A COURSE ON VALUATIONS

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Summary

This is an introductory course on valuation theory, prepared with the intention to try to present the tradition initiated by Hensel and the geometric tradition initiated by Dedekind and Weber in the same package.

Chapter 1

A little history

1.1 Dedekind-Weber and Hensel-Kürschák

Valuation theory has at least two origins. One usually ascribes its origin to reflexions on the work of Hensel by Kürschák in 1912. This gave birth to the "henselian" tradition in valuation theory, which revolutionized number theory essentially by proving the existence of roots of polynomials in suitable completions of number fields thanks to "Hensel's lemma", and also produced an important body of work in model theory, by way of an effort to understand and classify from a logical viewpoint the stucture of henselian valued fields. The beginning of this theory (up to Krull's 1928 paper) is beautifully explained and organized in the paper [] of P. Roquette.

One can also say, but only in hindsight, that an origin of valuations is in the notion of a "place" introduced by Dedekind and Weber in 1882 in the study of algebraic curves. Their purpose was to define algebraically the Riemann surface of a field of algebraic functions of one variable, e.g the field of rational functions on an affine plane algebraic curve, which may of course have singularities. The Riemann surface being by definition compact and non singular, we find here the source of two basic themes of valuation theory: compactify and desingularize.

So we seek to associate to the field K a set of points X such that K is a field of functions on X (possibly not everywhere defined). Over the complex numbers, the idea is that since, given a point x, one may associate to "many" functions in K a value $f(x) \in \mathbf{C}$, and since the rational functions defined at a point x form a subring K_x

of K having K as field of fractions, we are going to define our set X as a set of homomorphisms $K_{\mathcal{P}} \to \mathbf{C}$, where $K_{\mathcal{P}}$ satisfies certain axioms. More precisely one defines generally a *place* of a field K with values in another field L as a ring homomorphism:

$$\mathcal{P}\colon K_{\mathcal{P}} \to L$$

where $K_{\mathcal{P}}$ is a subring of K such that :
1) If $a \in K \setminus K_{\mathcal{P}}$, then $a^{-1} \in K_{\mathcal{P}}$ and $\mathcal{P}(a^{-1}) = 0$.
2) There exists $a \in K_{\mathcal{P}}$ such that $\mathcal{P}(a) \neq 0$.
Remark that condition 1) is equivalent to saying that the homomorphism $\mathcal{P}$ cannot be extended to a subring larger than $K_{\mathcal{P}}$. Another way to define a place (see [B]) is to define for any field L a new set $\tilde{L} = L \cup \infty$, and extend to $\tilde{L}$ the sum and product in the usual way, with $-\infty = \infty$, $0^{-1} = \infty$, etc... A place of K with values in Δ is then nothing but a homomorphism

$$\tilde{\mathcal{P}}\colon \tilde{K} \to \tilde{\Delta},$$

and $K_{\mathcal{P}}$ is just the set of elements with finite image. If K and Δ are extensions of a "base field" k, we say that $\mathcal{P}$ is a k-place if $\tilde{\mathcal{P}}(c) = c$ for all $c \in k$, and in particular $k \subset K_{\mathcal{P}}$.
Now we may note that the set $m_{\mathcal{P}}$ of elements of $a \in K_{\mathcal{P}}$ such that $\mathcal{P}(a) = 0$ is an ideal with the property that if $b \in K_{\mathcal{P}} \setminus m_{\mathcal{P}}$, then by axiom 1) we have $b^{-1} \in K_{\mathcal{P}}$. This shows that $K_{\mathcal{P}}$ is a local ring with maximal ideal $m_{\mathcal{P}}$. You can also see this by the maximality of $K_{\mathcal{P}}$; if $m_{\mathcal{P}}$ was not maximal, since it is in any case a prime ideal, the homomorphism $\mathcal{P}$ could be extended to the localization $(K_{\mathcal{P}})_{m_{\mathcal{P}}}$. Moreover, still by axiom 1), given two elements $(a, b) \in K_{\mathcal{P}}$, if $a^{-1}b$ is not in $K_{\mathcal{P}}$, then $b^{-1}a \in K_{\mathcal{P}}$, so that the ideal (a, b) of $K_{\mathcal{P}}$ is always principal, generated by either a or b. From this follows that every ideal $I = (a_1, \ldots, a_r)K_{\mathcal{P}}$ generated by finitely many elements is in fact principal and generated by one of these elements.
Let us agree to call any integral domain with this property a *valuation ring*; to each place of a field K we have thus associated a valuation ring having K as field of fractions.
Conversely, let V be a valuation ring with field of fractions K; we are going to associate to V a place of K. For this let us remark that V is necessarily a local ring: it suffices to show that non-invertible elements form an ideal, and since if ab is invertible, a and b both are, the only thing to prove is that the set of non-invertible elements is stable by addition . But if V is a valuation ring, $a + b$ is either

$a(1+b')$ or $(a'+1)b$, and if it is invertible, at least one of a and b must be. So V is a local ring with maximal ideal m_V, and the natural map $h_V\colon V \to V/m_V$ is a place of K as one immediately checks.
Conversely, given a place of K with values in a field L, there exist a valuation ring V of K and a morphism $j\colon \kappa(V) = V/m_V \to L$ such that $\mathcal{P} = j \circ h_V$. These conditions uniquely determine V and j. The valuation ring V is the ring $K_{\mathcal{P}}$ of elements of K with finite image, and its maximal ideal is the set of elements of $a \in K_{\mathcal{P}}$ such that $\mathcal{P}(a) = 0$. The homomorphism j is obtained from $\mathcal{P}$ by passing to the quotient by m_V.
This result reduces the study of places of K to that of valuation rings V with field of fractions equal to K and morphisms of fields $\kappa(V) \to L$.
Given a valuation ring V, we may associate to it a commutative group with a total order, as follows:
Define an preorder on V by saying that $a \leq b$ means that the ideal (a, b) may be generated by the element a. Since V is a valuation ring, this is a total preorder. The corresponding equivalence relation is that $a \cong b$ means that $(a, b) = aV = bV$. This equivalence relation is clearly compatible with the product of V. The element 0 of V is clearly greater than any nonzero element, so if we denote by $\Phi_+ \cup \infty$ the quotient $V/\cong$, we have a natural map

