TYPES AND ℓ_1-SUBSPACES

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The main result in this paper is the following:

<u>Theorem</u>. A separable (real) Banach space E contains a subspace isomorphic to ℓ_1 if and only if there exists $x^{**} \in E^{**}$ such that

$$x^{**} \neq 0 \quad \text{and} \quad \|x^{**}+x\| = \|x^{**}-x\|, \quad \forall x \in E .$$

This result is not true for non-separable Banach spaces; a counter-example due to Haydon will be presented in section 3.

Let us first show two "canonical" examples of the situation described in the theorem. If $E = L_1$, every element $x^{**} \in L_1^{**}$, disjoint from L_1, satisfies $\|x^{**}+x\| = \|x^{**}\| + \|x\|$, $\forall x \in L_1$. If $E = C[0,1]$ and if x^{**} is a Borel function on $[0,1]$ such that $\|x^{**}\|_\infty = 1$ and that the sets $\{x^{**}=1\}$ and $\{x^{**}=-1\}$ are dense in $[0,1]$, then $\|x^{**}+x\|_\infty = 1+\|x\|$, $\forall x \in C[0,1]$.

In the first part of the paper we prove the easiest implication, namely that the existence of x^{**} with the properties above implies that E contains an isomorph of ℓ_1. Actually a slightly stronger result is proved in Proposition 1.1.

In the second part we recall the notion of types, and introduce the notion of "strong types". This approach makes the proof perhaps unnecessarily long, but we feel that the notion of strong type could be useful.

The third section contains a counter-example due to Haydon which shows that our main result does not hold for non-separable Banach spaces, and a result for separable spaces containing c_o, somewhat similar to the ℓ_1-case.

1. The easiest implication

We say that $x^{**} \in E^{**}$ is "symmetric over E" if

$$\|x^{**}+x\| = \|x^{**}-x\|, \quad \forall x \in E .$$

Proposition 1.1. Let E be an arbitrary Banach space, and suppose there exists $x^{**} \in E^{**}$ symmetric over E, $x^{**} \neq 0$. For every increasing sequence (F_n) of finite dimensional subspaces of E and every decreasing sequence (ε_n) of positive real numbers one can find a sequence (x_n) in E such that $\lim\limits_n \|x_n\| > 0$ and $\forall x \in F_n$, $(1-\varepsilon_n)\|x+x_{n+1}\| \leq \|x + \sum\limits_{j=n+1}^{\infty} \alpha_j x_j\| \leq (1+\varepsilon_n)\|x+x_{n+1}\|$ for all scalars (α_j) such that $\sum\limits_{j=n+1}^{\infty} |\alpha_j| = 1$.

Proof. From the local reflexivity principle there exists a net (x_i) in E such that

$$\|x^{**}+x\| = \lim_{i\to\infty} \|x_i + x\| , \quad \forall x \in E$$

$$\text{and} \quad x^{**} = w^* \lim_{i\to\infty} x_i .$$

Then $\lim\limits_{i\to\infty} \lim\limits_{j\to\infty} \|x+\alpha x_i+\beta x_j\| = \lim\limits_{i\to\infty} \|x+\alpha x_i+\beta x^{**}\| = \lim\limits_{i\to\infty} \|x+\alpha x_i \pm \beta x^{**}\| \geq \|x+\alpha x^{**} \pm \beta x^{**}\| = \|x+(|\alpha|+|\beta|)x^{**}\|$.

On the other hand, assuming $|\alpha|+|\beta| = 1$

$$\lim_{i\to\infty} \lim_{j\to\infty} \|x+\alpha x_i+\beta x_j\| \leq |\alpha| \lim_{i\to\infty} \|x+\text{sign}\,\alpha\, x_i\| + |\beta| \lim_{j\to\infty} \|x+\text{sign}\,\beta\, x_j\|$$
$$= \|x+x^{**}\|, \quad \text{so that finally}$$

$$(*)\qquad \lim_{i\to\infty}\lim_{j\to\infty}\|x+\alpha x_i+\beta x_j\| = \lim_{i\to\infty}\|x+(|\alpha|+|\beta|)x_i\|\ ,\quad \lim_{i\to\infty}\|x+x_i\|>0$$

for all $\alpha,\beta\in\mathbb{R}$ and all $x\in E$.

For further use we state:

Lemma 1.2. Property (*) implies the conclusion of Proposition 1.1.

Select inductively $u_n = x_{i_n}$ so that

$\lim_{j\to\infty}\|z+\alpha u_n+\beta x_j\| \underset{\rho_n}{\sim} \lim_{i\to\infty}\|z+(|\alpha|+|\beta|)x_i\|$ for all $\alpha,\beta\in\mathbb{R}$ and $z\in F_n+[u_1,\dots,u_{n-1}]$, where we note $A\underset{\rho}{\sim}B$ means $\rho^{-1}A\le B\le\rho A$, assuming $\rho_n>1$.

