PREFACE

In october 2007, the "Abdus Salam" International Centre for Theoretical Physics (ICTP) organized a school in mathematics at the Biblioteca Alexandrina in Alexandria, Egypt. From the 3rd century B.C. until the 4th century A.C. Alexandria was a centre for mathematics. Euclid, Diophante, Eratostene, Ptolemy, Hypatia were among those who made the fame of Alexandria and its antique library. The choice of the Biblioteca Alexandria was symbolic. With the reconstruction of the library it was natural that one also resumes the universal intellectual exchange of the antique library. The will of the director of the Biblioteca, Ismael Seralgedin made that school possible.

The topic of the school was "Algebraic approach of differential equations". This special topic which is at the convergence of Algebra, Geometry and Analysis was chosen to gather mathematicians of different disciplines in Egypt. This topic arises from the pioneer work of E. Kolchin, L. Gårding, B. Malgrange and was formalized by the school of M. Sato in Japan. The techniques used are among the most recent and modern techniques of mathematics. In these lectures we give an elementary presentation of the subject. Applications are given and new areas of research are also hinted. This book allows to understand developments of this. We hope that this book which gathers most of the lectures given in Alexandria will interest specialists and show how linear differential systems are studied nowadays.

I especially thank the secretaries Alessandra Bergamo and Mabilo Koutou of the mathematics section of ICTP and Anna Triolo of the publications section of ICTP for all the help they gave for the publication of this book.

Lê Dung Tráng

ACKNOWLEDGMENTS

The school was made possible with the help of Mr Mohamed El Faham, Deputy Director of the Bibliotheca Alexandrina, Ms Sahar Aly in charge of the international meetings, Ms Mariam Moussa, Ms Marva Elwakie, Ms Yasmin Maamoun, Ms Omneya Kamel, Ms Asmaa Soliman and Ms Samar Seoud, all from Bibliotheca Alexandrina.

From the ICTP side, Ms Koutou Mabilo and Ms Alessandra Bergamo was in charge of the organisation of the school and Ms Anna Triolo was in charge of the publication of the proceedings.

Lê Dung Tráng
Head of the mathematics section
ICTP, Trieste, Italy

CONTENTS

Preface v

Acknowledgments vii

D-Modules in Dimension 1 1
L. N. Macarro

Modules Over the Weyl Algebra 52
F. J. Castro Jiménez

Geometry of Characteristic Varieties 119
L. D. Tráng and B. Teissier

Singular Integrals and the Stationary Phase Methods 136
E. Delabaere

Hypergeometric Functions and Hyperplane Arrangements 210
M. Jambu

Bernstein-Sato Polynomials and Functional Equations 225
M. Granger

Differential Algebraic Groups 292
B. Malgrange

D-MODULES IN DIMENSION 1

L. NARVÁEZ MACARRO*

Departamento de Álgebra & Instituto de Matemáticas (IMUS)
Universidad de Sevilla, P.O. Box 1160, 41080 Sevilla, Spain
**E-mail: narvaez@algebra.us.es*

Introduction

These notes are issued from a course taught in the I.C.T.P. School on Algebraic Approach to Differential Equations, held at Alexandria (Egypt) from November 12 through November 24, 2007.

These notes are intended to guide the reader from the classical theory of linear differential equations in one complex variable to the theory of D-modules. In the first four sections we try to motivate the use of sheaves, in very concrete terms, to state Cauchy theorem and to express the phenomena of analytic continuation of solutions. We also study multivalued solutions around singular points. In sections 5 and 6 we recall the classical result of Fuchs, the index theorem of Komatsu-Malgrange and Malgrange's homological characterization of regularity, which is a key point in understanding regularity in higher dimension. Section 7 is extracted from the very nice paper[2] of J. Briançon and Ph. Maisonobe. It contains the division tools on the ring of (germs of) linear differential operators in one variable. They allow us to prove "almost everything" on (complex analytic) D-module theory in dimension 1 from the classical results. Section 8 tries to motivate the point of view of higher solutions, a landmark in D-module theory. Sections 9 and 10 deal with holonomic D-modules and the general notion of regularity. Both sections are technically based on the division tools and so they are very specific for the one dimensional case, but they give a good flavor of the general theory. Section 11 is written in collaboration with F. Gudiel and it contains the local version of the Riemann-Hilbert correspondence in

*Partially supported by MTM2007–66929 and FEDER.

dimension 1 stated in the paper[13] with some complements. In section 12 we sketch the theory of D-modules on a Riemann surface.

We would like to thank the organizers of the I.C.T.P. school, specially M. Darwish who took care of all practical (and very important) details, and Lê Dũng Tráng who conceived the school and took the heavy task of editing the lecture notes.

