00}$. If B_n is defined, we add to it

— all (m, f)-forms $x^* = f(m)^{-1}\sum_{j=1}^{m} x_j^*$ using elements x_j^* from B_n, for any integer $m \ge 2$

— all functionals λEUx^* where $|\lambda| = 1$, E is an interval, $U \in \mathbf{S}$ and x^* is any special functional $x^* = (\sum_{j=1}^{k} g_j)/\sqrt{f(k)}$ with $g_j \in B_n$ for $j = 1, \ldots, k$;

we let finally B_{n+1} be the convex hull of the union of B_n and of the set of all these new functionals. We let $B = \bigcup_n B_n$ and we can see that the above defined norm is equal to

$$\|x\| = \sup\{|x^*(x)| : x^* \in B\}.$$

If we want to compute the norm of $x \in X$, either $\|x\| = \|x\|_{c_0}$ or, given $\varepsilon > 0$ such that $\|x\|_{c_0} < \|x\| - \varepsilon$, there exist a first $n \ge 0$ such that $|x^*(x)| > \|x\| - \varepsilon$ for some x^* that was adjoined to B_n in the construction of B_{n+1}, namely either an (m, f)-form or some EUy^*, with y^* special functional and $U \in \mathbf{S}$. It should be observed that if $g \in \mathcal{F}$ is such that $g = \sqrt{f}$ on K, then all these functionals of the form EUy^* are (k, g)-forms for some $k \ge 2$ (observe that the images of successive functionals by a spread are still successive). The next technical Lemma is taken from [GM1]. It is just a painful exercise using only elementary calculus.

Lemma 11.6. *Let $K_0 \subset K$. There exists a function $g \in \mathcal{F}$ such that $f \ge g \ge \sqrt{f}$, $g(k) = \sqrt{f(k)}$ whenever $k \in K_0$ and $g(x) = f(x)$ whenever $N \in J \setminus K_0$ and x is in the interval* $[\log N, \exp N]$.

Lemma 11.7. *Let $0 < \varepsilon \le 1$, $M \in L$ and let N be such that $N \in [\log M, \exp M]$. Assume that $x = x_1 + \cdots + x_N$ satisfies the RIS(ε) condition and let $x = x_1 + \cdots + x_N$. Then*

$$\|(f(N)/N)x\| \le 1 + 2\varepsilon.$$

Assume further that $0 \le \delta < 1$, and let n be such that $N/n \in [\log M, \exp M]$ and $f(N) \le (1 + \delta)f(N/n)$. Then $\|(f(N)/N)x\|_{(n)} \le (1 + \delta)(1 + 3\varepsilon)$.

Proof. Let g be the function given by Lemma 11.6 in the case $K_0 = K$. It is clear that every vector Ex such that $\|Ex\| > 1$ is normed by a (k, g)-form for some $k \geq 2$; furthermore if $k \geq 2^{N^2/\varepsilon^2}$, then $|x^*(x)| \leq 1 + 2\varepsilon$ by Lemma 11.3, so the conditions of Lemma 11.4 are satisfied for $x'_i = (1 + 2\varepsilon)^{-1} x_i$, and thus $\| \sum_{i=1}^N x'_i \| \leq N/g(N)$. Since $g(N) = f(N)$ we obtain the first estimate. The second follows by Lemma 11.5. ?????

We start now the proof of assertion 3 in Theorem 11.1. Let $\widetilde{X}_0$ be the weakly null part of $\widetilde{X}$, consisting of all classes $\widetilde{x}$ of weakly null sequences (x_n) in X. Similarly we shall consider $\widetilde{Y}_0$ for every subspace Y of X. Let $\Xi_0(Y)$ be the space of finitely supported sequences of elements of $\widetilde{Y}_0$. We shall use the following notation,

$$(\widetilde{y}_1, \ldots, \widetilde{y}_k, 0, 0, \ldots) = \sum_{j=1}^k \widetilde{y}_j \otimes \mathbf{f}_j \in \Xi_0(X),$$

where $(\mathbf{f}_n)$ is the canonical basis for the space of sequences (to avoid confusion, we chose a notation different from the previous $(\mathbf{e}_n)$). We define a norm on $Y \oplus \Xi_0(Y)$ by induction on k

$$\|y + \sum_{j=1}^k \widetilde{y}_j \otimes \mathbf{f}_j\| = \lim_{n,\mathcal{U}} \|y + y_{1,n} + \sum_{j=2}^k \widetilde{y}_j \otimes \mathbf{f}_j\|,$$

where $\widetilde{y}_1 = (y_{1,n})$. We can write this directly with an iterated limit,

$$\|y + \sum_{j=1}^k \widetilde{y}_j \otimes \mathbf{f}_j\| = \lim_{n_1,\mathcal{U}} \ldots \lim_{n_k,\mathcal{U}} \|y + y_{1,n_1} + \cdots + y_{k,n_k}\|.$$

We may also observe that for every integer $k \geq 1$, there exists an ultrafilter $\mathcal{U}^{\otimes k}$ on $\mathbb{N}^k$ defined by

$$A \in \mathcal{U}^{\otimes k} \Leftrightarrow \lim_{n_1,\mathcal{U}} \ldots \lim_{n_k,\mathcal{U}} 1_A(n_1, \ldots, n_k) = 1.$$

For every integer $m \geq 1$ we extend the norm $\|.\|_{(m)}$ to $Y \oplus \Xi_0(Y)$ in the following way

$$\|y + \sum_{j=1}^k \widetilde{y}_j \otimes \mathbf{f}_j\|_{(m)} = \lim_{n_1,\mathcal{U}} \ldots \lim_{n_k,\mathcal{U}} \|y + y_{1,n_1} + \cdots + y_{k,n_k}\|_{(m)}.$$

Letting $\mathbf{n} = (n_1, \ldots, n_k)$ and $y_{\mathbf{n}} = y + \sum_{j=1}^k y_{j,n_j}$ we may write this iterated limit as

$$\lim_{\mathbf{n},\mathcal{U}^{\otimes k}} \|y_{\mathbf{n}}\|_{(m)}.$$

Observe that since $(y_{j,n})_n$ is weakly null for each j, the vectors $y_{1,n_1}, \ldots, y_{k,n_k}$ are almost successive when $n_1 < \ldots < n_k$ is lacunary enough, which always happens when we consider the iterated limit. It follows from the lower-f estimate that

$$\|\sum_{j=1}^k \widetilde{y}_j \otimes \mathbf{f}_j\| \geq (\sum_{j=1}^k \|\widetilde{y}_j\|)/f(k).$$

For $\xi = \sum_j \widetilde{x}_j \otimes \mathbf{f}_j \in \Xi_0$, we define its support by as $\{j : \widetilde{x}_j \neq 0\}$. It is possible to generalize the above in the following way:

Lemma. If $\xi_1, \dots, \xi_k$ are successive in Ξ_0, then

$$\|\xi_1 + \cdots + \xi_k\| \geq (\sum_{j=1}^{k} \|\xi_j\|)/f(k).$$

Let Ξ_Y be the completion of $Y \oplus \Xi_0(Y)$. We do the same for X, writing simply Ξ. We will work now with the triple norm. Let $T \in \mathcal{L}(Y, X)$. Recall that for every $\varepsilon > 0$, there exists $m \geq 1$ such that

$$(*) \qquad \forall y \in Y, \ \ (\|P_m y\| \leq 1/m) \Rightarrow \|Ty\| \leq (|\!|\!|T|\!|\!| + \varepsilon)\|y\|_{(m)}.$$

For $\xi \in \Xi$ we define $|\!|\!|\xi|\!|\!|$ to be the (increasing) limit of $\|\xi\|_{(m)}$, when m tends to $+\infty$; this limit may be $+\infty$. For every bounded operator T from Y to X, we know that $\widetilde{T}(\widetilde{Y}_0) \subset \widetilde{X}_0$, and we define an operator $T_\Xi : \Xi_Y \to \Xi$ by the formula

$$T_\Xi(y + \sum_{j=1}^{k} \widetilde{y}_j \otimes \mathbf{f}_j) = Ty + \sum_{j=1}^{k} \widetilde{T}(\widetilde{y}_j) \otimes \mathbf{f}_j \in \Xi.$$

It is clear from the iterated limit formula that $\|T_\Xi\| = \|T\|$.