If $x\in F_n$, we have

$$\begin{aligned}\|x+\sum_{j=n}^{N}\alpha_j u_j\| &\underset{\rho_N}{\sim} \lim_{i\to\infty}\|x+\sum_{j=n}^{N-1}\alpha_j u_j+|\alpha_N|x_i\|\\ &\underset{\rho_{N-1}\rho_N}{\sim} \lim_{i\to\infty}\|x+\sum_{j=n}^{N-2}\alpha_j u_j+(|\alpha_{N-1}|+|\alpha_N|)x_i\|\\ &\underset{\rho}{\sim} \lim_{i\to\infty}\|x+(\sum_{j=n}^{N}|\alpha_j|)x_i\| \quad\text{with}\quad \rho=\prod_{j=n}^{N}\rho_j\ .\end{aligned}$$

We deduce easily the conclusion of Proposition 1.1, for a proper choice of ρ_n and with $x_n=u_n$.

Remark 1.3. Assume that (x_i) is a net such that

$$\lim_{i\to\infty}\lim_{j\to\infty}\|x+\alpha x_i+\beta x_j\| = \lim_{i\to\infty}\|x+(\alpha+\beta)x_i\|\ ,\quad \lim_{i\to\infty}\|x+x_i\|>0$$

for all $\alpha,\beta\ge 0$ and $x\in E$. Using the same proof one can find a subsequence $u_n=x_{i_n}$ such that

$$\forall x\in F_n,\quad (1-\varepsilon_n)\lim_{i\to\infty}\|x+x_i\|\le\|x+\sum_{j=n+1}^{\infty}\alpha_j u_j\|\le(1+\varepsilon_n)\lim_{i\to\infty}\|x+x_i\|$$

for all $(\alpha_j)\ge 0$ such that $\sum_{j=n+1}^{\infty}\alpha_j=1$.

We prove now a lemma which will be used for the reverse implication of the theorem.

<u>Lemma 1.4</u>. Let E be a Banach space, (F_n) an increasing sequence of finite subsets of E and (ε_n) a sequence of positive real numbers with $\lim_{n\to\infty} \varepsilon_n = 0$. Let (x_n) be a sequence in E such that

$$\forall x \in F_n, \quad \Big| \|x + x_{n+1}\| - \|x + \sum_{j=n+1}^{\infty} \alpha_j x_j\| \Big| \le \varepsilon_n$$

for all $(\alpha_j) \ge 0$ with $\sum_{j=n+1}^{\infty} \alpha_j = 1$. Then there exist $x^{**} \in E^{**}$ such that

$$\forall x \in \overline{\bigcup_n F_n}, \quad \|x + x^{**}\| = \lim_{n\to\infty} \|x + x_n\| .$$

<u>Proof</u>. Observe first that $\lim_{j\to\infty} \|x + x_j\| = \theta(x)$ exists for $x \in F_n$, with $|\theta(x) - \|x + x_{n+1}\|| \le \varepsilon_n$, and by continuity this limit exists for $x \in \overline{\bigcup_n F_n}$.

Let x^{**} be any w^*-cluster point of the sequence (x_n). If $x \in F_n$, the convex set $x + \text{conv}\{x_j \,;\, j > n\}$ is disjoint from the open ball of radius $\theta(x) - 2\varepsilon_n$ in E. By Hahn-Banach there exists a norm-one functional x^* such that

$$\langle x^*, x + x_j \rangle \ge \theta(x) - 2\varepsilon_n , \quad \forall j > n .$$

It follows that $\langle x^*, x + x^{**} \rangle \ge \theta(x) - 2\varepsilon_n$, and finally $\|x + x^{**}\| \ge \theta(x)$ since $\lim_{n\to\infty} \varepsilon_n = 0$. But $\theta(x) \ge \|x + x^{**}\|$ since the norm is w^* l.s.c so $\theta(x) = \|x + x^{**}\|$ for $x \in \bigcup_n F_n$, or $x \in \overline{\bigcup_n F_n}$ by continuity.

<u>Remark 1.5</u>. If C_n denotes the w^*-closure of $\text{conv}\{x_j \,;\, j > n\}$ in E^{**}, any point $x^{**} \in \bigcap_n C_n$ will give the same result.

2. Types, strong types and the reverse implication

Let E be a Banach space. For each $x \in E$, consider the function $\tilde{x}(y) = \|x+y\|$ on E, and denote by $\tilde{E}$ the set of these functions. The space of types on E, denoted by $\mathscr{C}(E)$ (or simply $\mathscr{C}$) is the pointwise closure of $\tilde{E}$ in $\mathbb{R}^E$. In other words, a type $\tau \in \mathscr{C}$ is a function on E of the form $\tau(y) = \lim_{i\to\infty} \|y+x_i\|$ for some net (x_i) in E. When E is separable, it is possible to use ordinary sequences instead of nets. The topology on $\mathscr{C}$ is the topology of pointwise convergence on E; it is locally compact, since the sets $\mathscr{C}_r = \{\tau \in \mathscr{C};\ \tau(0) \leq r\}$ are compact by Tychonov, and metrizable when E is separable.