$$\nu\colon V \to (V/\cong) = \Phi_+ \cup \infty$$

which is a morphism from the multiplicative semigroup of V to an ordered semigroup, which we note additively, and Φ_+ is an additive totally ordered semigroup, satisfying the usual relations with respect to ∞. so we setting $\nu(0) = \infty$, we have:
i) $\nu(ab) = \nu(a) + \nu(b)$
and because $a + b = a(1+b')$ or $(a'+1)b$ as we saw above, we have:
ii) $\nu(a+b) \geq \inf(\nu(a), \nu(b))$, with equality if the two valuations are different since then $1 + b'$ (resp. $a' + 1$) is not in the maximal ideal, hence is invertible.
This map ν may be extended to K by setting $\nu(\frac{a}{b}) = \nu(a) - \nu(b)$ if we consider the (totally) ordered abelian group Φ determined by Φ_+; it is the smallest group containing Φ_+, its symetrization. It is sometimes written $\Phi = \Phi_+ - \Phi_+$ to mark the fact that its elements are differences of elements of Φ_+. Sometimes $\{0\}$ is not considered

to be in Φ_+, but here it is. The ordering on Φ_+ gives a total ordering on Φ and hence a decomposition

$$\Phi = \Phi_- \cup \Phi_+ \text{ with } \Phi_- = -\Phi_+,\ \Phi_- \cap \Phi_+ = \{0\}$$

and we have a map

$$\nu\colon K^* \to \Phi$$

satisfying i) and ii) above, which satisfies also $\nu(1) = 0$ and can be extended to K by adding to Φ an element ∞ greater than all the elements of Φ and setting $\nu(0) = \infty$, so it is is a valuation according to the usual definition.

Conversely, given a valuation $\nu\colon K^* \to \Phi$ with values in an ordered group Φ, we may define a subring

$$V = \{0\} \cup \{a \in K^*/\nu(a) \geq 0\} \subset K,$$

and I leave it as an exercise to check that V is a valuation ring of K with maximal ideal $m_V = \{0\} \cup \{a \in K^*/\nu(a) > 0\}$.

If we assume that K contains a field k, we say that ν is a k-valuation if $\nu(c) = 0$ for all $c \in k^*$. It means that $k \subset V$ and we have $m_V \cap k = \{0\}$ in K, so that the inclusion $k \subset K$ gives rise to an inclusion $k \subset \kappa(V) = V/m_V$, and $\kappa(V)$ is an extension of k.

So the second avatar of the notion of a place is that of a valuation.

Now in each avatar there is a natural relation between objects: for places it is specialization, for valuation rings, it is inclusion, and for valuations it is composition. Before we can see that they are essentially the same, we have to define precisely these relations.

If we think of $K_{\mathcal{P}}$ as the ring of rational functions which do not have a pole at a certain point $\mathcal{P}$, and $\mathcal{P}'$ is a specialization of $\mathcal{P}$, by definition of what specialization means, we must have

$$K_{\mathcal{P}'} \subset K_{\mathcal{P}} \text{ and } \ m_{\mathcal{P}'} \subset m_{\mathcal{P}}$$

and conversely.

So we say that a place $\mathcal{P}'$ is a specialization of another place $\mathcal{P}$ if these inclusions holds. Of course this is also an inclusion of valuation rings as local rings. Now let us translate this into the language of valuations.

Let $V' \subset V$ be two valuation rings of the field K. If $a \cong b$ in V', we also have $a \cong b$ in V, since $a \cong b$ in V' means that $b = au$ where u is invertible in V', and therefore also in V. From this follows that the inclusion $V' \subset V$ induces an ordered map of groups $\lambda\colon \Phi' \to \Phi$,

which is surjective since an element of V is of the form ab^{-1} with $a, b \in V'$, and $\nu(ab^{-1}) = \lambda(\nu'(a) - \nu'(b))$. The kernel of λ consists of the images in Φ' of the elements of V' which are not invertible in V' but become invertible in V.
Conversely, suppose that we are given a valuation ring V' of K and a surjective homomorphism of ordered groups $\lambda\colon \Phi' \to \Phi$, where Φ' is $(V \setminus \{0\})/\cong$. Consider the map $\nu'\colon K^* \to \Phi'$ and set $\nu = \lambda \circ \nu'$. It is immediate to check that ν is also a valuation of K, and that the valuation ring V of ν contains V'.
In this case we says that the valuation ν' is a specialization of ν, in accordance with what happens for places.
Now the situation becomes more interesting: since every element of V' invertible in V' is invertible in V, we have the inclusion $m_V \cap V' \subset m_{V'}$, and so an injection

$$V'/m_V \cap V' \subset V/m_V \ = \kappa(V)$$

The first ring is an integral domain, and every ideal of finite type is principal since it is a quotient of a ring with that property. Note that if

specialization of places $\leftrightarrow$ inclusion of valuation rings $\leftrightarrow$ composition of valuations.

From the fact that a valuation ring is associated to a place it is an exercize to check that a valuation ring is a maximal subring of its field of fractions for the relation of *birational domination*: the local ring (B, n) birationally dominates the local ring (A, m) if we have $A \subset B$, they have the same field of fractions and $n \cap A = m$.

From the definition of a valuation ring one deduces that:

Proposition 1.1.1. *a) The set of ideals of a valuation ring is totally ordered by inclusion.*
b) This set of ideals is well ordered.

To prove a), let I and J be two ideals of the valuation ring V, and assume that $I \not\subset J$. Let $x \in I \setminus J$. For any $y \in J$, we have that $Vx \not\subseteq Vy$, which means $Vy \subseteq Vx$ since V is a valuation ring. This shows that J is contained in I, and the result. Assertion b) follows from the fact that an intersection of ideals is an ideal.

Definition 1.1.2. The *rank*, also called the *height* of a valuation is the ordinal type of the sequence of prime ideals of its valuation ring V. When it is finite, it is the Krull dimension of the local ring V.

Let V be a valuation ring with field of fractions K. If a valuation ring V' contains V, it is a localization of V at one of its prime ideals.

The Henselian tradition:

Chapter 2

The Riemann-Zariski manifold of a field

2.1 The definition

Zariski generalized to arbitrary dimensions the Dedekind-Weber construction of the Riemann surface of a field of algebraic functions of one variable. The result, however is very far from being an algebraic variety when the dimension is > 1.

2.2 Valuations and series

An ordered set is artinian in the sense of [?] if it contains no infinite strictly decreasing sequences and it is narrow if it does not contain infinite sets of incomparable elements. According to *loc.cit.*, we can define the k-algebra of formal power series $k[[X^{\mathbb{Q}^d_{\geq 0}}]]$ where $X = (X_1, \ldots, X_d)$, as follows:

It is the set of maps $c\colon \mathbb{Q}^d_{\geq 0} \to k$ which are such that $c^{-1}(k^*)$ is artinian and narrow, equipped with the addition coming from the addition of k and the convolution product $c * c'(\lambda) = \Sigma_{\mu+\sigma=\lambda} c(\mu)c'(\sigma)$. The elements of $k[[X^{\mathbb{Q}^d_{\geq 0}}]]$ can be thought of as formal sums $\Sigma c_\lambda X^\lambda$ with $c_\lambda = c(\lambda)$ or, writing in a less condensed form as the algebra of formal sums $\Sigma c_{\lambda^1,\ldots,\lambda^d} X_1^{\lambda^1} \ldots X_d^{\lambda^d}$ where $\lambda = (\lambda^1, \ldots, \lambda^d)$ with artinian and narrow sets of exponents in $\mathbb{Q}^d_{\leq 0}$. The properties just quoted imply that this product is well defined.