Lemma. Let $T \in \mathcal{L}(Y, X)$. For every $\varepsilon > 0$, there exists an integer $m \geq 1$ such that

$$\forall \xi \in \Xi_0(Y), \ \ \|T_\Xi(\xi)\| \leq (|\!|\!|T|\!|\!| + \varepsilon)\|\xi\|_{(m)}.$$

It follows that $\|T_\Xi(\xi)\| \leq |\!|\!|T|\!|\!| \, |\!|\!|\xi|\!|\!|$.

Proof. Given $\varepsilon > 0$, we find $m \geq 1$ such that $(*)$ holds. Let $\xi = \sum_{j=1}^{k} \widetilde{y}_j \otimes \mathbf{f}_j$ and let $\beta > \|\xi\|_{(m)}$. Let $\xi_{\mathbf{n}} = \sum_{j=1}^{k} y_{j,n_j}$. Since each $\widetilde{y}_j$ is weakly null, we get

$$\lim_{\mathbf{n}, \mathcal{U}^{\otimes k}} \|P_m(\xi_{\mathbf{n}})\| = 0.$$

It is therefore possible to find $A \in \mathcal{U}^{\otimes k}$ such that $\|P_m(\xi_{\mathbf{n}})\| \leq \beta/m$ for every $\mathbf{n} \in A$. There exists $B \in \mathcal{U}^{\otimes k}$ such that $\|\xi_{\mathbf{n}}\|_{(m)} \leq \beta$ for every $\mathbf{n} \in B$. It follows by $(*)$ that $\|T(\xi_{\mathbf{n}})\| \leq (|\!|\!|T|\!|\!| + \varepsilon)\beta$ for $\mathbf{n} \in A \cap B$, hence $\|T_\Xi(\xi)\| \leq (|\!|\!|T|\!|\!| + \varepsilon)\|\xi\|_{(m)}$.

Let $V(L) = \bigcup_{N \in L} [\log N, \exp N]$. Let $\widetilde{\mathcal{N}}_Y$ be the part of $\widetilde{Y}_0$ of all $\widetilde{x}$ such that

$$\frac{f(l)}{l}(\sum_{j=1}^{l} \widetilde{x} \otimes \mathbf{f}_j)$$

is bounded when $l \in V(L)$. For every $\widetilde{x} \in \widetilde{\mathcal{N}}$ we consider the family

$$m(\widetilde{x}) = \Big(\frac{f(l)}{l}(\sum_{j=1}^{l} \widetilde{x} \otimes \mathbf{f}_j)\Big)_{l \in L}$$

as representing a new vector in a further ultrapower $\widetilde{Y}_\omega$ of Ξ_Y, where the index set is L. By the lower-f estimate, we have $\|m(\widetilde{x})\| \geq \|\widetilde{x}\|$. If $\widetilde{x} \in \widetilde{\mathcal{M}}$, then $\|m(\widetilde{x})\| \leq 1$ by ???, so that $1 \geq \|m(\widetilde{x})\| \geq \|\widetilde{x}\|$. Given $T \in \mathcal{L}(Y, X)$, we can extend T_Ξ to an operator $T_\omega \in \mathcal{L}(\widetilde{Y}_\omega, \widetilde{X}_\omega)$ in the usual way. Then $T_\omega(m(\widetilde{x})) = m(\widetilde{T}\widetilde{x})$.

Lemma. Let $T \in \mathcal{L}(Y, X)$. Then

$$\forall \widetilde{y} \in \widetilde{Y}_0, \quad \left\| m(\widetilde{T}\widetilde{y}) \right\| \leq \|T\| \, \|\widetilde{y}\| \, .$$

Proof. Let $\varepsilon > 0$ and $m \geq 1$ satisfying $(*)$. For every $l \in V(L)$ let

$$\xi_l = \widetilde{y} \otimes \Big(\frac{f(l)}{l} \sum_{j=1}^{k} \mathbf{f}_j \Big).$$

Suppose $\|\widetilde{y}\| \leq 1$. Let $N \in L$ and let $M_1 > 2^{N^2/\varepsilon^2}$. We can find $A_1 \in \mathcal{U}$ such that $\|P_m y_{n_1}\| < 1/m$ and $\|y_{n_1}\|_{(M_1)} \leq 1$ for every $n_1 \in A_1$. For every $n_1 \in A_1$, let $M_2(n_1)$ be such that $\varepsilon\sqrt{f(N_2(n_1))} > |\operatorname{ran}(y_{n_1})|$; we can find $A_2(n_1) \in \mathcal{U}$ such that $\|P_{M_2(n_1)} y_{n_1}\| < \varepsilon/N$ and $\|x_{n_2}\|_{(M_2(n_1))} \leq 1$ for every $n_2 \in A_2(n_1)$; continuing in this way we construct

$$A = \{(n_1, \ldots, n_N) : n_j \in A_j(n_1, \ldots, n_{j-1}), j = 2, \ldots, N\} \in \mathcal{U}^{\otimes N}$$

such that $y_{n_1}, \ldots, y_{n_N}$ is a small perturbation of a RIS(ε) whenever $(n_1, \ldots, n_N) \in A$. This implies that

$$\| \frac{f(l)}{l} \sum_{j=1}^{l} y_{n_j} \|_{(m)} \leq 1 + \varepsilon$$

when $\log N \leq l \leq N$ and m ????? by Lemma 11.7, therefore by $(*)$ we get for every $l \in [\log N, N]$

$$\| \frac{f(l)}{l} \sum_{j=1}^{l} T y_{n_j} \| \leq \|T\| + \varepsilon.$$

We obtain by Lemma 11.5 that

$$\| \frac{f(l)}{l} \sum_{j=1}^{l} T y_{n_j} \|_{(p)} \leq \|T\| + \varepsilon$$

when $p =$????? Since this holds for every $\mathbf{n} \in A$ we obtain $\|T_\Xi \xi_l\|_{(p)} \leq \|T\| + \varepsilon$, so that finally $\|\widetilde{T}_\omega m(\widetilde{x})\|_{(p)} \leq \|T\| \, \|\widetilde{x}\|$ for every $p \geq 1$.

With these elements it is easy to prove property 3 of Theorem 11.1. Suppose that we pick $\widetilde{y}$ such that $\|\widetilde{y}\| = 1$ and

$$\|\widetilde{ST}\widetilde{y}\| = \|ST\| \, .$$

We obtain

$$\|\widetilde{ST}\widetilde{y}\| \le \|m(\widetilde{ST}\widetilde{y})\| = \|S_\omega(m(\widetilde{T}\widetilde{y}))\| \le \|\!|S|\!\|\, \left\|\!\left| m(\widetilde{T}\widetilde{y}) \right|\!\right\| \le \|\!|S|\!\|\, \|\!|T|\!\|\, \|\!|\widetilde{y}|\!\| \,.$$

The next Lemma is the main part of the analysis in [GM2], where the properties of special functionals are fully used, as well as the structure of $\mathbf{S}$. Note first that a proper set $\mathbf{S}$ of spreads must be countable, and if we write it as $\{U_1, U_2, \ldots\}$ and set $\mathbf{S}_m = \{U_1, \ldots, U_m\}$ for every m, then for any $x \in X(\mathbf{S})$, $x^* \in X(\mathbf{S})^*$, we have $\lim_m \sup\{|x^*(Ux)| : U \in \mathbf{S} \setminus \mathbf{S}_m\} \le \|x\|_\infty \|x^*\|_\infty$.