From now on we identify $\tilde{E}$ to E and write for example $\tau = \lim_{i\to\infty} x_i$ in $\mathscr{C}$. If $\alpha \in \mathbb{R}$ we define the dilation $\alpha\tau$ by

$$\alpha\tau = \lim_{i\to\infty} \alpha x_i \quad \text{if} \quad \tau = \lim_{i\to\infty} x_i \;, \quad \text{i.e.}$$

$$(\alpha\tau)(y) = |\alpha| \cdot \tau(y/\alpha) \quad \text{if} \quad \alpha \neq 0 \;.$$

A type is said to be symmetric when $\tau = (-1)\tau$. The set $\mathscr{C}_s$ of symmetric types is closed in $\mathscr{C}(E)$. Note that the functions $y \to \tau(y)$ and $\alpha \to \alpha\tau(y)$ are convex and Lipschitz.

<u>Remark 2.1</u>. Assume E separable and let F be an ultrapower of E. It is easy to see that the types are exactly the functions $y \to \|f+y\|$, for some $f \in F$. (The same is true for general E, taking the set of finite subsets of E as index set for the ultrapower.)

Consider now for $x \in E$ the function $\bar{x}[\tau] = \tau(x)$ on $\mathscr{C}$, and as before consider the set $\bar{E}$ of these functions on $\mathscr{C}$. The set $\bar{\mathscr{C}}(E)$ (or $\bar{\mathscr{C}}$) of strong types on E will be the pointwise closure of $\bar{E}$ in $\mathbb{R}^{\mathscr{C}}$. In other words, a strong type $\bar{\tau} \in \bar{\mathscr{C}}$ is a function on $\mathscr{C}$ of the form $\bar{\tau}[\theta] = \lim_{i\to\infty} \theta(x_i)$

for some net (x_i) in E. The topology on $\overline{\mathscr{C}}$ is the pointwise convergence on $\mathscr{C}$; it is again locally compact, but in general not metrizable even for separable E. If $\overline{\tau} = \lim_{i\to\infty} x_i$ in $\overline{\mathscr{C}}$, the dilation $\alpha\overline{\tau}$ will be $\lim_{i\to\infty} \alpha x_i$, i.e.

$$(\alpha\overline{\tau})[\theta] = |\alpha| \; \overline{\tau}[\alpha^{-1}\theta] \quad \text{for} \quad \alpha \neq 0 \; .$$

A symmetric strong type $\overline{\tau}$ is defined as before by $\overline{\tau} = (-1)\overline{\tau}$, and the set $\overline{\mathscr{C}}_s$ of symmetric strong types is closed in $\overline{\mathscr{C}}$.

Remark 2.2. Assume E separable and let F be an ultrapower of E. It is easy to identify $\overline{\mathscr{C}}(E)$ with the subset of $\mathscr{C}(F)$ consisting of limits in $\mathscr{C}(F)$ of nets (x_i) in E.

Remark 2.3. If $\overline{\tau} \in \overline{\mathscr{C}}$, its restriction to $E \subset \mathscr{C}$ defines a type τ, and the restriction mapping is continuous from $\overline{\mathscr{C}}$ to $\mathscr{C}$, maps $\overline{\mathscr{C}}_s$ into $\mathscr{C}_S$; it is clearly onto, because if $\tau = \lim_{i\to\infty} x_i$ in $\mathscr{C}$, any cluster point $\overline{\tau}$ for the net (x_i) in $\overline{\mathscr{C}}$ has τ as restriction; it is not in general 1-1, but that happens when E is stable. Recall that E is stable when

$$\lim_{i\to\infty} \lim_{j\to\infty} \|x_i + y_j\| = \lim_{j\to\infty} \lim_{i\to\infty} \|x_i + y_j\|$$

whenever all limits exist. If $\overline{\sigma}$ is a cluster point for the net (x_i) in $\overline{\mathscr{C}}$ and similarly $\overline{\tau}$ for (y_j), the preceding relation says

$$\overline{\sigma}[\tau] = \overline{\tau}[\sigma]$$

where σ, τ are the restrictions of $\overline{\sigma}$ and $\overline{\tau}$, so $\overline{\sigma}$ is determined from σ.