One can also define the algebra of formal power series with sets of exponents which are well ordered in $\mathbb{Q}^d_{\succcurlyeq 0}$ with respect to the total order $\preccurlyeq$. We will denote it by $k[[X^{\mathbb{Q}^d_{\succcurlyeq 0}}]]$.

A remark which will be useful in the sequel is that for any compatible total ordering $\preccurlyeq$ on $\mathbb{Q}^d$ we have, setting $X^{\frac{1}{m}} = (X_1^{\frac{1}{m}}, \ldots, X_d^{\frac{1}{m}})$, the strict inclusions:

$$k[[X]] \subset \widetilde{k[[X]]} := \bigcup_{N \geq 1} k[[X^{\frac{1}{N}}]] \subset k[[X^{\mathbb{Q}^d_{\geq 0}}]] \subset k[[X^{\mathbb{Q}^d_{\succcurlyeq 0}}]].$$

Since $\preccurlyeq$ is a total order and we we consider series with well ordered exponent sets, the smallest exponent of a series in $k[[X^{\mathbb{Q}^d_{\succcurlyeq 0}}]]$ is well defined. Since k is an integral domain this defines a valuation on the field of fractions of $k[[X^{\mathbb{Q}^d_{\succcurlyeq 0}}]]$, with values in $(\mathbb{Q}^d, \preccurlyeq)$.

This construction, which goes back to Krull, makes sense for any totally ordered abelian group. This means that for any totally ordered abelian group Φ and any field k we can define the k-algebra $k[[t^{\Phi_+}]]$, which is endowed with a natural k-valuation with values in Φ.

The next result tells us that the field of fractions of $k[[t^{\Phi_+}]]$ is the set of power series with a well ordered set of exponents in Φ.

The result goes back to Hahn ([]) , and we give here a proof due to B.H. Neumann ([]). He actually proves the generalization to non commutative groups. There are also a generalizations to partially ordered groups (see [],[]).

Theorem 2.2.1. *The ring $k[[t^{\Phi}]]$ of series with coefficients in k and exponents forming a well-ordered set in Φ is a field.*

Proposition 2.2.2. *Let $E \subset \Phi_+$ be a well ordered subset of the positive part of a totally ordered group Φ. Then the semigroup $\langle E \rangle$ generated by E is well ordered.*

Proof. Assume the contrary; there exists an infinite strictly decreasing sequence $f_1 > f_2 > \cdots$ of elements of $\langle E \rangle$. Let us write them:

$$\begin{array}{ll} f_1 & = e_{11} + e_{12} + \ldots + e_{1,\lambda(1)} \\ f_2 & = e_{21} + e_{22} + \ldots + e_{2,\lambda(2)} \\ \vdots & \qquad\qquad = \qquad \vdots \\ f_\mu & = e_{\mu 1} + e_{\mu 2} + \ldots + e_{\mu,\lambda(2)} \\ \vdots & \qquad\qquad = \vdots \end{array}$$

For each μ denote by E_μ the largest of the elements appearing in that line. If we denote by $\Psi(\phi)$ the smallest convex subgroup of Φ such that $\phi \in \Psi(\phi)$, we have $\Psi(E_\mu) = \Psi(f_\mu)$. The sequence of the $\Psi(u_\mu)$ is decreasing and since the sequence of convex subgroups is well ordered, there is a least element. Different sequences can give us different least elements, but among those, there is a smallest.

So we may assume that our sequence (u_μ) is such that the smallest convex subgroup to which its elements ultimately belong is as small as possible.

Forgetting finitely elements of the sequence we may assume that all u_μ are in a certain convex subgroup Ψ, and not in a smaller one. Let us denote by Ψ_+ the largest convex subgroup of Φ strictly contained in Ψ. The set $\Psi \setminus \Psi_+$ contains elements of E, for example the E_μ; we can denote by $\psi(E)$ the smallest element of E which is in $\Psi \setminus \Psi_+$ and we have in the quotient Ψ/Ψ_+, denoting the images in the usual way:

$$\overline{u_1} \geq \overline{E}_1 \geq \overline{\psi}(E) > 0.$$

But Ψ/Ψ_+ is archimedian so that for some integer q we have $q\overline{\psi}(E) \geq \overline{u_1}$ Note that $q \geq 2$ since $\overline{u_1} \geq \overline{\psi}(E)$. We have then $\overline{u_\mu} < \overline{u_1} \leq q\overline{\psi}(E)$ and if we have chosen q as small as possible we may assume $\overline{u_\mu} > (q-1)\overline{\psi(E)}$. Let us assume that we have chosen the sequence (u_μ) so that q is as small as possible.

Now $\overline{u_\mu}$ has to be of the form $\overline{u_\mu} = \overline{u'_\mu} + \overline{E_\mu}$. Only finitely many of the $\overline{u'_\mu}$ can be non zero since otherwise the E_μ, which are elements of S, would form an infinite decreasing sequence. Then among the u'_μ there must be an infinite decreasing sequence. But since by definition $E_\mu \geq \psi(E)$ we must have $\overline{u'_\mu} < (q-1)\overline{\psi}(E)$, and so we have found an infinite decreasing sequence for which either Ψ is smaller or q is smaller, contrary to our choices. This provides the desired contradiction. $\square$

Corollary 2.2.3. *An element of $\langle E\rangle$ lies in only finitely many of the sets $E, E+E, E+E+E, \ldots$.*

Proof. Let M be the set of elements of $\langle E\rangle$ which can be represented by arbitrarily long sums of elements of E. We want to prove that the set M is empty. Assume that it is not. Then, since $\langle E\rangle$ is well ordered, it has a smallest element, say f. By construction, f can be written

$$f = e_{1,1} + \cdots + e_{1,\lambda(1)} = e_{2,1} + \cdots + e_{1,\lambda(2)}$$
$$= \cdots = e_{i,1} + \cdots + e_{i,\lambda(i)} = \cdots$$

with $\lambda(1) < \lambda(2) < \cdots$ and all $e_{i,j}$ non zero.