Lemma 11.8. *Let $\mathbf{S}$ be a proper set of spreads, let $X = X(\mathbf{S})$, let $Y \subset X$ be an infinite-dimensional subspace and let T be a continuous linear operator from Y to X. Let $\mathbf{S} = \bigcup_{m=1}^\infty \mathbf{S}_m$ be a decomposition of $\mathbf{S}$ satisfying the condition just mentioned. Then for every $\varepsilon > 0$ there exists m such that, for every $x \in Y$ such that $\|x\|_{(m)} \le 1$ and $\|P_m x\| \le 1/m$,*

$$d(Tx, m\,\mathrm{conv}\{\lambda Ux : U \in \mathbf{S}_m, |\lambda| = 1\}) \le \varepsilon \,.$$

Proof. We may also assume that $\|T\| \le 1$. Suppose that the result is false. Then, for some $\varepsilon > 0$, we can find a sequence $(y_n)_{n=1}^\infty$ with $y_n \in Y$, $\|y_n\|_{(n)} \le 1$ and $\|P_n(y_n)\| \le 1/n$ such that, setting $C_n = n\,\mathrm{conv}\{\lambda U y_n : U \in \mathbf{S}_n, |\lambda| = 1\}$, we have $d(Ty_n, C_n) > \varepsilon$, and we can also find a sequence (E_n) of successive intervals such that if z_n is any one of y_n, Ty_n or Uy_n for some $U \in \mathbf{S}_n$ and z_{n+1} is any one of y_{n+1}, Ty_{n+1} or Vy_{n+1} for some $V \in \mathbf{S}_{n+1}$, then $\|(\mathbb{N} \setminus E_n) z_n\| \le \varepsilon 2^{-n}$ and $\|E_n z_{n+1}\| \le \varepsilon 2^{-n}$.

By the Hahn-Banach theorem, for every n there is a norm-one functional y_n^* such that

$$\sup\{|y_n^*(x)| : x \in C_n + \varepsilon B(X)\} < y_n^*(Ty_n) \,.$$

It follows that $y_n^*(Ty_n) > \varepsilon$ and $\sup |y_n^*(C_n)| \le 1$. Therefore $|y_n^*(Uy_n)| \le n^{-1}$ for every $U \in \mathbf{S}_n$. We may also assume that the support of y_n^* is contained in E_n (up to $1/n$) ???. (The case of complex scalars requires a standard modification.)

Given $N \in L$ define an *N-pair* to be a pair (x, x^*) constructed as follows. Let $y_{n_1}, y_{n_2}, \ldots, y_{n_N}$ be a subsequence of $(y_n)_{n=1}^\infty$ satisfying the RIS(1) condition, which implies that $n_1 > N^2$. Let $x = N^{-1} f(N)(y_{n_1} + \cdots + y_{n_N})$ and let $x^* = f(N)^{-1}(y_{n_1}^* + \cdots + y_{n_N}^*)$, where the $y_{n_i}^*$ are as above. Lemma 11.7 implies that $\|x\| \le 4$ and $\|x\|_{(\sqrt{N})} \le 8$.

If (x, x^*) is such an N-pair, then $x^* \in A_N^*(X)$ and, by our earlier assumptions about supports,

$$x^*(Tx) = N^{-1} \sum_{i=1}^N y_{n_i}^*(Ty_{n_i}) > \frac{\varepsilon}{2} \,.$$

Similarly, $|x^*(Ux)| \le N^{-2}$ for every $U \in \mathbf{S}_N$.

Let $k \in K$ be such that $(\varepsilon/24) f(k)^{1/2} > 1$. We now construct sequences $x_1, \ldots, x_k$ and $x_1^*, \ldots, x_k^*$ as follows. Let $N_1 = j_{2k}$ and let (x_1, x_1^*) be an N_1-pair. Let M_2 be such that $|x_1^*(Ux_1)| \le \|x_1\|_\infty \|x_1^*\|_\infty$ if $U \in \mathbf{S} \setminus \mathbf{S}_{M_2}$. The functional x_1^* can be perturbed so that it is in $\mathbf{Q}$ and so that $\sigma(x_1^*) > \max\{M_2, f^{-1}(4)\}$, while (x_1, x_1^*) is still an N_1-pair.

In general, after $x_1,\dots,x_{i-1}$ and $x_1^*,\dots,x_{i-1}^*$ have been constructed, let (x_i,x_i^*) be an N_i-pair such that all of x_i, Tx_i and x_i^* are supported (up to ???) after all of x_{i-1}, Tx_{i-1} and x_{i-1}^*, and then perturb x_i^* in such a way that, setting $N_{i+1}=\sigma(x_1^*,\dots,x_i^*)$, we have $|x_i^*(Ux_i)| \le \|x_i\|_\infty \|x_i^*\|_\infty$ whenever $U \in \mathbf{S}\setminus \mathbf{S}_{N_{i+1}}$ and we also have $f(N_{i+1}) > 2^{i+1}$ and $\sqrt{f(N_{i+1})} > 2|\operatorname{ran}(\sum_{j=1}^i x_j)|$.

Now let $x=(x_1+\dots+x_k)$ and let $x^*=f(k)^{-1/2}(x_1^*+\dots+x_k^*)$. Our construction guarantees that x^* is a special functional, and therefore of norm at most 1. We therefore have

$$\|Tx\| \ge x^*(Tx) > \varepsilon k f(k)^{-1/2} \ .$$

Our aim is now to get an upper bound for $\|x\|$ and to deduce an arbitrarily large lower bound for $\|T\|$. For this purpose we use Lemma 11.4.

Let g be the function given by Lemma 11.6 in the case $K_0 = K\setminus\{k\}$. It is clear that all vectors Ex are either normed by (M,g)-forms or by spreads of special functionals of length k, or they have norm at most 1. In order to apply Lemma 11.4 with this g, it is therefore enough to show that $|U^*z^*(Ex)| = |z^*(UEx)| \le 1$ for any special functional z^* of length k and $U\in\mathbf{S}$. Let $z^*=f(k)^{-1/2}(z_1^*+\ldots+z_k^*)$ be such a functional with $z_j^*\in A_{m_j}^*$. Suppose that $U\in \mathbf{S}_{M+1}\setminus\mathbf{S}_M$, and let j be such that $N_j\le M < N_{j+1}$. Let t be the largest integer such that $m_t=N_t$. Then $z_i^*=x_i^*$ for all $i<t$, because σ is injective. For such an i, $|z_i^*(Ux_i)| = |x_i^*(Ux_i)| < N_i^{-2}$, if $M<N_i$. If $M\ge N_{i+1}$, then $U\notin \mathbf{S}_{N_{i+1}}$, so $|x_i^*(Ux_i)| \le \|x_i\|_\infty \|x_i^*\|_\infty \le 2^{-i}$. If $N_i\le M<N_{i+1}$, the only remaining case, then $i=j$ and at least we know that $|z_i^*(Ux_i)|\le \|x_i\|\le 4$.

If $l\ne i$ or $l=i>t$, then $z_l^*(Ux_i) = U^*z_l^*(x_i)$, and we have $U^*z_l^*\in A_{m_l}^*$ for some m_l. Moreover, because σ is injective and by definition of t, in both cases $m_l\ne N_i$. If $m_l<N_i$, then, as we remarked above, $\|x_i\|_{(\sqrt{N_i})}\le 8$, so the lower bound of j_{2k} for m_1 tells us that $|U^*z_l^*(x_i)|\le k^{-2}$. If $m_l>N_i$, the same conclusion follows from Lemma ?. There are at most two pairs (i,l) for which $0\ne z_l^*(UEx_i)\ne z_l^*(Ux_i)$ and for such a pair $|z_l^*(UEx_i)|\le 1$.