We will now introduce several notions of convolution of types. If $x \in E$ and $\theta \in \mathscr{C}$ define $x * \theta$ on E by $(x * \theta)(y) = \theta(x+y)$. The mapping $\theta \rightsquigarrow x * \theta$ is continuous from $\mathscr{C}$ to $\mathbb{R}^E$, sends $\tilde{E}$ into $\tilde{E}$, thus $x * \theta \in \mathscr{C}$ by continuity and the associativity of addition in E extends by continuity to $x_1 * (x_2 * \theta) = (x_1 + x_2) * \theta = x_2 * (x_1 * \theta)$. If $\bar{\tau} \in \bar{\mathscr{C}}$, define $x * \bar{\tau}$ on $\mathscr{C}$ by $(x * \bar{\tau})[\theta] = \bar{\tau}[x * \theta]$. For the same reason, $x * \bar{\tau} \in \bar{\mathscr{C}}$ and $\bar{\tau} \rightsquigarrow x * \bar{\tau}$ is continuous on $\bar{\mathscr{C}}$. If $\theta \in \mathscr{C}$, define $\bar{\tau} * \theta$ on E by $(\bar{\tau} * \theta)(y) = \bar{\tau}[y * \theta]$. Note that it is compatible with the preceding when $\bar{\tau} = x$, that $\bar{\tau} \rightsquigarrow \bar{\tau} * \theta$ is continuous from $\bar{\mathscr{C}}$ to $\mathbb{R}^E$, sends $\bar{E}$ into $\mathscr{C}$, so $\bar{\tau} * \theta \in \mathscr{C}$. Note that $x * (\bar{\tau} * \theta) = (x * \bar{\tau}) * \theta = \bar{\tau} * (x * \theta)$ since the three expressions are continuous with respect to $\bar{\tau}$ and agree when $\bar{\tau} \in \bar{E}$.

Finally if $\bar{\sigma}, \bar{\tau} \in \bar{\mathscr{C}}$ define $\bar{\sigma} * \bar{\tau}$ on $\mathscr{C}$ by

$$(\bar{\sigma} * \bar{\tau})[\theta] = \bar{\sigma}[\bar{\tau} * \theta] .$$

If $x \in E$ is considered as a strong type the two definitions of $x * \bar{\tau}$ agree since $\bar{x}[\bar{\tau} * \theta] = (\bar{\tau} * \theta)(x) = \bar{\tau}[x * \theta]$. Now again the mapping $\bar{\sigma} \rightsquigarrow \bar{\sigma} * \bar{\tau}$ is continuous from $\bar{\mathscr{C}}$ to $\mathbb{R}^{\mathscr{C}}$ and sends $\bar{E}$ into $\bar{\mathscr{C}}$, therefore $\bar{\sigma} * \bar{\tau}$ is a strong type. Note that

$$\bar{\sigma} * (\bar{\tau} * \theta) = (\bar{\sigma} * \bar{\tau}) * \theta .$$

Since the two members are continuous in $\bar{\sigma}$, it is enough to check the case $\bar{\sigma} = x$, which has been done already. It follows that

$$\bar{\rho} * (\bar{\sigma} * \bar{\tau}) = (\bar{\rho} * \bar{\sigma}) * \bar{\tau} \quad \text{for} \quad \bar{\rho}, \bar{\sigma}, \bar{\tau} \in \bar{\mathscr{C}} .$$

Note finally that when $\bar{\sigma} = \lim_{i \to \infty} x_i$ and $\bar{\tau} = \lim_{j \to \infty} y_j$ in $\bar{\mathscr{C}}$, then $(\bar{\sigma} * \bar{\tau})[\theta] = \lim_{i \to \infty} \lim_{j \to \infty} \theta(x_i + y_j)$. Also note

$$(\alpha\bar{\sigma}) * (\alpha\bar{\tau}) = \alpha(\bar{\sigma} * \bar{\tau}) .$$

This shows that $\bar{\mathscr{C}}_s$ is stable under dilation and convolution.

We start now the actual proof of the reverse implication.

Lemma 2.4. If E contains an isomorph of ℓ_1 there exists a symmetric strong type $\overline{\tau} \in \overline{\mathscr{C}}_s(E)$ such that $\overline{\tau}[0] > 0$ and $\alpha\overline{\tau} * \beta\overline{\tau} = (|\alpha|+|\beta|)\overline{\tau}$, $\forall \alpha, \beta \in \mathbb{R}$.

Since $\overline{\tau}$ is symmetric the property of $\overline{\tau}$ reduces to $\alpha\overline{\tau} * (1-\alpha)\overline{\tau} = \overline{\tau}$ for every $\alpha \in [0,1]$.

Let C be a set of strong types. We say that C is stable under convex convolution (resp. symmetric convex convolution) if

$$\alpha \in [0,1],\ \overline{\sigma}, \overline{\tau} \in C \Rightarrow \alpha\overline{\sigma} * (1-\alpha)\overline{\tau} \in C$$

(resp. $|\alpha|+|\beta| = 1,\ \overline{\sigma}, \overline{\tau} \in C \Rightarrow \alpha\overline{\sigma} * \beta\overline{\tau} \in C$).

Lemma 2.5. If $C \subset \overline{\mathscr{C}}$ is closed non-empty and stable under symmetric convex convolution, then C contains a symmetric strong type.