Set $g_i = f - e_{i,1}$. Since E is well ordered, the sequence $(e_{i,1})_{i\geq 1}$ contains a nondecreasing subsequence, and so the sequence $(g_i)_{i\geq 1}$ contains a non increasing subsequence of elements of $\langle E\rangle$. Since $\langle E\rangle$ is well ordered, this subsequence is eventually constant, say equal to $g \in \langle E\rangle$. By construction we have $g \in S$ and $g < f$. This provides the desired contradiction. $\square$

2.3 The transcendental hypersurface and its approximations

Generalizing the classical definition of the quasi-ordinary hypersurface singularities (see the paragraph before definition **??** below and [?], [?]) we define a transcendental quasi-ordinary hypersurface singularity in the following manner:

Definition 2.3.1. An element $\zeta(X) = \sum c_\lambda X^\lambda \in k[[X^{\mathbb{Q}^d_{\geq 0}}]]$ is called a generalized quasi-ordinary series if there exists a totally ordered sequence $0 = \lambda_0 < \lambda_1 < \ldots < \lambda_i < \lambda_{i+1} < \ldots$ of elements of $(\mathbb{Q}^d, \leq)$ with respect to which $\zeta(X)$ satisfies the following conditions:

- $c_{\lambda_i} \neq 0$ for all $i \in \mathbb{N}$.
- Setting $Q_0 = \mathbb{Z}^d$ and $Q_i = \mathbb{Z}^d + \Sigma_{k=1}^i \mathbb{Z}\lambda_k \subset \mathbb{Q}^d$, we have that if $c_\lambda \neq 0$, then $\lambda \in \bigcup_{j=0}^\infty Q_j$.
- For each λ such that $c_\lambda \neq 0$ we have the equality

$$\min\{i/\lambda \in Q_i\} = \max\{j/\lambda_j \leq \lambda\}.$$

  We will denote this number by $\kappa(\lambda)$.
- For all $j \geq 0$ we have $\kappa(\lambda_j) = j$.

Note that if these conditions are satisfied, $\kappa(\lambda) = i$ implies that $\lambda_i \leq \lambda$ so that one can write: $\zeta(X) = \sum_{i=0}^\infty p_i,\ p_i \in k[[X^{\mathbb{Q}^d_{\geq 0}}]]$ where $p_i = \Sigma_{\kappa(\lambda)=i} c_\lambda X^\lambda$.

Given a generalized quasi-ordinary series $\zeta(X)$, one checks by induction that a minimal system of λ_i satisfying the conditions above is unique and the λ_i are then called distinguished exponents of the series. This terminology is justified in Definition **??**, in which we define for any $i \in \mathbb{N}$, irreducible quasi-ordinary hypersurfaces (see [**?**] or [**?**]) which are parametrized by $X = X, Y = \zeta^{(i)}(X)$ where $\zeta^{(i)}(X)$ is a fractional power series with distinguished exponents $\lambda_1, \ldots, \lambda_i$.

Definition 2.3.2. Given a generalized quasi-ordinary series as above, we define inductively the integers $n_j = [Q_j : Q_{j-1}]$ and $m^{(0)} = 1$, $m^{(i)} = n_1 \cdots n_i$. Note that by construction we have $n_j > 1$ for all j, so that the $m^{(i)}$ are integers which tend to infinity with i.

One then checks by induction that in the decomposition $\zeta(X) = \Sigma_{i=0}^{\infty} p_i$, we have $p_i \in k[[X^{\frac{1}{m^{(i)}}}]]$. This is due to the fact that a series in X or $X^{\frac{1}{N}}$ whose set of exponents is artinian and narrow is a formal power series in the usual sense (see [**?**], Example 3).

The definitions just given are a generalization of [**?**], subsection 4.4, where a "natural valuation" is attached to a "transcendental plane curve", studied through a series of examples from different perspectives: the sequence of point blow ups, the semigroup, the graded valuation ring, ... Moreover, the relations between these approaches are studied. In this text we follow the same approach.

Proposition 2.3.3. *If the sequence of distinguished exponents of a generalized quasi-ordinary series $\zeta(X)$ is infinite, the element $\zeta(X)$ is transcendental over the subring $k[X] \subset k[[X^{\mathbb{Q}^d_{\geq 0}}]]$. In other words, the morphism of $k-$algebras*

$$\begin{array}{rccc} \Theta_\zeta\colon & k[X,Y] & \rightarrow & k[[X^{\mathbb{Q}^d_{\geq 0}}]] \\ & X & \mapsto & X \\ & Y & \mapsto & \zeta(X) \end{array}$$

is injective.

Proof. Assume the contrary and let $\zeta(X)$ be the root of an irreducible polynomial $f \in k[X,Y]$. Consider the algebraically closed field $k((X^{\mathbb{Q}^d_{\succcurlyeq 0}}))$, (see [**?**], Chap. 6, Section 3, $n°$ 4, Exemple 6). We have $\zeta(X) \in k((X^{\mathbb{Q}^d_{\succcurlyeq 0}}))$. In the sequence λ_r of the distinguished exponents the denominators tend to infinity. Therefore, there is an index i such that the denominators of $\lambda_{r,i}$ tend to infinity with r. We can assume that this index is d. Consider the algebraically closed field $k' = k((X'^{\mathbb{Q}^{d-1}_{\geqslant 0}}))$, where $X' = (X_1, \ldots, X_{d-1})$. We can regard $f(X,Y)$ as a polynomial in the ring $k'[X_d, Y]$ and $\zeta(X)$ as an element of the ring $k'[[X_d^{\mathbb{Q}_{\geqslant 0}}]]$. By the Newton-Puiseux theorem (here we use the fact that k is of characteristic zero) all the roots of $f(X,Y)$ are in the ring $\widetilde{k'[[X_d]]}$. It implies that $\zeta(X) \in \widetilde{k'[[X_d]]}$ which is absurd. $\square$

A variant of this proof gives us the following statement: Given any $f \in k[X,Y]$, there does not exist a root $\eta(X) \in k[[X^{\mathbb{Q}^d_{\geqslant 0}}]]$ of f, such that the denominators of the terms of η tend to infinity (By denominator of a term $c_\beta X^\beta$ of η we mean: the least natural number n such that $n.\beta \in \mathbb{N}^d$).

We introduce a sequence of quasi-ordinary hypersurfaces $f^{(i)}$, which approximates the original element $\zeta(X)$.

Recall, see ([?]), that a polynomial $P(X,Y) \in k[[X]]Y]]$ is said to define a (formal germ of) quasi-ordinary hypersurface if its discriminant with respect to Y is of the form $X^\delta \mathrm{u}(X)$ where $\delta \in \mathbb{N}$ and $\mathrm{u}(X)$ is a unit in $k[[X]]$. When k is algebraically closed of characteristic zero this is known (Abhyankar-Jung Theorem) to imply that the roots of the polynomial can be expressed as power series in $X^{\frac{1}{m}}$ for some m, and these series are quasi-ordinary in the sense of Definition **??**, with a finite set of distinguished exponents.