Putting all these facts together, we get that $|z^*(UEx)|\le 1$, as desired. We also know that $(1/8)(x_1,\dots,x_k)$ satisfies the RIS(1) condition. Hence, by Lemma 11.4, $\|x\|\le 24kg(k)^{-1} = 24kf(k)^{-1}$. It follows that $\|T\|\ge(\varepsilon/24)f(k)^{1/2}>1$, a contradiction.

We can now explain how the main assertion 4 of Theorem 11.1 follows by a fixed point argument. Suppose that $m(\widetilde{x}) = m_1(\widetilde{x})$, $m_2(\widetilde{x})$,... are successive copies of $m(\widetilde{x})$. For example,

$$m_2(\widetilde{x}) = \Big(\frac{f(l)}{l}\sum_{j=1}^{l}\widetilde{x}\otimes \mathbf{f}_{l+j}\Big)_{l\in L}.$$

For every l, the vector

$$\xi_{1,l}+\dots+\xi_{k,l} = \sum_{i=1}^{k}\frac{f(l)}{l}\sum_{j=1}^{l}\widetilde{x}\otimes\mathbf{f}_{(i-1)l+j}$$

is the sum of kl successive vectors $\widetilde{x} \otimes \mathbf{f}_i$ in Ξ_0, therefore

$$\|\sum_{j=1}^{k} \xi_{j,l}\| \geq \frac{kl}{f(kl)} \frac{f(l)}{l} \|\widetilde{x}\|$$

and $f(kl)/f(l)$ tends to 1 for fixed k when $l \to \infty$, so that

$$\|m_1(\widetilde{x}) + \cdots + m_k(\widetilde{x})\| \geq k\|\widetilde{x}\|.$$

More generally, if $\widetilde{x}_1, \ldots, \widetilde{x}_k \in \widetilde{\mathcal{N}}$, we get

$$\|m_1(\widetilde{x}_1) + \cdots + m_k(\widetilde{x}_k)\| \geq \sum_{j=1}^{k} \|\widetilde{x}_j\|.$$

Indeed,...

Lemma. Suppose that $T \in \mathcal{L}(Y, X)$ and that m and ε are as in the above Lemma. Let

$$\mathcal{A}_m = \text{conv}\{\lambda U : U \in \mathbf{S}_m, |\lambda| = 1\}.$$

Then there exists $U \in \mathcal{A}_m$ such that $|\!|\!|T - U \circ i_{Y,X}|\!|\!| < 8\varepsilon$.

Proof. If the Lemma is false, then for every $U \in \mathcal{A}_m$ there is a sequence $\widetilde{x}_U \in \widetilde{Y}_0$ such that $|\!|\!|\widetilde{x}_U|\!|\!| \leq 1$ and $\|(\widetilde{T - U})\widetilde{x}_U\| > 17\varepsilon$. Our first aim is to show that these $\widetilde{x}_U$ can be chosen continuously in U. Let $(\mathcal{U}_j)_{j=1}^k$ be a covering of $\mathcal{A}_m$ by open sets of diameter less than ε in the operator norm. For every $j = 1, \ldots, k$, let $U_j \in \mathcal{U}_j$ and let $\widetilde{x}_j$ be a sequence with the above property with $U = U_j$. By the condition on the diameter of $\mathcal{U}_j$, we have $\|(\widetilde{T - U})\widetilde{x}_j\| > 16\varepsilon$ for every $U \in \mathcal{U}_j$. Let $(\phi_j)_{j=1}^k$ be a partition of unity on $\mathcal{A}_m$ with ϕ_j supported inside $\mathcal{U}_j$ for each j.

Now let us consider in $\widetilde{Y}_\omega$ the vector $y(U) = \sum_{j=1}^k \phi_j(U) m_j(\widetilde{x}_j)$. We shall show that $y(U)$ is a "bad" vector for U, by showing that $\|(T_\omega - U_\omega)y(U)\| > 8\varepsilon$. To do this, let $U \in \mathcal{A}_m$ be fixed and let $J = \{j : \phi_j(U) > 0\}$. Note that $\left\|(\widetilde{T - U})\widetilde{x}_j\right\| > 16\varepsilon$ for every $j \in J$, hence

$$\|(T_\omega - U_\omega)y(U)\| = \|\sum_{j=1}^{k} \phi_j(U) m_j((\widetilde{T - U})\widetilde{x}_j)\| \geq \sum_{j=1}^{k} \phi_j(U)\|(\widetilde{T - U})x_j\| > 16\varepsilon.$$

The function $U \mapsto y(U)$ is clearly continuous. We now apply a fixed-point theorem. For every $U \in \mathcal{A}_m$, let $\Gamma(U)$ be the set of $V \in \mathcal{A}_m$ such that $\left\|(\widetilde{T - V})y(U)\right\| \leq 8\varepsilon$. Clearly $\Gamma(U)$ is a compact convex subset of $\mathcal{A}_m$. By the previous lemma, $\Gamma(U)$ is non-empty for every U. The continuity of $U \mapsto y(U)$ gives that Γ is upper semi-continuous, so there exists a point $U \in \mathcal{A}_m$ such that $U \in \Gamma(U)$. But this is a contradiction.

12. Applications to some specific examples

In this section we present some specific examples which are special cases of Theorem 11.1.

Construction of a H.I. space

Let $\mathbf{S} = \{Id\}$, let $X = X(\mathbf{S})$, let Y be any subspace of X and let $i_{Y,X}$ be the inclusion map from Y to X. Then given any operator T from Y to X, there exists by Theorem 11.1, for every $\varepsilon > 0$, some λ such that $|\!|\!|T - \lambda i_{Y,X}|\!|\!| < \varepsilon$. Since $|\lambda| \le \|T\| + \varepsilon$, an easy compactness argument then shows that there exists λ such that $|\!|\!|T - \lambda i_{Y,X}|\!|\!| = 0$ and thus that $T - \lambda i_{Y,X}$ is strictly singular, which is one of the main results of [GM1]. It implies easily that X is hereditarily indecomposable. Recall (Exercise 10.1) that X^n is isomorphic to X^m if and only if $m = n$.

Shift space X_s

Let $\mathbf{S}$ be the proper set generated by the right shift R on c_{00}. This set $\mathbf{S}$ consists of all maps of the form $S_{A,B}$ where $A = [m, \infty)$ and $B = [n, \infty)$. Let $X_s = X(\mathbf{S})$. We will write L for the left shift, which is (formally) the adjoint of R, and Id for the identity on c_{00}. Then $LR = Id$ and every operator in $\mathcal{S}$ is of the form $R^m L^n$. Since $RL - Id$ is of rank one, every operator V in $\mathcal{A}$ is a finite-rank perturbation of an operator of the form

$$U = \sum_{n=-N}^{-1} a_n L^{-n} + \sum_{n=0}^{N} a_n R^n,$$

so the difference $V - U$ is of $|\!|\!|.|\!|\!|$-norm zero, hence strictly singular.

For every such $U \in \mathcal{A}$ we define a fonction φ_U on the unit circle $\mathbb{T}$ by

$$\varphi_U(\lambda) = \sum_{i=-N}^{N} a_i \lambda^i.$$

For every $\lambda \in \mathbb{T}$ it is easy to find a normalized sequence $(x_n^{(\lambda)})$ in $\mathcal{M}$ such that $Rx_n^{(\lambda)} - \lambda x_n^{(\lambda)} \to 0$ (such vectors are simply obtained by Lemma 11.1 by normalizing a sum like $\sum_{j=k}^{k+N} \lambda^{-j} \mathbf{e}_j$, where N is much larger than k). This implies for any such sequence $(x_n^{(\lambda)})$ that $Ux_n^{(\lambda)} \simeq \varphi_U(\lambda) x_n^{(\lambda)}$, and we know that $\limsup_n \|Ux_n^{(\lambda)}\| \le |\!|\!|U|\!|\!|$. It follows that $\|\varphi_U\|_\infty \le |\!|\!|U|\!|\!|$ (the uniform norm is taken on the unit circle $\mathbb{T}$).