The proof uses a variant of an argument due to H. Rosenthal for finding "universally symmetric types". Let $\overline{\tau} \in C$ and set

$$K = \{\overline{\sigma} \in \overline{C};\ \overline{\sigma}[0] \leq \overline{\tau}[0]\}\ .$$

Since K is compact it is enough to find for every finite subset F of $\mathscr{C}$ an element $\overline{\sigma}$ of K such that $\overline{\sigma}[\theta] = \overline{\sigma}[-\theta]$ for every $\theta \in F$. Let $F = \{\theta_1, \ldots, \theta_n\}$ and define a continuous function φ from the unit sphere S_n of ℓ_1^{n+1} to $\mathbb{R}^n$ in the following way: for $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n) \in S_n$ consider $\overline{\tau}_\alpha = \alpha_0\overline{\tau} * \alpha_1\overline{\tau} * \ldots * \alpha_n\overline{\tau} \in K$ and

$$\varphi(\alpha) = (\overline{\tau}_\alpha[\theta_1], \overline{\tau}_\alpha[\theta_2], \ldots, \overline{\tau}_\alpha[\theta_n])\ .$$

From the Borsuk-Ulam antipodal theorem, there exists $\alpha \in S_n$ such that $\varphi(\alpha) = \varphi(-\alpha)$, which is just what we want.

Lemma 2.6. If $C \subset \overline{\mathscr{C}}$ is closed, non-empty and stable under convex convolution, then C contains $\overline{\tau}$ such that $\alpha\overline{\tau} * (1-\alpha)\overline{\tau} = \overline{\tau}$ for every $\alpha \in [0,1]$.

Since the sets $\{\overline{\sigma} \in C;\ \overline{\sigma} \leq \overline{\tau}\}$ are compact (the order being pointwise order on $\mathscr{C}$) the set C must contain a minimal element $\overline{\tau}$. But by convexity, if $\overline{\tau} = \lim_{i\to\infty} x_i$ in $\overline{\mathscr{C}}$ and $\alpha \in [0,1]$

$$(\alpha\overline{\tau} * (1-\alpha)\overline{\tau})[\theta] = \lim_{i\to\infty}\ \lim_{j\to\infty} \theta(\alpha x_i + (1-\alpha)x_j)$$

$$\leq \alpha \lim_{i\to\infty} \theta(x_i) + (1-\alpha) \lim_{j\to\infty} \theta(x_j) = \overline{\tau}[\theta]\ ,$$

and we must have $\alpha\overline{\tau} * (1-\alpha)\overline{\tau} = \overline{\tau}$.

We return now to the proof of lemma 2.4. Let (e_n) be a sequence in E such that $\|e_n\| = 1$ and

$$\Big\|\sum_{k=1}^{\infty} c_k e_k\Big\| \geq \delta \cdot \sum_{k=1}^{\infty} |c_k|, \quad \text{for all scalars } (c_k), \text{ with } \delta > 0\ .$$

Consider the set K of all strong types $\overline{\tau}$ on E satisfying: for every n and every neighborhood V of $\overline{\tau}$ in $\overline{\mathscr{C}}$, there exists $N > n$ and $x = \sum_{k=n}^{N} c_k e_k$, with $\sum_{k=n}^{N} |c_k| = 1$ and $x \in V$.

It is clear that K is closed in $\overline{\mathscr{C}}$, and that $\delta \leq \overline{\tau}[0] \leq 1$ for every $\overline{\tau} \in K$ (so K is compact and $\overline{0} \notin K$). Furthermore, K is stable under symmetric convex convolution. To see this, pick $\overline{\sigma}, \overline{\tau} \in K$ and $\alpha, \beta \in \mathbb{R}$ with $|\alpha| + |\beta| = 1$, and consider $\overline{\rho} = \alpha\overline{\sigma} * \beta\overline{\tau}$. Let V be an open subset of $\overline{\mathscr{C}}$ containing $\overline{\rho}$ and $n \in \mathbb{N}$; there exists an open subset W containing $\overline{\sigma}$ such that $\alpha y * \beta\overline{\tau} \in V$ whenever $y \in W$. Since $\overline{\sigma} \in K$, we can choose $y = \sum_{k=n}^{N} c_k e_k$, with $\sum_{k=n}^{N} |c_k| = 1$. We can find now an open neighborhood U of $\overline{\tau}$ in $\mathscr{C}$ such that $\alpha y + \beta z \in V$ for every $z \in U$, and we can choose z of the form $z = \sum_{k=N+1}^{M} d_k e_k$. Finally $\alpha y + \beta z \in V$ has the desired form.

By lemma 2.5 K contains symmetric strong types, hence $K_s = K \cap \mathscr{C}_s$ is again non-empty closed and stable under symmetric convex convolution. Applying lemma 2.6 and recalling that $\overline{0} \notin K$ finishes the proof of 2.4.

Consider the strong type $\overline{\tau}$ given by lemma 2.4. In order to conclude the proof of the reverse implication, it is enough to find $x^{**} \in E^{**}$ such that $\overline{\tau}[x] = \|x^{**}+x\|$, $\forall x \in E$, because then $\|x^{**}+x\| = \overline{\tau}[x] = \overline{\tau}[-x] = \|x^{**}-x\|$, and $\|x^{**}\| = \overline{\tau}[0] > 0$.