1. Cauchy Theorem

Let $U \subset \mathbb{C}$ be an open set. A complex linear differential equation on U is given by

$$a_n \frac{d^n y}{dz^n} + \cdots + a_1 \frac{dy}{dz} + a_0 y = g, \tag{1}$$

where the a_i and g are holomorphic functions on U and y is an unknown holomorphic function on U, which in case it exists is called a *solution* (on U) of the equation (1). If the function a_n does not vanishes identically, we say that equation (1) has order n.

When $g = 0$ in (1), we call it an *homogeneous* complex linear differential equation. In such a case, the solutions form a complex vector space, i.e.
-) the product of any constant and any solution is again a solution.
-) The sum of two solutions is again a solution.

Remark 1.1. A very basic (and obvious) remark is that a complex linear differential equation on U as (1) determines, by restriction, a complex linear differential equation on any open subset $V \subset U$ and we may be interested in searching its solutions, not only on the whole U, but on any open subset $V \subset U$.

If $a_n(x) \neq 0$ for all $x \in U$, then equation (1) is equivalent (in the sense that they have the same solutions) to

$$\frac{d^n y}{dz^n} + a'_{n-1} \frac{dy}{dz} + \cdots + a'_1 \frac{dy}{dz} + a'_0 y = g', \tag{2}$$

where $a'_i = \frac{a_i}{a_n}$ and $g' = \frac{g}{a_n}$.

Equation (2) is still equivalent to a linear system of order 1

$$\frac{dY}{dz} = AY + B, \quad Y = \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix}, \ B = \begin{pmatrix} B_1 \\ \vdots \\ B_n \end{pmatrix} \tag{3}$$

with $B_1 = \cdots = B_{n-1} = 0$, $B_n = b$ and

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a'_0 & -a'_1 & -a'_2 & \cdots & -a'_{n-1} \end{pmatrix}.$$

Correspondences

$$y \mapsto \begin{pmatrix} Y_1 = y \\ Y_2 = y^{(1)} \\ \vdots \\ Y_n = y^{(n-1)} \end{pmatrix}, \qquad \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix} \mapsto y = Y_1$$

establish a bijection between the solutions of (2) and the solutions of (3). When $b = 0$ this bijection is an isomorphism of complex vector spaces.

The basic existence theorem for solutions of a linear system of type (3) is the following result, which ca be found on almost any book of differential equations (see for instance the book[5] n^o 384).

Theorem 1.1. *Let $U \subset \mathbb{C}$ be an open disc centered at the origin, A a $(n \times n)$ matrix of holomorphic functions on U and B a n-column vector of holomorphic functions on U. Let us call $\mathcal{S}$ the set of solutions of the system $\frac{dY}{dz} = AY + B$. Then, the map*

$$Y \in \mathcal{S} \mapsto Y(0) \in \mathbb{C}^n$$

is bijective. Moreover, when $B = 0$ the application above is an isomorphism of complex vector spaces.

Corollary 1.1. *Let $U \subset \mathbb{C}$ be an open disc centered at the origin and let $a_0, \dots, a_n$ holomorphic functions on U with $a_n(z) \neq 0$ for all $z \in U$. Then, for any holomorphic function g on U and any "initial conditions" $v_0, \dots, v_{n-1} \in \mathbb{C}$ there is a unique holomorphic function y on U, which is a solution of the linear differential equation*

$$a_n \frac{d^n y}{dz^n} + \cdots + a_1 \frac{dy}{dz} + a_0 y = g,$$

and such that

$$y(0) = v_0, y^{(1)}(0) = v_1, \dots, y^{(n-1)}(0) = v_{n-1}.$$

2. Sheaves of Holomorphic Functions

Theorem 1.1 can be rephrased in terms of sheaf theory and local systems, which is in principle nothing but an enlargement of our mathematical language. However, this enlargement becomes fundamental in order to understand higher dimensional phenomena and the global behaviour of solutions of differential equations. Let us start by introducing some provisional[a] definitions.

For each open set $V \subset \mathbb{C}$ let us denote by $\mathcal{O}(V)$ the complex vector space of holomorphic functions defined on V.