For a general $V \in \mathcal{A}$ we notice that $|\!|\!|V - U|\!|\!| = 0$ implies that $\lim_n V(x_n^{(\lambda)}) - \varphi_U(\lambda) x_n^{(\lambda)} = 0$. We may thus define $\varphi_V(\lambda)$ to be the only scalar μ such that $\lim_n V(x_n^{(\lambda)}) - \mu x_n^{(\lambda)} = 0$ for any sequence $(x_n^{(\lambda)}) \in \mathcal{M}$ such that $\lim(R - \lambda I_X)(x_n^{(\lambda)}) = 0$; then $|\!|\!|V - U|\!|\!| = 0$ implies that $\varphi_V = \varphi_U$; it follows easily that $\varphi_{V_1}\varphi_{V_2} = \varphi_{V_1 V_2}$. By properties 4 and 5 we can therefore extend the map φ to an algebra homomorphism $\varphi : U \to \varphi_U$ from $\mathcal{L}(X)$ to $C(\mathbb{T})$.

Proposition 12.1. *Let $T \in \mathcal{L}(X_s)$; the operator T is finitely singular iff φ_T does not vanish on $\mathbb{T}$.*

Proof. Suppose that $\lambda \in \mathbb{T}$ and $\varphi_T(\lambda) = 0$, choose $U \in \mathcal{A}$ such that $\|\|T - U\|\| \simeq 0$. Then $\varphi_U(\lambda) \simeq 0$ which implies that $U(x_n^{(\lambda)}) \simeq 0$ therefore $T(x_n^{(\lambda)}) \simeq 0$ and T is infinitely singular.

In the other direction, assume T infinitely singular. We can find a block subspace Y such that $\|T_{|Y}\| < \varepsilon$ by Proposition 3.2, hence a vector $x \in Y$ such that $1 = \|x\| \leq \|x\|_n \leq 1 + \varepsilon$ (by Lemma 11.1) and $\|Tx\| < \varepsilon$. Next we get a normalized weakly null sequence $\widetilde{x} = (x_n)$ in $\mathcal{M}(Y)$ such that $Tx_n \to 0$. We have $\|\widetilde{x}\| = 1$, $\widetilde{x} \in \widetilde{X}_0$ (the weakly null part of the ultrapower $\widetilde{X}$) and $\widetilde{T}\widetilde{x} = 0$. Let $\widetilde{T}_0$ denote the restriction of $\widetilde{T}$ to $\widetilde{X}_0$. Let $U \in \mathcal{A}$ be such that $\|\|T - U\|\| < \varepsilon$. Then $\widetilde{R}\widetilde{U} - \widetilde{U}\widetilde{R} = 0$ on $\widetilde{X}_0$ because $RU - UR$ has finite rank. Now $\|\widetilde{U}\widetilde{x}\| \leq \varepsilon$ because $\widetilde{x} \in \mathcal{M}$. Since R is an isometry preserving successiveness, we see that $\widetilde{R}\widetilde{x} \in \mathcal{M}$, and $\widetilde{R}\widetilde{U}\widetilde{x} = \widetilde{U}\widetilde{R}\widetilde{x}$, so $\|\widetilde{T}\widetilde{R}\widetilde{x}\| \leq 2\varepsilon$. We get $\widetilde{T}\widetilde{R}\widetilde{x} = 0$, and similarly for every $k \geq 1$ we have $\widetilde{T}\widetilde{R}^k\widetilde{x} = 0$. It follows that we can find an invariant space for $\widetilde{R}$ on which $\widetilde{T}_0 = 0$. We can then find an approximate eigenvector $\widetilde{y}$ for $\widetilde{R}$ such that $\widetilde{T}\widetilde{y} = 0$, and the eigenvalue must be some $\lambda \in \mathbb{T}$. Then $\varphi_T(\lambda) = 0$.

Projections in X_s

Suppose that P is a projection on X_s. Since φ is an algebra homomorphism, it follows from $P^2 = P$ that $\varphi_P^2 = \varphi_P$, hence 0 and 1 are the only possible values for $\varphi_P(\lambda)$. By continuity we get either $\varphi_P = 0$ or $\varphi_P = 1$. In the second case P is finitely singular by Proposition 12.1, hence has finite dimensional kernel. In the other case we get a finite dimensional range. We see that X_s is indecomposable. However, X_s is not H.I. For every $\lambda \in \mathbb{T}$, we can find an H.I. subspace X_λ of X_s by considering a subspace generated by a normalized basic subsequence (x'_n) of the sequence $(x_n^{(\lambda)})$ such that $Rx'_n - \lambda x'_n \to 0$ rapidly. If $\lambda \neq \mu$, it is easy to see that $Y = X_\lambda + X_\mu$ is closed, which implies that Y is decomposable and X_s is not H.I.

Remark-Exercise. These spaces X_λ are pairwise non-isomorphic for $\lambda \in \mathbb{T}$. We have an uncountable family of different germs indexed by the unit circle $\mathbb{T}$.

This space X_s is a new prime space. The only known examples before [GM2] were c_0 and ℓ_p ($1 \leq p \leq \infty$). The space X_s is prime by virtue of having no non-trivial complemented subspaces and being isomorphic to its subspaces of finite codimension. Indeed, we know that every projection P on X_s is of finite rank or corank. Thus, if PX_s is infinite-dimensional, then it has finite codimension. Since the shift on X_s is an isometry, it follows that X_s and PX_s are isomorphic, which proves the following theorem (using next Exercise).

Theorem 12.1. *The space X_s is prime.*

Exercise. The hyperplanes of a given Banach space X are mutually isomorphic. More generally, all subspaces of X of a fixed finite codimension are isomorphic.

We note here that the argument in the above proof can be generalized to show that if m and n are integers with $m > n$, then X_s^n does not contain a family $P_1, \ldots, P_m$ of infinite-rank projections satisfying $P_iP_j = 0$ whenever $i \neq j$. Indeed, given any projection $P \in \mathcal{L}(X^n)$, we can regard it as an element of $M_n(\mathcal{L}(X))$. Acting on each entry with φ, we get a function $h \in M_n(C(\mathbb{T}))$. The map taking P to h is an algebra homomorphism so h is an idempotent. Regarding h as a continuous function from $\mathbb{T}$ to $M_n(\mathbb{C})$, we have that $h(t)$ is an idempotent in $M_n(\mathbb{C})$ for every $t \in \mathbb{T}$. By the continuity of rank for idempotents, we have that if $h(t) = 0$ for some t, then h is identically zero. But then P is strictly singular and hence of finite rank. Applying this reasoning to the family $P_1, \ldots, P_m$ above, we obtain $h_1, \ldots, h_m$ such that, for every $t \in \mathbb{T}$, $h_1(t), \ldots, h_m(t)$ is a set of non-zero idempotents in $M_n(\mathbb{C})$ with $h_i(t)h_j(t) = 0$ when $i \neq j$. But this is impossible if $m > n$. It follows that X_s^n and X_s^m are isomorphic if and only if $n = m$.

Another simple consequence of properties 4 and 5 is that, up to strictly singular perturbations, any two operators on X_s commute. Indeed, $U_1U_2 - U_2U_1$ has finite rank when $U_1, U_2 \in \mathcal{A}$, hence $\|\!|U_1U_2 - U_2U_1|\!\| = 0$. By approximation we get $\|\!|T_1T_2 - T_2T_1|\!\| = 0$ for every pair of operators on X_s.

One can actually get better estimates relating X_s to the Wiener algebra.

Lemma. (Lemma 11 of [GM2].) Let $U = \sum_{n=0}^{N} \lambda_n R^n + \sum_{n=1}^{N} \lambda_{-n} L^n$. Then

$$\|U\| = \|\!|U|\!\| = \sum_{n \in \mathbb{Z}} |\lambda_n|.$$

It follows that the homomorphism φ takes values in the Wiener algebra W and it is possible to improve in this case property 5 by saying that for every $T \in \mathcal{L}(X_s)$, there exists $U = \sum_{n=1}^{\infty} a_{-n} L^n + \sum_{n=0}^{\infty} a_n R^n$ such that $\sum_{n \in \mathbb{Z}} |a_n| < \infty$ and $\|\!|T - U|\!\| = 0$. Using this and the properties of invertible elements in W we get in [GM2] an easier approach to Proposition 12.1.