We will see in the next lemma that this is a consequence of the property $\alpha\overline{\tau} * (1-\alpha)\overline{\tau} = \overline{\tau}$, $\forall\alpha \in [0,1]$, or even of the slightly weaker property $\alpha\overline{\tau} * (1-\alpha)\tau = \tau$, $\forall\alpha \in [0,1]$, where $\tau \in \mathscr{C}$ is the restriction of $\overline{\tau}$ to E.

<u>Lemma 2.7</u>. Let E be a separable Banach space, and $\overline{\tau}$ a strong type on E such that $\alpha\overline{\tau} * (1-\alpha)\tau = \tau$, $\forall\alpha \in [0,1]$, where τ is the restriction of $\overline{\tau}$ to E. There exists $x^{**} \in E^{**}$ such that

$$\tau(x) = \|x^{**}+x\|, \ \forall x \in E \ .$$

<u>Proof</u>. Let (x_i) be a net in E such that $\overline{\tau} = \lim_{i\to\infty} x_i$ in $\mathscr{C}$. The property of $\overline{\tau}$ yields:

$$\forall x \in E, \ \forall\alpha \in [0,1], \ \lim_{i\to\infty} \ \lim_{j\to\infty} \|x+\alpha x_i + (1-\alpha)x_j\|$$
$$= \lim_{i\to\infty} \|x+x_i\| = \tau(x) \ ,$$

which is the same as the hypothesis in remark 1.3. Since E is separable, we can find an increasing sequence (F_n) of finite subsets of E with $E = \overline{\bigcup_n F_n}$ and by remark 1.3 find a subsequence (x_n) of (x_i) such that

$$\forall x \in F_n \ , \quad \left| \left\|x + \sum_{j=n+1}^{\infty} \alpha_j x_j\right\| - \tau(x) \right| \leq 2^{-n}$$
$$\text{for all } (\alpha_j) \geq 0 \text{ with } \sum_{j=n+1}^{\infty} \alpha_j = 1 \ .$$

Apply then lemma 1.4.

Remark 2.8. If we put together the different elements of the proof, we see that the final x^{**} has been obtained as a w^*-cluster point of some sequence of ℓ_1-normalized blocks of the ℓ_1-basis (e_n). We can rephrase the main theorem in the following way: if $T: \ell_1 \to E$ is an embedding of ℓ_1 in a separable Banach space E, there exists $x^{**} \in \ell_1^{**}$ such that

$$T^{**}x^{**} \neq 0 , \quad \|T^{**}x^{**}+y\| = \|T^{**}x^{**}-y\| , \quad \forall y \in E .$$

But it is not possible to find $x^{**} \in \ell_1^{**}$ satisfying this property for every embedding T of ℓ_1 in a separable Banach space: this is related to Haydon's counterexample (see below).

3. Further comments

We begin with an example due to Haydon which shows that the separability assumption can't be removed in the main theorem (actually, lemma 2.7 is the only place where it was used).

Consider $\ell_2(\mathcal{P}(\mathbb{N}))$, where $\mathcal{P}(\mathbb{N})$ is the set of subsets of $\mathbb{N}$. An element $y \in \ell_2(\mathcal{P}(\mathbb{N}))$ will be denoted by (y_A), $A \in \mathcal{P}(\mathbb{N})$. Consider on $E = \ell_1 \oplus \ell_2(\mathcal{P}(\mathbb{N}))$ the norm

$$\|(x,y)\| = \max\{\|x\|_1 , \|y\|_2 , \sup\{|y_A + \sum_{k\in A} x_k| ; A \subset \mathbb{N}\}\} .$$

The bidual ℓ_1^{**} is the set of finitely additive measures μ on $\mathbb{N}$. It can be shown that the bidual norm on $E^{**} = \ell_1^{**} \oplus \ell_2(\mathcal{P}(\mathbb{N}))$ is given by

$$\|(\mu,y)\| = \max\{\|\mu\|, \|y\|_2, \sup\{|y_A + \mu(A)| ; A \subset \mathbb{N}\}\} .$$

It follows easily that no element (μ,y) can be symmetric over E.

We pass now to a result for spaces containing c_o, analogous to the main theorem. For the statement we need the notion of "canonical reproductions" of an element $x^{**} \in E^{**}$ in the duals $E^{(2k)}$ of even order $(k \geq 1)$. To this end consider the canonical embedding $i_k E$ in $E^{(2k-2)}$ and define x_k^{**} as the image of x^{**} in $E^{(2k)}$ by i_k^{**}. In other words, if $x^{**} = w^* \lim_{i \to \infty} x_i$ for some net (x_i) in E, then x_k^{**} is the limit of the same net (x_i) for the topology $\sigma(E^{(2k)}, E^{(2k-1)})$.

