Banach spaces with few operators

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1 Introduction

Several natural questions about the linear structure of infinite-dimensional normed spaces, that were asked since the early days of the theory, remained without answer for many years. Here are two examples (in this article, *Banach space* means *infinite-dimensional* Banach space, real or complex):

A. *Is it true that every Banach space is isomorphic to its (closed) hyperplanes?*

B. *If a Banach space X is isomorphic to every infinite-dimensional subspace of itself, does it follow that X is isomorphic to ℓ_2?*

These two questions come from Banach's book [Ba]. Question **B** was called the *homogeneous Banach space problem* in modern times. Question **A** is attributed to Banach; actually, Banach's book contains a weaker question, formulated in terms of *linear dimension*, which amounts to asking whether every Banach space *embeds* isomorphically in its hyperplanes; this simply asks whether X is isomorphic to some proper subspace of itself. Let us formulate two other general questions, less ancient than the first two. Recall that a sequence $(e_n)_{n\geq 0}$ of non zero vectors is *unconditional* if there exists a constant C such that

$$\|\sum_{i=1}^{m} \pm a_i e_i\| \leq C \,\|\sum_{i=1}^{m} a_i e_i\|$$

for every $m \geq 0$, all scalars $(a_i)_{i=1}^m$ and every choice of signs ± 1; the best constant C is called *unconditionality constant* of $(e_n)_{n\geq 0}$. An unconditional sequence is a basis for its closed linear span; unconditional bases were earlier called *absolute bases*, see [BP].

C. *Does every Banach space contain an infinite unconditional basic sequence?*

D. *Is it possible to decompose every Banach space as a topological sum of two infinite-dimensional subspaces?*

Question **D** was formulated around 1970 by Lindenstrauss [L2]. Question **C** appears in Bessaga-Pełczyński [BP] in 1958, but was considered several years before, since it asks for a natural improvement of the classical result from Banach's book, according to which every Banach space contains a subspace with basis.

All these questions have been answered during the last decade, most of them in the *negative* direction that seems to indicate that there is no hope for a structure theory of general Banach spaces. There is one notable example though of a positive answer, the homogeneous space problem; interestingly enough, one of the "negative" objects discovered during the period 1990–95 plays a little rôle in the positive solution of Question **B**: at some point in the proof, one has to exclude the possibility that the homogeneous space could be *hereditarily indecomposable.* Despite my rather pessimistic comments above, the results and examples obtained since 1990 represent a significant progress of our understanding of infinite-dimensional Banach spaces. The solutions of the different problems have various points of contact, and introduce new notions that underlie several of the constructions and proofs.

Let us agree that for the rest of this paper, the word *subspace* will always indicate an infinite-dimensional vector subspace of a Banach space (but not necessarily a closed subspace). Beside the preceding problems, one of the main questions that remained a mystery for years is the following: does every Banach space contain a subspace isomorphic to some ℓ_p (for $1 \le p < +\infty$) or to c_0? This question was settled negatively in 1974 by the famous example of Tsirelson [Ts], who constructed a reflexive Banach space that does not contain any ℓ_p. Tsirelson's solution has had an enormous influence on most of what is discussed here. After being disappointed by Tsirelson's example, the structure-seekers had to look for more modest questions, for example the following.

E. *Does every Banach space contain a reflexive subspace, or else a subspace isomorphic to either* c_0 *or* ℓ_1*?*

This question conforms to the common experience that non reflexivity is often related to the presence of c_0 or ℓ_1; indeed, a theorem of James [J1] says that a Banach space X with unconditional basis is reflexive if and only if it does not contain isomorphs of c_0 or ℓ_1. There is therefore a loose connection between Questions **E** and **C**; any counterexample to **E** has to be a counterexample to **C** as well.

As we have said before, it is obvious when we look back that the first giant step in the direction of all the results mentioned in this article was done around 1974 by Tsirelson, (in his only paper about Banach spaces, as he likes to point out!). As far as I know, Tsirelson's space was the first example of a space where the norm is defined by an inductive procedure that "forces" some specific property to hold, but somehow, nothing more than the desired property. The same year, Krivine proved the finite-dimensional counterpart, that goes in the opposite (positive) direction and says roughly that every Banach space contains ℓ_p^n's of arbitrary large finite dimension n. Almost 20 years passed before Tsirelson's breakthrough was extended to a solution of the above mentioned problems; during these years, it was still hoped by many that techniques using Krivine's ideas could lead, for example, to a positive solution of question **C**.

A difficulty common to these questions is that one has to analyze whether or not some particular phenomenon will occur in *every* subspace of a given Banach space X; this rather vague question can be put to precise terms as follows: what do we know about subsets A of the unit sphere of X that meet every infinite-dimensional vector subspace? Here, I want to call such a set A a (linearly) *inevitable* set (these sets were called in [GM1] by the inexpressive term *asymptotic*; I feel that one should reform this poor terminology). Can we say that two inevitable sets A_1, A_2 are in a sense so big that they must intersect, or at least *almost intersect*, meaning that $\mathrm{dist}(A_1, A_2) = 0$? An equivalent question asks whether every enlargement A_ε of an inevitable set A must contain an infinite-dimensional vector subspace. This type of problem reminds Ramsey theory, but nobody could exploit this analogy before Gowers, in his *dichotomy theorem* (see below). It was realized by V. Milman (see [M1,M2]), a few years before Tsirelson's example, that if we can prove that any two inevitable sets A_1 and A_2 in the unit sphere of a given space X almost intersect, then X must contain some ℓ_p, $p \in [1, \infty)$, or c_0. In [M1], Milman defines a notion of *spectrum* for a uniformly continuous function on the unit sphere, and shows that when the spectrum of every such function is non-empty, then X contains some ℓ_p or c_0; if inevitable sets almost intersect, then this spectrum is non empty. This question about intersecting inevitable sets is not totally absurd, since the answer is positive in one case: the result of Gowers' paper [G1] implies that any two inevitable subsets of the unit sphere of c_0 almost intersect.

It follows indirectly that Tsirelson gave the first example of two inevitable sets A_1 and A_2 that are separated by some $\delta > 0$, that is $\|a_1 - a_2\| \geq \delta$ for all $a_1 \in A_1$, $a_2 \in A_2$. Tsirelson's space thus gives us a first clue, but the real start was the modification of Tsirelson's space constructed by Schlumprecht [Sc], which provides us with an infinite sequence $(A_n)_{n \geq 0}$ of such separated inevitable sets, giving a lot of possibilities for *coding* things in the sphere. This will be explained in section 4.

For Gowers and the author of this article, this was the decisive information for constructing a space with no infinite unconditional sequence; during the summer of 1991, after hearing Schlumprecht at the Banach space conference in Jerusalem, we both constructed an example of a space with no unconditional sequence; the two examples $X(G)$ and $X(M)$ were very similar. Gowers' example was the first to be presented to a few specialists. The construction will be indicated in section 7; it is rather intricate, but the fundamental principle, that seems to me very clear, is presented in a separated section (section 4) containing the following partial result: if we have a sequence of well separated inevitable sets in a Banach space X, then for any given $C \geq 1$ we can renorm X in such a way that any infinite basic sequence in X has unconditionality constant larger than this C in the new norm. Obviously we are on the way to a negative answer to question **C**. The reader who wants to get an idea about what is going on with the failure of unconditionality *must* read section 4. The technical problem for solving question **C** itself is to mix the simple idea from section 4 with an inductive definition of a norm *a la* Tsirelson-Schlumprecht. This will be done in section 7.

Schlumprecht's example is an *ad hoc* example of a space containing a sequence of separated inevitable subsets (A_n), but this strange situation did not seem likely to happen in the most regular of all spaces, namely ℓ_2. It was therefore a big surprise when Odell and Schlumprecht showed that one can move the sets (A_n) from Schlumprecht's space S to ℓ_2 by a non linear procedure, and get a sequence (B_n) of inevitable subsets in the unit sphere of ℓ_2 that are somewhat orthogonal [OS]. On the other hand, it is still unknown whether Tsirelson's space T contains a *sequence* of well separated inevitable subsets, or whether T satisfies the opposite property of having *bounded distortion*. This subject of distortion will not be discussed here. The reader may consult [BLi, Chapter 13] for a complete description, that includes the results of [OS].

The space $X(G)$ (or $X(M)$) was basically intended to be a counterexample to the unconditional sequence problem **C**, but it quickly appeared to have a very radical property: when seeing Gowers' preprint about $X(G)$, W.B. Johnson observed that the space had the additional property that for every pair (Y, Z) of subspaces, we have that $\inf \|y - z\|$ is zero, when y and z run in the unit spheres of Y and Z. This means that Y and Z can never form a topological sum; in other words, every subspace of $X(G)$ is indecomposable. The paper [GM1] therefore introduced the first example X_{gm} of a *hereditarily indecomposable space* (in short: HI), thus solving negatively question **D**. As it happens sometimes, question **D** was solved by proving much more than asked. Obviously, a HI space cannot contain an infinite unconditional sequence, since the span of such a sequence is clearly decomposable (in odd and even, for example), and X_{gm} of course solves negatively question **C**. Finding examples of indecomposable spaces that are not HI took some more time (the shift space X_s mentioned later in this introduction is such an example; other examples are given by non-HI duals of HI spaces, as given by Ferenczi or Argyros-Felouzis, see below); at this point of the story, it seems easier to get a HI space than a genuine indecomposable space (one that is not HI)!

The HI spaces are very rigid, in the following sense: every bounded linear operator T on a complex HI space X has the form $\lambda Id + S$, where $\lambda \in \mathbb{C}$ and S is a *strictly singular* operator. Let us recall that a famous open question is the existence of a Banach space X such that every operator on X would have the form $\lambda Id + K$, with K compact. The existence of HI spaces is far from solving this "$\lambda Id + K$ problem", but it does give comparable spectral consequences, because it is known that strictly singular operators allow to extend the classical Riesz spectral theory of compact operators (see [LT, 2.c] for example). It follows that operators on a complex HI space X have a discrete spectrum, consisting of a converging sequences of eigenvalues, together with the limit (that ought to be λ, if $T = \lambda Id + S$); if T is Fredholm on a HI space X, it has always index 0 (this result also holds in the real case). This implies easily that a HI space is not isomorphic to any proper subspace, thus solving negatively question **A** (it solves exactly the question from Banach's book, but the question had become more popular under the weaker form that asks whether *hyperplanes* are isomorphic to the whole space; as for question **D**, this weaker form was solved by proving much more). These relatively easy spectral consequences of the HI property are presented in section 5; the interested reader can jump directly to that section.

Gowers has constructed several further examples; the most striking is a space X_g not containing any reflexive subspace, and containing no c_0 or ℓ_1 [G3]; this is a counterexample to question **E**. The construction is partly similar to that of [GM1], but in a much more difficult context which requires new ideas; I like to think that this example X_g is a sort of generalized James-tree space (see [J3,LS]), where every vector in the unit ball of X^* is a potential node for the tree. To some extent, constructing this space X_g is like building X_{gm} on this abstract tree.

It was very tempting to relate the HI property of a Banach space X to the fact that X does not contain any infinite unconditional sequence. This was done by T. Gowers in his beautiful "dichotomy Theorem" [G5], see also [G6].

Let X be an arbitrary infinite-dimensional Banach space. Either X contains an infinite unconditional sequence, or X contains a HI subspace.

For proving this theorem, Gowers has found a very satisfactory way to extend Ramsey theory to a linear setting. This result also explains why it is not so surprising to get a HI space when one simply looks for a space with no infinite unconditional sequence. It also gives a profound reason for introducing HI spaces. Some questions about general Banach spaces can then be divided into two cases: the "usual" case, where unconditional bases exist, and the "exotic" case, where we may find a HI space in the middle of our road. This dichotomy was the missing piece for the solution of question **B**, for which Komorowski and Tomczak had proved the following result: a Banach space X with unconditional basis, non trivial type and not containing ℓ_2, contains a subspace not isomorphic to X (see [KT,KTb]). They deduce the following partial solution to question **B**: *if a homogeneous Banach space X contains an infinite unconditional sequence, then X is isomorphic to ℓ_2*. Now, by the dichotomy theorem, a homogeneous space X must contain an unconditional sequence: otherwise, it contains a HI subspace, hence it is HI itself by homogeneity, but clearly a HI space is not homogeneous! Combining [KT] and [G5] solves question **B**: *every homogeneous space X is isomorphic to ℓ_2.*

As we have said, every operator on a complex HI space X has the form $\lambda Id+S$, where S is strictly singular. Ferenczi [F2] proved a more general result (which was previously checked by hand in [GM1] for the specific example X_{gm}): every operator from a subspace $Y \subset X$ to X has the form $\lambda i + S$, where i is the inclusion map and S is strictly singular. Actually, this property of a complex Banach space X is clearly equivalent to the fact that X is HI. One could be fully satisfied to have examples of spaces that have as few operators as possible. However, there are other questions which assume some structure for the space and then ask whether further structure follows. Roughly speaking, the results of [GM2] state that given an algebra of maps satisfying certain conditions, one can replace the multiple of the inclusion map in the statement above by the restriction to Y of some element of the algebra. These examples illustrate the following principle: you will find inside the space constructed in this way, nothing more than what you decide to put at the start; we obtain in this way a space X_s with an isometric right shift S but no infinite unconditional sequence; every operator $T \in \mathcal{L}(X_s)$ is a perturbation of an absolutely summable series of iterates of the shift S and its adjoint. In this space all complemented subspaces are trivial (finite-dimensional or finite-codimensional), which makes this space a bizarre example of a *prime space* (a Banach space isomorphic to every infinite-dimensional complemented subspace; here the shift and its iterates provide isomorphisms between X_s and the finite-codimensional subspaces). Recall that the "normal" known prime spaces are c_0, ℓ_p, for $p \in [1, +\infty)$ (see [Pe]) and also ℓ_∞ (see [L1]). These results from [GM2] are described in section 9. This paper [GM2] also gives a space isomorphic to its subspaces of codimension two but not to its hyperplanes and a space isomorphic to its cube but not to its square (Gowers had previously given in [G4] the first example of this *cube-not-square* phenomenon).

The shift space X_s provides a good illustration for Gowers' dichotomy theorem. Indeed, this space X_s does not contain any infinite unconditional sequence; but X_s is not HI, because it has a non trivial operator, the shift S; one can also see directly that for every $\lambda \in \mathbb{C}$ with modulus one, we may find a subspace Y_λ of X on which the shift S is almost equal to λId (we generate Y_λ from a sequence of almost eigenvectors of S, corresponding to the spectral value λ); when $\mu \neq \lambda$, the two subspaces Y_λ and Y_μ can be chosen to form a topological direct sum, and this explains why X is not HI. By Gowers' dichotomy theorem, every subspace of X_s must contain a further subspace which is HI. One can check that the subspaces Y_λ are examples of such HI subspaces.

Let us mention further results in the HI direction. Kalton [Ka] has constructed an example of a quasi-Banach space X with the very strange property that there is a vector $x \neq 0$ such that every closed infinite-dimensional subspace of X contains x. It follows that this quasi-Banach space does not contain any infinite basic sequence. This example is related to an example of Gowers [G2] of a space with unconditional basis not isomorphic to its hyperplanes; Kalton's construction uses the technique of *twisted sums* together with the properties of the space in [G2]. Argyros and Deliyanni [AD] constructed HI spaces without using Schlumprecht's space, by a technique called *mixed Tsirelson's norms*; their example is also an *asymptotically-ℓ_1* space. V. Ferenczi [F1] constructed a uniformly convex HI space, by adding to the tools from [GM1] the tool of complex interpolation for families of Banach spaces developed by Coifman, Cwikel, Rochberg, Sagher and Weiss. P. Habala [Ha] constructed a space such that no infinite-dimensional subspace has the Gordon-Lewis property (this property is a weak form of unconditional structure for a Banach space), and Ferenczi-Habala unite in [FH].

Ferenczi showed that the dual of the example X_{gm} in [GM1] is also HI and that every quotient of this space is still HI [F4]. This type of question is not yet clarified in general. What is clear is that the dual of a reflexive indecomposable space is indecomposable; therefore, if every quotient of a reflexive space is HI, then every subspace of a quotient is indecomposable and this property clearly passes to the dual. However Ferenczi [F4] gave an example of a HI space such that the dual is not HI. This phenomenon was widely extended by Argyros and Felouzis [AF], who showed that for every $p > 1$, the space ℓ_p (or c_0 when $p = +\infty$) is isomorphic to a quotient of some HI space X_p (the corresponding result is obviously false for ℓ_1). The dual space X_p^*, which contains ℓ_q, is clearly not a HI space. This sheds more light on the non stability of the HI property under duality. Argyros and Felouzis obtain this quotient result as a consequence of a factorization theorem through HI spaces: every operator which is *thin* in some sense factors through a HI space. This is, in a way, a very perverse result, as one is normally trying to factor operators through nice spaces, like what Davis, Figiel, Johnson and Pełczyński did in [DFJP] for weakly compact operators! A sketch for the results of [AF] is presented in section 8. A variant of the interpolation method of [DFJP] will be used for the general factorization result. Obtaining ℓ_p as quotient of a HI space requires a clever construction which is also sketched in section 8.

1.1 Other spaces with few operators

As we have said, complex HI spaces have few operators. But HI spaces are not the first examples of spaces that have, in some sense, few operators, and there is a series of works on this theme. The space constructed in [GM1] could be regarded as the infinite-dimensional analogue of the random finite-dimensional spaces introduced by Gluskin [Gl1,Gl2]. It was shown by Szarek, that operators on Gluskin's spaces all approximate, in a certain precise sense, multiples of the identity. Although the proofs in [GM1] and [Gl1] are very different, there are some points of contact, such as the idea of constructing a unit ball with just a few "spikes". Since these spikes must, under a well-bounded operator, map to other spikes, if they can be chosen in a very non-symmetrical way, a well-bounded operator is forced to approximate a multiple of the identity. Gluskin achieved the lack of symmetry by choosing the spikes randomly, whereas in [GM1] they were constructed directly (in the dual space) using some infinite (and not too difficult) combinatorics.

Gluskin's spaces were "glued" together to produce several infinite-dimensional examples of interest by, amongst others, Bourgain [Bo], Szarek [Sz1,Sz2], Mankiewicz [Ma] and Read [Re]. Some of these gluing methods were not at all straightforward. Several of the properties of these spaces are shared by the spaces constructed in [GM1] and [GM2]. For instance, Bourgain's example is a complex Banach space X such that X and its opposite space $\overline{X}$ are not isomorphic complex spaces. Szarek's example is a real reflexive Banach space X that does not admit a complex structure (because X does not have any operator T such that $T^2 = -Id$). The complex HI space from [GM1] has Bourgain's property, while its real version satisfies Szarek's conclusion (but these two facts are not stated in [GM1]).

A related direction is the search for examples in Banach algebras. One particular question is the existence of non-zero homomorphisms from the algebra of bounded linear operators $\mathcal{L}(X)$ to $\mathbb{C}$, or equivalently of closed ideals I such that $\mathcal{L}(X)/I$ is $\mathbb{C}$, or more generally a commutative Banach algebra. The result on the "shift space" in [GM2] can be compared to those of Mankiewicz [Ma]: we have in [GM2] another example of a complex Banach space such that there exists a (unital) algebra homomorphism from $\mathcal{L}(X)$ into a commutative Banach algebra. It follows (as is recalled in [Ma]) that X is not isomorphic to any power Y^n of a Banach space Y, for any $n \geq 2$ (see also Figiel [Fi] for an early related example of a reflexive space not isomorphic to its square). Indeed, if φ is a non zero multiplicative functional from $\mathcal{L}(X)$ to $\mathbb{C}$, and if $X = Y^n$, there is a natural homomorphism i from M_n to $\mathcal{L}(X)$. But then $\varphi \circ i$ would be a non zero multiplicative functional on M_n, which is not possible, as soon as $n \geq 2$.

Let us mention two results that have little in common with the present paper, apart from their statements. Kalton and Roberts [KR] have given results for quasi-Banach spaces. They constructed subspaces of L_p, $0 < p < 1$, where the only continuous linear operators are the multiples of the identity. Shelah [Sh] has constructed examples of non-separable Banach spaces for which every bounded linear operator has the form $\lambda Id + S$, where S has separable range. The proof used some heavy machinery from logic (diamond axiom, or $V = L$) which was avoided later in [ShS]. In the fall of 1999, Argyros and Tolias announced that they can obtain the same conclusion as Shelah, from a space X_a that is a relative of the HI family (see [AT]). This new example lies somewhere between the spaces from [AD] and [G3]. As in the case of the James tree space, the space X_a is a space of sequences indexed by a tree. The dual X_a^* is non separable, but every operator on X_a^* has the form $\lambda Id + S$, where S has separable range.

At the end of this introduction, I must confess that part of this paper has been realized by the well-known cut-and-paste technique, applied to the two papers [GM1] and [GM2]; as a result, some portions of the present paper may sound curiously too English to the reader: they were stolen from T. Gowers' writing of our papers.

2 Ancestors

In order that the reader sees where our methods come from, we have to say a few words about the ancestor of this story, namely the space T constructed by Tsirelson [Ts] (see also [FJ,CS]). Let us first fix some notation about sequence spaces. Let c_{00} denote the space of finitely supported scalar sequences. Given two subsets $E, F \subset \mathbb{N}$, we say that $E < F$ if $\max E < \min F$. Let $(\mathbf{e}_n)_{n=0}^{\infty}$ be the standard basis of c_{00}. Given a vector $x = \sum_{n=0}^{+\infty} x_n \mathbf{e}_n$ its *support*, denoted $\mathrm{supp}(x)$, is the set of n such that $x_n \neq 0$. If $x, y \in c_{00}$, we write $x < y$ when $\mathrm{supp}(x) < \mathrm{supp}(y)$. We also write $n < x$ when $n \in \mathbb{N}$ and $n < \min \mathrm{supp}(x)$. If $x_1 < \ldots < x_n$, then we say that the vectors $x_1, \ldots, x_n$ are *successive*.

Let $C > 1$ be fixed and let B_T^* be the smallest convex subset of $B(c_0) \cap c_{00}$ that contains $\pm\mathbf{e}_i$ for each $i \geq 0$ and such that

$$x_1^* + \cdots + x_n^* \in C\, B_T^*$$

whenever $x_1^*, \ldots, x_n^*$ are successive in B_T^* and $n < x_1^*$. The Tsirelson norm is then defined on c_{00} by

$$\|x\|_T = \sup\{|x^*(x)| : x^* \in B_T^*\}$$

and T is the completion of c_{00} under this norm. This point of view agrees with Tsirelson's original presentation. A dual formulation, given by Figiel and Johnson [FJ], introduces the Tsirelson norm as the smallest norm on c_{00} satisfying $\|\mathbf{e}_i\| = 1$ for every $i \geq 0$ and

$$\|x_1 + \cdots + x_n\| \geq \frac{1}{C} \sum_{i=1}^{n} \|x_i\|$$

for every $n \geq 2$ and all $n < x_1 < \ldots < x_n$. The original choice of C was $C = 2$; very interesting effects can be achieved by varying the constant C and mixing the norms obtained in this way (see [AD] as one example). Recall briefly why T is a counterexample to the ℓ_p-containment problem: since $\|x_1 + \cdots + x_n\| \geq \frac{1}{2} n$ whenever n norm one vectors satisfy $n < x_1 < \ldots < x_n$, it is quite clear that among ℓ_p spaces, only ℓ_1 can embed into T; Tsirelson excluded this possibility by showing that T is reflexive; Figiel and Johnson gave a quantitative proof, which is closer to the spirit of this paper; they showed that T does not contain a $(9/8)$-isomorph of ℓ_1 (see also [LT, 2.e.1]). But James [J2] proved (by the well-known James' blocking technique) that if X contains an isomorph of ℓ_1, then X contains a $(1+\varepsilon)$-isomorph of ℓ_1 for every $\varepsilon > 0$. Therefore T does not contain ℓ_1, and as a consequence, T does not contain any subspace isomorphic to an ℓ_p space or to c_0.