Definition 2.3.4. Let $\zeta(X)$ be a generalized quasi-ordinary series with distinguished exponents $\lambda_1, \ldots \lambda_i, \ldots$. For each integer i we define, using the notations introduced in definition **??**, the i-th approximation

$$\zeta^{(i)}(X) = \sum_{j=1}^{i} p_j.$$

It is a quasi-ordinary series in the sense we have just recalled. As we saw, it is an element of $k[[X^{\frac{1}{m^{(i)}}}]]$. Note that $\zeta(X) - \zeta^{(i)}(X)$ is of the form $X^{\lambda_{i+1}} \times$ unit and also that for $0 \leq j < i$, the difference $\zeta^{(i)}(X) - \zeta^{(j)}(X)$ is of the form $X^{\lambda_{j+1}} \times$ unit .

Since $\zeta^{(i)}(X) \in k[[X^{\frac{1}{m^{(i)}}}]]$ is algebraic over $k((X))$ we can now introduce the irreducible polynomials having $\zeta^{(i)}(X)$ as a root:

Definition 2.3.5. Set $f^{(0)}(X,Y)=Y$, and for any $i\in\mathbb{N}$ define an irreducible quasi-ordinary polynomial $f^{(i)}(X,Y)\in k[[X]][Y]$ as the unitary minimal polynomial of $\zeta^{(i)}(X)$ over $k((X))$. It is shown in [?] that it a polynomial of degree $m^{(i)}$ in Y dividing the polynomial

$$\Pi_{\omega\in(\mu_{m^{(i)}})^d}(Y-\zeta^{(i)}(\omega X).$$

Here $\mu_{m^{(i)}}$ is the group of roots of unity and $\omega X=(\omega_1X_1,\dots,\omega_dX_d)$.

Definition 2.3.6. Using the notations of Definition **??**, it can be proved that $m^{(i)}=\deg_Y(f^{(i)})$ (see [?] or [?]). Moreover, we define the following vectors (originally defined and studied in [?]):

$$\gamma_1=\lambda_1,\ \gamma_j=n_{j-1}\gamma_{j-1}+\lambda_j-\lambda_{j-1},\ j>1.$$

Remark 2.3.7. The subgroups of $\mathbb{Q}^d$ generated by $(\gamma_1\dots,\gamma_j)$ and $(\lambda_1,\dots,\lambda_j)$ are equal.

By $\widetilde{R(f)}$, for a quasi-ordinary f, we mean the set of the roots of f in $k[[X]]$. Following [?], we define the notion of the intersection index of two "comparable" quasi-ordinary hypersurfaces.

In the case where the process terminates after finitely many steps, we make use of the following lemma, which shows that any finite truncation of a quasi-ordinary series, corresponding to a finite initial set of generators $(\gamma_1,\dots,\gamma_i)$ of the associated semigroup, can be viewed as a truncation of a transcendental quasi-ordinary series whose associated semigroup is generated by $(\gamma_1,\dots,\gamma_i)$.

Lemma 2.3.8. [1] *Let $\gamma_1,\dots,\gamma_i$ be an increasing sequence of elements of $\mathbb{Q}^d_{\geq 0}$ satisfying the conditions of Lemma* **??** *and generating a subgroup $G_i\subset\mathbb{Q}^d$. Let $\lambda_1,\dots,\lambda_i$ be the increasing sequence of elements of $\mathbb{Q}^d_{\geq 0}$ corresponding as in definition* **??** *to the γ_j. Given a finite sum*

$$\zeta_F(X)=\Sigma_{\lambda\in F}c_\lambda X^\lambda$$

having $\lambda_1,\dots,\lambda_i$ as distinguished exponents in the sense of Definition **??**, *there exists a series of the form*

$$\zeta(X)=\zeta_F(X)+\sum c_{\tilde\lambda}X^{\tilde\lambda}$$

which is transcendental over $k(X)$ and such that $\zeta_F(X)$ is a finite truncation of $\zeta(X)$ and the group of the valuation associated to $\zeta(X)$ by a choice of a compatible order on $\mathbb{Q}^d$ is G_i.

[1] This lemma and its proof were given by Bernard Teissier

Proof. Without loss of generality we may assume that G_i is not contained in any coordinate hyperplane of $\mathbb{Q}^d$. Let us take an infinite increasing sequence of exponents $(\tilde{\lambda}_j)_{j\geq 1}$ tending to ∞ in $(\mathbb{Q}^d, \leq)$, with $\tilde{\lambda}_1$ larger than all the exponents of ζ_F, and with $\tilde{\lambda}_1$ and $\tilde{\lambda}_j - \tilde{\lambda}_{j-1}$ in the semigroup generated by the $(\gamma_k)_{k\leq i}$ for all $j > 1$. In addition we ask the following "large gaps" condition:
1) $\tilde{\lambda}_{j+1} > (j+1)\tilde{\lambda}_j$ for all j
and remark that because we assume that the $(\tilde{\lambda}_j)$ increase and tend to ∞ in $(\mathbb{Q}^d, \leq)$ we have:
2) for any positive integer s we have $\tilde{\lambda}_\ell > s\mathbf{1}$ for sufficiently large ℓ, where $\mathbf{1} = (1, \ldots, 1) \in \mathbb{Q}^d_{\geqslant 0}$.
Now the claim is that for arbitrary nonzero coefficients $c_{\tilde{\lambda}_j}$ the series

$$\zeta(X) = \zeta_F(X) + \Sigma_{j=i+1}^{\infty} c_{\tilde{\lambda}_j} X^{\tilde{\lambda}_j} \in k[[X^{\mathbb{Q}^d_{\geq 0}}]]$$

satisfies the conditions of the Lemma.
By construction, $\zeta_F(X)$ is a finite truncation of $\zeta(X)$. To check the transcendance, let $Q(X,Y)$ be a polynomial of degree $\leq s$ in Y and of total degree $\leq s$ in the X_k; by our assumptions we can choose an ℓ such that $\ell > s$ and $\tilde{\lambda}_\ell > s\mathbf{1}$. Set $u = \zeta_F(X) + \Sigma_{j=1}^{\ell} c_{\tilde{\lambda}_j} X^{\tilde{\lambda}_j}$ and $v = \Sigma_{j=\ell+1}^{\infty} c_{\tilde{\lambda}_j} X^{\tilde{\lambda}_j}$; we have $\zeta(X) = u + v$ in $k[[X^{\mathbb{Q}^d_{\geqslant 0}}]]$ and the equality:

$$Q(X, u+v) = Q(X,u) + vA_1(X) + v^2 A_2(X) + \cdots$$

with $A_k(X) \in k[[X^{\mathbb{Q}^d_{\geq 0}}]]$ and therefore by our choice of $\tilde{\lambda}_j$, denoting by ν the X-adic order, we have:

$$\nu(vA_1(X) + v^2 A_2(X) + \cdots) \geq \nu(v) = \tilde{\lambda}_{\ell+1} > (\ell+1)\tilde{\lambda}_\ell.$$

On the other hand, since we consider finite sums, we can speak of the degree in X of a polynomial such as $Q(X,u) = Q_0(X) + Q_1(X)u + Q_2(X)u^2 + \cdots + Q_m(X)u^m$, with $m \leq s$ and $Q_m(X) \neq 0$; this degree is an element of $\mathbb{Q}^d_{\geq 0}$.
Since if $\Sigma_1^d a_i \leq s$ we have the inequality $(a_1, \ldots, a_d) \leq s\mathbf{1} < \tilde{\lambda}_\ell$, for $i < m$, the degree of $Q_i(X)u^i$ is $\leq s\mathbf{1} + i\tilde{\lambda}_\ell < (i+1)\tilde{\lambda}_\ell \leq m\tilde{\lambda}_\ell$. On the other hand, the degree of the last term $Q_m(X)u^m$ is $\geq m\tilde{\lambda}_\ell$.