Second theorem. Let E be a separable Banach space. The following conditions are equivalent:

a) E contains a subspace isomorphic to c_o.

b) There exists a symmetric strong type $\bar{\tau}$ on E such that $\bar{\tau}[0] > 0$ and $\bar{\tau} * \bar{\tau} = \bar{\tau}$.

c) For every incerasing sequence (F_n) of finite dimensional subspaces of E and every sequence (ε_n) of positive real numbers, there exists a normalized sequence (x_n) in E such that

$$\forall x \in F_n, \quad (1-\varepsilon_n)\|x + x_{n+1}\| \leq \|x + \sum_{k=n+1}^{N} \alpha_k x_k\| \leq (1+\varepsilon_n)\|x + x_{n+1}\|$$

for all $N > n$ and scalars (α_k) such that $\max_{n<k\leq N} |\alpha_k| = 1$.

d) There exists $x^{**} \in E^{**}$ such that $x^{**} \neq 0$ and

$$\forall x \in E, \quad \|x + x_1^{**} - x_2^{**}\| = \|x - x_1^{**} + x_2^{**}\| = \|x + x_1^{**} - x_2^{**} + x_3^{**} - x_4^{**}\|$$

where $x_1^{**} = x^{**}, x_2^{**}, \ldots$ are the canonical reproductions of x^{**} in $E^{**}, E^{(4)}, \ldots$.

Sketch of the proof. a) $\Rightarrow$ b). If (e_n) is equivalent to the c_o-basis, consider the set $K \subset \bar{\mathscr{C}}$ of all $\bar{\tau}$ such that for every n and every neighborhood V of $\bar{\tau}$ in $\bar{\mathscr{C}}$ there exists $x = \sum_{k=n}^{N} \alpha_k e_k$ with $\max_{n\leq k\leq N} |\alpha_k| = 1$ and $x \in V$. Then K is compact, $\bar{0} \notin K$ and $\alpha\bar{\sigma} * \beta\bar{\tau} \in K$ whenever $\bar{\sigma}, \bar{\tau} \in K$ and $\max\{|\alpha|, |\beta|\} = 1$. It is easy to modify lemma 2.5, replacing ℓ_1^n-spheres by

ℓ_∞^n-spheres to see that $K_s = K \cap \overline{\mathscr{C}}_s$ is not empty. Now since K_s is compact it must contain maximal elements, but if $\overline{\tau}$ is such a maximal symmetric element, with $\overline{\tau} = \lim_{i\to\infty} x_i$

$$\overline{\tau}[\theta] = \lim_{i\to\infty} \theta(x_i) \leq \lim_{i\to\infty} \lim_{j\to\infty} \{\tfrac{1}{2}\,\theta(x_i + x_j) + \tfrac{1}{2}\,\theta(x_i - x_j)\} = \overline{\tau} * \overline{\tau}[\theta] ,$$

and hence $\overline{\tau} = \overline{\tau} * \overline{\tau}$.

b) $\Rightarrow$ c) Assume the strong type in b) is $\overline{\tau} = \lim_{i\to\infty} x_i$, then

$$\forall x \in E,\ \lim_{i\to\infty} \lim_{j\to\infty} \|x + \alpha x_i + \beta x_j\| = \lim_{i\to\infty} \|x + \max(|\alpha|, |\beta|) x_i\| ,$$

and we may assume $\overline{\tau}[0] = 1 = \|x_i\|$. The proof is then identical to the proof of lemma 1.2.

c) $\Rightarrow$ d) Set $\theta(x) = \lim_{n\to\infty} \|x + x_n\|$ and consider for $n < m$ $y_{n,m} = x_{n+1} + x_{n+2} + \ldots + x_m$, and $y_n^{**} = w^* \lim_{m\to\mathscr{U}} y_{n,m} \in E^{**}$ ($\mathscr{U}$ an ultrafilter). If $x \in F_n$ the convex set $x + \mathrm{conv}\{y_{n,m};\ m > n\}$ is disjoint from the ball of radius $(1-\varepsilon_n)^2\theta(x)$ in E (use c)) so we conclude $(1-\varepsilon_n)^{-2}\theta(x) \geq \|x + y_n^{**}\| \geq (1-\varepsilon_n)^2\theta(x)$. Actually the same argument works for $y^{**} = \sum_{k=1}^{N} \alpha_k y_{n_k}^{**} = w^* \lim_{m\to\mathscr{U}} \sum_{k=1}^{K} \alpha_k y_{n_k,m}$ where $n \leq n_1 < n_2 < \ldots < n_K$. (Consider $x + \mathrm{conv}\{\sum_{k=1}^{K} \alpha_k y_{n_k,m}\ ;\ m > n_K\}$).

If $y^{(4)} = \lim_{n\to\mathscr{U}} y_n^{**}$ in $\sigma(E^{(4)}, E^{(3)})$, the Hahn-Banach argument of lemma 1.4 shows that $\theta(x) = \|x + y^{(4)}\|$ for $x \in \overline{\bigcup_n F_n}$, and we could choose $\overline{\bigcup_n F_n} = E$. But

$$y_n^{**} = y_0^{**} - y_{0,n} , \quad \text{and} \quad \lim_{n\to\mathscr{U}} y_{0,n} \ \text{ in } \sigma(E^{(4)}, E^{(3)})$$

is the canonical reproduction of y_0^{**} in $E^{(4)}$. We got the first part of d) with $x^{**} = y_0^{**}$, and more precisely $\theta(x) = \|x + x_1^{**} - x_2^{**}\| = \theta(-x)$ (symmetry of θ follows from c)).