A second example, extremely important for us, is the space S constructed by Schlumprecht [Sc], which is a very useful variation of the construction of T. The constant factor C in Tsirelson's construction

$$x_1^* + \cdots + x_n^* \in C\, B_T^*$$

is replaced by a variable value $f(n)$ depending upon the number n of vectors in the sum; this function f should tend to infinity, but slowlier than any power n^α, $\alpha > 0$. Schlumprecht chooses $f(n) = \log_2(n+1)$. Let B_S^* be the smallest convex subset of $B(c_0) \cap c_{00}$ containing $\pm \mathbf{e}_i$ for each $i \geq 0$ and such that

$$x_1^* + \cdots + x_n^* \in f(n)\, B_S^*$$

whenever $x_1^*, \ldots, x_n^*$ are successive in B_S^* and $n \geq 2$. The norm is then defined on c_{00} by

$$\|x\|_S = \sup\{|x^*(x)| : x^* \in B_S^*\}$$

and S is the completion of c_{00} for this norm. It is useful to observe that B_S^* is obtained as the union of an increasing family of convex sets $(B_n)_{n\geq 0}$, where B_0 is the intersection of c_{00} with the unit ball of ℓ_1, and B_{n+1} is obtained from B_n by adding all vectors x of the form $x = f(m)^{-1}(x_1 + \cdots + x_m)$ with $m \geq 2$ arbitrary and $x_1, \ldots, x_m$ successive elements of B_n, and letting B_{n+1} be the convex hull of this extended set. This remark helps to show several properties of the space, for example the fact that the unit vector basis is 1-unconditional in S, by checking this inductively for $\|\,.\,\|_n = \sup\{|x^*(x)| : x^* \in B_n\}$.

An alternative description of S defines the norm as the solution of some implicit equation. We could say that the Schlumprecht norm on c_{00} is the smallest norm $\|\,.\,\|$ on c_{00} such that the unit vector basis is normalized and

$$\|x_1 + \cdots + x_n\| \geq \Big(\sum_{i=1}^n \|x_i\|\Big)/f(n)$$

for every integer $n \geq 2$ and every sequence of n successive vectors in S. In other words, the Schlumprecht norm is the solution to the implicit equation

$$\|x\|_S = \max\Big(\|x\|_{c_0}, \sup\{\, f(n)^{-1}\Big(\sum_{i=1}^n \|x_i\|_S\Big) : \\ n \geq 2,\ x = \sum_{i=1}^n x_i,\ x_1 < \ldots < x_n\}\Big).$$

3 Inevitable behaviours

In order to support some intuition for the notion of *inevitable set* (formally defined in the next section), we recall some easy facts, together with a few words about Schlumprecht's space S. We begin by a well known blocking procedure for constructing ℓ_1^n, originating in James [J2].

Lemma 1 *Let $n \geq 2$ be an integer and $0 < \varepsilon < 1$; suppose that N is an integer that can be written as $N = n^k$ for some $k \geq 1$, and let $(x_i)_{i=1}^N$ be norm one vectors in a normed space X, such that*

$$\|\sum_{i=1}^{N} \pm x_i\| \geq (1 - \varepsilon/n)^k N = (n - \varepsilon)^k$$

for all signs ± 1. There exists a sequence of n blocks $y_1, \ldots, y_n$ from $(x_i)_{i=1}^N$ that is $(1-\varepsilon)^{-1}$-equivalent to the unit vector basis of ℓ_1^n.

Let us briefly sketch the proof: we consider successive generations of blockings, the first one being the obvious blocks x_i of length one, $i = 1, \ldots, n^k$, and for $\ell = 0, \ldots, k-1$ the next generation (numbered $\ell + 1$) consists of $n^{k-\ell-1}$ new vectors $z = \sum_{j=1}^n \varepsilon_j z_j$, that are blocks of n consecutive elements $z_1, \ldots, z_n$ from the preceding generation, with some signs $\varepsilon_j = \pm 1$ chosen such that $\| \sum_{j=1}^n \varepsilon_j z_j\|$ is minimal. The last generation is just one single block using all n^k vectors. If on our way from $\ell = 0$ to $\ell = k$ we never encounter an inequality of the form

$$\|\sum_{j=1}^{n} \varepsilon_j z_j\| \geq (n - \varepsilon) \max_{1 \leq j \leq n} \|z_j\|,$$

we get at the end a choice of signs $(\varepsilon_i)_{i=1}^{n^k}$ such that $\| \sum_{i=1}^N \varepsilon_i x_i\| < (n-\varepsilon)^k$, which contradicts the hypothesis. We must therefore get at some stage a family of n blocks (z_j) that give, after we rescale them into (y_j)

$$\|y_j\| \leq 1,\ j = 1, \ldots, n; \quad \|\sum_{j=1}^{n} \pm y_j\| \geq n - \varepsilon$$

for all signs. It is easy to conclude, using simple convexity arguments, that this sequence (y_j) is well equivalent to the unit vector basis of ℓ_1^n.

Let us give an application of this lemma. Suppose that X is a Banach space with basis, and that f is a non-decreasing function on $[1, +\infty)$ that tends to $+\infty$ at infinity, but slowlier than any power function t^α ($\alpha > 0$, for example $f(t) = \log t$). Suppose that for every $N \geq 2$, all sequences of successive norm one vectors $x_1, \dots, x_N$ in X satisfy $\| \sum_{i=1}^N \pm x_i \| \geq N/f(N)$. This is of course true for ℓ_1, or for an Orlicz sequence space ℓ_M with $M(t) \sim t/\log(1/t)$ as $t \to 0$; it is also true, and more interesting in our context, for the Tsirelson space T or Schlumprecht's space S.

If $\varepsilon > 0$ and $n \geq 2$ are given, we can choose k large enough so that $f(n^k) \leq (1-\varepsilon/n)^{-k}$. If $N = n^k$, we are in a position to apply Lemma 1. It follows easily that every subspace Y of X contains almost isometric copies of ℓ_1^n, spanned by small perturbations of successive vectors. We can describe a *scheme* $\mathcal{L}_1$ for getting subsets of the unit sphere of X which intersect every subspace, namely we can describe a decreasing sequence of sets L_1^n that intersect every subspace: let L_1^n consist of all norm one vectors $x \in X$ such that there exist successive vectors $y_1 < \ldots < y_n$ that are $(1 + 2^{-n})$-equivalent to the unit vector basis of ℓ_1^n, and

$$\|x - \frac{1}{n} \sum_{i=1}^n y_i\| < 2^{-n}.$$

Then L_1^n intersects every subspace of X. Of course the intersection of the classes $(L_1^n)_{n \geq 1}$ is in general empty; what we call $\mathcal{L}_1$ is not a class of vectors, but a symbolic notation that is meant to represent the "idea" of any L_1^n with large n. In the present case, we say that the scheme $\mathcal{L}_1$ is *inevitable* in X.

The next natural thing to do, if we want to know whether X contains, not only ℓ_1^ns, but also ℓ_1, is to look for sums $x + y$, where x is in L_1^m and y in some L_1^n, with n much larger than m and much larger than the "size" of x; if $(x + y)/\|x + y\|$ is again in some L_1^k, with k large, then we are in the right direction for building ℓ_1. Let us write symbolically this set of vectors $x + y$, with $x \in L_1^m$, $y \in L_1^n$, as $L_1^m * L_1^n$, and let the notation $\mathcal{L}_1 * \mathcal{L}_1$ represent the scheme $\lim_m \lim_n L_1^m * L_1^n$. This scheme $\mathcal{L}_1 * \mathcal{L}_1$ is related to the limit behaviour of the norm of sums $x + y$, where a first large m is given, with a vector $x \in L_1^m$, and where n is then chosen much larger than m and than the "size" of x, with a vector $y \in L_1^n$; the notion of size of a vector x as to be precised: it could be for example the ℓ_1-norm of x, or the product of the ℓ_∞-norm of x by the smallest N such that $\operatorname{supp}(x) \leq N$. When the scheme $\mathcal{L}_1$ is inevitable, then obviously $\mathcal{L}_1 * \mathcal{L}_1$, $\mathcal{L}_1 * \mathcal{L}_1 * \mathcal{L}_1$, and so on, are also inevitable (extending here the use of the word *inevitable* from subsets of the unit sphere to bounded sets $A \subset X$ on which the norm function is bounded below by some $\kappa > 0$). When we look for ℓ_1, our first step is to check whether the schemes $2\,\mathcal{L}_1$ and $\mathcal{L}_1 * \mathcal{L}_1$ have something in common. Indeed, when x and y are blocks of a given ℓ_1 basis, with $x \in L_1^m$, $y \in L_1^n$, $x < y$ and $m < n$, then $\frac{1}{2}(x + y)$ belongs to $L_1^{k(m)}$, with $k(m) \to +\infty$.

This is precisely this first step that goes wrong with T. When we look for the scheme $\mathcal{S}_2 = \mathcal{L}_1 * \mathcal{L}_1$, we get a new inevitable scheme, distinct from $2\,\mathcal{L}_1$.

Indeed, the Figiel-Johnson argument for proving that T does not contain ℓ_1 shows that the inevitable set $L_1^m * L_1^n$ is well separated from the inevitable sets $2\, L_1^k$, when n is very large and $k > 20$, say. We may go further, and look for $\mathcal{S}_3 = \mathcal{L}_1 * \mathcal{L}_1 * \mathcal{L}_1$, and so on; in T, we do not seem to get much more by having this infinity of possibilities; what we do get is that, in some sense, we cannot get more from T than what was input at the start: if we have $x_j \in L_1^{n_j}$, $j = 1, \ldots, k$, with n_{j+1} very large with respect to the "size" of $x_1, \ldots, x_j$, then (taking the defining constant C equal to 1/2)

$$\frac{k}{2} \le \|\sum_{j=1}^{k} x_j\| \le (1+\varepsilon)\, \frac{k}{2}.$$

Let us turn now to Schlumprecht's space S. Since $f(n) = \log_2(n+1)$ grows slowlier than any power of n, we get ℓ_1^ns everywhere, hence the scheme $\mathcal{L}_1$ is inevitable in S, and so are the next schemes $\mathcal{S}_k$, $k \ge 2$. But now we get something new and very interesting.

Lemma 2 Schlumprecht's lemma. *If $x_1, \ldots, x_k$ is a successive sequence in S, such that $x_j \in L_1^{n_j}$ for $j = 1, \ldots, k$, with n_{j+1} very large compared to ε^{-1}, to the size of $x_1, \ldots, x_j$ and to $n_1, \ldots, n_j$, then*

$$\frac{k}{f(k)} \le \|x_1 + \cdots + x_k\| \le (1+\varepsilon)\, \frac{k}{f(k)}.$$

A sequence such as $x_1, \ldots, x_k$ will be called a *rapidly increasing sequence* (of ℓ_1^ns), in short RIS. Schlumprecht's Lemma states that for a RIS, we almost get an equality between the norm of the sum $\| \sum_{i=1}^k x_i\|$ and the lower bound $k/f(k)$ that was imposed by the definition of the space. Recall the observation made before the statement of the Lemma, that implies that for every $k \ge 1$ and every subspace $Y \subset S$, we may find a RIS of length k consisting of small perturbations of vectors in Y. In other words, the normalized sums of RIS of length k form a nearly inevitable set A_k in S.

Since $f(k)$ tends to infinity with k, it is clear that we get now infinitely many different inevitable classes in S. This was the first known example of this situation, and it was used (implicitly) by Schlumprecht in order to show that S is *arbitrarily distortable.* We obtain in this way what is for us the most important feature of Schlumprecht's example: on one hand, we can find ℓ_1^n in every subspace; on the other hand, we can always combine very different $\ell_1^{n_i}$ in a RIS and get a behaviour arbitrarily far from the ℓ_1 behaviour. There is an endless interplay between these classes of vectors: we see that $\mathcal{S}_k$ deviates more and more from the ℓ_1^k behaviour; if $x_1, \ldots, x_k$ is as above then clearly it is not a good ℓ_1^k basis, since $f(k)$ can get as big as we want, but by Lemma 1 it can be blocked to give a good $\ell_1^{\sqrt{k}}$, say. Applying this blocking procedure with successive RIS of increasing lengths k_j will give a new RIS built from pieces in $L_1^{\sqrt{k_j}}$; the sum of this RIS can again be blocked to ℓ_1^n, and so on... This remark is crucial for the construction of the example X_{gm} in [GM1], and it is used intensively in section 7.

We can precise what we mean by “well separated” inevitable sets, in the framework of S. Schlumprecht's Lemma does not only say that RIS deviate more and more from the ℓ_1^n behaviour, but it gives a precise estimate for the norm of the sum. With this information, it is possible to associate a class of functionals that almost norm the sums of RIS. For every $n > 1$, let A_n denote the set of normalized vectors which are multiple of the sum of a RIS sequence of length n; we define a class A_n^* of functionals on S, consisting of all functionals of the form $f(n)^{-1}(x_1^* + \cdots + x_n^*)$, with $\|x_i^*\| \le 1$, $i = 1, \ldots, n$. If $x \in A_n$, then we may easily find $x^* \in A_n^*$ such that $x^*(x) > 1/2$. Indeed, this vector x has the form $c\,n^{-1} f(n) \sum_{i=1}^n x_i$, where (x_i) is a RIS and $\frac{1}{2} < c \le 1$ (apply Lemma 2 with $\varepsilon < 1$); we select for each i a norming functional x_i^* for x_i, with $\mathrm{supp}(x_i^*) = \mathrm{supp}(x_i)$, and we set $x^* = f(n)^{-1}(x_1^* + \cdots + x_n^*)$. We have $\|x^*\| \le 1$ by the definition of S, and $x^*(x) \ge c > 1/2$. Furthermore, it can be shown (see [GM1,OS], or [BLi, Theorem 13.30]) that $|x_m^*(x_n)|$ is small –depending upon $\min(m, n)$– when $x_m^* \in A_m^*$, $x_n \in A_n$ and m, n are very different. This gives a weak form of orthogonality, which is what we name *almost biorthogonal inevitable system* in the next section.

4 Coding with inevitable sets

Let X be a normed space and let $S(X)$ be its unit sphere. We shall say that a subset $A \subset S(X)$ is *inevitable* if $A \cap S(Y) \ne \emptyset$ for every infinite-dimensional (not necessarily closed) subspace $Y \subset X$. It is sometimes more convenient to work with the notion of a *nearly inevitable set* $A \subset S(X)$, that has the property that $\inf\{d(y, A) : y \in Y\} = 0$ for every infinite-dimensional subspace $Y \subset X$.

When X has a basis, it is easy to check that every subspace contains a further subspace which is spanned by a perturbation of a block sequence (see [LT, 1.a.11]). It follows that A is nearly inevitable in X when the above condition is true for Y an arbitrary block subspace. Observe that if we replace a nearly inevitable set A by the enlargement A_ε consisting of all $x \in S(X)$ such that $\mathrm{dist}(x, A) < \varepsilon$, we get an inevitable set A_ε.

The most obvious example of an inevitable set is the unit sphere $S(Y)$ of a finite-codimensional subspace Y. On the sphere of ℓ_2, I would not be able to show any interesting example that can be described and proved inevitable using only pre-'90s ideas. The discussion of the preceding section shows that in a space with basis, close to ℓ_1 in the sense above, all the classes L_1^n are inevitable; we also said that in Schlumprecht's space S, we can even find a *sequence* of distinct inevitable sets.

A key observation made in [GM1] is that if a space X contains infinitely many inevitable sets that are all well disjoint from one another, then these can be used to construct an equivalent norm on X such that no sequence is C-unconditional in this norm. The idea is to use a certain coding, that has some similarity with what was done in [MR] for getting a statement about every subsequence of a given sequence; this is extended now to general vectors (*general*, as opposed to vectors from a given weakly null sequence); here, the *coding action* will be related to the numbering of a sequence of inevitable sets, while the coding action in [MR] was simply related to the numbering of the given sequence; we shall get in this way a statement about every subspace of a given space. Let us explain this obvious coding principle in a simplified setting: let Δ be a countable set, and let $(B_n)_{n\geq 0}$ be a sequence of disjoint non empty subsets of Δ; a *coding function* is an injective map σ, from the countable set of finite sequences of elements of Δ, to the natural numbers; a coding number $N = \sigma(d_1, \ldots, d_m)$ is thus associated in a $1-1$ way to every finite sequence $(d_1, \ldots, d_m)$ of elements of Δ, where m varies in $\mathbb{N}$; the *coding action* builds a tree of finite sequences, by saying that $(d_1, \ldots, d_m, d)$ is a successor of $(d_1, \ldots, d_m)$ if and only if d belongs to B_N with $N = \sigma(d_1, \ldots, d_m)$, and saying that $(d_1, \ldots, d_m)$ is a node of the tree if $(d_1, \ldots, d_j)$ is a successor of $(d_1, \ldots, d_{j-1})$ for $j = 2, \ldots, m$. If $(d_1, \ldots, d_m)$ and $(e_1, \ldots, e_n)$ are two nodes of this tree and if d_j, e_k belong to the same set B_ℓ, then it follows that $j = k$ and that $d_1 = e_1$, $\ldots$, $d_{j-1} = e_{j-1}$. If $\mathcal{C}$ is a class of subsets of Δ such that every $C \in \mathcal{C}$ intersects every set B_n, then clearly, for every $C \in \mathcal{C}$ we can construct arbitrarily long nodes $(d_1, \ldots, d_n)$ such that $d_j \in C$ for $j = 1, \ldots, n$. In this way, we have a tool that can affect every $C \in \mathcal{C}$. Below, Δ will be a rich enough countable subset of the unit ball of the dual of a separable Banach space X and each $C \in \mathcal{C}$ will be a family of functionals which are norming for some subspace Y_C of X.

Let $A_1, A_2, \ldots$ be a sequence of subsets of the unit sphere of a normed space X and let $A_1^*, A_2^*, \ldots$ be a sequence of subsets of the unit ball of X^*. We shall say that $A_1, A_2, \ldots$ and $A_1^*, A_2^*, \ldots$ are an *almost biorthogonal inevitable system with constant* $\delta > 0$ if the following conditions hold for every integer $n \geq 1$:

- (i) the set A_n is inevitable;
- (ii) for every $x \in A_n$ there exists $x^* \in A_n^*$ such that $x^*(x) > 1/2$;
- (iii) for every $m \geq 1$ with $n \neq m$, every $x \in A_n$ and every $x^* \in A_m^*$, we have $|x^*(x)| < \delta$.

The definition is interesting only when $\delta > 0$ is small. It is not at all obvious that any Banach space contains an almost biorthogonal inevitable system with constant $\delta < 0.01$ say. As far as I know, no example was known before the

Schlumprecht space S appeared. The main result of this section is the following theorem, whose proof is taken almost *verbatim* from [GM1].

Theorem 3 *Let r be an integer > 9 and let X be a separable normed space containing an almost biorthogonal inevitable system with constant $\delta = r^{-2}$. Then there is an equivalent norm on X such that no sequence is $r/9$-unconditional.*

PROOF. We shall write the proof in the real case. Let $\|.\|$ be the original norm on X and let $A_1, A_2, \ldots$ and $A_1^*, A_2^*, \ldots$ be the almost biorthogonal inevitable system with constant $\delta = r^{-2}$. For each $n \geq 0$ let Z_n^* be a countable subset of $\bigcup_{0<\lambda<1} \lambda A_n^*$ such that for every $x \in A_n$ there exists $x^* \in Z_n^*$ with $0 < x^*(x) - 1/2 < \delta$. Obviously the almost orthogonality relations (iii) still hold between the sets Z_m^* and A_n when $m \neq n$. Let $\Delta = \bigcup_{n=1}^{\infty} Z_n^*$. Next, let σ be an injection to the natural numbers from the countable collection of finite sequences of elements of Δ.

We shall now define a collection of functionals which we call *special* functionals. A *special sequence* of functionals is a sequence of the form $z_1^*, z_2^*, \ldots, z_r^*$, where $z_1^* \in Z_1^*$ and, for $1 \leq i < r$, $z_{i+1}^* \in Z_{\sigma(z_1^*,\ldots,z_i^*)}^*$. A *special functional* is simply the sum $z^* = z_1^* + \cdots + z_r^*$ of a special sequence of functionals. We shall let Γ stand for the collection of special functionals. Let us define an equivalent norm $|||.|||$ on X by

$$|||x||| = \max(\|x\|, r \sup\{|z^*(x)| : z^* \in \Gamma\}).$$

Let $x_1, x_2, \ldots$ be any sequence of linearly independent vectors in X. We shall show that it is not $r/9$-unconditional in the norm $|||.|||$. We shall do this by constructing a sequence of vectors $z_1, \ldots, z_r$, generated by $x_1, x_2, \ldots$ and disjointly supported with respect to these vectors, with the property that

$$r\, |||\sum_{i=1}^{r} (-1)^i z_i||| < 9\, |||\sum_{i=1}^{r} z_i|||.$$

Let X_1 be the algebraic subspace generated by $(x_i)_{i=1}^{\infty}$. Since A_1 is an inevitable set, we can find $z_1 \in A_1 \cap X_1$. This implies that z_1 has norm 1 and is generated by finitely many of the x_i. Next we can find $z_1^* \in Z_1^*$ such that $0 < z_1^*(z_1) - 1/2 < \delta$. Now let X_2 be the algebraic subspace generated by all the x_i not used to generate z_1. Since $A_{\sigma(z_1^*)}$ is inevitable, we can find $z_2 \in A_{\sigma(z_1^*)} \cap X_2$ of norm 1. We can then find $z_2^* \in Z_{\sigma(z_1^*)}^*$ such that $0 < z_2^*(z_2) - 1/2 < \delta$.