This shows that $Q(X,u)$ is not zero and since its X-adic order cannot exceed the highest power of X which appears, applying to

$Q_m(X)u^m$ the same argument as for $i < m$, we see that it is $< (m+1)\tilde{\lambda}_\ell$.

So $Q(X, u+v) = Q(X, \zeta(X))$ is the sum of a polynomial with fractional exponents of X-adic order $< (m+1)\tilde{\lambda}_\ell \leq (s+1)\tilde{\lambda}_\ell$ and a series of X-adic order $> (\ell+1)\tilde{\lambda}_\ell > (s+1)\tilde{\lambda}_\ell$. It follows that $Q(X, \zeta(X))$ is of X-adic order $< (s+1)\tilde{\lambda}_\ell$ whenever ℓ satisfies the conditions stated above with respect to the degree of the polynomial $Q(X, Y)$. This proves that the series $\zeta(X)$ is not algebraic over $k(X)$. Moreover by construction $\zeta(X)$ is a quasi-ordinary series with distinguished exponents $\lambda_1, \ldots, \lambda_i$ so that one checks using proposition ?? that the value group is G_i.

$\square$

2.4 Key polynomials and the valuative tree of Favre-Jonsson

2.5 Key polynomials and the work of Vaquié

2.6 Key polynomials and toric geometry

We need to approximate graded algebras of the form $\mathrm{gr}_\nu R_\nu$ by polynomial rings. The key ingredient is the Jacobi-Parron algorithm.

Let $(\tau_1. \ldots, \tau_m)$ be m rationally independent positive real numbers. The algorithm consists in writing

$$\tau_1 = \tau_m^{(1)}, \tau_2 = \tau_1^{(1)} + a_2^{(0)}\tau_m^{(1)}, \ldots, \tau_m = \tau_{m-1}^{(1)} + a_m^{(0)}\tau_m^{(1)},$$

where

$$a_j^{(0)} = [\tau_j/\tau_1], \quad j = 2, \ldots, m,$$

and repeating this operation after replacing $(\tau_1, \ldots, \tau_m)$ by $(\tau_1^{(1)}, \ldots, \tau_m^{(1)})$, and so on.

One checks that the $(\tau_1^{(j)}, \ldots, \tau_m^{(j)})$ are also rationally independant and positive. Moreover, since

$$\frac{\tau_i^{(1)}}{\tau_m^{(1)}} = \frac{\tau_{i+1}}{\tau_1} - [\tau_{i+1}/\tau_1],$$

we see that $\tau_i^{(1)} < \tau_m^{(1)}$. After h steps, one has written

$$\tau_i = A_i^{(h)}\tau_1^{(h)} + \cdots + A_i^{(h+m-1)}\tau_m^{(h)}$$

or, if we denote by w the (weight) vector $(\tau_1, \ldots, \tau_m) \in \mathbf{R}^m$ and by $A^{(h)}$ the vector $(A_1^{(h)}, \ldots, A_m^{(h)})$,

$$w = \tau_1^{(h)} A^{(h)} + \tau_2^{(h)} A^{(h+1)} + \cdots + \tau_m^{(h)} A^{(h+m-1)}$$

where the $\tau_j^{(h)}$ are positive, the coefficients $A_i^{(j)}$ are non negative integers, and the matrix of the vectors

$$A^{(h)}, A^{(h+1)}, \ldots, A^{(h+m-1)}$$

has determinant $(-1)^{h(m-1)}$. Moreover, as h grows the directions in $\mathbf{P}^{m-1}(\mathbf{R})$ of the vectors $A^{(h)}$ tend to the direction of w.

The details of the proof can be found in [H-P], Vol. III.

So we have a sequence of vectors $A^{(h)}$ with positive integral coordinates whose directions in $\mathbf{P}^{m-1}(\mathbf{R})$ spiral to the direction of w and such that any consecutive m of them as above form a basis of the integral lattice such that w is contained in the convex cone $\sigma^{(h)} = \langle A^{(h)}, A^{(h+1)}, \ldots, A^{(h+m-1)} \rangle$ which they generate. The convex dual $\check{\sigma}^{(h)}$ of $\sigma^{(h)}$ (see [Cox], §2, [E], V, 2, p. 149) is contained in the half space $\sum_{i=1}^m a_i\tau_i \geq 0$, the integral points of which form the semigroup Φ_+. The algebra of the semigroup $\check{\sigma}^{(h)} \cap \mathbf{Z}^m$ is a polynomial algebra $k[x_1^{(h)}, \ldots, x_m^{(h)}]$ (*loc.cit.*, VI,2) contained in $k[t^{\Phi_+}]$, and since by assumption there are no integral points on the hyperplane $\sum_{i=1}^m a_i\tau_i = 0$ except the origin, the semigroup Φ_+ is the union of the $\check{\sigma}^{(h)} \cap \mathbf{Z}^m$ as $h \to \infty$. This proves that $k[t^{\Phi_+}]$ is the union, or direct limit, of these polynomial subalgebras.

Note that by construction we have for each $h \geq 1$ the equality

$$A^{(h)} - A^{(h+m)} + a_2^{(h)} A^{(h+1)} + \cdots + a_m^{(h)} A^{(h+m-1)} = 0,$$

which shows, since the $a_j^{(h)} = [\tau_j^{(h)}/\tau_1^{(h)}]$ are non-negative, that we have

$$A^{(h+m)} \in \sigma^{(h)} = \langle A^{(h)}, \ldots, A^{(h+m-1)} \rangle$$

and therefore $\sigma^{(h+1)} \subset \sigma^{(h)}$, that is $\check{\sigma}^{(h)} \subset \check{\sigma}^{(h+1)}$ and

$$k[x_1^{(h)}, \ldots, x_m^{(h)}] \subset k[x_1^{(h+1)}, \ldots, x_m^{(h+1)}],$$

so that our direct system is in fact a nested sequence of polynomial subalgebras. The morphisms between these polynomial algebras correspond by duality to the expressions of the $A^{(h+m)}$ as linear combinations with non negative integral coefficients of $(A^{(h)}, \dots, A^{(h+m-1)})$ and are therefore monomial as announced.