Exercise. Compute $K_0(\mathcal{L}(X_s))$ and $K_1(\mathcal{L}(X_s))$.

The result in this section can be compared to those of Mankiewicz [Mz]; we have here another example of a Banach space such that there exists an algebra homomorphism from $\mathcal{L}(X)$ into a commutative Banach algebra. It follows that X is not isomorphic to any power Y^n, for $n \geq 2$. Indeed, if φ is a non zero multiplicative functional from $\mathcal{L}(X)$ to $\mathbb{C}$, and if $X = Y^n$, there is a natural homomorphism i from M_n to $\mathcal{L}(X)$. But then $\varphi \circ i$ would be a non zero multiplicative functional on M_n, which is not possible for $n \geq 2$.

Double shift space X_d

This example X_d is isomorphic to its codimension 2 subspaces but not to its hyperplanes. Let $\mathbf{S}$ be the proper set generated by the double shift R^2. That is, $\mathbf{S}$ is as in the previous example but m and n are required to be even. We show that every Fredholm operator T on $X_d = X(\mathbf{S})$ has even index. By property 4, and by the fact that every operator in $\mathbf{S}$ differs by a finite-rank operator from some even shift, we can find, for any $\varepsilon > 0$, some linear combination U of even shifts such that $\|\!|T - U|\!\| < \varepsilon$. Then $s(T-U) < \varepsilon$ and we know that $\mathrm{ind}(T) = \mathrm{ind}(U)$ when ε is small by Corollary 6.1. Hence it is enough to show that every $U \in \mathcal{A}$ has even index.

Lemma. Let V be a Fredholm isometry on a Banach space X with a left inverse W, and let $T : X \to X$ be a Fredholm operator which can be written in the form $P(V) + Q(W)$ for polynomials P and Q. Then the index of T is a multiple of the index of V.

Proof. Suppose first that the scalars are complex. It is clear since V is isometric that $V - \lambda I_X$ is an into isomorphism when $|\lambda| \neq 1$ (and it is onto when $|\lambda| > 1$). By Lemma 4.6, we get $\mathrm{ind}(V - \lambda I_X) = \mathrm{ind}(V)$ when $|\lambda| < 1$ (because we can connect $V - \lambda I_X$ to V by a path of semi-Fredholm operators) and $\mathrm{ind}(V - \lambda I_X) = 0$ when $|\lambda| > 1$. If $V - \lambda I_X$ is finitely singular for some $\lambda \in \mathbb{T}$, then $\mathrm{ind}(V - \lambda I_X) = 0$ for the same reason. In all cases, the only possible values for the index are 0 and $\mathrm{ind}(V)$.

Now suppose that T is as in the statement of the lemma. For sufficiently large N, TV^N can be written $F(V)$ for some polynomial F and is still Fredholm. Writing $F(V) = c\prod_i (V - \lambda_i I_X)$, we must have $V - \lambda_i I_X$ finitely singular for TV^N to be Fredholm, so $\mathrm{ind}(V - \lambda_i I_X)$ is either 0 or $\mathrm{ind}(V)$. It follows from the composition formula, Proposition 4.2, that the index of $F(V)$, and hence that of T, is a multiple of the index of V as stated.

When the scalars are real we may complexify V to an isometry $V_{\mathbb{C}}$ of $X_{\mathbb{C}}$, for example using the injective norm $\mathbb{C} \otimes_\varepsilon X$ on $X_{\mathbb{C}}$.

Remark. When $\mathbb{K} = \mathbb{C}$ and V is not invertible, the spectrum of V contains $\mathbb{T}$.

Putting these facts together, we find that no continuous operator on X_d can be Fredholm with odd index. We therefore have the following result.

Theorem 12.2. *The space X_d is isomorphic to its subspaces of even codimension while not being isomorphic to those of odd codimension. In particular, it is isomorphic to its subspaces of codimension two but not to its hyperplanes.*

Remark. The proof of Theorem 12.1 gives for this space also that every complemented subspace has finite dimension or codimension. Combining this observation with Theorem 12.2, we see that the space X_d has exactly two infinite-dimensional complemented subspaces, up to isomorphism. It is true for this space as well that it is isomorphic to no subspace of infinite codimension. Note that the methods of this section generalize easily to proper sets generated by larger powers of the shift.

Exercise. Compute $K_0(\mathcal{L}(X_d))$.

Ternary space

This application is more complicated than the previous ones. The aim is to construct a space X_t which is isomorphic to $X_t \oplus X_t \oplus X_t$ but not to $X_t \oplus X_t$. This question is related to the Schröder-Bernstein problem for Banach spaces, first solved (by the negative) by T. Gowers [G3]. We have seen that in some cases, we can deduce that $X \simeq Y$ from the fact that X and Y embed complementably in each other. The Schröder-Bernstein problem for Banach spaces is the question whether this is true in general. Constructing a Banach space X such that $X \simeq X^3$ but $X \not\simeq X^2$ gives a strong negative question to the problem, because X and X^2 are in this case complementably embeddable in each other.

There is a very natural choice of $\mathbf{S}$ in this case, strongly related to the algebra $\mathcal{P}$ from section 7 (or to the Cuntz' algebra $\mathcal{O}_3$). For $i = 0, 1, 2$ let A_i be the set of positive integers equal to $i + 1 \pmod 3$, let U'_i be the spread from $\mathbb{N}$ to A_i and let $\mathbf{S}'$ be the semigroup generated by U'_0, U'_1 and U'_2 and their adjoints. It is shown in [GM2] that this is a proper set. The space $X_t = X(\mathbf{S}')$ is easily seen to be isomorphic to its cube, and this isomorphism is achieved in a "minimal" way. (The primes in this paragraph are to avoid confusion later.) This is one of the models introduced in section 7 for the algebra $\mathcal{P}$. We equipped it here with the norm of $\mathcal{L}(X)$, where X is an exotic Banach space given by Theorem 11.1.

We shall indeed consider the space $X(\mathbf{S}')$ defined above. However, we define it slightly less directly, which helps with the proof later that it is not isomorphic to its square. The algebra $\mathcal{A}'$ arising from the above definition is, if completed in the ℓ_2-norm, isometric to the Cuntz algebra $\mathcal{O}_3$ ([C1], see section 7). Our proof is inspired by his paper [C2]. Recall some notation from section 7: $\mathcal{T}$ is the ternary tree $\bigcup_{n=0}^{\infty}\{0,1,2\}^n$, Y_{00} is the vector space of finitely supported scalar sequences indexed by $\mathcal{T}$ and the canonical basis for Y_{00} is denoted by $(e_t)_{t\in\mathcal{T}}$; we write e for $e_\emptyset$, denote the length of a word $t\in\mathcal{T}$ by $|t|$ and (s,t) stands for the concatenation of $s,t\in\mathcal{T}$. Let Id denote the identity operator on the space of sequences. Let V_i and T_i, for $i=0,1,2$ be defined by their action on the basis as follows:

$$V_i e_t = e_{(t,i)},\ T_i e_t = e_{(i,t)}.$$

Thus T_i takes the whole tree $\mathcal{T}$ on to the i^{th} branch. The adjoints V_i^* and T_i^* act in the following way: $V_i^* e_t = e_s$ if t is of the form $t=(s,i)$, and $V_i^* e_t = 0$ otherwise, while $T_i^* e_t = e_s$ if $t=(i,s)$, and $T_i^* e_t = 0$ otherwise. The following facts are easy to check: $V_iT_j = T_jV_i$, $V_i^*V_j = T_i^*T_j = \delta_{i,j}Id$; $V_iV_i^*$ and $T_iT_i^*$ are projections; if Q denotes the natural rank one projection on the line $\mathbb{C}e$, then $\sum_{i=0}^{2} V_iV_i^* = \sum_{i=0}^{2} T_iT_i^* = Id - Q$. Let $\mathbf{S}$ and $\mathcal{A}$ be respectively the proper set generated by V_0, V_1 and V_2, and the algebra generated by this proper set. (Strictly speaking, $\mathbf{S}$ is not a proper set, but it is easy to embed $\mathcal{T}$ into $\mathbb{N}$ so that the maps V_0, V_1 and V_2 become spreads as defined earlier. Note that $\mathbf{S}$ is the semigroup generated by the V_i and the V_i^*, that it contains Id and that $\mathcal{A}$ contains Q, as we have just shown.