Continuing this process, we obtain sequences $z_1, \ldots, z_r$ and $z_1^*, \ldots, z_r^*$ with the following properties. Let $n_1 = 1$, and $n_{i+1} = \sigma(z_1^*, \ldots, z_i^*)$ for $1 \leq i < r$. First, $z_i \in A_{n_i}$ (thus $\|z_i\| = 1$) for each i. Second, $z_i^* \in Z_{n_i}^*$ for each i (i.e. $z_1^*, \ldots, z_r^*$ is a special sequence of functionals). Third, $z_i^*(z_i) \sim 1/2$ for each i. Fourth, since σ is an injection, the z_i^* belong to different Z_n^*s, so $|z_i^*(z_j)| < \delta$ when $i \neq j$ since $z_i^* \in Z_{n_i}^*$ and $z_j \in A_{n_j}$. Let us now estimate $|||\sum_{i=1}^r z_i|||$. Since $z_1^*, \ldots, z_r^*$ is a special sequence, the triple norm is at least

$$r\Big(\sum_{i=1}^r z_i^*\Big)\Big(\sum_{i=1}^r z_i\Big) > r(r/2 - \delta r^2) = r(r/2 - 1) > r^2/3$$

(since $r > 6$). On the other hand, if $(w_i^*)_{i=1}^r$ is any special sequence, let t be the maximal index i such that $w_i^* = z_i^*$ (or let $t = 0$ if $w_1^* \neq z_1^*$). Then

$$\Big|\sum_{i=1}^r (-1)^i w_i^*(z_i)\Big| \leq \Big|\sum_{i=1}^t (-1)^i w_i^*(z_i)\Big| + |w_{t+1}^*(z_{t+1})| + \sum_{i=t+2}^r |w_i^*(z_i)|.$$

Since σ is an injection, w_i^* and z_j^* are chosen from different sets Z_n^* whenever $i \neq j$ or $i = j > t+1$. By property (iii) this tells us that $|w_i^*(z_j)| < \delta$. In particular, $\sum_{i=t+2}^r |w_i^*(z_i)| < \delta r = 1/r$. When $i \leq t$ we know that $w_i^* = z_i^*$, hence $|1/2 - w_i^*(z_i)| \leq \delta$, so $\Big|\sum_{i=1}^t (-1)^i w_i^*(z_i)\Big| \leq 1/2 + t/r^2$. It follows that

$$\Big|\sum_{i=1}^r (-1)^i w_i^*(z_i)\Big| \leq 1/2 + 1 + 2/r < 2.$$

We also know that $\sum_{i \neq j} |w_i^*(z_j)| \leq \delta r^2 = 1$. Putting all these estimates together, we find that

$$r \sup_{w^* \in \Gamma} |w^*(\sum_{i=1}^r (-1)^i z_i)| \leq 3r.$$

Finally, by the triangle inequality, $\| \sum_{i=1}^r (-1)^i z_i\| \leq r$ and $||| \sum_{i=1}^r (-1)^i z_i||| \leq 3r$, from which it follows that the sequence $x_1, x_2, \ldots$ was not $r/9$-unconditional in the equivalent norm.

5 HI spaces, spectral properties and consequences

Definition 4 *Let X be an infinite-dimensional Banach space, real or complex. We say that X is* indecomposable *if X cannot be written as the topological direct sum of two infinite-dimensional closed subspaces Y_1 and Y_2. We say that X is* hereditarily indecomposable *(in short, HI) if every closed infinite-dimensional subspace Y of X is indecomposable, that is if no subspace Y of X can be written as the topological direct sum of two infinite-dimensional closed subspaces Y_1 and Y_2 of X.*

Obviously, if X is HI then every subspace $Y \subset X$ is HI. It is easy to see that a Banach space X is HI if and only if for all subspaces Y and Z of X, we have

$$\inf\{\|y-z\| : y \in Y,\, z \in Z,\, \|y\| = \|z\| = 1\} = 0.$$

We see that X is HI when the "angle" between any two subspaces Y and Z of X is equal to 0. Since every Banach space contains a subspace with basis, it is formal from the existence of any HI space that there exist HI spaces with monotone basis; the example X_{gm} in [GM1] has a basis, and it is also reflexive; much more difficult is another example due to Gowers [G3] of a space X_g without any reflexive subspace, and not containing c_0 or ℓ_1; it follows by James [J1] that X_g does not contain any subspace with unconditional basis, hence by the dichotomy theorem, X_g must contain a HI subspace Y_g; of course Y_g has no reflexive subspace; saying that

there exist a HI space Y_g with no reflexive subspace

seems to be a clean way to present things, but [G3] left open the point of deciding whether the space X_g is already HI.

Definition 5 *A bounded linear operator $T \in \mathcal{L}(X, Y)$ is* strictly singular *when there is no infinite-dimensional subspace $X_0 \subset X$ such that T restricted to X_0 is an into isomorphism; in other words, for every X_0 and $\varepsilon > 0$ there exists $x \in X_0$ such that $\|Tx\| < \varepsilon\,\|x\|$.*

It is standard to deduce that when T is strictly singular, we can construct in any subspace X_0 of X a further subspace $X_1 \subset X_0$, spanned by a normalized basic sequence, such that $\|T_{|X_1}\| \le \varepsilon$ (see [LT, 2.c.4]). Observe that X is HI if and only if for every subspace $Y \subset X$, the quotient map $\pi_Y : X \to X/Y$ is strictly singular.

Let X be a complex Banach space and let T be a bounded linear operator on X. Let Ω_T be the set of all $\mu \in \mathbb{C}$ such that there exist $c > 0$ and a finite-codimensional subspace $Z \subset X$ such that $\|Tx - \mu x\| \ge c\,\|x\|$ for every $x \in Z$. Clearly Ω_T is open in $\mathbb{C}$ and contains the complement of the spectrum $\mathrm{Sp}(T)$ of T. Saying that $\mu \in \Omega_T$ implies that $\ker(T - \mu Id)$ is finite-dimensional and that the range of $T - \mu Id$ is closed. This indicates that $T_1 = T - \mu Id$ is *semi-Fredholm* when $\mu \in \Omega_T$, with generalized index $\mathrm{ind}(T_1) = \dim(\ker T_1) - \mathrm{codim}(T_1(X))$ finite or equal to $-\infty$.

Let $F_T = \mathbb{C} \setminus \Omega_T$; it is not difficult to show that this closed subset of $\mathrm{Sp}(T)$ is not empty. Let us give a rather simple argument for this. It is well known that $U \in \mathcal{L}(X)$ is Fredholm if and only if the class $\widehat{U}$ of U in the quotient algebra $\mathcal{L}(X)/\mathcal{K}(X)$ is invertible (where $\mathcal{K}(X)$ denotes the ideal of compact operators); this quotient Banach algebra (the *Calkin algebra*) is not trivial when X is infinite-dimensional. It follows that the spectrum of $\widehat{T}$ in the Calkin algebra is not empty; this spectrum is the essential spectrum $\sigma_{\text{ess}}(T)$ of T. Let λ be a point of $\sigma_{\text{ess}}(T)$ with maximal modulus. Then $T - \mu Id$ is Fredholm for every $\mu \in \mathbb{C}$ such that $|\mu| > |\lambda|$; by continuity of the index, all operators $T - \mu Id$ for $|\mu| > |\lambda|$ have index 0, because this is true when μ is large enough to make $T - \mu Id$ invertible. If $T - \lambda Id$ was semi-Fredholm, then all $T - \mu Id$ for μ close to λ would be also semi-Fredholm, with the same (generalized) index; but this index must be 0 by the preceding argument, and $T - \lambda Id$ would then be Fredholm, which was excluded from the beginning by the fact that λ belongs to the essential spectrum. This show that $\lambda \in F_T$.

If $\lambda \in F_T$, that is if $\lambda \notin \Omega_T$, we may construct by induction a normalized basic sequence (x_n) such that $\|Tx_n\| \leq 2^{-n}\varepsilon$, and get a subspace $Y_\varepsilon \subset X$ such that $\|Ty - \lambda y\| \leq \varepsilon \|y\|$ for every $y \in Y_\varepsilon$ (see again [LT, 2.c.4]).

Suppose now that X is a complex HI space. It is easy to see that F_T contains exactly one element λ in this case: indeed, if $\lambda, \mu \in F_T$, we may find two subspaces Y and Z in X such that $T \sim \lambda Id$ on Y and $T \sim \mu Id$ on Z; since the unit spheres of Y and Z almost meet, it follows that $\lambda = \mu$. Let λ_0 be this unique value, and let $U = T - \lambda_0 Id$. Let Z be any subspace of X. Since $\lambda_0 \in F_T$, there exists for every $\varepsilon > 0$ a subspace Y such that $\|U_{|Y}\| < \varepsilon$. Since X is HI, the unit spheres of Z and Y almost meet, thus we may find z in the unit sphere of Z such that $\|Uz\| < \varepsilon$. This shows that U is strictly singular. We already obtained most of the next theorem.

Theorem 6 *Let X be a complex HI space. Then every $T \in \mathcal{L}(X)$ can be written as $T = \lambda Id + S$, where $\lambda \in \mathbb{C}$ and S is strictly singular. Thus every $T \in \mathcal{L}(X)$ is either strictly singular or Fredholm with index 0. Furthermore, the spectrum of T is either finite, or consists of a sequence (λ_n) converging to λ. In this second case, each $\lambda_n \neq \lambda$ is an eigenvalue of T with finite multiplicity.*

PROOF. We proved that there exists a number $\lambda \in \mathbb{C}$ such that $T - \lambda Id = S$ is strictly singular. If $\lambda \neq 0$, it is classical that $T = \lambda Id + S$ is Fredholm with index 0, because it is a strictly singular perturbation of λId (see for example [LT, 2.c.10]). The property of eigenvalues is also well known. By the discussion above, if $\mu \neq \lambda$ belongs to the spectrum of T, then μ is not in the essential spectrum of T, hence $T - \mu Id$ is Fredholm and not invertible, therefore μ is an eigenvalue of T with finite multiplicity.

In the real case, we have:

Corollary 7 *Let X be a real HI space. Every $T \in \mathcal{L}(X)$ is either strictly singular or Fredholm with index 0.*

PROOF. Let X be a real HI space and consider its complexification $X_\mathbb{C} = X \oplus X$. This space $X_\mathbb{C}$ need not be a complex HI space, but it has the property that given three subspaces, the sphere of one of them almost meets the sum of the two other subspaces. This implies that the essential spectrum of any $U \in \mathcal{L}(X_{\mathbb{C}})$ contains at most two points, say λ and μ (using the three subspaces property from above).

Let $T \in \mathcal{L}(X)$, and consider its complexification $U = T_{\mathbb{C}}$ on $X_{\mathbb{C}}$. The spectrum of the complexified operator $T_{\mathbb{C}}$ is invariant under complex conjugation, therefore $\mu = \overline{\lambda}$. Either λ is real, and $T - \lambda Id$ is strictly singular; then $T = \lambda Id + (T - \lambda Id)$ is Fredholm with index 0 if $\lambda \neq 0$, or T is strictly singular when $\lambda = 0$. Otherwise, we have $\lambda \in \mathbb{C} \setminus \mathbb{R}$; the part of the spectrum of $T_{\mathbb{C}}$ contained in the upper half plane is finite or consists of a convergent sequence of eigenvalues with finite multiplicity, together with its limit λ or $\overline{\lambda}$. Then $T_{\mathbb{C}} - \nu Id$ is Fredholm with index 0 for every $\nu \notin \{\lambda, \overline{\lambda}\}$; in particular, $T_{\mathbb{C}}$ is Fredholm with index 0, and the same holds for T.

Corollary 8 *Let X be a HI space, real or complex. Then X is not isomorphic to any proper subspace. In particular, X is not isomorphic to its hyperplanes.*

PROOF. Let T be an isomorphism from X into itself; then T is not strictly singular, hence it must be Fredholm with index 0 by Theorem 6 or Corollary 7 and thus $TX = X$.

These properties of operators on HI spaces were not immediately seen by the authors of [GM1]; Gowers actually constructed, before [GM1] was written, a modification X_u of X_{gm} and he showed that X_u is not isomorphic to its hyperplanes [G2]; this space X_u was thus the first *declared* example of a space not isomorphic to its hyperplanes. This example X_u is a very intriguing example of space with unconditional basis.

If X is a reflexive HI space, then the facts that the spectrum is countable and that $T - \mu Id$ is Fredholm with index 0 for all but one value $\mu = \lambda \in \mathbb{C}$ hold true for operators on X^*, although X^* need not be HI. As a consequence, the dual of a reflexive HI space is not isomorphic to any proper subspace, and a reflexive HI space is not isomorphic to any proper quotient. However, these results do not seem to answer the following question: does there exist a HI space X isomorphic to its dual X^*? We can say that such an X must be reflexive, because $X^{**} \simeq X$ is HI and $X \subset X^{**}$. An obvious remark is that the usual way to get simple examples of spaces isomorphic to their dual, namely $X = Y \oplus Y^*$, Y reflexive, has no chance to yield a HI space X!

Ferenczi [F2] has shown that, given a complex HI space X and a bounded linear operator T from a subspace Y of X to X, one can write $T = \lambda i_{Y,X} + S$, where $\lambda \in \mathbb{C}$, S is strictly singular and $i_{Y,X}$ denotes the inclusion map from Y to X. Clearly, Ferenczi's result is a characterization of complex HI spaces.

Let us sketch Ferenczi's proof. In some sense all subspaces of a HI space X intersect; we may define a net of subspaces of X that captures a good part of the structure of this HI space; the order of this net is not the inclusion, as it is not *strictly* true that any two subspaces have an infinite-dimensional intersection, but almost.... We say that $Y \le Z$ if there exists a compact operator $K : Y \to X$ such that $(i_{Y,X} + K)(Y) \subset Z$. Given $Z_1, Z_2 \subset X$ there exists Y such that $Y \le Z_1$ and $Y \le Z_2$. We could call "germ" of HI space an equivalence class of such nets, in a way to be made precise. An interesting class of examples is the family of spaces containing no infinite unconditional sequence but having only a finite set of germs of HI spaces [F3]. The example X_g of Gowers [G3] of a HI space with no reflexive subspace gives an example of "non reflexive germ".

Ferenczi's proof consists essentially in showing that the space of "germs" of operators on X is a Banach field, hence isomorphic to $\mathbb{C}$. A *germ of operator* is an equivalence class for the relation where $T_1 \in \mathcal{L}(Z_1, X)$ and $T_2 \in \mathcal{L}(Z_2, X)$ are equivalent if we can find a subspace $Y \le Z_1, Z_2$ such that the operator $T_1 \circ (i_{Y,X} + K_1) - T_2 \circ (i_{Y,X} + K_2)$ is compact on Y, where K_1 and K_2 are the compact operators that appear in the definition of the order.

6 Sequence spaces

The scalar field is $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Since we want to include some of the results of Argyros-Felouzis, we shall work in an extended setting where the space of sequences is a space of sequences of vectors taken from some separable space V, or even more generally, from a sequence $\mathbf{V} = (V_n)$ of separable spaces. We let $\mathbf{K}$ denote the sequence (V_n) where $V_n = \mathbb{K}$ for every $n \ge 0$. The reader may decide that $\mathbf{V} = \mathbf{K}$ until he wants to study the section about [AF]. The norm on V_n is denoted $\| \,.\, \|_n$. The exposition is slightly simpler if we assume that each V_n is reflexive, but one can modify the construction in order to take care of the case V_n separable but not reflexive.

Let us denote by $c_{00}(\mathbf{V})$ the space of vector sequences $x = (v_n)_{n \ge 0}$, where $v_n \in V_n$ for every $n \ge 0$, and such that $v_n \ne 0$ for only finitely many values of n. If $x \in c_{00}(\mathbf{V})$, we call *support* of x the (finite) set $\operatorname{supp} x$ of integers n such that $v_n \ne 0$. If $x, y \in c_{00}(\mathbf{V})$, we say that $x < y$ if $\operatorname{supp}(x) < \operatorname{supp}(y)$. We also write $n < x$ when $n \in \mathbb{N}$ and $n < \min \operatorname{supp}(x)$. If $x_1 < \ldots < x_n$, then we say that the vectors $x_1, \ldots, x_n$ are *successive.* An infinite sequence of successive non-zero vectors is also called a *block basis* and a subspace generated by a block basis is a *block subspace.* An *interval* of integers is a set of the form $\{n, n+1, \ldots, m\}$ and the *range* of a vector x, written $\operatorname{ran}(x)$, is the smallest interval containing $\operatorname{supp}(x)$. It is convenient to write $x = (v_n) \in c_{00}(\mathbf{V})$ as $x = \sum_{n=0}^{+\infty} v_n \otimes \mathbf{e}_n$. Given a subset $E \subset \mathbb{N}$ and a vector x as above, we write

Ex for the vector $\sum_{n\in E} v_n \otimes \mathbf{e}_n$. We let P_n denote the projection corresponding to the set $E_n = \{0, \ldots, n\}$; thus P_n is the natural projection from $c_{00}(\mathbf{V})$ onto $\sum_{k=0}^{n} V_k \otimes \mathbf{e}_k$.

If $\mathbf{V}^*$ is the sequence of duals (V_n^*), we have a natural duality between $c_{00}(\mathbf{V})$ and $c_{00}(\mathbf{V}^*)$; the second space will be considered as *space of functionals*; we extend our terminology to functionals, for example we shall talk about *successive functionals.*

Let $\mathcal{X}(\mathbf{V})$ stand for the set of Banach spaces X obtained as the completion of $c_{00}(\mathbf{V})$ for a norm $\| \,.\, \|$ such that $\|Ex\| \le \|x\|$ for every interval E, and $\|v_n \otimes \mathbf{e}_n\| = \|v_n\|_n$ for every $n \ge 0$ and $v_n \in V_n$; when $\mathbf{V} = \mathbf{K}$, this means that $(\mathbf{e}_n)$ is a normalized bimonotone basis for X.

The reader must pay attention to the following fact: when $\mathbf{V} = \mathbf{K}$, every subspace Y of X contains a normalized sequence (y_n) which is a perturbation of a successive sequence $(x_n) \subset X$. This is clearly not the case in the vector case, as Y could be equal to $V_0 \otimes \mathbf{e}_0$ for example. However, if we assume that Y is a subspace of X such that for every $n \ge 0$, the projection P_n is not an into isomorphism from Y to $\sum_{k=0}^{n} V_k \otimes \mathbf{e}_k$, then the standard gliding hump procedure can be extended to the vector setting in an obvious way. We shall say in this case that Y is *a GH-subspace.* We shall obtain interesting information only about GH-subspaces. This is a vacuous limitation in the scalar case, but it is an important one in the vector case. When the restriction to Y of every P_n, $n \ge 0$, is strictly singular, we get that every subspace of Y is a GH-subspace.

Given a "slowly increasing" function f from $[1, +\infty)$ to $[1, +\infty)$ and a space $X \in \mathcal{X}(\mathbf{V})$, we shall say that X *satisfies a lower f-estimate* if, given any vector $x \in X$ and any sequence of intervals $E_1 < \ldots < E_n$, we have that $\|x\| \ge f(n)^{-1} \sum_{i=1}^{n} \|E_i x\|$. In the dual formulation, this property means that whenever $x_1^*, \ldots, x_n^*$ are successive functionals with norm ≤ 1, then

$$\|(x_1^* + \cdots + x_n^*)/f(n)\|_{X^*} \le 1.$$

Let $X \in \mathcal{X}(\mathbf{V})$ and $x \in X$. For every $n \ge 1$, let

$$\|x\|_{(n)} = \sup \sum_{i=1}^{n} \|E_i x\|$$

where the supremum is extended to all families $E_1 < \ldots < E_n$ of successive intervals. This quantity is clearly increasing with n, and $\|x\| = \|x\|_{(1)}$ by the monotonicity property of $(\mathbf{e}_i)$, since $X \in \mathcal{X}(\mathbf{V})$. Observe that the basis $(\mathbf{e}_i)$ satisfies $\|v_i \otimes \mathbf{e}_i\|_{(n)} = \|v_i\|_i$ for every $n \ge 1$. Clearly, $\| \,.\, \|_{(n)}$ is an equivalent norm on X, with $\|x\| \le \|x\|_{(n)} \le n\, \|x\|$; also, $\|Ex\|_{(n)} \le \|x\|_{(n)}$ for every interval E.

The value of this norm at x is related to the fact that x can be broken into blocks that look like the unit vector basis of ℓ_1^n, or more accurately, into blocks that look like a ℓ_{1+}^n basis; this norm will be used in the definition of RIS as a substitute for the notion of ℓ_{1+}^n-average used in [GM1]. It is easy to check that when $(x_i)_{i=1}^{n^2}$ are successive and have norm ≤ 1, then $x = n^{-2}\sum_{i=1}^{n^2} x_i$ satisfies $\|x\|_{(n)} \leq 1 + 1/n$. When (x_i) is a $\ell_{1+}^{n^2}$-sequence with constant C, we get the additional fact that $\|x\| \geq 1/C$. This type of vectors x for which the original norm is well equivalent to $\|x\|_{(n)}$ for some large n will play an essential rôle later. Not surprisingly, the proof of the next lemma, which states the existence of such vectors x when $X \in \mathcal{X}(\mathbf{V})$, is essentially identical to that of Lemma 1.

Lemma 9 *Let $X \in \mathcal{X}(\mathbf{V})$ satisfy a lower f-estimate. Given a positive integer n and $\varepsilon > 0$, there exists an integer $N(n,\varepsilon)$ such that for every sequence $x_1, \ldots, x_N$ of successive norm one vectors with $N \geq N(n,\varepsilon)$, we can find x of the form $x = \lambda \sum_{i\in A} x_i$, where A is some subinterval of $\{1, \ldots, N\}$, such that $\|x\| = 1$ and $\|x\|_{(n)} \leq 1 + \varepsilon$.*

Corollary 10 *Let $X \in \mathcal{X}(\mathbf{V})$ satisfy a lower f-estimate. Then for every $n \geq 1$ and $\varepsilon > 0$, every GH-subspace Y of X contains a vector y such that $\|y\| = 1$, $\|y\|_{(n)} \leq 1 + \varepsilon$ and $\|P_n(y)\| < \varepsilon$.*

PROOF. By the gliding hump procedure, we may select for every integer N a normalized sequence $y_1, \ldots, y_N$ of vectors in Y and successive vectors $n < x_1 < \ldots < x_N$ in X such that $\|y_i - x_i\| < \varepsilon/nN$. The result follows from Lemma 9 and an obvious approximation argument.

7 A class of examples

The contents of this section come from [GM1] and [GM2]. The general strategy is as follows: we want to use the coding principle from section 4 and build special functionals that will somehow distinguish between sums like $x + y + z$ and $x - y + z$; the difference in this section is that the inevitable sets are not given in advance, but must be constructed together with the norm, by an inductive procedure similar to the construction of Schlumprecht's space; in short, our example is a Schlumprecht space with special functionals.

Another difference with section 4 is that in order to kill unconditionality, we want to push the unconditionality constant beyond C, not only for a given big C, but for every C. In section 4, we used special functionals of a fixed length, depending upon C. Here, we shall need special sequences with different lengths k, tending to infinity. Each length k will be used to prove that every basic sequence has unconditionality constant $\geq C_k$, with $\lim_k C_k = +\infty$. In order to make the behaviour of the space easier to understand, we try to ensure that the different types of special functionals, of lengths k_1 and $k_2 \neq k_1$, have as little interaction as possible; for this we shall prove various Lemmas that give almost orthogonality of several classes of vectors and functionals. These Lemmas are easy. Also, special functionals should not ruin the possibility of having some form of Schlumprecht's Lemma in our space X; this will require some harder work.