If we now consider a group with one more generator $\tau_{m+1} > 0$ which is rationally dependent on $\tau_1, \dots, \tau_m$, Zariski shows in ([Z1], B. I, p. 862) that the new weight vector $w = (\tau_1, \dots, \tau_m, \tau_{m+1}) \in \mathbf{R}^{m+1}$ is contained in a rational simplicial cone $\sigma \subset \mathbf{R}^{m+1}$ generated by m integral vectors $v_1, \dots, v_m$ of the first quadrant forming part of a basis of the integral lattice. Indeed w is contained in a unique rational hyperplane. The dual cone $\check{\sigma} \subset \check{\mathbf{R}}^{m+1}$ is the product of an m dimensional strictly convex cone generated by vectors $e_1, \dots, e_m$ by a 1-dimensional vector space (see [E], V, 2), generated by a primitive integral vector e_{m+1}, which is the dual of the rational hyperplane containing w. The vectors $e_1, \dots, e_{m+1}$ are a basis of the integral lattice, and correspond to the variables $x_1, \dots, x_{m+1}$ generating a polynomial ring. Note that the map $f : \mathbf{Z}^{m+1} \to \mathbf{R}$ defined by $(a_1, \dots, a_{m+1}) \mapsto \sum_{i=1}^{m+1} a_i\tau_i$ is no longer injective; the primitive vector e_{m+1} corresponding to the variable x_{m+1} is in the kernel. Let us set $\widetilde{\Phi_+} = f^{-1}(\Phi_+ \cup \{0\})$. By refining as above by the Perron algorithm for w inside the linear span $< \sigma >$ of σ, starting with the coordinates of w in σ, we find a sequence of regular simplicial cones $\sigma^{(h)} \subset \sigma$ whose duals $\check{\sigma}^{(h)} \subset \widetilde{\Phi_+}$ correspond ([E], VI, Th. 2.12) to algebras of the form $k[x_1^{(h)}, \dots, x_m^{(h)}, x_{m+1}^{\pm 1}] \subset k[t^{\widetilde{\Phi_+}}]$. The free semigroups $\check{\sigma}^{(h)} \cap \mathbf{Z}^{m+1}$ fill up $\widetilde{\Phi_+}$ as $h \to \infty$ since the only rational points of the hyperplane $\sum_{i=1}^{m+1} a_i\tau_i = 0$ are on the dual of the hyperplane containing w, which is contained in all the $\check{\sigma}^{(h)}$. So the direct limit of the images of the maps $k[t^f]\colon k[x_1^{(h)}, \dots, x_m^{(h)}, x_{m+1}^{\pm 1}] \to k[t^{\Phi_+}]$ is $k[t^{\Phi_+}]$. But these images are isomorphic to $k[x_1^{(h)}, \dots, x_m^{(h)}, x_{m+1}^{\pm 1}]/(x_{m+1} - 1)$ so that they are again polynomial rings $k[x_1^{(h)}, \dots, x_m^{(h)}]$. If we have more generators rationally dependent on $\tau_1, \dots, \tau_m$, we can repeat the argument after taking as new generators the coordinates of the weight vector with respect to the m primitive vectors of σ. In both examples, the fact that k is a field plays no role, so we have proved:

Lemma 2.6.1. *Let Φ be a totally ordered finitely generated group of height one (i.e., archimedian), or $\mathbf{Z}^d$ with the lexicographic order. For any commutative ring A the semigroup algebra $A[t^{\Phi_+}]$ of Φ_+ with coefficients in A is the direct limit of a direct system of graded* *subalgebras which are polynomial algebras* $A[x_1,\ldots,x_m]$ *over* A *with* $m = \mathrm{r}(\Phi)$.

In addition, the maps between these algebras are toric maps, i.e., each variable of one is sent to a monomial in the variables of the other, and there is a cofinal subsystem which is a chain of nested subalgebras.

Remarks 2.6.2. 1) In both cases, the smooth subalgebras are produced by an algorithm.
2) The result, especially in view of its algorithmic nature, is much more useful than its consequence that $\mathrm{Spec}A[t^{\Phi_+}]$ is a pro-object in the category of smooth affine toric schemes over $\mathrm{Spec}A$ with toric maps.
3) The proof also shows that the semigroup algebra of the semigroup

$$\widetilde{\Phi_+} = \{(a_1,\ldots,a_m,a_{m+1},\ldots,a_{m+r}) \in \mathbf{Z}^{m+r} \mid \sum_{i=1}^{m+r} a_i\tau_i \geq 0\},$$

where $\tau_1,\ldots,\tau_m$ are rationally independent and the others are rationally dependent upon them, is a toric direct limit of toric subalgebras of the form

$$A[x_1,\ldots,x_m,x_{m+1}^{\pm 1},\ldots,x_{m+r}^{\pm 1}].$$

4) The result for subgroups of $\mathbf{R}$ holds without the finiteness assumption provided the rational rank is finite, since any abelian group is a direct limit of its subgroups of finite type. If we allow m to vary, even that last assumption is unnecessary.

Let now Φ be a totally ordered group of finite height $h > 1$. We have a surjective monotone non-decreasing map $\lambda\colon \Phi \to \Phi_1$ where Φ_1 is of height $h-1$, and the kernel Ψ of λ is of height 1. By induction on the height we may assume that Φ_{1+} is the union of sub-semigroups isomorphic to $\mathbf{N}^m$, and we know from the lemma above that the same is true for Ψ_+.

Let us denote the free semigroups that fill Φ_{1+} by F_i, and let $\tilde{F}_i \subset \Phi_+$ be the subsemigroup generated by elements $e_1,\ldots,e_{r_i}$ which lift to $\Phi_+ \setminus \Psi$ the generators of F_i. Similarly let us denote by $G_j \subset \Psi_+$ free semigroups which fill Ψ_+, generated say by $f_1,\ldots,f_{s_j}$. Note that for $\phi \in \Phi_+ \setminus \Psi$, $\psi \in \Psi$, $\phi+\psi \in \Phi_+$, and consider for r_is_j-tuples $n = (n_{s,t}, 1 \leq s \leq r_i,\ 1 \leq t \leq s_j)$ of non negative integers, the free semigroups $\tilde{F}_i(n) \subset \Phi_+ \cup \{0\}$ generated by $e_1 - \sum_t n_{1t} f_t, \ldots, e_{r_i} - \sum_t n_{r_i t} f_t$. Let us check that the direct sums of free semigroups $\tilde{F}_i(n) \oplus G_j$ fill up Φ_+; the proof generalizes that given in the case of the lexicographic $\mathbf{Z}^d$. Given $\phi \in \Phi_+$, there exists an index i and a $\phi_1 \in \tilde{F}_i$ such that $\phi - \phi_1 \in \Psi$. If we have $\phi - \phi_1 \in \Psi_+$, there exists an index j such that $\phi - \phi_1 \in G_j$ so that indeed $\phi \in \tilde{F}_i \oplus G_j$. This happens in particular if $\phi \in \Psi$. If $\phi - \phi_1 \in \Psi_-$, there exists an index j such that we can write:

$$\phi = \sum_{s=1}^{r_i} k_s e_s - \sum_{t=1}^{s_j} \ell_t f_t \text{ with non negative integers } k_s,\ \ell_t \text{ and } f_t \in G_j.$$