Definition 2.1. The *sheaf of holomorphic functions* on an open set $U \subset \mathbb{C}$ is the data consisting of all the complex vector spaces $\mathcal{O}(V)$, when V runs into the set of open subsets of U. It will be denoted by $\mathcal{O}_U$, and for each open set $V \subset U$ we will write $\mathcal{O}_U(V) := \mathcal{O}(V)$. The following properties clearly hold:

(a) If $V' \subset V \subset U$ are open sets and $f \in \mathcal{O}_U(V)$, then $f|_{V'} \in \mathcal{O}_U(V')$.
(b) If $V \subset U$ is an open set, $\{V_i\}_{i \in I}$ is an open covering of V and $f : V \to \mathbb{C}$ is a function, we have: $f \in \mathcal{O}_U(V) \Leftrightarrow f|_{V_i} \in \mathcal{O}_U(V_i) \quad \text{for all} \quad i \in I$.

Property (b) above means that for a function, being holomorphic is a local property.

Definition 2.2. A *subsheaf*[b] of $\mathcal{O}_U$ is the data $\mathcal{F}$ consisting of a vector subspace $\mathcal{F}(V) \subset \mathcal{O}_U(V)$ for each open set $V \subset U$ satisfying the following properties:

(a) If $V' \subset V \subset U$ are open sets and $f \in \mathcal{F}(V)$, then $f|_{V'} \in \mathcal{F}(V')$.
(b) If $V \subset U$ is an open set, $\{V_i\}_{i \in I}$ is an open covering of V and $f \in \mathcal{O}_U(V)$, we have $f \in \mathcal{F}(V) \Leftrightarrow f|_{V_i} \in \mathcal{F}(V_i) \quad \text{for all} \quad i \in I$.

If the data $\mathcal{F}$ satisfies property (a) and not necessarily property (b), then we say that it is a *subpresheaf* of $\mathcal{O}_U$. If $\mathcal{F}$ is a subpresheaf of $\mathcal{O}_U$, we will simply write $\mathcal{F} \subset \mathcal{O}_U$.

If $\mathcal{F}, \mathcal{F}'$ are subpresheaves of $\mathcal{O}_U$, we say that $\mathcal{F} \subset \mathcal{F}'$ if $\mathcal{F}(V) \subset \mathcal{F}'(V)$ for any open set $V \subset U$.

Let us note that if $\mathcal{F} \subset \mathcal{O}_U$ is a subsheaf and $U' \subset U$ is an open subset, then the data $\mathcal{F}|_{U'}$ defined by $\mathcal{F}|_{U'}(V) = \mathcal{F}(V)$ for any open set $V \subset U'$ is

[a]Later, we will need the general notion of sheaf, but in this section we only study the sheaf of holomorphic functions and its subsheaves.
[b]Here, we only consider subsheaves of complex vector spaces.

a subsheaf of $\mathcal{O}_{U'}$, that we call *the restriction of* $\mathcal{F}$ *to* U'. Let us also note that $\mathcal{O}_U|_{U'} = \mathcal{O}_{U'}$.

Exercise 2.1. (1) Let $\mathcal{F}$ be the data defined by

$$\mathcal{F}(V) = \{f : V \to \mathbb{C} \mid f \quad \text{is a constant function}\} \subset \mathcal{O}_U(V),$$

for each open set $V \subset U$. Prove that $\mathcal{F}$ is a *subpresheaf* of $\mathcal{O}_U$ which is not a subsheaf. (Hint: what happens with property (b) every time V is not connected?)
(2) Prove that the data $\mathbb{C}_U$ defined by

$$\mathbb{C}_U(V) = \{f : V \to \mathbb{C} \mid f \quad \text{is a locally constant function}\} \subset \mathcal{O}_U(V),$$

for each open set $V \subset U$, is a subsheaf of $\mathcal{O}_U$.

Exercise 2.2. Let $U \subset \mathbb{C}$ be an open set, $\Sigma \subset U$ a closed discrete set and let us denote by $j : U \setminus \Sigma \hookrightarrow U$ the inclusion.
(1) Let $\mathcal{F}$ be the data defined by

$$\mathcal{F}(V) = \{f \in \mathcal{O}_U(V) \mid f = 0 \text{ on a neighborhood of any point } p \in \Sigma \cap V\}.$$

Prove that $\mathcal{F}$ is a subsheaf of $\mathcal{O}_U$, which will be denoted by $j_!\mathcal{O}_{U\setminus\Sigma}$.
(2) Let $\mathcal{F}$ be the data defined by

$$\mathcal{F}(V) = \{f \in \mathbb{C}_U(V) | \ f = 0 \text{ on a neighborhood of any point } p \in \Sigma \cap V\}.$$

Prove that $\mathcal{F}$ is a subsheaf of $\mathcal{O}_U$, which will be denoted by $j_!\mathbb{C}_{U\setminus\Sigma}$.