The definition of special sequences is essentially taken from section 4, except that we must use sets (A^*_ℓ) which are not given in advance, but are enriched step by step as the induction proceeds; we have to guess from the beginning that the chosen sets (A^*_ℓ) will satisfy the needed properties at the end of the construction. However our choice is simple and inspired by what we saw about Schlumprecht's space S at the end of section 3: the set A^*_ℓ will consist of functionals of the form $f(\ell)^{-1}(x^*_1 + \cdots + x^*_\ell)$, where f is some logarithmic function, fixed in the whole chapter. But in order to make sure that some set A_ℓ (consisting of normalized sums of RIS of length ℓ) will be inevitable and normed by A^*_ℓ, we have to have some sort of Schlumprecht's Lemma here (as explained at the end of section 3, it is important to have a precise estimate for the norm of the sum of a RIS, in order to be able to predict a class of almost norming functionals for the vectors in A_ℓ). An obvious start is to force a lower f-estimate, which ensures that the scheme $\mathcal{L}_1$ is inevitable in X, as well as all sets A_ℓ obtained from RIS of length ℓ. But the presence of the special functionals will ruin the possibility that X has a behaviour as regular as that of the Schlumprecht space, and this regular behaviour seemed important for getting Schlumprecht's Lemma.

This difficulty will be solved in the following way: the usual Schlumprecht Lemma will hold for RIS that have length ℓ in some thin subset L of $\mathbb{N}$, and the corresponding norming functionals will be of the form $f(\ell)^{-1}(x^*_1+\cdots+x^*_\ell)$ as before. On the other hand, the special sequences $y^*_1, \ldots, y^*_k$ will have lengths k in another thin subset K, chosen "very far" from L. Then, the "special normalization" puts the special functional $f(k)^{-1/2}(y^*_1 + \cdots + y^*_k)$ in the dual unit ball, so that a RIS sequence of length $k \in K$ may have a sum whose norm is as big as $k/\sqrt{f(k)}$ (to be compared to the "usual" smaller value $k/f(k)$).

The failure of unconditionality will appear exactly as in section 4, by constructing together two sequences $(x_j)_{j=1}^k$, $(x^*_j)_{j=1}^k$, in such a way that $x_j \in A_{n_j}$, where n_j is the coding number assigned to the beginning $x^*_1, \ldots, x^*_{j-1}$ of the special sequence $x^*_1, \ldots, x^*_k$, and (x_j) is also a RIS vector (a notion to be defined, essentially the normalized sum of a RIS); the special functional $f(k)^{-1/2}(x^*_1 + \cdots + x^*_k)$ will then give to the sum $\sum_{j=1}^k x_j$ an abnormally large norm, that would not be achieved by the alternate sum $\sum_{j=1}^k (-1)^j x_j$, for which the special functionals will fail to push the norm far beyond the usual $k/f(k)$ estimate. In this way we show that the unconditionality constant is larger than $\sqrt{f(k)}$, and this can be done with arbitrarily large $k \in K$.

In order to generalize Schlumprecht's Lemma, we analyze what properties of a slowly increasing function g are needed. We shall show that the class of possible functions is flexible enough to allow the construction of a function g which agrees with f on L and with $\sqrt{f}$ on K (or on part of K), and stays between $\sqrt{f}$ and f everywhere. To this end we introduce the family $\mathcal{F}$ of functions $g : [1, \infty) \to [1, \infty)$ satisfying the following conditions:

(i) $g(1) = 1$ and $g(t) < t$ for every $t > 1$;
(ii) g is strictly increasing and tends to infinity;
(iii) $\lim_{t\to\infty} t^{-\varepsilon} g(t) = 0$ for every $\varepsilon > 0$;
(iv) the function $t/g(t)$ is concave and non-decreasing;
(v) $g(st) \leq g(s)g(t)$ for every $s, t \geq 1$.

We shall give a convenient representation formula for a subclass $\mathcal{F}_0$ of $\mathcal{F}$. Let us denote by $\mathcal{L}$ the class of real functions on $[0, +\infty)$ that are non-decreasing and 1-Lipschitz on $[0, +\infty[$, and tend to $+\infty$ at $+\infty$. Suppose that M belongs to $\mathcal{L}$; the reader will easily check that the formula

$$(F) \qquad g_M(t) = \exp\Big(\int_0^{\ln t} \frac{du}{1 + \mathrm{e}^{M(u)}}\Big)$$

defines a function $g_M \in \mathcal{F}$ provided g_M tends to infinity at infinity, which means that $\int_0^{+\infty} (1 + \mathrm{e}^{M(u)})^{-1}\, du = +\infty$. The only unpleasant point is to check that $t/g(t)$ is concave, see the appendix. The idea of using this subclass $\mathcal{F}_0$ is taken from Habala's paper [Ha].

One nice point about this subclass $\mathcal{F}_0$ is that it is extremely easy to glue together different functions from the class $\mathcal{L}$: if we divide $[0, +\infty)$ into successive intervals (I_n), it is obvious that M belongs to $\mathcal{L}$ if and only if M is continuous, tends to $+\infty$ at infinity and coincides on each interval I_n with a function $M_n \in \mathcal{L}$. Therefore, a function g is in $\mathcal{F}_0$ when it is C^1, tends to $+\infty$ at infinity and coincides on each interval e^{I_n} with a function $g_n \in \mathcal{F}_0$.

If we let $M(u) = \ln(a + bu)$, with $0 < b \leq a$, we get a function in $\mathcal{L}$, for which $g_M(t) = (1 + \frac{b}{1+a} \ln t)^{1/b}$. We shall use the two special cases $M_0(u) = \ln(1 + u)$, with $f_0(t) = 1 + \frac{1}{2} \ln t$ (corresponding to $a = b = 1$) and $M_1(u) = \ln(3 + 2u)$, with $f_1(t) = (1 + \frac{1}{2} \ln t)^{1/2} = \sqrt{f_0(t)}$ (corresponding to $b = 2$, $a = 3$). For the rest of the paper we set $f(t) = f_0(t) = 1 + \frac{1}{2} \ln t$; then $\sqrt{f} = f_1$ also belongs to the class $\mathcal{F}_0$. Notice that $t^{-1/4} \ln(t)$ decreases when $t > \mathrm{e}^4$. Checking the value at $t = \mathrm{e}^{16}$, we obtain that

$$(F_1) \qquad 4t^{-1/4} f(t) \leq 1 \quad \text{when} \quad t \geq \mathrm{e}^{16}\,.$$

We need a technical lemma.

Lemma 11 *For every* $t_0 = \mathrm{e}^{u_0}$ *with* $u_0 \geq 5$ *and* $t_1 = \mathrm{e}^{4u_0^2}$ *there exists a function* $g \in \mathcal{F}_0$ *such that* $\sqrt{f} \leq g \leq f$ *on* $[1, +\infty)$*,* $g = f$ *on* $[1, t_0]$ *and*

$g = \sqrt{f}$ *on* $[t_1, +\infty)$. *Similarly, there exists a function* $g \in \mathcal{F}_0$ *such that* $\sqrt{f} \le g \le f$ *on* $[1, +\infty)$, $g = \sqrt{f}$ *on* $[1, t_0]$ *and* $g = f$ *on* $[t_1, +\infty)$.

For the proof, see the appendix.

We begin by gathering some lemmas that do not make use of any "special" construction. Amongst them is a version of Schlumprecht's lemma (Lemma 14); Lemma 16 shows how to rebuild an ℓ_{1+}^n from a RIS, and Lemma 12 deals with almost orthogonality.

The first ingredient is the notion of a *rapidly increasing sequence*, in short RIS; it was already mentioned in sections 2 and 3, but not in precise terms. Let $X \in \mathcal{X}(\mathbf{V})$. We say that a sequence $x_1, \ldots, x_r$ of successive non-zero vectors in X *satisfies the RIS condition* if there is a sequence $n_1 < \ldots < n_r$ of integers such that $\mathrm{e}^{2r^3} < n_1$, $\|x_i\|_{(n_i)} \le 1$ for each $i = 1, \ldots, r$ and

$$\sqrt{f(n_i)} > |\mathrm{ran}(\sum_{j=1}^{i-1} x_j)|, \quad i = 2, \ldots, r.$$

If E is an interval, then the non-zero vectors of the sequence $Ex_1, \ldots, Ex_r$ clearly form a new RIS of length $r_1 \le r$. Also, $\lambda x_1, \ldots, \lambda x_r$ is a RIS when $0 < |\lambda| \le 1$ and every subsequence of a RIS is again a RIS.

Given $g \in \mathcal{F}$, $q \ge 1$ and $X \in \mathcal{X}(\mathbf{V})$, a (q, g)-*form on* X is defined to be a functional x^* of norm at most one which can be written as $\sum_{j=1}^q x_j^*$ for a sequence $x_1^* < \ldots < x_q^*$ of successive functionals all of which have norm at most $g(q)^{-1}$. Observe that if x^* is a (q, g)-form then $\|x^*\|_\infty \le g(q)^{-1}$ and $|x^*(x)| \le g(q)^{-1}\|x\|_{(q)}$ for any $x \in X$. Observe the obvious fact that a (q, g_1)-form is a (q, g_2)-form when $g_1 \ge g_2$.

We are looking for two kinds of orthogonalities: the first says that a long q-form x^* has a small action on a much longer ℓ_{1+}^n-average x, where $q << n$; this is simply given by the preceding relation $|x^*(x)| \le g(q)^{-1}\|x\|_{(q)}$. The second kind says that a very long form has a moderate action on the sum of a RIS (thus, by Lemma 14 below, a small action on the normalized sum of this RIS).

Lemma 12 *Suppose that* $(x_1, \ldots, x_r)$ *satisfies the RIS condition in a space* $X \in \mathcal{X}(\mathbf{V})$. *Let* $g \in \mathcal{F}$ *satisfy* $g \ge \sqrt{f}$. *If* x^* *is a* (q, g)*-form on* X *and* $q \ge \mathrm{e}^{2r^2}$, *then we have*

$$|x^*(x_1 + \cdots + x_r)| \le 3.$$

PROOF. Let $n_1 < n_2 < \ldots < n_r$ be the sequence of integers associated to the RIS property of the sequence $x_1, \ldots, x_r$. Let $i \in \{0, \ldots, r\}$ be such that $n_i < q \le n_{i+1}$ (consider that $n_0 = 0$ and that n_{r+1} is larger than q

and n_r). Observe that the RIS condition implies $\|x_j\|_\infty \le \|x_j\| \le 1$ for each $j = 1, \ldots, r$. The result follows from three easy inequalities,

$$\Big|x^*(\sum_{j=1}^{i-1} x_j)\Big| \le \|x^*\|_\infty \, |\mathrm{ran}(\sum_{j=1}^{i-1} x_j)| \le g(q)^{-1} f(n_i)^{1/2} \le 1,$$

$|x^*(x_i)| \le \|x_i\| \le 1$, and for $j \ge i+1$,

$$|x^*(x_j)| \le g(q)^{-1} \|x_j\|_{(q)} \le g(q)^{-1} \|x_j\|_{(n_j)} \le g(q)^{-1},$$

so that $|x^*(x_1 + \cdots + x_r)| \le 2 + rg(q)^{-1}$. When we have $q \ge \mathrm{e}^{2r^2}$, we get that $g(q) \ge \sqrt{f(q)} = (1 + \frac{1}{2} \ln q)^{1/2} \ge r$.

Remark 13 *It is possible to prove a $(1+\varepsilon)$-version of Lemma 12 (see [GM2]), as well as $(1+\varepsilon)$-versions of all lemmas that follow; we make the deliberate choice of giving simpler proofs, to the cost of introducing ridiculous constants* $5, 15, 45, 75 \ldots$ *in what follows.*

The next Lemma is a variation on Schlumprecht's Lemma. We need a more general statement than the one from section 3, that allows to play with different functions from the family $\mathcal{F}$. In our example, the next lemma will be applied either to $g = f$ or to $g = \sqrt{f}$, or actually to a variety of functions g between $\sqrt{f}$ and f.

Lemma 14 Variant of Schlumprecht's lemma. *Let $X \in \mathcal{X}(\mathbf{V})$, $g \in \mathcal{F}$, and let $p \ge 2$ be an integer. Suppose that $x_1 < \ldots < x_r$ in X satisfy $\|x_i\|_{(p^r)} \le 1$ for every $i = 1, \ldots, r$. Let $x = \sum_{i=1}^r x_i$ and suppose that*

$$\|Ex\| \le 1 \vee \sup\{|x^*(Ex)| : x^* \textit{ is a } (q,g)\textit{-form},\ 2 \le q \le p\}$$

for every interval E. Then

$$\|x\| \le rg(r)^{-1}.$$

The painful proof is deferred to the appendix.

Corollary 15 *Suppose that $\mathbf{V} = \mathbf{K}$. Let $X \in \mathcal{X}(\mathbf{K})$ and $g \in \mathcal{F}$, $g \ge \sqrt{f}$; suppose that X satisfies a lower f-estimate and that*

$$\|x\| \le \|x\|_{c_0} \vee \sup\{|x^*(x)| : x^* \textit{ is a } (q,g)\textit{-form},\ 2 \le q\}$$

for every $x \in X$. Then X is reflexive. When $\mathbf{V} \ne \mathbf{K}$, this is also true if each space V_n is reflexive.

PROOF. It follows immediately from the fact that X satisfies a lower f-estimate that the standard basis $\mathbf{e}_0, \mathbf{e}_1, \ldots$ is boundedly complete. Now suppose that it is not a shrinking basis. Then we can find $\varepsilon > 0$, a norm-1 functional $x^* \in X^*$ and a sequence of successive normalized blocks $y_1, y_2, \ldots$ such that $x^*(y_n) \geq 2\varepsilon$ for every $n \geq 1$. It is easy to see that if $x = \frac{1}{2} |A|^{-1} \sum_{i \in A} y_i$, then $\|x\|_{(n)} \leq 1/2 + 1/n$ for every $A \subset \mathbb{N}$ such that $|A| \geq n^2$, and $x^*(x) \geq \varepsilon$. It is clear that for every r, we may construct a RIS $x_1, \ldots, x_r$ with vectors x_i of the preceding form. By Lemma 12, we have $x^*(\sum_{i=1}^r x_i) \leq 3$ for every long form, therefore Lemma 14 can be applied to the sequence $\frac{1}{3}(x_i)$, proving that $\|\sum_{i=1}^r x_i\| \leq 3\, r/g(r)$. For r sufficiently large, this contradicts $x^*(\sum_{i=1}^r x_i) > r\,\varepsilon$. For the vector setting $\mathbf{V}$, the reader may consult [AF].

The next simple Lemma is useful in conjunction with Lemma 14. It explains how to "rebuild" a good ℓ_{1+}^n basis from a RIS.

Lemma 16 *Let $X \in \mathcal{X}(\mathbf{V})$, let m, r be integers such that $\mathrm{e}^{16} < m < r \leq m^4$; let $x_1 < \ldots < x_r$ in X be such that*

$$\|x_i\| \leq 1,\ i = 1, \ldots, r, \quad \textit{and} \quad \|\sum_{i \in A} x_i\| \leq \frac{|A|}{f(|A|)}$$

for every interval $A \subset \{1, \ldots, r\}$ such that $|A| \geq m$. Then for every integer n such that $mn \leq 2r^{3/4}$ we have

$$\|\sum_{i=1}^r x_i\|_{(n)} \leq 5\, \frac{r}{f(r)}.$$

PROOF. Let $x = \sum_{i=1}^r x_i$ and let $(E_h)_{h=1}^n$ be a sequence of n successive intervals. For every h, let E'_h be the largest interval contained in E_h such that $E'_h x = \sum_{i \in A_h} x_i$ for some interval $A_h \subset \{1, \ldots, r\}$; then, E_h is the union of E'_h and two small intervals at both ends of E_h, and $\|E_h x\| \leq 2 + \|E'_h x\|$. Let $H = \{h : |A_h| \geq m\}$. We get $\|E_h x\| \leq 2 + (m-1)$ when $h \notin H$, and

$$\begin{aligned}\sum_{h=1}^n \|E_h x\| &\leq n(m+1) + \sum_{h \in H} \|E_h x\| \leq 2nm + \sum_{h \in H} \frac{|A_h|}{f(|A_h|)} \\ &\leq 4r^{3/4} + \frac{\sum_h |A_h|}{f(m)} \leq 4r^{3/4} + \frac{r}{f(m)} \leq 4r^{3/4} + 4\,\frac{r}{f(r)}\end{aligned}$$

since $f(r) \leq f(m^4) \leq 4\, f(m)$ (using the conditions for the family $\mathcal{F}$). The result follows, because the condition $r \geq \mathrm{e}^{16}$ implies that $4r^{3/4} \leq r/f(r)$, by formula (F_1).

The construction of our examples uses a lacunary subset J of $\mathbb{N}$. Let us write J in increasing order as $\{j_1, j_2, \ldots\}$. We assume that each j_n is the fourth power of some integer. This set J should be such that between two successive elements j_n and j_{n+1} of J, there is enough room to apply Lemma 11; to be on the safe side, let us assume that

$$(L1) \qquad j_1 \geq \mathrm{e}^{256} \quad \text{and} \quad \forall n \geq 1, \quad \mathrm{e}^{256\, j_n^2} < j_{n+1}.$$

This implies by induction that

$$(L2) \qquad j_n \geq \mathrm{e}^{n^3}$$

for every $n \geq 1$, because $j_1 > \mathrm{e}$ and $\mathrm{e}^{(n+1)^3} \leq \mathrm{e}^{8n^3} \leq \mathrm{e}^{8j_n} < j_{n+1}$ for $n \geq 1$. Now we check that Lemma 11 can be applied between j_n^4 and $j_{n+1}^{1/4}$. If we let $u = 4j_n$, then $u \geq 5$ and $j_n^4 \leq \mathrm{e}^{4j_n} = \mathrm{e}^u$; by $(L1)$, $j_{n+1}^{1/4} \geq \mathrm{e}^{64j_n^2} = \mathrm{e}^{4u^2}$. With Lemma 11 and the conditions on J it is rather clear that

Lemma 17 *Let $K_0 \subset K$. There exists a function $g \in \mathcal{F}_0$ such that $f \geq g \geq \sqrt{f}$, $g(k) = \sqrt{f(k)}$ whenever $k \in K_0$ and $g(t) = f(t)$ whenever $j \in J \setminus K_0$ and t is in the interval $[j^{1/4}, j^4]$.*

Since we want to include some of the results of Argyros-Felouzis, we shall work in the extended setting $\mathbf{V}$. Also, we want to present a part of [GM2], so we are trying to build a space with a given algebra of operators generated by some set $\mathcal{S}$ of basic operators on our space. Here again, the reader may decide to consider that $\mathcal{S} = \{Id\}$ (the trivial case), which is what is needed for constructing a HI space.

Given two infinite sets $A, B \subset \mathbb{N}$, define the *spread from A to B* to be the map on c_{00} defined as follows. Let the elements of A and B be written in increasing order respectively as $\{a_0, a_1, \ldots\}$ and $\{b_0, b_1, \ldots\}$. Then $\mathbf{e}_n$ maps to zero if $n \notin A$, and $\mathbf{e}_{a_i}$ maps to $\mathbf{e}_{b_i}$ for every $i \geq 0$. Denote this map by $S_{A,B}$. Note that $S_{B,A}$ is (formally) the adjoint of $S_{A,B}$. Observe that for every interval projection E and any $U \in \mathcal{S}$, there exist two intervals F_1 and F_2 such that $EU = UF_1$, $UE = F_2U$.

Given any set $\mathcal{S}$ of spreads containing the identity map, we shall say that it is a *semi-group* if it is closed under composition (note that this applies to all compositions and not just those of the form $S_{B,C}S_{A,B}$). If we want to define extensions of the $S_{A,B}$'s in the $\mathbf{V}$ setting, we need to assume more about the relations between the different spaces V_n. We say that $\mathbf{V}$ and $\mathcal{S}$ are *compatible* if whenever $U \in \mathcal{S}$ and $U\mathbf{e}_m = \mathbf{e}_n$ for some m and n, then $V_m \subset V_n$ and $\|v\|_n \leq \|v\|_m$ for every vector $v \in V_m$. The trivial semi-group $\mathcal{S} = \{Id\}$ is of course compatible with any family $\mathbf{V}$. Given any set $\mathcal{S}$ of spreads, compatible with $\mathbf{V}$, all maps $S_{A,B}$ in $\mathcal{S}$ are extended to $c_{00}(\mathbf{V})$ in the usual way, by tensoring with the identity of V_n, giving the set $\mathcal{S}(\mathbf{V})$. For example, $S_{A,B}(v_n \otimes \mathbf{e}_n) = v_n \otimes S_{A,B}(\mathbf{e}_n)$. The compatibility assumption means that $\|S_{A,B}\| \leq 1$ for the $c_{00}(\mathbf{V})$ norm (or $\ell_1(\mathbf{V})$), for every $S_{A,B} \in \mathcal{S}$.

The main tool for our construction, already seen in section 4, is the notion of *special sequence*. We split the lacunary set J into two disjoint parts K and L: let $K, L \subset J$ be the sets $\{j_1, j_3, j_5, \ldots\}$ and $\{j_2, j_4, j_6, \ldots\}$. The set K is used for the lengths of special sequences, while $m \in L$ or m "close" to L is used for lengths of "regular" RIS $x_1, \ldots, x_m$ that will satisfy the *ordinary inequality* $\|\sum_{i=1}^m x_i\| \leq 3\,m/f(m)$; the sum of such RIS will be therefore normed by (m, f)-forms; this explains our choice for the members of special sequences below. For every $n \geq 0$, let us choose a countable subset $\Delta_n \subset V_n^*$ dense in V_n^* (this is where assuming V_n reflexive makes our life easier; if not, we would have to work with countable norming sets for V_n). Let $\mathbf{Q} \subset c_{00}(\mathbf{V}^*)$ be the (countable) set of sequences with the nth coordinate in Δ_n for every n and maximum at most 1 in V_n^*-norm. Let σ be an injection from the collection of finite sequences of successive elements of $\mathbf{Q}$ to the set L introduced above. Given $X \in \mathcal{X}(\mathbf{V})$ and given an integer $m \geq 1$, let $A_m^*(X)$ be the set of (m, f)-forms on X, i.e. the set of all functionals x^* of norm at most 1 of the form $x^* = f(m)^{-1} \sum_{i=1}^m x_i^*$, where $x_1^* < \ldots < x_m^*$ and $\|x_i^*\| \leq 1$ for each $i = 1, \ldots, m$. If $k \in K$, let Γ_k^X be the set of sequences $y_1^* < \ldots < y_k^*$ such that $y_i^* \in \mathbf{Q}$ for each i, $y_1^* \in A_{j_{2k}}^*(X)$ and $y_{i+1}^* \in A_{\sigma(y_1^*, \ldots, y_i^*)}^*(X)$ for each $1 \leq i \leq k-1$. We call these *special sequences*. Let $B_k^*(X)$ be the set of all functionals y^* of the form

$$y^* = \frac{1}{\sqrt{f(k)}} \sum_{j=1}^k y_j^*$$

such that $(y_1^*, \ldots, y_k^*) \in \Gamma_k^X$ is a special sequence. These, when $k \in K$, are the *special functionals* (on X of size k). The idea behind this notion of special functionals is that their normalization is different from the usual normalization of functionals obtained by the "Schlumprecht operation" $(x_1^* + \cdots + x_n^*)/f(n)$, so that they produce "spikes" in the unit ball of X^*; but special functionals are extremely rare, and they are easy to trace, as it was explained in section 4. Now, given a semi-group $\mathcal{S}$ of spreads, compatible with $\mathbf{V}$, we consider the smallest norm on $c_{00}(\mathbf{V})$ satisfying the following equation,

$$\begin{aligned} \|x\| = \|x\|_{c_0} &\vee \sup\Big\{ f(n)^{-1} \sum_{i=1}^n \|E_i x\| : n \geq 2,\ E_1 < \ldots < E_n \text{ intervals} \Big\} \\ &\vee \sup\Big\{ |x^*(Ex)| : k \in K,\ x^* \in B_k^*(X),\ E \subset \mathbb{N} \text{ an interval} \Big\} \\ &\vee \sup\{ \|Ux\| : U \in \mathcal{S} \}. \end{aligned}$$

We define the space $X(\mathcal{S}, \mathbf{V})$ as the completion of $c_{00}(\mathbf{V})$ under this norm. In the case $\mathcal{S} = \{Id\}$ the fourth term drops out and the definition reduces — when $\mathbf{V} = \mathbf{K}$ — to that of the space constructed in [GM1]. The fourth term is there to force every spread $U = S_{A,B} \in \mathcal{S}$ to define a bounded operator on X

(actually, we get $\|S_{A,B}\| \le 1$. This restrictive choice could perhaps be relaxed to produce more examples). The second term ensures that X satisfies a lower f-estimate.