Since $\phi \notin \Psi$, at least one of the k_s is not zero, say k_1. Choosing positive integers $\tilde{\ell}_t$ such that $k_1 \tilde{\ell}_t > \ell_t$, we may rewrite ϕ as follows

$$\phi = k_1(e_1 - \sum_{t=1}^{s_j} \tilde{\ell}_t f_t) + \sum_{s=2}^{r_i} k_s e_s + \sum_{t=1}^{s_j} (k_1 \tilde{\ell}_t - \ell_t) f_t.$$

This shows that ϕ is indeed in $\tilde{F}_i(n) \oplus G_j$ with $n = \tilde{\ell}$.

Now if we assume, as we may by induction, that the F_i and G_j are nested sequences, and choose $n(i)$ given by $(n_{s,t} = i, 1 \leq s \leq r_i,\ 1 \leq t \leq s_j)$, we see that the corresponding groups $\tilde{F}_i(n(i)) \oplus G_i$ form a nested sequence which is cofinal in the direct system. So we have:

Proposition 2.6.3. *For any totally ordered group Φ of finite height, and any commutative ring A, the semigroup algebra $A[t^{\Phi_+}]$ is a direct limit of a direct system of graded polynomial subalgebras over A with monomial maps, and there are cofinal nested subsystems of such polynomial subalgebras. If Φ is of finite rational rank $\mathrm{r}(\Phi)$, all the polynomial subalgebras may be chosen isomorphic to $A[x_1, \ldots, x_{\mathrm{r}(\Phi)}]$.*

Proof. There remains only to prove the last sentence. This is done by induction on the height; we have shown above that the result is true for valuations of height one. Assume that the result is true for Ψ and Φ_1, and so the rank of the free monoid $\tilde{F}_i(n) \oplus G_j$ is the sum of the rational ranks of Ψ and Φ_1. But since the rational rank is additive in an exact sequence because $\mathcal{Q}$ is a flat $\mathbf{Z}$-module, this is the rational rank of Φ. □

Corollary 2.6.4. *If the rational rank of Φ is finite, the semigroup algebra $A[t^{\Phi_+}]$ endowed with its natural grading is a quotient of a polynomial ring over A in countably many indeterminates $A[(V_j)_{j\in J}]$ graded by Φ_+ by a homogeneous binomial ideal of the form*

$$(V_j - V^{m(j)})_{j\in J'},$$

where J' is a subset of J and $|m(j)| \geq 2$ for $j \in J'$.

Proof. We may choose as a system of homogeneous generators of the A-algebra $A[t^{\Phi_+}]$, the union of the generators of the polynomial subalgebras of which $A[t^{\Phi_+}]$ is the direct limit. The only relations between these generators are those corresponding to the toric inclusions $A[x_1, \dots, x_r] \to A[y_1, \dots, y_r]$, and they are of the announced type after we discard the trivial relations $x_i = y_j$ by removing some generators. □

Remarks 2.6.5. 1) There is in general no minimal system of generators for $A[t^{\Phi_+}]$ and we can of course discard an arbitrary number of those generators which appear as V_j in a relation $V_j - V^{m(j)} = 0$. For example consider $\Phi = \mathbf{Z}^2_{lex}$.
2) We can apply this to the subalgebra $k[v^{\Phi_+}]$ of $\mathbf{A}_\nu(R)$, and view the v^ϕ as coordinates for Spec$k[v^{\Phi_+}]$ subjected to the binomial relations described in this Corollary.

Proposition 2.6.6. *Given the ring R_ν of a valuation of finite rational rank $r(\nu)$:*
a) the graded k_ν-algebra $\mathrm{gr}_\nu R_\nu$ is a quotient of a polynomial ring $k_\nu[(V_j)_{j\in J}]$ in countably many indeterminates over the residue field k_ν, graded by Φ_+, by a homogeneous binomial ideal of the form

$$(V_j - \lambda_j V^{m(j)})_{j\in J'}\ ,\ \lambda_j \in k_\nu^*$$

where J' is a subset of J and $|m(j)| \geq 2$ for $j \in J'$.
b) The graded k_ν-algebra $\mathrm{gr}_\nu R_\nu$ is the union of a nested sequence of graded polynomial subalgebras $k_\nu[x_1^{(h)}, \dots, x_{\mathrm{r}(\nu)}^{(h)}]$, where the inclusions are given by maps sending each variable $x_i^{(h)}$ to a constant times a monomial in the $x_j^{(h+1)}$, $1 \leq j \leq \mathrm{r}(\nu)$.

Proof. See Proposition **??** and the previous Corollary. We have seen in Proposition **??** that $\mathrm{gr}_\nu R_\nu$ is isomorphic to a quotient of a polynomial algebra $k[(V_i)_{i\in I}]$ by a binomial ideal whose generators are of the form $V^m - \lambda_{mn} V^n$. Setting all the constants λ_{mn} equal to one gives the semigroup algebra $k[t^{\Phi_+}]$, hence assertion a). Assertion b) follows from this and the correspondence between the direct system of polynomial subalgebras and the binomial equations exhibited in the proof of the Corollary to Proposition **??**. $\square$

Corollary 2.6.7. *Given any finite set of homogeneous elements* $\overline{x}_1, \overline{x}_2, \cdots, \overline{x}_s$ *in* $\mathrm{gr}_\nu R_\nu$ *and a cofinal nested system of polynomial subalgebras as above, there is an algebra in our nested system such that not only do we have* $\overline{x}_i \in A[x_1^{(h)}, \ldots, x_{\Gamma(\Phi)}^{(h)}]$ *for* $1 \leq i \leq s$*, but the element of least degree divides all the others in this subalgebra.*

Chapter 3

Completion

3.1 The case of height one valuations

3.2 The case of valuations of finite height

3.3 Pseudo-Cauchy sequences

Chapter 4

Applications

4.1 Valuations and growth conditions

4.2 Valuations and non oscillating trajectories

4.3 Valuations and arc spaces