But in fact $y^{(4)}$ can be obtained from any y_n^{**} since $y_n^{**} - y_0^{**} \in E$. We have, using the ℓ.s.c. character of the norms in the duals:

$$\|x + x_1^{**} - x_2^{**} + x_3^{**} - x_4^{**}\| \leq \lim_{m \to \mathcal{U}} \lim_{p \to \mathcal{U}} \|x + x_1^{**} - x_2^{**} + y_{n,p} - y_{n,m}\|$$

$$= \lim_{m \to \mathcal{U}} \lim_{p \to \mathcal{U}} \theta(x + y_{n,p} - y_{n,m})$$

$$= \lim_{m \to \mathcal{U}} \lim_{p \to \mathcal{U}} \lim_{k \to \infty} \|x + y_{n,p} - y_{n,m} + x_k\| \leq (1 + \varepsilon_n)\theta(x)$$

so $\|x + x_1^{**} - x_2^{**} + x_3^{**} - x_4^{**}\| \leq \|x + x_1^{**} - x_2^{**}\|$ since n is arbitrary, the other inequality can be proved by using the norm-one projection of $E^{(8)}$ onto $E^{(6)}$ that maps $x_3^{**} - x_4^{**}$ to zero.

d) $\Rightarrow$ a) Using the local reflexcivity principle applied to $E \oplus [x_1^{**}, x_2^{**}, x_3^{**}]$ it is possible to find a net (x_i) in E such that $x_4^{**} = \lim_{i \to \infty} x_i$ in $\sigma(E^{(8)}, E^{(7)})$ and

$$\|x + \alpha x_1^{**} + \beta x_2^{**} + \gamma x_3^{**} + \delta x^{**}\| = \lim_{i \to \infty} \|x + \alpha x_1^{**} + \beta x_2^{**} + \gamma x_3^{**} + \delta x_i\| ,$$

for all $x \in E$, $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. If j denotes the canonical isometric embedding from $E^{(4)}$ into $E^{(6)}$ we have $j^{**}(x_3^{**}) = x_4^{**}$ and j^{**} leaves $E^{(4)}$ invariant, therefore $\|x + \alpha x_1^{**} + \beta x_2^{**} + \gamma x_3^{**}\| = \|x + \alpha x_1^{**} + \beta x_2^{**} + \gamma x_4^{**}\|$ and similarly $\|x + \alpha x_1^{**} + \beta x_2^{**}\| = \|x + \alpha x_1^{**} + \beta x_4^{**}\|$, $\|x + \alpha x_1^{**}\| = \|x + \alpha x_4^{**}\|$. It follows that

$$\|x + \alpha(x_1^{*} - x_2^{**}) + \beta(x_3^{**} - x_4^{**})\| = \lim_{i \to \infty} \lim_{j \to \infty} \|x + \alpha(x_1^{**} - x_2^{**}) + \beta(x_j - x_i)\|$$

$$= \lim_{i \to \infty} \lim_{j \to \infty} \lim_{k \to \infty} \lim_{\ell \to \infty} \|x + \alpha(x_\ell - x_k) + \beta(x_j - x_i)\| .$$

If we set $u_1 = (x_1^{**} - x_2^{**})$, $u_2 = (x_3^{**} - x_4^{**})$ and remark that the norm-one projections from $E^{(4)}$ onto E^{**} and from $E^{(8)}$ onto $E^{(6)}$ map respectively u_1 and u_2 to zero, we get $\|x\| \le \|x + \alpha u_1\| = \|x - \alpha u_1\| \le \|x \pm \alpha u_1 + \beta u_2\|$. It also follows that

$$\|x + \beta u_2\| \le \|x + \alpha u_1 + \beta u_2\| = \|x - \alpha u_1 + \beta u_2\|, \ \forall x \in E, \quad \forall \alpha, \beta \in \mathbb{R} .$$

We deduce by convexity

$$\|x + \alpha u_1 + \beta u_2\| = \|x + \max(|\alpha|, |\beta|) u_1\|, \quad \forall x \in E, \quad \forall \alpha, \beta \in \mathbb{R} .$$

Finally we see that the net $(x_j - x_i)$ has the property

$$\lim_{i\to\infty} \lim_{j\to\infty} \lim_{k\to\infty} \lim_{\ell\to\infty} \|x + \alpha(x_\ell - x_k) + \beta(x_j - x_i)\|$$
$$= \lim_{i\to\infty} \lim_{j\to\infty} \|x + \max(|\alpha|, |\beta|)(x_j - x_i)\|$$

for all $x \in E$ and $\alpha, \beta \in \mathbb{R}$, which is exactly what we need for proving c), and we finish with the obvious c) $\Rightarrow$ a).