It is useful to understand the construction of X in a way similar to what we have said about the spaces T and S in section 2: we construct a norming subset in X^* in a sequence of steps, producing an increasing sequence (B_n) of convex subsets of $B_{c_0(\mathbf{V}^*)}$. We start with $B_0 = B_{\ell_1(\mathbf{V}^*)} \cap c_{00}(\mathbf{V}^*)$. After B_n is defined, we enlarge it as follows:

— for any integer $m \ge 2$, we add all functionals $x^* = f(m)^{-1} \sum_{j=1}^m x_j^*$ built from elements $x_j^* \in B_n$,

— for every $k \in K$, we add all functionals $\lambda E U^* x^*$ where $|\lambda| = 1$, E is an interval, $U \in \mathcal{S}$ and x^* is any special functional $x^* = f(k)^{-1/2} \sum_{j=1}^k y_j^*$ with $y_j^* \in B_n$ for $j = 1, \ldots, k$;

we obtain in this way an expanded set $\widetilde{B}_n \supset B_n$, and we let finally B_{n+1} be the convex hull of $\widetilde{B}_n$. We let $B = \bigcup_n B_n$ and we can check that the above defined norm is equal to

$$\|x\| = \sup\{|x^*(x)| : x^* \in B\}.$$

Observe that the images of successive functionals by a spread are still successive; observe also that the adjoint operation of a spread is again a spread, and that for every interval F and spread U, the operator UF can also be expressed as EU for some interval E; it follows by induction that B_n is stable under the adjoints of the spreads in $\mathcal{S}$ and under projections on intervals, and that B_n is contained in the unit ball of $c_0(\mathbf{V}^*)$, for every $n \ge 0$.

All this implies that X belongs to the family $\mathcal{X}(\mathbf{V})$. We summarize the preceding discussion in the following statement.

Proposition 18 *Let $\mathcal{S}$ be a semi-group of spreads, compatible with the family $\mathbf{V}$. The space $X(\mathcal{S}, \mathbf{V})$ belongs to $\mathcal{X}(\mathbf{V})$ and satisfies a lower f-estimate. Every spread $U \in \mathcal{S}$ verifies $\|U\| \le 1$.*

If we want to compute the norm of $x \in X = X(\mathcal{S}, \mathbf{V})$, either $\|x\| = \|x\|_{c_0}$ or, given $\varepsilon > 0$ such that $\|x\|_{c_0} < \|x\| - \varepsilon$, there exists a first $n \ge 0$ such that $|x^*(x)| > \|x\| - \varepsilon$ for some $x^* \in \widetilde{B}_n$ that was adjoined to B_n in the construction of B_{n+1}, namely either an (m, f)-form or some EU^*y^*, with y^* a special functional of some length $k \in K$, E an interval and $U \in \mathcal{S}$. Let us call *surface functional* any functional x^* on X which is either a (m, f)-form for some $m \ge 2$ or a $(k, \sqrt{f})$-form EU^*y^*, with $k \in K$ and y^* a special functional. We may summarize the lines above by saying that for every vector x in X, either x has the $c_0(\mathbf{V})$-norm, or $\|x\|$ is the supremum of $|x^*(x)|$, when x^* runs over the set of surface functionals. Note that if $g \in \mathcal{F}$ and $g(k) = f(k)^{1/2}$, then a special functional y^* of size $k \in K$ and norm ≤ 1 is also a (k, g)-form, and the same is true for each EU^*y^*, for every interval E and every $U \in \mathcal{S}$. A trick which will be repeated several times is that, when $g \in \mathcal{F}$ satisfies $\sqrt{f} \leq g \leq f$ and $g = \sqrt{f}$ on K, then all surface functionals are g-forms of a certain length ≥ 2, either because $g \leq f$ or because $g = \sqrt{f}$ on K. This remark explains why the generalized Schlumprecht Lemma (Lemma 14), applied to suitable functions $g \in \mathcal{F}$, will be our main tool for estimating norms in $X(\mathcal{S}, \mathbf{V})$.

The preceding paragraph illustrates an aspect of our class of examples that is simpler than what happens with the example of [AD], produced by mixed Tsirelson norms. When working with Tsirelson norms, it is often necessary to analyze how a vector was constructed, in a tree of operations corresponding to the inductive definition of the space. This is not the case here. We will not need to look "below the surface".

Let $\widehat{L} = \bigcup_{\ell \in L} [\ell^{1/4}, \ell^4]$. The next lemma talks about "regular" RIS. For them, the norm of the sum behaves essentially as in Schlumprecht's space.

Lemma 19 *Let $x_1, \ldots, x_r$ be a RIS in $X(\mathcal{S}, \mathbf{V})$ with $r \in \widehat{L}$. Then*

$$\|\sum_{i=1}^{r} x_i\| \leq 3 \, \frac{r}{f(r)}.$$

PROOF. Use Lemma 17 to get $g \in \mathcal{F}_0$ equal to f on $\widehat{L}$, to $\sqrt{f}$ on K, and satisfying $\sqrt{f} \leq g \leq f$. Then all surface functionals are g-forms of a certain length. Let $n_1, \ldots, n_r$ be the integers associated to $x_1, \ldots, x_r$ by the definition of a RIS. We have $\mathrm{e}^{2r^3} < n_1$ and $\|x_i\|_{(n_i)} \leq 1$. Let p be the integral part of e^{2r^2}. Then $p^r \leq \mathrm{e}^{2r^3} < n_1 \leq n_i$, thus $\|x_i\|_{(p^r)} \leq \|x_i\|_{(n_i)} \leq 1$ for every $i = 1, \ldots, r$. Let $x = \sum_{i=1}^{r} x_i$; by Lemma 12, we know that $|x^*(Ex)| \leq 3$ for every interval E, when x^* is a (q, g)-form with $q > p$, and by our remarks, we know that $\|Ex\|$ is less than the supremum of $|x^*(Ex)|$, when x^* runs in the set of (q, g)-forms (unless Ex has the c_0 norm, in which case $\|Ex\| \leq 1$). We see that Lemma 14 applies to the RIS $(\frac{1}{3}x_i)_{i=1}^{r}$, hence $\|x\| \leq 3 \, r \, g(r)^{-1} = 3 \, r \, f(r)^{-1}$.

Lemma 20 *Let $\ell \in L$ and let $x_1, \ldots, x_r$ be a RIS in $X(\mathcal{S}, \mathbf{V})$ with $\ell \leq r \leq \ell^4$. Then*

$$\|\sum_{i=1}^{r} x_i\|_{(\sqrt{\ell})} \leq 15 \, \frac{r}{f(r)}.$$

PROOF. Let m denote the smallest integer larger than $r^{1/4}$. Then $\ell^{1/4} \leq m \leq \ell$ and we know that $[m, r] \subset \widehat{L}$ by construction, hence $\|\sum_{i \in A} x_i\| \leq 3|A|/f(|A|)$ when $|A| \geq m$, by Lemma 19. The result follows from Lemma 16, applied to the sequence $\frac{1}{3}(x_i)$ with $n = \sqrt{\ell}$, because $\mathrm{e}^{16} < j_2^{1/4} \le m \le r \le m^4$, $m \le 2\,r^{1/4}$, hence $mn = m\sqrt{\ell} \le 2\,r^{3/4}$.

A *RIS vector* is a vector x of the form

$$x = r^{-1} f(r) \sum_{i=1}^{r} x_i,$$

where $x_1, \ldots, x_r$ is a RIS. We say that r is the *length of the RIS vector* x. By definition of a RIS, $\|x_i\| \le \|x_i\|_{(n_i)} \le 1$ for some $n_i > \mathrm{e}^{2r^3}$. The most interesting case is when the vectors (x_i) have norms bounded below, by $1/2$ say. In this case the lower f-estimate gives $\|x\| \ge 1/2$. If furthermore $r \in \widehat{L}$, then we know that $\|x\| \le 3$ by Lemma 19.

The next Lemma is absolutely crucial for understanding what happens in the space $X(\mathcal{S}, \mathbf{V})$. This Lemma says roughly the following. Suppose that we construct together a special sequence $(x_j^*)_{j=1}^k$ of length $k \in K$ and a sequence $(x_j)_{j=1}^k$ of RIS vectors, in such a way that the number $\ell_j \in L$ such that x_j^* is a (ℓ_j, f)-form, coincides with the length of the RIS vector x_j. Then the special functional $x^* = f(k)^{-1/2} \sum_{i=1}^k x_i^*$ and its images $U^* x^*$ by the adjoints of the spreads $U \in \mathcal{S}$ will be essentially the only functionals that can force the norm of the vector $x = \sum_{i=1}^k x_i$ to exceed significantly the usual bound $k/f(k)$. The corresponding lemma was not correctly stated in [GM1] (the condition $\operatorname{ran}(x_i) \subset \operatorname{ran}(x_i^*)$ was missing there).

Lemma 21 *Let $k \in K$ and let $x_1^*, \ldots, x_k^*$ be a special sequence of length k; let $\ell_1 = j_{2k}$, and for $2 \le i \le k$ let $\ell_i = \sigma(x_1^*, \ldots, x_{i-1}^*)$. Assume that $f(\sqrt{\ell_i}) > |\operatorname{ran}(\sum_{j=1}^{i-1} x_j^*)|^2$ for $i = 2, \ldots, k$. Let $x_1, \ldots, x_k$ be a sequence of successive vectors in $X(\mathcal{S}, \mathbf{V})$ such that every x_i is a RIS vector of length ℓ_i and* $\operatorname{ran}(x_i) \subset \operatorname{ran}(x_i^*)$, *$i = 1, \ldots, k$. Suppose that*

$$|(\sum_{i=1}^{k} U^* x_i^*)(\sum_{i=1}^{k} E x_i)| \le 16$$

for every interval E and every $U \in \mathcal{S}$. Then

$$\|\sum_{i=1}^{k} x_i\| \le 45\,\frac{k}{f(k)}.$$

PROOF. We know by Lemma 20, applied to the decomposition of x_i into a RIS, that $\|x_i\|_{(m_i)} \le 15$, where $m_i = \sqrt{\ell_i}$. Also $\ell_1 = j_{2k} \ge \mathrm{e}^{8k^3}$ by the lacunarity condition (L_2), thus $\ell_1^{1/2} = m_1 \geq \mathrm{e}^{2k^3}$. Since $\operatorname{ran} x_i \subset \operatorname{ran} x_i^*$ for each i, this implies that

$$\sqrt{f(m_i)} \geq |\mathrm{ran}(\sum_{j=1}^{i-1} x_j^*)| \geq |\mathrm{ran}(\sum_{j=1}^{i-1} x_j)|, \quad i = 2, \ldots, k.$$

This and the lower bound for m_1 ensure that $\frac{1}{15}(x_1, \ldots, x_k)$ is a RIS of length k, preparing thus to apply Lemma 12 to the sequence $(x'_i) = (x_i/15)$. Let $x = \sum_{j=1}^k x_j$ and $x' = \sum_{j=1}^k x'_j = x/15$.

By Lemma 17, we may select a function $g \in \mathcal{F}$ equal to f on $\widehat{L} \cup \{k\}$, to $\sqrt{f}$ on $K \setminus \{k\}$ and such that $\sqrt{f} \leq g \leq f$. Then all f-forms and all $*$-spreads of special forms of length $\neq k$ are g-forms (by *$*$-spread*, we mean the adjoint of a spread in $\mathcal{S}$). In order to prove Lemma 21 we shall apply Lemma 14 to the sequence (x'_i). We observe that all vectors Ex are either normed by (q, g)-forms or by $*$-spreads of special functionals of length k, or they have norm at most 1. The first observation is that "long" forms have a small action on x': indeed by Lemma 12 we know that

$$(*) \qquad |z^*(Ex')| \leq 3$$

whenever z^* is a (q, g)-form with $q > p$, where p denotes the integral part of e^{2k^2}. On the other hand, each vector x_i satisfies $\|x_i\|_{(m_i)} \leq 15$ with $m_i \geq \mathrm{e}^{2k^3} \geq p^k$, which puts us in a position to apply Lemma 14 to x'. Before this, we need to show that $\|Ex'\|$ is not given by $|z^*(Ex')|$, where z^* is a $*$-spread of a special functional of length k (the only kind of surface functionals which are not g-forms, in the present situation).

We shall show that if $z_1^*, \ldots, z_k^*$ is any special sequence of functionals of length k and E is any interval, then $|U^* z^*(Ex')| \leq 1$ for every $U \in \mathcal{S}$, where z^* is the $(k, \sqrt{f})$-form $f(k)^{-1/2} \sum_{i=1}^k z_i^*$. Indeed, let t be maximal such that $z_t^* = x_t^*$ or zero if no such t exists. Suppose $i \neq j$ or one of i, j is greater than $t+1$. We show that $|(U^* z_i^*)(Ex'_j)| < k^{-2}$. Since σ is an injection, we can find $\lambda_1 \neq \lambda_2 \in L$ such that z_i^* is a (λ_1, f)-form, x_j is a RIS vector of length λ_2 and $\|x_j\|_{(m'_2)} \leq 15$, where $m'_2 = \sqrt{\lambda_2}$. If $\lambda_1 < \lambda_2$, it follows from the lacunarity properties of the set $J \supset L$ that $\lambda_1 < m'_2$. This yields that $|(U^* z_i^*)(Ex'_j)| \leq \|x'_j\|_{(\lambda_1)}/f(\lambda_1) \leq f(\lambda_1)^{-1}$. We know that $\lambda_1 \geq j_{2k}$ since λ_1 appears in a special sequence of length k. The conclusion in this case now follows from the fact that $f(\ell) \geq k^2$ when $\ell \geq j_{2k}$ (implied by the lacunarity condition (L_2)).

If $\lambda_2 < \lambda_1$, we apply Lemma 12 to the vector $x''_j = \lambda_2 f(\lambda_2)^{-1} x_j$ equal to the sum of the vectors in the RIS defining x_j. The definition of L gives us that $\mathrm{e}^{256\,\lambda_2^2} < \lambda_1$, so Lemma 12 gives $|(U^* z_i^*)(Ex''_j)| \leq 3$. It follows that $|(U^* z_i^*)(Ex'_j)| \leq f(\lambda_2)/\lambda_2$. The conclusion follows because $\ell \geq j_{2k}$ implies that $f(\ell)/\ell \leq \ell^{-3/4} \leq \mathrm{e}^{-6k^3} \leq k^{-2}$ (by condition (F_1)).

Now choose an interval F (depending upon U) such that

$$\Big|(\sum_{i=1}^{t} U^* z_i^*)(Ex)\Big| = \Big|(\sum_{i=1}^{k} U^* x_i^*)(Fx)\Big| \le 16.$$

It follows (since $\|x_{t+1}\| \le 3$ by Lemma 19) that

$$\Big|\Big(\sum_{i=1}^{k} U^* z_i^*\Big)(Ex)\Big| \le 16 + |(U^* z_{t+1}^*)(Ex_{t+1})| + 15\, k^2.k^{-2} \le 45.$$

We finally obtain that $|U^* z^*(Ex')| \le (45/15) f(k)^{-1/2} < 3 f(j_1)^{-1/2} < 1$ as claimed. It follows from $(*)$ and from what we have just shown about special sequences of length k that

$$\|Ex'\| \le 3 \vee \sup\{|x^*(Ex')| : 2 \le q \le p,\ x^* \text{ is a } (q,g)\text{-form}\}$$

whenever E is an interval. Since $\|x'_j\|_{(p^k)} \le 1$ for each $j = 1, \ldots, k$, Lemma 14 applied to $x'/3$ implies that $\|x'\| \le 3\, k/g(k) = 3\, k/f(k)$.

7.1 We have a HI space!

In this paragraph, we assume that $\mathcal{S} = \{Id\}$, so that any family $\mathbf{V}$ is compatible with $\mathcal{S}$. We are primarily interested in the case $\mathbf{V} = \mathbf{K}$, and we shall prove that the resulting space $X = X(\{Id\}, \mathbf{K})$ is HI. However, we shall keep the $\mathbf{V}$ framework and prove at the same time that any two GH-subspaces of $X(\mathbf{V}) = X(\{Id\}, \mathbf{V})$ almost intersect. When $\mathbf{V} = \mathbf{K}$, every subspace Y of X is a GH-subspace, and we get that X is HI. Let Y, Z be two GH-subspaces of X. Let us choose $\delta > 0$ and let $k \in K$ be an integer such that $f(k)^{-1/2} < \delta/360$. We want to show that the distance between the unit spheres of Y and Z is less than δ.

By the gliding hump property of Y and Z, we may assume that both Y and Z are spanned by block bases. Since X satisfies a lower f-estimate, Corollary 10 tells us that every block subspace of X contains, for every $n \ge 1$, a vector x such that $\|x\|_{(n)} \le 1$ and $\|x\| > 1/2$. We already observed that every vector Ex either has the supremum norm or satisfies the inequality

$$\|Ex\| \le \sup\Big\{|x^*(Ex)| : q \ge 2, x^* \text{ is a } (q,g)\text{-form}\Big\}$$

where g is the function obtained from Lemma 17 in the case $K_0 = K$. This allows us to make the following construction.

Using Corollary 10, we may find in Y (or in Z) for any given integer $\ell \in L$, a sequence $y_1, \ldots, y_\ell$ which satisfies the RIS condition, and also such that $\|y_j\| > 1/2$ for $j = 1, \ldots, \ell$. We are going to apply this fact k times, with increasing values of ℓ, and alternating our choice between Y and Z at each step. Let first $\ell_1 = j_{2k} \in L$, and let $x_1 = \ell_1^{-1} f(\ell_1) \sum_{j=1}^{\ell_1} y_j \in Y$ be a RIS vector of length ℓ_1, with $\|y_j\| > 1/2$ for every j. For each j between 1 and ℓ_1 let y_j^* be a functional such that $\|y_j^*\| \leq 1$ and $0 < y_j^*(y_j) - 1/2 < k^{-1}$; let ${x'}_1^*$ be the (ℓ_1, f)-form $f(\ell_1)^{-1} \sum_{j=1}^{\ell_1} y_j^*$. Then $0 < {x'}_1^*(x_1) - 1/2 < k^{-1}$. By continuity and the density of Δ_n in the unit ball of V_n^*, we may assume that there exists an (ℓ_1, f)-form $x_1^* \in \mathbf{Q}$ such that $0 < x_1^*(x_1) - 1/2 < k^{-1}$ and $\operatorname{ran}(x_1^*) = \operatorname{ran}(x_1)$ (in the real case; in the complex case, we ask for $1/2 < \Re x_1^*(x_1)$ and $|x_1^*(x_1) - 1/2| < k^{-1}$); since we have an infinite sequence of possible choices for $x_1^* \in \mathbf{Q}$ and since σ is injective, we may choose x_1^* such that $\ell_2 = \sigma(x_1^*)$ satisfies $f(\sqrt{\ell_2}) > |\operatorname{ran}(x_1^*)|^2$. Also, note that there is no difference between an (ℓ_1, g)-form and an (ℓ_1, f)-form, because $g = f$ on L.

Now let $\ell_2 = \sigma(x_1^*)$ and pick a RIS vector $x_2 \in Z$ of length ℓ_2 such that $x_1 < x_2$, similarly to the first step. As above, we can find an (ℓ_2, g)-form $x_2^* \in \mathbf{Q}$ such that $0 < x_2^*(x_2) - 1/2 < k^{-1}$, $\operatorname{ran}(x_2^*) = \operatorname{ran}(x_2)$ and $f(\sqrt{\ell_3}) > |\operatorname{ran}(x_1^* + x_2^*)|^2$, where $\ell_3 = \sigma(x_1^*, x_2^*)$.

Continuing in this manner, we obtain a pair of sequences $x_1, \ldots, x_k$ and $x_1^*, \ldots, x_k^*$ with various properties we shall need. First, $x_i \in Y$ when i is odd and $x_i \in Z$ when i is even. We also know that $0 < x_i^*(x_i) - 1/2 < 1/k$ for each i. Finally, and perhaps most importantly, the sequence $x_1^*, \ldots, x_k^*$ has been carefully chosen to be a special sequence of length k. It follows immediately from the implicit definition of the norm and from the fact that $\operatorname{ran}(x_i^*) = \operatorname{ran}(x_i)$ for each i that

$$\|\sum_{i=1}^{k} x_i\| \geq f(k)^{-1/2} \sum_{i=1}^{k} x_i^*(x_i) > \frac{1}{2}\, k\, f(k)^{-1/2}.$$

The proof will be complete if we can find a suitable upper bound for the norm of the alternate sum $\sum_{i=1}^{k} (-1)^{i-1} x_i$. For this we apply Lemma 21. The conditions on $f(\sqrt{\ell_i})$ and on the inclusions of ranges have been taken care of during the construction of the sequences (x_i) and (x_i^*). It remains to show that $|(\sum_{i=1}^{k} x_i^*)(\sum_{i=1}^{k} (-1)^i E x_i)| \leq 16$ for every interval E. This follows easily from the fact that $x_i^*(x_i)$ is almost exactly $1/2$ for every i; there are possibly two incomplete terms, one at the beginning and one at the end of E, for which we use $|x_i^*(x_i)| \leq \|x_i\| \leq 3$ (this follows from Lemma 19 applied to the RIS corresponding to x_i). Lemma 21 therefore shows that $\|\sum_{i=1}^{k} (-1)^{i-1} x_i\| \leq 45\, k f(k)^{-1}$.