In order to obtain the space X_t, consider the subset $\mathcal{T}_0$ of $\mathcal{T}$ consisting of all words $t\in\mathcal{T}$ that do not start with 0 (including the empty sequence). We modify the definition of V_0 slightly when defining U_0, by letting U_0e equal e instead of e_0. Operators U_1 and U_2 are defined exactly as V_1 and V_2 were. We still have that the $U_i{U_i}^*$ are projections and that ${U_i}^*U_j = \delta_{i,j}Id$, but this time $\sum_{i=0}^{2} U_i{U_i}^* = Id$. We noticed in section 7 that we can associate the integer $n_s = 3^{n-1}i_1 + \cdots + 3i_{n-1} + i_n + 1$, (with $n_\emptyset = 1$), and this defines a bijection between $\mathcal{T}_0$ and $\mathbb{N}$. The operators U_0, U_1 and U_2 then coincide with the spreads on c_{00} defined earlier, so we can define $\mathbf{S}'$ to be the proper set they generate and obtain the space $X_t = X(\mathbf{S}')$. Let $\mathcal{A}'$ be the algebra generated by $\mathbf{S}'$.

For $t\in\mathcal{T}$, we defined V_t inductively by $V_{(t,i)} = V_iV_t$. Let V_t^* be the adjoint of V_t. We now from section 7 that every $W\in\mathcal{A}$ has a decomposition

$$W = \sum_{l=1}^{N} c_l\, V_{\alpha_l} V_{\beta_l}^*\ ,$$

where α_l and β_l are words in $\mathcal{T}$. Define $\beta(W)$ to be the smallest value of $\max_l |\beta_l|$ over all such representations of W. We make the obvious modifications to the above definitions for $\mathcal{A}'$. The remarks are still valid, except that the actions of V_t and U_t on e will be different if the word t begins with 0. The next lemma is similar to Lemma 11 in [GM2]. The notation $\|x\|_1$ is for the norm in ℓ_1.

Lemma 12.1. (Lemma 20 of [GM2].) *Let $U \in \mathcal{A}'$. Then for $|t| > \beta(U)$, we have the inequality $\|Ue_t\|_1 \leq \|\|U\|\|$.*

Proof. Let $|t| > \beta(U)$ and suppose that $Ue_t = \sum_{k=1}^M c_k e_{s_k}$, where the s_ks are distinct. Since $|t| > \beta(U)$, we have $Ue_{u,t} = \sum_{k=1}^M c_k e_{u,s_k}$ for every $u \in \mathcal{T}$. Pick a sequence $(u_j)_{j=1}^\infty$ lacunary enough to guarantee that the sequences $(e_{u_j,t})_{j=1}^\infty$ and $(Ue_{u_j,t})_{j=1}^\infty$ are successive. Then by the construction of X_t, we obtain the inequality

$$\|U(\sum_{j=1}^N e_{u_j,t})\| \geq \frac{N}{f(MN)} \sum_{k=1}^M |c_k|.$$

By Lemma 11.7 we know that for some infinite subset $L \subset \mathbb{N}$ and for every $N \in L$

$$\|\sum_{j=1}^N e_{u_j,t}\| \leq \frac{N}{f(N)}.$$

Letting $N \to \infty$, this gives $\sum_{k=1}^M |c_k| \leq \|U\|$. For the inequality for $\|\|U\|\|$, see [GM2] or Lemma 11.7. ?????

We now consider the algebra $\mathcal{A}$. Let $Y = \ell_1(\mathcal{T})$ be the completion of Y_{00} equipped with the ℓ_1 norm and let $\mathcal{E}$ denote the norm closure of $\mathcal{A}$ in $\mathcal{L}(Y)$. Note that every V_i or T_i is an isometry on Y, and $\|V_i^*\| \leq 1$, $\|T_i^*\| \leq 1$.

Lemma 12.2. *Every Fredholm operator in $\mathcal{E}$ has index 0. More generally, every Fredholm operator $T : Y^n \to Y^n$ given by a matrix in $M_n(\mathcal{E})$ has index* 0.

Proof. Since the Fredholm index is stable under small perturbations, it is enough to consider operators in $\mathcal{A}$ (as operators on Y). For any such operator W we associate the operator

$$W^\# = \sum_{i=0}^2 T_i W T_i^*.$$

We claim that $W^\#$ is a finite rank perturbation of W. It is enough to show that $V_i^\#$ is a rank-one perturbation of V_i (and to observe that $(W^*)^\# = (W^\#)^*$). But

$$V_j^\# = \sum_{i=0}^2 T_i V_j T_i^* = V_j \sum_{i=0}^2 T_i T_i^* = V_j(I - Q) = V_j - V_j Q \ ;$$

(instead of using approximation of elements in $\mathcal{E}$ by elements in $\mathcal{A}$, we could observe directly that for every $V \in \mathcal{E}$, $V - V^\#$ is compact). Consider the projections $Q_i = T_i T_i^*$. Then $Q_i Q_j = 0$ for $i \neq j$ and

$$Y = \mathbb{C}e \oplus Q_0 Y \oplus Q_1 Y \oplus Q_2 Y.$$

Each $T_i W T_i^*$ represents an operator on $Q_i Y$, equivalent (in the obvious sense) to W on Y, so that $\operatorname{ind}(T_i W T_i^*) = \operatorname{ind}(W)$ (the first operator is obtained from W by composition with onto isomorphisms) and $W^{\#}$ is 0 on the component $\mathbb{C}e$. It follows that $\operatorname{ind}(W^{\#}) = 3\operatorname{ind}(W)$. On the other hand $\operatorname{ind}(W^{\#}) = \operatorname{ind}(W)$ since it is a finite rank perturbation of W. It follows that $\operatorname{ind}(W) = 0$.

The proof is essentially the same for the more general statement. Given $T \in \mathcal{L}(Y^n)$, represented by a matrix $A \in M_n(\mathcal{E})$, use the #-operation on each entry. The resulting matrix is equivalent to three copies of A plus the zero matrix in M_n. This zero matrix contributes n to the dimension of the kernel and n to the codimension of the image, from which we obtain the equation

$$\operatorname{ind}(T) = \operatorname{ind}(T^{\#}) = 3\operatorname{ind}(T) + n - n.$$

Remark. In the rectangular case, if $T : Y^m \to Y^n$ is Fredholm, then $2\operatorname{ind}(T) + m - n = 0$. This shows that there is no Fredholm operator from Y^m to Y^n when $m - n$ is odd.

Let $\mathcal{I}$ be the closed two-sided ideal in $\mathcal{E}$ generated by Q. This ideal contains all rank-one operators of the form $e_s^* \otimes e_t$ with $s, t \in \mathcal{T}$. Hence, every finite rank operator on Y^n which is w^*-continuous (considering Y^n as the dual of $(c_0)^n$) belongs to $M_n(\mathcal{I})$. Indeed, the matrix of such an operator consists of entries which are finite sums of the form $\sum_k y_k \otimes x_k$, with $y_k \in c_0$. We can approximate y_k and x_k by finitely supported sequences y'_k and x'_k, and $\sum_k y'_k \otimes x'_k$ certainly belongs to $\mathcal{I}$. (In fact, $\mathcal{I}$ consists exactly of the compact w^*-continuous operators on ℓ_1.)