We have now constructed two vectors $y \in Y$, the sum of the odd-numbered x_is, and $z \in Z$, the sum of the even-numbered x_is, such that $\|y + z\| >$

$(1/90)\, f(k)^{1/2}\|y-z\|$. If a is the maximum of $\|y\|$ and $\|z\|$, then y/a and z/a have distance $< 180/\sqrt{f(k)} < \delta/2$, and one of them belongs to the unit sphere of Y or Z. It follows that the distance from the unit sphere of Y to the unit sphere of Z is less that δ.

Suppose that $\mathbf{V} = \mathbf{K}$. The above proof shows that X is HI. We also observe that X is reflexive. This follows from Corollary 15. In the vector case, the conclusion is that any two GH-subspaces have distance 0. In particular, we obtain that every subspace Y of X such that all projections P_n, $n \ge 0$ are strictly singular on Y, is a HI space (because all subspaces of Y are GH-subspaces of X).

Theorem 22 *Let $X = X(\mathbf{V})$ be the space constructed in section 7 when $\mathcal{S} = \{Id\}$. Then any two GH-subspaces of X have distance* 0. *Every subspace Y of X such that all projections P_n, $n \ge 0$ are strictly singular on Y, is a HI space. When $\mathbf{V} = \mathbf{K}$, then X is a reflexive HI space.*

8 Factorization through a HI space

In this section we present some of the results of Argyros and Felouzis [AF, Theorems 2.3 and 2.4]. We shall use a variant of the interpolation spaces of Lions-Peetre (see [BL] for instance), following the spirit of the exposition of [DFJP] rather than that of interpolation theory. Let W be a bounded symmetric closed convex subset of a Banach space V. Suppose that $\mathbf{a} = (a_n)$ is a decreasing sequence of positive numbers, such that $\lim_n a_n = 0$. For every $n \ge 0$, we consider the bounded symmetric convex set $C_n = 2^n W + a_n B_V$ and we let $j_{\mathbf{a},n}$ be the gauge of C_n. For every n, the gauge $j_{\mathbf{a},n}$ defines an equivalent norm on V, and we shall call V_n the space V equipped with the equivalent norm $j_{\mathbf{a},n}$. We shall be interested in the (usually unbounded, possibly infinite) gauge $j_{\mathbf{a}} = \sup_{n\ge 0} j_{\mathbf{a},n}$ on V. It is clear that $j_{\mathbf{a}}$ is finite on W (because $j_{\mathbf{a}}$ is less than the gauge j_W of W, since $j_{\mathbf{a},n} \le 2^{-n} j_W$ for every $n \ge 0$). It is also clear that $j_{\mathbf{a}}$ is larger than a multiple of the norm of V (because C_0 is bounded, say). The classical construction of factorization spaces in [DFJP] uses the "quadratic" gauge $q_{\mathbf{a}}(x) = (\sum_{n=0}^{+\infty} j_{\mathbf{a},n}(x)^2)^{1/2}$, but we shall instead use a non standard way of mixing the norms of the V_ns, namely the construction from the preceding section 7.

We need a notion of *thinness* of W that guarantees that there is no subspace Y of V on which the $j_{\mathbf{a}}$ norm is finite and equivalent to the V-norm. This was done by Neidinger (see [N1,N2]) in a similar setting (Neidinger was looking for hereditarily-ℓ_p interpolation spaces, while we are looking for hereditarily indecomposable interpolation spaces). Let us say that a bounded symmetric closed convex subset W is $\mathbf{a}$-*thin in* V when there is no infinite-dimensional subspace Y of V such that $j_{\mathbf{a}}$ is finite and bounded on the unit ball $B_Y = Y \cap B_V$ of Y. This means that the inclusion from V_1 to V is strictly singular, where V_1 is the space of those $v \in V$ such that $\|v\|_{V_1} = j_{\mathbf{a}}(v) < +\infty$.

The set W is *thin* in Neidinger's sense when for every subspace Y of V, there exists $\varepsilon > 0$ such that for every C, the unit ball B_Y of Y is not contained in $CW + \varepsilon B_V$. If W is thin in Neidinger's sense, then it is $\mathbf{a}$-thin for every sequence $\mathbf{a}$: there exists $\varepsilon > 0$ such that $B_Y \not\subset 2^m W + \varepsilon B_V$ for every m, in particular for any given ℓ, when $a_n < \varepsilon 2^{-\ell}$ there exists $y \in B_Y$ such that $y \notin 2^{n+\ell} W + 2^\ell a_n B_V$, thus $j_{\mathbf{a},n}(y) \geq 2^\ell$ and W is $\mathbf{a}$-thin. On the other hand, if $\mathbf{a}$ is the sequence $a_n = \mathrm{e}^{-8^n}$, then $W = B_{L_\infty}$ is $\mathbf{a}$-thin in $V = L_1 = L_1[0,1]$ (see [AF, Proposition 2.2]), but the inclusion $L_\infty \to L_1$ is not thin, as mentioned by Neidinger. This shows that the statement of the factorization theorem below is slightly more general when formulated with $\mathbf{a}$-thin instead of thin. We say that a bounded linear operator $T : U \to V$ is $\mathbf{a}$-*thin* if the closure $\overline{T(B_U)}$ of $T(B_U)$ is $\mathbf{a}$-thin in V.

Theorem 23 *Suppose that $T : U \to V$ is $\mathbf{a}$-thin for some $\mathbf{a}$. Then T factors through a HI space.*

PROOF. We know that for some $\mathbf{a}$, the set $W = \overline{T(B_U)}$ is $\mathbf{a}$-thin in V. For every $n \geq 0$, let V_n denote V with the norm $j_{\mathbf{a},n}$, and consider the family $\mathbf{V} = (V_n)_{n \geq 0}$. Let $X = X(\mathbf{V})$ be the space constructed in section 7, and let Z denote the diagonal of this space, in other words Z is the vector subspace of V consisting of those $v \in V$ such that $\mathbf{v} = (v, v, \ldots)$ belongs to X. We define a norm on Z by $\|v\|_Z = \|\mathbf{v}\|_X$. We shall check that T factors through Z, and that Z is a HI space.

For every $u \in U$ such that $\|u\| \leq 1$, the vector $w = Tu \in V$ belongs to W, therefore $\|w\|_{V_n} = j_{\mathbf{a},n}(w) \leq 2^{-n}$; the series $\sum_n w \otimes \mathbf{e}_n$ is normally convergent in X and defines an element $T_1(u) \in Z$. On the other side, the norm of $\mathbf{v}$ in X is larger than $j_{\mathbf{a}}$ (by the general assumptions about $\mathcal{X}(\mathbf{V})$), thus larger than $\|v\|_V$, and this shows that there is a natural inclusion map i from Z to V. We have therefore obtained the factorization $T = i \circ T_1$.

In order to prove that Z is HI, we have only to check that Z, regarded as a subspace of X via the diagonal map, is such that P_{n_0} is strictly singular on Z for every $n_0 \geq 0$. Given any subspace $Y \subset Z$, it is possible to find $y \in Y$ such that $j_{\mathbf{a},n}(y) < 2^{-n_0}$ for $n = 0, \ldots, n_0$ but $\|\mathbf{y}\|_X = 1$. This is clear since each of the first gauges $j_{\mathbf{a},n}$ is equivalent to the V-norm while $j_{\mathbf{a}}$ is unbounded on B_Y and less than the norm of X. This shows that P_{n_0} is strictly singular on Z. The result then follows from Theorem 22 (we are a little bit cheating, since we wrote the proof under the additional hypothesis that V is reflexive).

Theorem 24 *For every $p \in (1, +\infty)$, the space ℓ_p is a quotient of some HI space. This is also true for c_0.*

It is obvious that this cannot hold for ℓ_1, by the lifting property of ℓ_1. We shall only sketch the case of ℓ_p, $p \in (1, +\infty)$; see [AF] for a much more general result, but also much more difficult to prove. The strategy for the proof is the following: we shall construct a space V and a symmetric **a**-thin closed convex subset $W \subset V$ which is norming for an ℓ_q-subspace L of V^*. Let U be the Banach space whose unit ball is W, and let us apply the preceding factorization result to the inclusion map $T : U \to V$. Then $T = i \circ T_1$, with $T_1 : U \to Y$ and Y a HI space with an embedding $i : Y \to V$. We need only show that i^* induces an isomorphism from L to a subspace of Y^*. This is easy since the set $T_1(B_U) = W_1$, which is smaller than B_Y, is already norming for the space $i^*(L)$.

The construction of V uses a tree; (very) roughly speaking, we introduce an infinite branch γ of the tree for every vector z^* of L, and a sequence $w_\gamma \in V$ supported on that branch, which norms z^*. Next, we define W to be the symmetric closed convex hull of the set of all elements w_γ. For every $n \geq 0$, let D_n be the subset of $[-1, 1]$ consisting of all numbers of the form $i\, 2^{-n-3}$, $|i| \leq 2^{n+3}$. Let $\mathcal{T}_0$ be the set of all $\nu = (d_0, \ldots, d_n)$, for $n \geq 0$, such that $d_j \in D_j$ for $j = 0, \ldots, n$. We say that $|\nu| = n$ is the length of ν. We say that $\nu' \leq \nu$ if ν' is an initial segment $(d_0, \ldots, d_m)$, $m \leq n$, of the sequence $(d_0, \ldots, d_n) = \nu$. We shall restrict our attention to the subtree $\mathcal{T} \subset \mathcal{T}_0$ consisting of those nodes ν such that $\sum_{i=0}^{|\nu|} |d_i|^p < 1$. On the space $c_{00}(\mathcal{T})$ of finitely supported scalar sequences indexed by $\mathcal{T}$, we consider the $\ell_p((\ell_1^{k_n}))$ norm,

$$\|v\|_V = \Bigl(\sum_{n=0}^{+\infty} \bigl(\sum_{\nu \in R_n} |v_\nu|\bigr)^p\Bigr)^{1/p}$$

where $R_n \subset \mathcal{T}$ is the set of nodes ν such that $|\nu| = n$ and $k_n = |R_n|$.

Let $(\mathbf{f}_\nu)_{\nu \in \mathcal{T}}$ be the natural unit vector basis for V. It is clearly 1-unconditional. For every $\nu = (d_0, \ldots, d_n) \in \mathcal{T}$, we set $c_\nu = d_n$; if b is a segment in the tree we set $x(b) = \sum_{\nu \in b} c_\nu \, \mathbf{f}_\nu$. Let W be the symmetric closed convex set generated by all vectors $x(b)$; by the definition of $\mathcal{T}$, we know that $\|x(b)\|_V < 1$ for every segment b of $\mathcal{T}$. Let L be the subspace of V^* generated by the sequence $g_n = \sum_{\nu \in R_n} \mathbf{f}_\nu^*$, for $n \geq 0$. It is clear that (g_n) is isometrically equivalent to the unit vector basis of ℓ_q, $1/q + 1/p = 1$. It is easy to prove that W is $\frac{1}{4}$-norming for L. Indeed, given $z^* = \sum_{n \geq 0} v_n g_n$ such that $\|z^*\| = 1$, we choose the branch $\gamma = (\nu_0, \ldots, \nu_n, \ldots)$, $|\nu_n| = n$, such that $d_n = c_{\nu_n}$ is as close as possible to $u_n = |v_n|^{q-1} \operatorname{sign}(v_n)$: we choose $d_n \in D_n$ such that $\operatorname{sign} d_n = \operatorname{sign} v_n$, and if $|u_n| \geq 2^{-n-2}$, we may also make sure that $\frac{1}{2}\,|u_n| \leq |d_n| \leq |u_n|$. We get

$$z^*(x(\gamma)) = \sum_{n=0}^{+\infty} c_{\nu_n} v_n \geq \frac{1}{2} \sum \{|v_n|^q : |v_n| \geq 2^{-n-2}\} \geq \frac{1}{4}.$$

We shall check that W is thin in V. If not, we can find a subspace Y of V, such that choosing $0 < \varepsilon < 1/4$, we have $B_Y \subset C\, W + \varepsilon B_V$ for some C. By a standard gliding hump argument, we can find in Y a normalized sequence (y_n) which is a small perturbation of a sequence supported on disjoint subsets (E_n) of the tree, say $\|y_n - E_n y_n\| < \varepsilon$, where E_n is the set of all nodes ν such that $k_n \le |\nu| \le \ell_n$, with $\ell_n < k_{n+1}$ for every $n \ge 0$. We need the following lemma, whose proof is sketched in the appendix. Let us call *band* in $\mathcal{T}$ any subset E consisting of all nodes t such that $k \le |t| \le \ell$, for some $k \le \ell$.

Lemma 25 *Let $\alpha > 0$, let (w_n) be a sequence of elements of W and (E_n) be a sequence of successive bands in $\mathcal{T}$. There exists an infinite subset $M \subset \mathbb{N}$ and for every $m \in M$ a decomposition of E_m in E'_m, E''_m, such that: $\|E''_m w_m\| < \alpha$, and for every $m_1 \ne m_2$ in M, the nodes in E'_{m_1} and E'_{m_2} are incomparable.*

Let us finish the proof of Theorem 24. We assumed that $B_Y \subset C\,W + \varepsilon B_V$, with $0 < \varepsilon < 1/4$; for every n, there exists $w_n \in W$ such that $\|y_n - C\,w_n\| < \varepsilon$; passing to a subsequence, we may assume that E_n is decomposed into E'_n and E''_n satisfying the properties cited in Lemma 25, with $\alpha = \varepsilon/C$. We have $\|E''_n(y_n - C\,w_n)\| < \varepsilon$ and $\|E''_n(C\,w_n)\| < C\,\alpha = \varepsilon$, therefore $\|E''_n y_n\| < 2\varepsilon$ and $\|E'_n y_n\| > 1 - 3\varepsilon$. We may thus find a normalized sequence of functionals (y^*_n) in V^*, with y^*_n supported on E'_n, and norming for $E'_n y_n$. This sequence (y^*_n) is isometrically equivalent to the unit vector basis of ℓ_q. Consider $y^* = N^{-1/q}(y^*_1 + \cdots + y^*_N)$; since the supports of the (y^*_j) are incomparable, every vector $x(b)$ from the family generating W acts on at most one y^*_i; it follows that $\sup_{w \in W} |y^*(w)| \le N^{-1/q}$. On the other hand, let $y = N^{-1/p}(y_1 + \cdots + y_n) \in Y$. We have that $y^*(y) > 1 - 3\varepsilon$ and by assumption, there exists $w \in W$ such that $\|y - C\,w\| < \varepsilon$. This implies that $|y^*(C\,w)| > 1 - 4\varepsilon$; this is a contradiction when N is large enough.

9 Additional results

In this section we present some of the results of [GM2]. We assume that $\mathbf{V} = \mathbf{K}$. Given a semi-group $\mathcal{S}$ of spreads, we denote by $X = X(\mathcal{S}) = X(\mathcal{S}, \mathbf{K})$ the space constructed in section 7, with $\mathbf{V} = \mathbf{K}$. In [GM2], three examples are given, corresponding to three semi-groups of spreads. We shall concentrate here on the example X_s, which is a space with an isometric right shift operator S, on which every bounded operator is a strictly singular perturbation of a normally converging series of powers of S and its adjoint, the left shift on X_s. Before presenting this example, we need to study some general properties of a larger class of examples.

Given any set $\mathcal{S}$ of spreads containing the identity map, we shall say that it is a $*$-*semi-group* if it is a semi-group closed under taking adjoints. An example of such a set is the collection of all spreads $S_{A,B}$ where $A = \{m, m+1, m+2, \ldots\}$ and $B = \{n, n+1, n+2, \ldots\}$ for some $m, n \ge 0$. This is the $*$-semi-group generated by the shift operator. Given any $*$-semi-group $\mathcal{S}$ of spreads, we shall say that it is a *proper $*$-semi-group* if, for every $(i,j) \neq (k,l)$, there are only finitely many spreads $S \in \mathcal{S}$ for which $\mathbf{e}_i^*(S\mathbf{e}_j) \neq 0$ and $\mathbf{e}_k^*(S\mathbf{e}_l) \neq 0$. The $*$-semi-group generated by the shift operator is proper. Let $x = \sum v_j\,\mathbf{e}_j$ and $x^* = \sum v_i^*\,\mathbf{e}_i^*$ be two elements of c_{00}. If $\mathcal{S}$ is proper, then except for finitely many $S \in \mathcal{S}$, the sum $\sum_{i,j} v_i^*\,v_j\,\mathbf{e}_i^*(S\mathbf{e}_j) = x^*(Sx)$ has only one non zero term, and is therefore bounded by $\|x\|_\infty\,\|x^*\|_\infty$. Note that a proper set $\mathcal{S}$ of spreads must be countable, and if we write it as $\{S_1, S_2, \ldots\}$ and set $\mathcal{S}_m = \{S_1, \ldots, S_m\}$ for every m, then for any $x \in X(\mathcal{S})$, $x^* \in X(\mathcal{S})^*$, we have

$$(P) \qquad \lim_m \sup\{|x^*(Ux)| : U \in \mathcal{S} \setminus \mathcal{S}_m\} \leq \|x\|_\infty \|x^*\|_\infty.$$

Let $X = X(\mathcal{S}, \mathbf{K})$ be the space constructed in section 7, with $\mathbf{V} = \mathbf{K}$; by Corollary 15, we know that X is reflexive. Let $y \in X$. Recall that for every integer $n \geq 1$,

$$\|y\|_{(n)} = \sup \sum_{i=1}^n \|E_i\,y\|$$

where the supremum is extended to all families $E_1 < \ldots < E_n$ of successive intervals. Observe that $\|\mathbf{e}_i\|_{(n)} = 1$ for every $n \geq 1$.

Given a subspace $Y \subset X$, we will be interested in a seminorm $|\!|\!|.|\!|\!|$ defined on $\mathcal{L}(Y, X)$ as follows. Given $T \in \mathcal{L}(Y, X)$ let $|\!|\!|T|\!|\!|$ be the supremum of those numbers κ such that for every $n \geq 1$, there exists a vector $y \in Y$ such that $\|P_n y\| \leq 2^{-n}$, $\|y\|_{(n)} \leq 1$ and $\|Ty\| > \kappa$. Clearly, $|\!|\!|T|\!|\!| \leq \|T\|$. Let us say the same thing in a slightly different way. The number $|\!|\!|T|\!|\!|$ is the smallest number with the following property: for every $\varepsilon > 0$, there exists an integer $n \geq 1$ such that, for every $y \in Y$, the conditions $\|P_n\,y\| \leq 2^{-n}$ and $\|y\|_{(n)} \leq 1$ imply that $\|Ty\| \leq |\!|\!|T|\!|\!| + \varepsilon$. We may also write that for every $y \in Y$ and $n \geq N(\varepsilon)$,

$$\|Ty\| \leq (|\!|\!|T|\!|\!| + \varepsilon)\,\|y\|_{(n)} + 2^n\,\|P_n y\|.$$

We say that a bounded sequence $(y_n) \subset Y$ is a sequence of *almost successive* vectors in X if there exists a sequence (x_n) of successive vectors such that $\lim_n \|x_n - y_n\| = 0$. If $(y_n) \subset Y$ is a sequence such that $\|P_n y_n\| \leq 2^{-n}$ for every $n \geq 1$, then clearly we may find almost successive subsequences (y_{n_k}). Let $\mathcal{M}_Y$ be the set of sequences $\mathbf{y} = (y_n)_{n=1}^{+\infty}$ of almost successive vectors in Y such that $\limsup_n \|y_n\|_{(n)} \leq 1$. Now, given $T \in \mathcal{L}(Y, X)$ it is clear that

$$|\!|\!|T|\!|\!| = \sup_{\mathbf{y} \in \mathcal{M}_Y} \limsup_n \|Ty_n\|.$$

Lemma 26 *For every infinite-dimensional subspace Y of $X = X(\mathcal{S}, \mathbf{K})$ and every $T \in \mathcal{L}(Y, X)$, we have*

(*i*) *if $|\!|\!|T|\!|\!| = 0$, then T is strictly singular;*

(*ii*) *if* T *is compact, then* $|||T||| = 0$*;*
(*iii*) *if for every* z *in some infinite-dimensional subspace* Z *of* Y*, we have* $\|Tz\| \geq \|z\|$*, then* $|||T||| \geq 1$.

PROOF. By Corollary 10, every subspace Y contains normalized sequences in $\mathcal{M}_Y$. Hence every subspace of Y contains a norm one vector y such that $\|Ty\| \leq |||T||| + \varepsilon$; in particular, if $|||T||| = 0$, then T is strictly singular.

It is clear that $\lim \|Tx_n\| = 0$ if T is compact and (x_n) almost successive (because X is reflexive), hence $|||T||| = 0$. Lastly, suppose that $\|Tz\| \geq \|z\|$ for every z in some subspace Z of Y. We know from Corollary 10 that Z contains a normalized sequence (z_n) of almost successive vectors with $\lim \|z_n\|_{(n)} = 1$. By definition, $|||T||| \geq \lim_n \|Tz_n\| \geq 1$.

Theorem 27 *Let* $\mathcal{S}$ *be a proper* $*$*-semi-group of spreads. The Banach space* $X = X(\mathcal{S}, \mathbf{K})$ *from section 7 satisfies a lower* f*-estimate and the following three properties.*

(*i*) *For every* $x \in X$ *and every* $S_{A,B} \in \mathcal{S}$*,* $\|S_{A,B}\, x\| \leq \|x\|$*, (and therefore* $\|S_{A,B}\, x\| = \|x\|$ *if* $\mathrm{supp}(x) \subset A$*);*
(*ii*) *If* Y *is an infinite-dimensional subspace of* X*, then every operator from* Y *to* X *is in the* $|||.|||$*-closure of the set of restrictions to* Y *of operators in the algebra* $\mathcal{A}$ *generated by* $\mathcal{S}$*. In particular, all operators on* X *are* $|||.|||$*-perturbations of operators in* $\mathcal{A}$*.*
(*iii*) *The seminorm* $|||.|||$ *on* $\mathcal{L}(X)$ *satisfies the algebra inequality* $|||UV||| \leq |||U|||\,|||V|||$*.*

Notice a straightforward consequence of this result. If we write $\mathcal{G}$ for the $|||.|||$-completion of $\mathcal{A}$ (after quotienting by operators with $|||.|||$ zero) then $\mathcal{G}$ is a Banach algebra. Given $T \in \mathcal{L}(X)$, we can find by (*ii*) a $|||.|||$-Cauchy sequence $(T_n)_{n=1}^{+\infty}$ of operators in $\mathcal{A}$ such that $|||T - T_n||| \to 0$. Let $\phi(T)$ be the limit of $(T_n)_{n=1}^{+\infty}$ in $\mathcal{G}$. This map is clearly well-defined. It follows easily from (*iii*) that it is also a unital algebra homomorphism. The kernel of ϕ is the set of T such that $|||T||| = 0$. We have $\mathcal{K}(X) \subset \ker \phi \subset \mathcal{S}(X)$. The restriction of ϕ to $\mathcal{A}$ is the identity (or more accurately the embedding of $\mathcal{A}$ into $\mathcal{G}$). If $\mathcal{A}$ is small, then, since the kernel of ϕ consists of small operators, $\mathcal{L}(X)$ is also small.

Let us indicate why X does not contain an infinite unconditional sequence. Let $Y \subset X$ be a subspace with an unconditional basis (y_n). Let $(M_\alpha)_\alpha$ denote an uncountable family of subsets of $\mathbb{N}$, such that any two of them differ by an infinite set. For every α, let P_α denote the projection from Y on to the span of $(y_n)_{n \in M_\alpha}$. For $\alpha \neq \beta$, there exists a subspace $Z \subset Y$ such that $\|(P_\alpha - P_\beta)z\| \geq \|z\|$ for every $z \in Z$. This implies that $|||P_\alpha - P_\beta||| \geq 1$ by Lemma 26, but this contradicts the separability of $\mathcal{L}(X)$ in the $|||\,.\,|||$-norm, that follows from (*ii*) and the countability of $\mathcal{S}$.

Lemma 28 *Let $\mathcal{S}$ be a proper $*$-semi-group of spreads, let $X = X(\mathcal{S})$, let $Y \subset X$ be an infinite-dimensional subspace and let T be a continuous linear operator from Y to X. Let $\mathcal{S} = \bigcup_{m=1}^{\infty} \mathcal{S}_m$ be a decomposition of $\mathcal{S}$ satisfying condition (P). Then for every $\varepsilon > 0$ there exists m such that, for every $x \in Y$ such that $\|x\|_{(m)} \leq 1$ and $\|P_m x\| \leq 2^{-m}$,*

$$d(Tx, m\,\mathrm{conv}\{\lambda Ux : U \in \mathcal{S}_m, |\lambda| = 1\}) \leq \varepsilon.$$

PROOF. Suppose that the result is false. Then, for some $\varepsilon > 0$, we can find a sequence $(y_n)_{n=1}^{+\infty}$ with $y_n \in Y$, $\|y_n\|_{(n)} \leq 1$ and $\|P_n y_n\| \leq 2^{-n}$ such that, setting $C_n = n\,\mathrm{conv}\{\lambda U y_n : U \in \mathcal{S}_n, |\lambda| = 1\}$, we have $d(Ty_n, C_n) > 2\,\varepsilon$. This yields that (y_n) is bounded away from 0. We may pass to an almost successive subsequence, still denoted (y_n), such that $\sum \|y_n - y'_n\| < +\infty$ for some successive sequence $(y'_n) \subset X$ satisfying $\|y'_n\|_{(n)} \leq 1$ for every integer $n \geq 1$. Then for some n_0, $(y_n)_{n \geq n_0}$ and $(y'_n)_{n \geq n_0}$ are equivalent basic sequences (see [LT, 1.a.9]), with the additional property that for every $\alpha > 0$, there exists $n_1 = n_1(\alpha) \geq n_0$ such that every norm one vector $y = \sum_{n \geq n_1} a_n y_n$ satisfies $\|y - \sum_{n \geq n_1} a_n y'_n\| < \alpha$. If we replace Y by the block subspace Y' generated by $(y'_n)_{n \geq n_0}$ and T by T' defined on Y' by $T' y'_n = T y_n$ for $n \geq n_0$, we still get the conclusion that $d(T' y'_n, C'_n) > \varepsilon$, where C'_n is defined from (y'_n) as C_n is defined from (y_n), provided n_0 was chosen large enough. This argument shows that if the result is false, then it is already false for some block subspace Y and some operator T from Y to X.

In this case, it is not hard to show that T can be perturbed (in the operator norm) to an operator whose matrix (with respect to the natural bases of X and Y) has only finitely many non-zero entries in each row and column. We may therefore assume that T has this property. We also assume $\|T\| \leq 1$.

Since we assumed that the result is false for Y and T, then for some $\varepsilon > 0$, we can find a sequence $(y_n)_{n=1}^{+\infty}$ with $y_n \in Y$, $\|y_n\|_{(n)} \leq 1$ and $\mathrm{supp}(y_n) > \{n\}$ such that $d(Ty_n, C_n) > \varepsilon$, and also such that if z_n is any one of y_n, Ty_n or Uy_n for some $U \in \mathcal{S}_n$ and z_{n+1} is any one of y_{n+1}, Ty_{n+1} or Vy_{n+1} for some $V \in \mathcal{S}_{n+1}$, then $z_n < z_{n+1}$. By the Hahn-Banach theorem, for every $n \geq 1$ there is a norm-one functional y_n^* such that

$$\sup\{y_n^*(x) : x \in C_n + \varepsilon B(X)\} < y_n^*(Ty_n).$$

It follows that $y_n^*(Ty_n) > \varepsilon$ and $\sup |y_n^*(C_n)| \leq 1$. Therefore $|y_n^*(Uy_n)| \leq n^{-1}$ for every $U \in \mathcal{S}_n$. We may also assume that the support of y_n^* is contained in the smallest interval containing the supports of y_n, Ty_n and Uy_n for $U \in \mathcal{S}_n$. (The case of complex scalars requires a standard modification.)

Given $\ell \in L$ define an *ℓ-pair* to be a pair (x, x^*) constructed as follows. Let $y_{n_1}, y_{n_2}, \ldots, y_{n_\ell}$ be a subsequence of $(y_n)_{n=1}^{+\infty}$ satisfying the RIS condition, which implies that $n_1 > e^{2\ell^3} > \ell^2$. Let $x = \ell^{-1} f(\ell)(y_{n_1} + \cdots + y_{n_\ell})$ and let $x^* = f(\ell)^{-1}(y^*_{n_1} + \cdots + y^*_{n_\ell})$, where the $y^*_{n_i}$ are as above. Lemma 19 implies that $\|x\| \le 3$ and Lemma 20 that $\|x\|_{(\sqrt{\ell})} \le 15$.

If (x, x^*) is such an ℓ-pair, then $x^* \in A^*_\ell(X)$ and, by our earlier assumptions about supports,

$$x^*(Tx) = \ell^{-1} \sum_{i=1}^{\ell} y^*_{n_i}(Ty_{n_i}) > \varepsilon.$$

Similarly, $|x^*(Ux)| \le n_1^{-1} < \ell^{-2}$ for every $U \in \mathcal{S}_\ell$.

Let $k \in K$ be such that $\varepsilon\, f(k)^{1/2} > 45$. We now construct sequences $x_1, \ldots, x_k$ and $x^*_1, \ldots, x^*_k$ as follows. Let $\ell_1 = j_{2k}$ and let (x_1, x^*_1) be an ℓ_1-pair. Let m_2 be such that $|x^*_1(Ux_1)| \le \|x_1\|_\infty \|x^*_1\|_\infty$ if $U \in \mathcal{S} \setminus \mathcal{S}_{m_2}$. The functional x^*_1 can be perturbed so that it is in $\mathbf{Q}$ and so that $\operatorname{ran}(x^*_1) \supset \operatorname{ran}(x_1)$, $\ell_2 = \sigma(x^*_1)$ is larger than m_2 and $f(\sqrt{\ell_2}) > |\operatorname{ran}(x^*_1)|^2 + 4$, while (x_1, x^*_1) is still an ℓ_1-pair. In general, after $x_1, \ldots, x_{i-1}$ and $x^*_1, \ldots, x^*_{i-1}$ have been constructed, let (x_i, x^*_i) be an ℓ_i-pair such that all of x_i, Tx_i and x^*_i are supported after all of x_{i-1}, Tx_{i-1} and x^*_{i-1}, and then perturb x^*_i in such a way that, setting $\ell_{i+1} = \sigma(x^*_1, \ldots, x^*_i)$, we have $|x^*_i(Ux_i)| \le \|x_i\|_\infty \|x^*_i\|_\infty$ whenever $U \in \mathcal{S} \setminus \mathcal{S}_{\ell_{i+1}}$ and we also have $\operatorname{ran}(x^*_i) \supset \operatorname{ran}(x_i)$, $f(\sqrt{\ell_{i+1}}) > |\operatorname{ran}(\sum_{j=1}^{i} x^*_j)|^2 + 2^{i+1}$. This yields that $\|x^*_{i+1}\|_\infty \le f(\ell_{i+1})^{-1} \le 2^{-i-1}$.

Now let $x = x_1 + \cdots + x_k$, let $z^* = x^*_1 + \cdots + x^*_k$ and $x^* = f(k)^{-1/2}\, z^*$. Our construction guarantees that x^* is a special functional, and therefore of norm at most 1. We have

$$\|Tx\| \ge x^*(Tx) > \varepsilon k f(k)^{-1/2}.$$

Our aim is now to get an upper bound for $\|x\|$ and to deduce an arbitrarily large lower bound for $\|T\|$. For this purpose we use Lemma 21. In order to apply this Lemma, it is enough to show that $|(U^*z^*)(Ex)| = |z^*(UEx)| \le 16$ for any interval E and $U \in \mathcal{S}$. We have $x^*_i \in A^*_{\ell_i}$ for $i = 1, \ldots, k$. Suppose that $U \in \mathcal{S}_{m+1} \setminus \mathcal{S}_m$, and let t be such that $\ell_t \le m < \ell_{t+1}$. If $i > t$, then $U \in \mathcal{S}_{\ell_i}$ and $|x^*_i(Ux_i)| < \ell_i^{-2}$. If $i < t$, then $U \notin \mathcal{S}_{\ell_{i+1}}$, so $|x^*_i(Ux_i)| \le \|x_i\|_\infty \|x^*_i\|_\infty \le 2^{-i}$. If $i = t$, then at least we know that $|x^*_i(Ux_i)| \le \|x_i\| \le 3$.

Putting all these facts together, we get that $|z^*(UEx)| \le 16$, as desired (we may have two incomplete terms at both ends of E). Hence, by Lemma 21, $\|x\| \le 45 k f(k)^{-1}$. It follows that $\|T\| \ge (\varepsilon/45) f(k)^{1/2} > 1$, a contradiction.

Lemma 29 *Let $\mathcal{S}$, X, Y, T and ε be as in the previous lemma, let m be as given by that lemma and let $\mathcal{A}_m = m \operatorname{conv}\{\lambda \mathcal{S}_m : |\lambda| = 1\}$. Then there exists $U \in \mathcal{A}_m$ such that $\|T - U\| \le 64\, \varepsilon$.*

PROOF. If the statement of the lemma is false, then for every operator $U \in \mathcal{A}_m$ there is a sequence $\underline{x}_U = (x_n) \in \mathcal{M}_Y$ of vectors in Y such that $\lim_n \|(T-U)x_n\| > 64\varepsilon$. We write this symbolically as $\|(T-U)\underline{x}_U\| > 64\varepsilon$. This yields that $\liminf \|x_n\| > \delta > 0$, with δ depending only upon T and ε. At this point, we may argue as in Lemma 28 in order to reduce the situation to the case of a block subspace Y'. Since $\mathcal{A}_m$ is compact in operator norm, we may find a finite set $(\underline{x}_\alpha) \subset \mathcal{M}_Y$ such that for every $U \in \mathcal{A}_m$, we have $\|(T-U)\underline{x}_\alpha\| > 64\,\varepsilon$ for some α. Passing to subsequences, we may assume that the sequences $(\underline{x}_\alpha)$ can be arranged to be subsequences of a single sequence $(y_n) \subset Y$ such that $\delta \leq \|y_n\| \leq \|y_n\|_{(n)} \leq 1$ for every n, and such that we can find a successive sequence (y'_n) in X satisfying $\|y'_n\|_{(n)} \leq 1$ for every n and $\sum \|y_n - y'_n\| < +\infty$. Let n_0 be chosen so that $(y_n)_{n \geq n_0}$ and $(y'_n)_{n \geq n_0}$ are equivalent basic sequences. Recall that for every $\alpha > 0$, there exists $n_1 = n_1(\alpha) \geq n_0$ such that every norm one vector $y = \sum_{n \geq n_1} a_n y_n$ satisfies $\|y - \sum_{n \geq n_1} a_n y'_n\| < \alpha$. Let $Y_1 \subset Y$ be the subspace generated by the sequence $(y_n)_{n \geq n_0}$; the conclusion of Lemma 28 is obviously still true for the restriction T_1 of T to Y_1. Now, let Y'_1 be the block subspace generated by the sequence $(y'_n)_{n \geq n_0}$; we may assume that all vectors in Y'_1 sit after m. Let us define T'_1 on Y'_1 by $T'_1 y'_n = T y_n$ for every $n \geq n_0$. For every vector $y' \in Y'_1$ such that $\|y'\|_{(m)} \leq 1$, we may find –if n_0 was chosen large enough– a vector $y \in Y_1$ such that $T'_1 y' = Ty$, $\|y - y'\| < \varepsilon/2$ and $\|y - y'\|_{(m)} \leq 2^{-m}$, hence $\|P_m y\| \leq 2^{-m}$ and $\|y\|_{(m)} \leq 3/2$. We see that the conclusion of Lemma 28 is still true for T'_1, provided we lose an additional ε. Furthermore, for every $U \in \mathcal{A}_m$, we have $\|(T'_1 - U)\underline{x}\| > 62\,\varepsilon$ for some $\underline{x} \in \mathcal{M}_{Y'_1}$. This shows that we may assume that Y is a block subspace such that

$$(*) \qquad \forall y \in Y, \quad d(Ty, \{Uy : U \in \mathcal{A}_m\}) \leq 2\,\varepsilon\,\|y\|_{(m)}$$

and that for every $U \in \mathcal{A}_m$, we have $\|(T-U)\underline{x}\| > 62\,\varepsilon$ for some $\underline{x} \in \mathcal{M}_Y$.

Let $\mathcal{U}_{i=1}^s$ be a covering of $\mathcal{A}_m$ by open sets of diameter less than ε in the operator norm. For every $i = 1, \ldots, s$, let $U_i \in \mathcal{U}_i$ and let $\underline{x}_i = (x_{i,n})_n$ be a successive sequence in $\mathcal{M}_Y$ such that $\|(T - U_i)\underline{x}_i\| > 62\,\varepsilon$. By the condition on the diameter of $\mathcal{U}_i$, we have $\|(T-U)\underline{x}_i\| > 60\,\varepsilon$ for every $U \in \mathcal{U}_i$. As in the last lemma, we can assume that the matrix of T has only finitely many non-zero entries in each row and column.

Our first aim is to show that the vectors $\underline{x}_U$ can be chosen continuously in U. (This statement will be made more precise later.) Let $(\phi_i)_{i=1}^r$ be a partition of unity on $\mathcal{A}_m$ with ϕ_i supported inside $\mathcal{U}_i$ for each i. Let $\ell \in L$ be greater than s and m^2. For each $i \leq s$, let $x_{i,n_1}, \ldots, x_{i,n_\ell}$ satisfy the RIS condition and let $m < x_{i,n_1}$. Let $y_i = \ell^{-1} f(\ell)(x_{i,n_1} + \cdots + x_{i,n_\ell})$. Let this be done in such a way that $y_1 < \ldots < y_s$ and also $(T-U)x_{i,n_1} < \ldots < (T-U)x_{i,n_\ell}$ for every i and every $U \in \mathcal{A}_m$. Finally, let the x_{i,n_j} be chosen so that $\|(T-U)x_{i,n_j}\| > 60\,\varepsilon$ for every $U \in \mathcal{U}_i$.

Now let us consider the vector $y(U) = \sum_{i=1}^{s} \phi_i(U) y_i$. By Lemma 20 we know that $\|y_i\|_{(\sqrt{\ell})} \leq 15$ for each $i = 1, \ldots, s$, from which it follows by the triangle inequality that $\|y(U)\|_{(\sqrt{\ell})} \leq 15$. We shall show that $y(U)$ is a "bad" vector for U, by showing that $\|(T - U)y(U)\| > 30\,\varepsilon$.

To do this, let $U \in \mathcal{A}_m$ be fixed and let $I = \{i : \phi_i(U) > 0\}$. Recall that $\|(T - U)x_{i,n_j}\| > 60\,\varepsilon$ for every $i \in I$ and $j = 1, \ldots, \ell$. For such an i and for $j \leq \ell$ let $z^*_{i,j}$ be a norm-one functional such that $z^*_{i,j}((T - U)x_{i,n_j}) > 60\,\varepsilon$. Let these functionals be chosen to be successive. Let $z^*_i = f(\ell)^{-1}(z^*_{i,1} + \cdots + z^*_{i,\ell})$ and $z^* = \sum_{i \in I} z^*_i$. Then $z^*_i(T - U)y_i > 60\,\varepsilon$, so

$$z^*\Big((T - U)y(U)\Big) = z^*\Big(\sum_{i \in I} \phi_i(U)(T - U)y_i\Big) > 60\,\varepsilon.$$

However, $\|z^*\| \leq f(s\ell)/f(\ell) \leq 2$, proving our claim.

The function $U \to y(U)$ is clearly continuous on $\mathcal{A}_m$. The vector $y(U)$ satisfies $\|y(U)\|_{(m)} \leq 15$ and $\|(T - U)y(U)\| > 30\,\varepsilon$. We now apply a fixed-point theorem. For every $U \in \mathcal{A}_m$, let $\Gamma(U)$ denote the set of $V \in \mathcal{A}_m$ such that $\|(T - V)y(U)\| \leq 30\,\varepsilon$. Clearly $\Gamma(U)$ is a compact convex subset of $\mathcal{A}_m$. By property $(*)$, we know that $\Gamma(U)$ is non-empty for every U. The continuity of $U \to y(U)$ gives that Γ is upper semi-continuous, so there exists a point $U \in \mathcal{A}_m$ such that $U \in \Gamma(U)$. But this is a contradiction.

Lemma 29 shows in particular that any operator $T : Y \to X$ can be approximated arbitrarily well in the $|\!|\!|.|\!|\!|$-norm by the restriction of some operator $U \in \mathcal{A}$. We have therefore finished the proof of property (ii). The proof of (iii) is much easier, and will complete the proof of Theorem 27.

Lemma 30 *The seminorm $|\!|\!|.|\!|\!|$ on $\mathcal{L}(X)$ satisfies the algebra inequality*

$$|\!|\!|UV|\!|\!| \leq 75\,|\!|\!|U|\!|\!|\,|\!|\!|V|\!|\!|.$$

PROOF. In [GM2], this statement is proved with the constant 1 in place of 75 (see Remark 13). Pick $c > 1$ and let $(x_n)_{n=1}^{+\infty} \in \mathcal{M}_X$ be a sequence such that $\|UVx_n\| \geq c^{-1}|\!|\!|UV|\!|\!|$ for every n. After suitable perturbations and selections of subsequences we may assume that x_n, Vx_n, UVx_n have supports before $x_{n+1}, Vx_{n+1}, UVx_{n+1}$, and that $x_{n+1}, \ldots, x_{2n}$ is a RIS for every $n \geq 1$. Let $\mathbf{u} > |\!|\!|U|\!|\!|$, $\mathbf{v} > |\!|\!|V|\!|\!|$ and pick $\ell \in L$ large enough so that

$$(*) \qquad \|Ux\| \leq \mathbf{u}\,\|x\|_{(\sqrt{\ell})}, \quad \|Vx\| \leq \mathbf{v}\,\|x\|_{(\sqrt{\ell})}$$

whenever $x \in X$ and $\ell < x$. We consider the vectors $x_{n+1} < \ldots < x_{n+\ell^4}$, for some $n > \ell^4$ such that $x_{n+1}, Vx_{n+1}, UVx_{n+1}$ have supports after ℓ. For every subset $A \subset \{1, \ldots, \ell^4\}$ such that $|A| \geq \ell$, we know by Lemma 20 applied with $r = |A|$ to the RIS $(x_{n+i})_{i \in A}$ that $\| \sum_{i \in A} x_{n+i} \|_{(\sqrt{\ell})} \leq 15\, |A|/f(|A|)$, hence we get by $(*)$ that $\| \sum_{i \in A} V x_{n+i} \| \leq 15\, \mathbf{v}\, |A|/f(|A|)$; by Lemma 16 applied with $m = \ell$, $n = \sqrt{\ell}$ and $r = \ell^4$ to the successive sequence $(\kappa\, V x_{n+i})_{i=1}^r$, $\kappa = (15\, \mathbf{v})^{-1}$, this yields

$$\| \sum_{i=1}^{r} V x_{n+i} \|_{(\sqrt{\ell})} \leq 75\, r\, f(r)^{-1}\, \mathbf{v};$$

by $(*)$ it follows that $\| \sum_{i=1}^r UV x_{n+i} \| \leq 75\, r\, f(r)^{-1}\, \mathbf{u}\, \mathbf{v}$; but

$$\| \sum_{i=1}^{r} UV x_{n+i} \| \geq c^{-1}\, r f(r)^{-1} \| UV \|$$

by the lower f-estimate, and finally $\| UV \| \leq 75\, c\, \mathbf{u}\, \mathbf{v}$.

9.1 The shift space

Let $\mathcal{S}$ be the proper $*$-semi-group mentioned earlier, generated by the shift, which we denote by S. That is, $\mathcal{S}$ consists of all maps of the form $S_{A,B}$ where $A = [m, \infty)$ and $B = [n, \infty)$. We will write L for the left shift, which is (formally) the adjoint of S. Then every operator in $\mathcal{S}$ is of the form $S^m L^n$, because $LS = Id$. Since $SL - Id$ is of rank one, every operator in $\mathcal{A}$ is a finite-rank perturbation of an operator of the form $\sum_{n=0}^N \lambda_n S^n + \sum_{n=1}^N \mu_n L^n$, so the difference is of $\|.\|$-norm zero. Let X_s denote the space obtained from Theorem 27 in this case.

Lemma 31 *Let* $U = \sum_{n=0}^N \lambda_n S^n + \sum_{n=1}^N \mu_n L^n$. *Then*

$$\|U\| = \|U\| = \sum_{n=0}^{N} |\lambda_n| + \sum_{n=1}^{N} |\mu_n|.$$

PROOF. For notational convenience, let $\lambda_{-n} = \mu_n$ for $1 \leq n \leq N$. Clearly it is enough to prove that $\|U\| \geq \sum_{n=-N}^N |\lambda_n|$. In this paper, we did not write almost isometric versions of the basic Lemmas, so we will only prove this inequality up to some multiplicative constant.