Lemma. If $V \in M_n(\mathcal{E})$ is Fredholm then there exists $W \in M_n(\mathcal{I})$ such that $V + W$ is invertible in $M_n(\mathcal{E})$.

Proof. By Lemma 12.2 the index of V is zero. Let $x_1, \ldots, x_N$ and $z_1, \ldots, z_N$ be bases for the kernel and cokernel. We can construct a w^*-continuous projection $\sum_{k=1}^N y_k \otimes x_k$ on the kernel. Then $W = \sum_{k=1}^N y_k \otimes z_k$ will do.

Let $\mathcal{O}$ denote the quotient algebra $\mathcal{E}/\mathcal{I}$. Since $\mathcal{I}$ consists of compact operators, we know that every operator on Y (or on Y^n) which is invertible modulo $\mathcal{I}$ (or mod $M_n(\mathcal{I})$) is Fredholm on Y or on Y^n by Corollary 4.3. Hence any lifting in $M_n(\mathcal{E})$ of an invertible element in $M_n(\mathcal{O})$ is Fredholm on Y^n. As an immediate consequence of the preceding discussion we have the following statement.

Corollary 12.1. *Every invertible element of $M_n(\mathcal{O})$ can be lifted to an invertible element of $M_n(\mathcal{E})$.*

It follows easily from Lemma 12.1 that $|||.|||$ is actually a norm on $\mathcal{A}'$. Let $\mathcal{G}$ be the Banach algebra $|||.|||$-completion of $\mathcal{A}'$. Recall that by properties 3 and 4 of Theorem 11.1 there is a unital algebra homomorphism $\varphi : \mathcal{L}(X) \to \mathcal{G}$.

Lemma. There is a norm-one algebra homomorphism θ from $\mathcal{G}$ to $\mathcal{O}$.

Proof. Define a map $\theta_0 : \mathcal{A}' \to \mathcal{O}$ as follows. Given $U \in \mathcal{A}'$, write $U = \sum_{l=1}^N c_l \, U_{\alpha_l} {U_{\beta_l}}^*$ in some way, consider the corresponding sum $\sum_{l=1}^N c_l V_{\alpha_l} V_{\beta_l}^*$ as an element of $\mathcal{E}$ and let $\theta_0(U)$ be the image of this operator under the quotient map from $\mathcal{E}$ to $\mathcal{O}$. To see that this map is well defined, observe that for any pair of words α and β we have the equation $U_\alpha {U_\beta}^* = \sum_{i=0}^{2} U_{(i,\alpha)} {U_{(i,\beta)}}^*$. If n is sufficiently large, we can therefore write U as above in such a way that all the α_l are words of length n. Let W_n be the set of all words of length n. Then what we have said implies that U can be written as a sum $\sum_{\alpha \in W_n} U'_\alpha T^*_\alpha$, where each T^*_α is some linear combination of distinct operators of the form ${U_\beta}^*$. It is easy to see now that $U = 0$ if and only if $T_\alpha = 0$ for every $\alpha \in W_n$, and moreover that distinct ${U_\beta}^*$ are linearly independent. Therefore any $U \in \mathcal{A}'$ has at most one representation in the above form. In $\mathcal{A}$ we know that for any pair of words α and β the images in $\mathcal{O}$ of the operators $V_\alpha V^*_\beta$ and $\sum_{i=0}^{2} V_{(i,\alpha)} V^*_{(i,\beta)}$ are the same. It follows that θ_0 is well defined. Similarly, one can show that it is a unital algebra homomorphism.

We may want to argue in this way: $\mathcal{E}/\mathcal{I}$ is an algebra with six elements which are the classes w_i of U_i and w^*_i of U^*_i, and these elements satisfy the defining properties of our algebra $\mathcal{P}$ from section ???, because $U^*_i U_i = Id$ and $Id - \sum U_i U^*_i \in \mathcal{I}$, therefore there exists an algebra homomorphism ρ from $\mathcal{P}$ to $\mathcal{E}/\mathcal{I}$ such that $\rho(u_i) = w_i$ and $\rho(u^*_i) = w^*_i$. ....

Let P_n denote the projection on to the first n levels of the tree $\mathcal{T}$, so that $P_n \in \mathcal{I}$ for every n. If $U \in \mathcal{A}'$, then Lemma 12.1 implies that

$$\lim_n \|U(I - P_n)\|_{\mathcal{L}(Y)} \leq \|U\| \ .$$

It follows that we may extend θ_0 to a norm-one homomorphism $\theta : \mathcal{G} \to \mathcal{O}$, as claimed.

We work with complex scalars for the rest of this section. The proof given in [GM2] works also in the real case, but we want to apply here directly K-theoretic results that are proved in the complex case.

Theorem 12.3. *The spaces X_t and $X_t \oplus X_t$ are not isomorphic.*

Proof. If X_t and X_t^2 are isomorphic we know that $[I_{X_t}] = 0$ in $K_0(\mathcal{L}(X_t))$ by ?????. Taking the image under $\theta \circ \phi : \mathcal{L}(X_t) \to \mathcal{O}$, this yields $[1_\mathcal{O}] = 0$ in $K_0(\mathcal{O})$. All we have to show now is that $[1_\mathcal{O}] \neq 0$. For this we follow the proof given by Cuntz for Theorem 3.7 of [C2]. By the definition of equivalence for idempotents, $1_\mathcal{E} = {V_i}^* V_i$ and $V_i V^*_i$ are equivalent. The relation $Id - Q = \sum_{i=0}^{2} V_i V^*_i$ implies in $K_0(\mathcal{E})$ that

$$[1_\mathcal{E}] - [Q] = 3[1_\mathcal{E}],$$

and therefore that $[Q] = -2[1_\mathcal{E}]$. Now consider the short exact sequence

$$0 \to \mathcal{I} \xrightarrow{j} \mathcal{E} \xrightarrow{\pi} \mathcal{O} \to 0$$

and the corresponding exact sequence in K-theory

$$K_1(\mathcal{O}) \xrightarrow{\partial_1} K_0(\mathcal{I}) \xrightarrow{j_*} K_0(\mathcal{E}) \xrightarrow{\pi_*} K_0(\mathcal{O}) \xrightarrow{\partial_0} K_1(\mathcal{I}).$$

It is easy to see that $K_1(\mathcal{I}) = 0$ and $K_0(\mathcal{I}) \simeq \mathbb{Z}$ as they are for the ideal of compact operators. Corollary 12.1 and the definition of ∂_1 (see section 9) immediately imply that $\partial_1 = 0$, so we get an exact sequence

$$0 \to K_0(\mathcal{I}) \xrightarrow{j_*} K_0(\mathcal{E}) \xrightarrow{\pi_*} K_0(\mathcal{O}) \to 0.$$

Now, we know that $r = [Q]$ generates $j_*(K_0(\mathcal{I})) = \ker \pi_* \simeq \mathbb{Z}$. If $0 = [1_{\mathcal{O}}] = \pi_*([1_{\mathcal{E}}])$, it follows by exactness that $[1_{\mathcal{E}}] = nr$ for some integer $n \in \mathbb{Z}$. But we know that $r = -2[1_{\mathcal{E}}]$, so $(2n+1)r = 0$, contradicting the fact that r generates a group isomorphic to $\mathbb{Z}$.

The proof of Theorem 12.3 generalizes in a straightforward way to give, for every $k \in \mathbb{N}$, an example of a space X such that X^n is isomorphic to X^m if and only if $m = n \pmod k$. It is likely that every Fredholm operator on the space X of this section has zero index, so that X is not isomorphic to its hyperplanes. Working with a dyadic tree may then give an example of a space X isomorphic to X^2 but not isomorphic to its hyperplanes.

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