Let $m \geq 1$ be given. For an integer $r \in L$ consider the vector $x_r = \sum_{j=r+1}^{2r} \mathbf{e}_{3jN}$. Since every unit vector $\mathbf{e}_i$ satisfies $\|\mathbf{e}_i\|_{(n)} = 1$ for every n, we have $\|x_r\| \leq r/f(r)$ by Lemma 14; let $y_r = r^{-1} f(r) x_r$; by Lemma 16, there exists r such that $\|y_r\|_{(m)} \leq 5$ and we may choose r as big as we like, in particular $r \geq 3N$. This shows that the sequence $\frac{1}{5}(y_r)$ has subsequences in $\mathcal{M}$, thus $\|U y_r\| \leq$

$(5+\varepsilon)\,|||U|||$ for large r. On the other hand, splitting Ux_r into $3rN$ singleton pieces from $3rN+1$ to $(6r+1)N$ gives that $\|Ux_r\| \geq r\, f(3Nr)^{-1} \sum_{n=-N}^{N} |\lambda_n|$ by the lower f-estimate. This lower bound on $\|Ux_r\|$ gives

$$10\,|||U||| \geq 5 f(3Nr)\, f(r)^{-1}\,|||U||| \geq \sum_{n=-N}^{N} |\lambda_n|.$$

Let W denote the Wiener convolution algebra $\ell_1(\mathbb{Z})$. The preceding Lemma gives an isometric embedding i from W into $\mathcal{L}(X_s)$. Indeed, since all powers of S and L have norm 1, we may associate to any $\mathbf{a} = (a_n) \in \ell_1(\mathbb{Z})$ the operator $i(\mathbf{a}) = \sum_{n=0}^{\infty} a_n S^n + \sum_{n=1}^{\infty} a_{-n} L^n \in \mathcal{L}(X_s)$. The next result gives, up to a strictly singular perturbation, the converse of this fact. We call *Toeplitz operators* on X_s the elements from the subspace $\mathcal{T} = i(W)$.

Corollary 32 *There is an algebra homomorphism and projection ϕ from the space $\mathcal{L}(X_s)$ onto the subspace $\mathcal{T}$ consisting of Toeplitz operators with absolutely summable coefficients. If $T \in \mathcal{L}(X_s)$ then $|||\phi(T) - T||| = 0$.*

PROOF. Recall the remark following the statement of Theorem 27. In this case, by Lemma 31, the algebra $\mathcal{G}$, the $|||.|||$-completion of $\mathcal{A}$, is the same as the completion in $\mathcal{L}(X_s)$ and also the completion in the operator norm on ℓ_1. Therefore $\mathcal{G}$ can be regarded as a subalgebra of $\mathcal{L}(X_s)$ consisting of Toeplitz operators with absolutely summable coefficients. If we do this, then the algebra homomorphism ϕ defined after Theorem 27 is also a projection. The equation $|||\phi(T) - T||| = 0$ follows easily from the definition of ϕ.

Theorem 33 *The space X_s is prime.*

PROOF. Let $P : X_s \to X_s$ be a projection. By the previous corollary the operator $\phi(P)$ is a convolution by some absolutely summable sequence $(a_n)_{n\in\mathbb{Z}}$. Moreover, $\phi(P)^2 = \phi(P)$. But the Fourier transform of the sequence $(a_n)_{n\in\mathbb{Z}}$ is a continuous function on the circle squaring to itself. Hence it is constantly zero or one. It follows that a_0 is zero or one and all the other a_n are zero. That is, $\phi(P)$ is zero or the identity. Since $P - \phi(P)$ is strictly singular, it follows that P is of finite rank or corank. Thus, if PX_s is infinite-dimensional, then it has finite codimension. Since the shift on X_s is an isometry, it follows that X_s is isometric to its range, namely an hyperplane; using powers of S, we see that X_s and PX_s are isomorphic, which proves the theorem.

A simple consequence of Corollary 32 is that, up to strictly singular perturbations, any two operators on X_s commute. Indeed, if V and W are two operators, then $\phi(V)$ and $\phi(W)$ commute, so $\phi(VW - WV) = 0$, from which it follows that $\|VW - WV\| = 0$. For the rest of this section, we assume that X_s has complex scalars. Let $\psi : \mathcal{L}(X_s) \to C(\mathbb{T})$ be the composition of ϕ with the Fourier transform. Then ψ is also a continuous algebra homomorphism. Given an operator T on X_s, let K_T be the compact set of $\mu \in \mathbb{C}$ such that μ is *infinitely singular* for T, which means for us that for every $\varepsilon > 0$ there is an infinite-dimensional subspace $Y \subset X_s$ such that $\|Ty - \mu y\| \leq \varepsilon \|y\|$ for every $y \in Y$. Since $T - \phi(T)$ is strictly singular, $K_{\phi(T)} = K_T$.

Lemma 34 *The function $\psi(T)$ takes the value zero at some $\exp(i\theta)$ if and only if 0 is infinitely singular for T.*

PROOF. If $\psi(T)$ takes the value zero at $\exp i\theta$, we can construct an approximate eigenvector for $\phi(T)$ with eigenvalue zero as follows. Suppose that $\phi(T)$ is convolution by the sequence $(a_n)_{n\in\mathbb{Z}}$, and let $\varepsilon > 0$. We know that $\psi(T)(\theta) = \sum_{n\in\mathbb{Z}} a_n \exp(in\theta) = 0$. Let $\ell \in L$ and let

$$x_\ell = f(\ell^2)\,\ell^{-2} \sum_{n=\ell^2}^{2\ell^2} \exp(in\theta)\mathbf{e}_n.$$

By Lemma 14 we have $\|x_\ell\| = 1$, because $\|\mathbf{e}_n\|_{(p)} = 1$ for every $p \geq 1$. Let U be convolution by the sequence $(a_n)_{n=-\ell}^{\ell}$. If ℓ is large enough, then $\|U - \phi(T)\| \leq \varepsilon/2$, since $(a_n)_{n\in\mathbb{Z}}$ is absolutely summable. Moreover, all but at most 4ℓ of the possible $\ell^2 + 2\ell$ non-zero coordinates of Ux_ℓ are equal to $f(\ell^2)\,\ell^{-2}\sum_{n=-\ell}^{\ell} a_n \exp(in\theta)$. Taking ℓ sufficiently large, we can therefore make $\|\phi(T) - U\|$ and $\|Ux_\ell\|$ as small as we like. Therefore zero is infinitely singular for $\phi(T)$. Since $\|T - \phi(T)\| = 0$, the same is true for T.

Conversely, if $\psi(T)$ never takes the value zero, then it can be inverted in $C(\mathbb{T})$. A classical result states that the Fourier transform of this inverse will also be in $\ell_1(\mathbb{Z})$, so in particular $\phi(T)$ has an inverse U which is continuous when considered as an operator on X_s and satisfies $U = \phi(U)$. Therefore $\phi(UT - Id) = 0$, so $UT - Id$ is strictly singular and 0 is not infinitely singular for T.

Corollary 35 *The set K_T is the image under $\psi(T)$ of the unit circle $\mathbb{T}$.*

PROOF. This follows from Lemma 34 applied to the operator $T - \lambda Id$.

Theorem 36 *A subspace Y of X_s is isomorphic to X_s if and only if it has finite codimension.*

PROOF. Let $T : X_s \to Y$ be an isomorphism. Then 0 is not infinitely singular for T, so, as in the proof of Lemma 34, we can find U such that

TU, UT and Id are the same, up to a strictly singular perturbation. Since $TU - Id$ is strictly singular, TU is Fredholm with index zero. In particular $\operatorname{codim} Y = \operatorname{codim} TX_s \leq \operatorname{codim} TUX_s < \infty$. As we have already mentioned, the if part follows from the existence of the isometric shift.

10 Appendix

Let us check that when g is defined by the formula (F) from section 7, then $t/g(t)$ is concave. We have that

$$k(t) = t/g(t) = \exp\Big(\int_0^{\ln(t)} \frac{\mathrm{e}^{M(u)}}{1 + \mathrm{e}^{M(u)}} \, du \Big),$$

hence after some computations

$$k''(t) = \frac{k'(t)}{t(1 + \mathrm{e}^{M(\ln t)})} \left(M'(\ln t) - 1\right) \leq 0$$

because M is 1-Lipschitz and $k'(t) \geq 0$.

Proof of Lemma 11. We shall explain how to go down from the "big" function f_0 at $t_0 = \mathrm{e}^{u_0}$ to the "small" function f_1 at $t_1 = \mathrm{e}^{4u_0^2}$, or equivalently, how to go from the small function $M_0(u) = \ln(1+u)$ at u_0 to the large function $M_1(u) = \ln(3 + 2u)$ at $u_1 = 4u_0^2$. More precisely, we need to build a function $M \in \mathcal{L}$, that coincides with M_0 on $[0, u_0]$ and with M_1 on $[u_1, +\infty)$, and such that $g = g_M$ satisfies the conclusions of the lemma, which reduce then to $g(t_0) = f_0(t_0)$, $g(t_1) = f_1(t_1)$ and $f_1 \leq g \leq f_0$ on $[t_0, t_1]$. The proof of the other case is similar.

It is clear that $M \leq N$ on $[0, u]$ implies $g_N \leq g_M$ on $[1, \mathrm{e}^u]$, and since $tg'_N(t)/g_N(t) = \left(1 + \mathrm{e}^{N(\ln(t))}\right)^{-1}$ we see that the inequality $M \leq N$ on some interval $[u, v] \subset [0, +\infty)$ implies that g_M/g_N is non-decreasing on the interval $[\mathrm{e}^u, \mathrm{e}^v]$.

We first define a function $N \in \mathcal{L}$ by letting $N(u) = M_0(u)$ for $u \leq u_0$ and $N(u) = M_0(u_0) + u - u_0 = \ln(1 + u_0) + u - u_0$ for $u \geq u_0$. There is a unique value $v_0 > u_0$ such that $N(v_0) = M_1(v_0)$ and a v_1 such that $N(v_1) = M_1(u_1) = \ln(3+2u_1)$, given by $v_1 = \ln(3+2u_1) + u_0 - \ln(1+u_0)$; one can check that $v_1 < u_1$ since $M_1(u_1) - M_0(u_0) < u_1 - u_0$ (exercise); it follows that $u_0 < v_0 < v_1 < u_1$ (the reader must draw a picture). Let $N_0 \in \mathcal{L}$ be equal to N on $[0, v_0]$ and to M_1 on $[v_0, +\infty)$, and $N_1 \in \mathcal{L}$ be equal to N on $[0, v_1]$, equal to the constant value $M_1(u_1)$ on $[v_1, u_1]$ and equal to M_1 on $[u_1, +\infty)$. For every $s \in [0,1]$ let $N_s = (1-s)N_0 + sN_1$, and let $g_s = g_{N_s} \in \mathcal{F}_0$ be the corresponding function. Notice that $g_s(t_0) = f_0(t_0)$ for every s. Our problem is to make sure that for some $s = s_0 \in (0,1)$, the value of g_s at the other end t_1 is what we expect, namely $g_s(t_1) = f_1(t_1)$. It will follow that $g_{s_0} = f_1$ on $[t_1, +\infty)$, and it will only remain to check that $f_1 \le g_{s_0} \le f_0$. We shall first check that $g_1(\mathrm{e}^{u_1}) < f_1(\mathrm{e}^{u_1})$, and next that $g_0(\mathrm{e}^{u_1}) > f_1(\mathrm{e}^{u_1})$: we have

$$\begin{aligned}
\ln\Big(\frac{g_1(\mathrm{e}^{v_1})}{f_0(\mathrm{e}^{u_0})}\Big) &= \ln\Big(\frac{g_1(\mathrm{e}^{v_1})}{g_1(\mathrm{e}^{u_0})}\Big) = \int_{u_0}^{v_1} \frac{du}{1+\mathrm{e}^{N(u)}} = \int_{u_0}^{v_1} \frac{du}{1+(1+u_0)\,\mathrm{e}^{u-u_0}} \\
&\le \frac{1}{1+u_0} \int_{u_0}^{+\infty} \mathrm{e}^{u_0-u}\,du \le \frac{1}{1+u_0} < \frac{1}{6}, \\
\ln\Big(\frac{g_1(\mathrm{e}^{u_1})}{g_1(\mathrm{e}^{v_1})}\Big) &= \int_{v_1}^{u_1} \frac{du}{1+\mathrm{e}^{M_1(u_1)}} = \int_{v_1}^{u_1} \frac{du}{4+2u_1} < \frac{1}{2}
\end{aligned}$$

so that finally $g_1(\mathrm{e}^{u_1})/f_0(\mathrm{e}^{u_0}) \le \mathrm{e}^{2/3} < 2$. On the other hand

$$\begin{aligned}
f_1(\mathrm{e}^{u_1})/f_0(\mathrm{e}^{u_0}) &= (1+\frac{1}{2}\ln t_1)^{1/2}/(1+\frac{1}{2}\ln t_0) \\
&= (1+2u_0^2)^{1/2}/(1+\frac{u_0}{2}) > 2,
\end{aligned}$$

(because $u_0 \ge 5$) so that $f_1(\mathrm{e}^{u_1}) > g_1(\mathrm{e}^{u_1})$. Next we see that $g_0(\mathrm{e}^{u_1}) > f_1(\mathrm{e}^{u_1})$: since $N_0 = M_1$ on $[v_0, u_1]$, the quotient g_0/f_1 is constant on the interval $[\mathrm{e}^{v_0}, \mathrm{e}^{u_1}]$, thus $g_0(\mathrm{e}^{u_1})/f_1(\mathrm{e}^{u_1}) = g_0(\mathrm{e}^{v_0})/f_1(\mathrm{e}^{v_0}) > 1$ because $N_0 < M_1$ on the interval $[0, v_0)$.

We have that $g_1(\mathrm{e}^{u_1}) < f_1(\mathrm{e}^{u_1}) < g_0(\mathrm{e}^{u_1})$. By continuity there exists $s \in (0,1)$ such that $g_s(\mathrm{e}^{u_1}) = f_1(\mathrm{e}^{u_1})$. It only remains to check that $f_1 \le g_s \le f_0$. Since $N_s < M_1$ on $[u_0, v_0)$ and $N_s > M_1$ on $(v_0, u_1]$ we know that g_s/f_1 is increasing on $[t_0, \mathrm{e}^{v_0})$ and decreasing on $(\mathrm{e}^{v_0}, t_1]$. Since the quotient is > 1 at t_0 and equal to 1 at t_1, it follows that $f_1 \le g_s$ on $[t_0, t_1]$. Using similar but simpler arguments one can see that $g_s \le f_0$ on the same interval, because $N_s \ge M_0$ on $[0, u_1]$.

Proof of Lemma 14. Let $G(t) = t/g(t)$ when $t \ge 1$ and $G(t) = t$ when $0 \le t \le 1$. This function G is concave and increasing on $[0, +\infty)$. For every interval E and every integer $\ell \ge 0$, let

$$\sigma_\ell(E) = \sum_{i=1}^{r} \|Ex_i\|_{(p^\ell)}.$$

This expression is increasing with ℓ, and

$$\sigma_0(E) = \sum_{i=1}^{r} \|Ex_i\|_{(1)} = \sum_{i=1}^{r} \|Ex_i\| \geq \|Ex\|.$$

We shall prove by induction on κ, $1 \leq \kappa \leq r$ that whenever E is an interval such that $Ex_i \neq 0$ for at most κ indices i, then

$$(*) \qquad \|Ex\| \leq G(\sigma_\kappa(E)).$$

Once this is done, we obtain the result for $\kappa = r$ and $E = \mathrm{ran}(x)$,

$$\|x\| \leq G(\sigma_r(\mathrm{ran}(x))) = G\Big(\sum_{i=1}^{r} \|x_i\|_{(p^r)}\Big) \leq G(r) = \frac{r}{g(r)}.$$

Let $\kappa(E)$ denote the number of indices $i \in \{1, \ldots, r\}$ such that $Ex_i \neq 0$ (if $\kappa(E) = 0$, then $Ex = 0$ and this case is obvious). Observe first that when $\|Ex\| \leq 1$, we have $\|Ex\| = G(\|Ex\|) \leq G(\sum_{i=1}^{r} \|Ex_i\|) \leq G(\sigma_\ell(E))$ for every $\ell \geq 0$. This shows in particular that $(*)$ is true when $\kappa(E) = 1$, since $\|Ex\| \leq 1$ in this case. Assume $(*)$ true when $\kappa(E) \leq \ell < r$, and suppose there exists an interval E such that $\kappa(E) = \ell + 1$ and $\|Ex\| > G(\sigma_{\ell+1}(E))$; since $(*)$ is not true for E we know that $\ell \geq 1$ and $\|Ex\| > 1$. By assumption there exists a (q, g)-form $x^* = g(q)^{-1}(\sum_{j=1}^{q} A_j x_j^*)$, $2 \leq q \leq p$, where $A_j = \mathrm{ran}(x_j^*)$, $A_1 < \ldots < A_q$ and $\|x_j^*\| \leq 1$, such that

$$G(\sigma_{\ell+1}(E)) < |x^*(Ex)|.$$

Assume first that $\kappa(A_j E) \leq \ell$ for every $j = 1, \ldots, q$. We have $\|A_j Ex\| \leq G(\sigma_\ell(A_j E))$ by the induction hypothesis, and using the concavity of G and the relation $q \leq p$ we obtain

$$\begin{aligned}
|x^*(Ex)| &\leq \frac{1}{g(q)} \sum_{j=1}^{q} \|A_j Ex\| \leq \frac{q}{g(q)} \sum_{j=1}^{q} \frac{1}{q}\, G(\sigma_\ell(A_j E)) \\
&\leq \frac{q}{g(q)}\, G\Big(\frac{1}{q} \sum_{j=1}^{q} \sigma_\ell(A_j E)\Big) = \frac{q}{g(q)}\, G\Big(\frac{1}{q} \sum_{i=1}^{r} \sum_{j=1}^{q} \|A_j Ex_i\|_{(p^\ell)}\Big) \\
&\leq \frac{q}{g(q)}\, G\Big(\frac{1}{q} \sum_{i=1}^{r} \|Ex_i\|_{(p^{\ell+1})}\Big) = \frac{q}{g(q)}\, G\Big(\frac{\sigma_{\ell+1}(E)}{q}\Big).
\end{aligned}$$

If $\sigma_{\ell+1}(E) \leq q$, this last expression is equal to

$$\sigma_{\ell+1}(E)/g(q) \leq \sigma_{\ell+1}(E)/g(\sigma_{\ell+1}(E)) = G(\sigma_{\ell+1}(E)),$$

otherwise it is equal to

$$\frac{\sigma_{\ell+1}(E)}{g(q)g(\sigma_{\ell+1}(E)/q)} \leq \frac{\sigma_{\ell+1}(E)}{g(\sigma_{\ell+1}(E))} = G(\sigma_{\ell+1}(E)),$$

so that we have reached a contradiction.

In the remaining case there exists $j_0 \in \{1,\ldots,q\}$ such that $A_{j_0}Ex_i \neq 0$ for every i such that $Ex_i \neq 0$. Assume for example $j_0 < q$ (otherwise $1 < j_0$ deserves a similar treatment). Let m be the last integer i such that $Ex_i \neq 0$. Let $B_{j_0} = A_{j_0} \setminus \operatorname{ran}(Ex_m)$, $B'_{j_0+1} = A_{j_0} \cap \operatorname{ran}(Ex_m)$, $B''_{j_0+1} = A_{j_0+1}$, $B_{j_0+1} = B'_{j_0+1} \cup B''_{j_0+1}$ and $B_j = A_j$ otherwise. We see that

$$\begin{aligned}\|A_{j_0}Ex\| + \|A_{j_0+1}Ex\| &\leq \|B_{j_0}Ex\| + \|B'_{j_0+1}Ex_m\| + \|B''_{j_0+1}Ex_m\| \\ &\leq \|B_{j_0}Ex\| + \|B_{j_0+1}Ex_m\|_{(2)}.\end{aligned}$$

Every B_j satisfies $\kappa(B_jE) \leq \ell$, so that the induction hypothesis applies and since $p^\ell \geq 2$ we obtain

$$\begin{aligned}\sum_{j=1}^{q} \|A_jEx\| &\leq \|B_{j_0+1}Ex_m\|_{(2)} + \sum_{j\neq j_0+1} \|B_jEx\| \\ &\leq G(\sigma_\ell(B_{j_0+1}E)) + \sum_{j\neq j_0+1} G(\sigma_\ell(B_jE)),\end{aligned}$$

and the conclusion follows as before.

Proof of Lemma 25. Let t be a node in $\bigcup_{n\geq 0} E_n$. If m is the integer such that $t \in E_m$, let b_t denote the segment consisting of those nodes $s \in E_m$ such that $s \leq t$. We let F'_m denote the set of nodes $t \in E_m$ such that $\|x(b_t)\| \geq \alpha/2$ and we let F''_m be the complement of F'_m in E_m. It is clear that $b \cap F''_m$ is a segment, whenever b is a segment.

Let $w = \sum_{b\in A} \lambda_b\, x(b)$ be an element of W, with A a finite family of segments and $\sum_{b\in A} |\lambda_b| \leq 1$. Let us check that $\|F''_m\, w\| < \alpha/2$ for every m. For each $b \in A$, let b'' denote the segment $b \cap F''_m$. If t is the longest node in b'', then $\|x(b'')\| \leq \|x(b_t)\| < \alpha/2$. Therefore

$$\|F''_m\, w\| = \|\sum_{b\in A} \lambda_b\, x(b'')\| < \alpha/2.$$

For every n let us write $E_n w_n = \sum_{b\in A_n} \lambda_b\, x(b)$, where A_n is a finite set of segments b contained in E_n, and $\sum_{b\in A_n} |\lambda_b| \leq 1$. Let Γ be the compact set of infinite branches of the tree $\mathcal{T}$, with the topology of pointwise convergence at nodes. For every $b \in A_n$, let γ_b be an infinite branch containing the segment b and consider the non-negative measure $\mu_n = \sum_{b \in A_n} |\lambda_b| \, \delta_{\gamma_b}$ on Γ. Passing to a subsequence we may assume that (μ_n) is weak-$*$ convergent to a finite non-negative measure μ on Γ.

Let B_m denote the set of $\gamma \in \Gamma$ that intersect F'_m. We want to show that $\mu(B_m)$ tends to 0. If not, we may find for every k a branch γ and $m_1 < \ldots < m_k$ such that $\gamma \in B_{m_j}$ for $j = 1, \ldots, k$. Let b_j be the segment $\gamma \cap E_{m_j}$. It follows from the definition of F'_{m_j} that $\|x(b_j)\| \geq \alpha/2$ for $j = 1, \ldots, k$. But this is impossible when k is large, because $\sum_j \|x(b_j)\|^p \leq \|x(\gamma)\|^p \leq 1$, by the definition of the space.

Suppose that $m = m_0$ satisfies $\mu(B_m) < \alpha/4$. This implies that $\mu_n(B_m) < \alpha/4$ when n is large, which means that most of the vectors $x(b)$ used in the construction of $E_n \, w_n$ sit on branches that do not meet F'_m. Taking $n > m$ large enough, we may also have $\mu(B_n) < \alpha/8$. We let F'''_n denote the set of nodes in E_n that are above some node in F'_m. Then $\|F'''_n w_n\| < \alpha/4$ and the nodes in $E'_n = F'_n \setminus F'''_n$ and in $E'_m = F'_m$ are incomparable. Furthermore, if we let $E''_n = F''_n \cup F'''_n$, then $\|E''_n \, w_n\| < \alpha$.

We have just explained the beginning of a construction by induction of a sequence $(m_j)_{j=0}^{\infty}$ satisfying the properties asked in Lemma 25. We let $m_0 = m$ and $m_1 = n$, where m and n are as in the preceding paragraph. The next value of n, which will be chosen as m_2, must satisfy $\mu_n(B_{m_0}) < \alpha/4$, $\mu_n(B_{m_1}) < \alpha/8$ and $\mu(B_n) < \alpha/16$. The reader will easily complete the missing steps.

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