Exercise 2.3. Let $\mathcal{F} \subset \mathcal{O}_U$ be a subpresheaf. Prove that:

(1) There is a unique subsheaf $\mathcal{F}^+ \subset \mathcal{O}_U$ such that:
(a) $\mathcal{F} \subset \mathcal{F}^+$.
(b) If $\mathcal{F}' \subset \mathcal{O}_U$ is a subsheaf with $\mathcal{F} \subset \mathcal{F}'$, then $\mathcal{F}^+ \subset \mathcal{F}'$.
The sheaf $\mathcal{F}^+$ is called the *associated sheaf* to $\mathcal{F}$.

(2) Prove that $\mathcal{F}$ is a subsheaf of $\mathcal{O}_U$ if and only if $\mathcal{F} = \mathcal{F}^+$.

(3) Prove that $(\mathcal{F}|_{U'})^+ = \mathcal{F}^+|_{U'}$ for any open subset $U' \subset U$.

Definition 2.3. An endomorphism of $\mathcal{O}_U$, $L : \mathcal{O}_U \to \mathcal{O}_U$, is the data consisting of a family of $\mathbb{C}$-linear maps $L(V) : \mathcal{O}_U(V) \to \mathcal{O}_U(V)$ such that for any open subsets $V' \subset V \subset U$ and any $f \in \mathcal{O}_U(V)$ we have $L(V)(f)|_{V'} = L(V')(f|_{V'})$.

Let us denote by $\mathrm{End}(\mathcal{O}_U)$ the set of endomorphisms $L : \mathcal{O}_U \to \mathcal{O}_U$. The definition of "composition" and "addition" inside $\mathrm{End}(\mathcal{O}_U)$ is clear and they define a non-commutative ring structure on $\mathrm{End}(\mathcal{O}_U)$. Composition in $\mathrm{End}(\mathcal{O}_U)$ will be denoted by $\circ$ or simply by juxtaposition, and addition by the usual "+". Moreover, we have an obvious ring homomorphism $\mathbb{C} \to \mathrm{End}(\mathcal{O}_U)$, and so $\mathrm{End}(\mathcal{O}_U)$ is a non-commutative $\mathbb{C}$-algebra.

If $L : \mathcal{O}_U \to \mathcal{O}_U$ is an endomorphism and $U' \subset U$ is an open set, then we define the *restriction* of L to U' as the endomorphism $L|_{U'} : \mathcal{O}_{U'} \to \mathcal{O}_{U'}$ given by $L|_{U'}(V) = L(V) : \mathcal{O}_{U'}(V) = \mathcal{O}_U(V) \to \mathcal{O}_{U'}(V) = \mathcal{O}_U(V)$ for any open set $V \subset U'$. It is clear that the map

$$L \in \mathrm{End}(\mathcal{O}_U) \mapsto L|_{U'} \in \mathrm{End}(\mathcal{O}_{U'})$$

is a homomorphism of $\mathbb{C}$-algebras.

Example 2.1. (a) The family of linear maps

$$f \in \mathcal{O}_U(V) \mapsto \frac{df}{dz} \in \mathcal{O}_U(V), \quad V \subset U \quad \text{open subset},$$

is an endomorphism of $\mathcal{O}_U$ that will be denoted by $\frac{d}{dz} : \mathcal{O}_U \to \mathcal{O}_U$.

(b) If $h \in \mathcal{O}_U(U)$, then the family of linear maps

$$f \in \mathcal{O}_U(V) \mapsto (h|_V)f \in \mathcal{O}_U(V), \quad V \subset U \quad \text{open subset},$$

is an endomorphism that will be denoted by $h : \mathcal{O}_U \to \mathcal{O}_U$.

(c) Example (b) gives rise to a ring homomorphism $\mathcal{O}_U(U) \to \mathrm{End}(\mathcal{O}_U)$, which is injective.

Exercise 2.4. Let $\{U_i\}_{i\in I}$ be an open covering of U and $L_i \in \mathrm{End}(\mathcal{O}_{U_i})$ for each $i \in I$, such that $L_i|_{U_i\cap U_j} = L_j|_{U_i\cap U_j}$ for all $i, j \in I$. Prove that there is a unique $L \in \mathrm{End}(\mathcal{O}_U)$ such that $L|_{U_i} = L_i$ for all $i \in I$.

Remark 2.1. The above exercise indicates that, for a given open set $U \subset \mathbb{C}$, the family $\mathrm{End}(\mathcal{O}_V), V \subset U$ open subset, satisfies the same formal properties as subsheaves of $\mathcal{O}_U$ (see definition 2.2). In fact, $\mathcal{O}_U$, subsheaves of $\mathcal{O}_U$, and $\{\mathrm{End}(\mathcal{O}_V), V \subset U \text{ open subset}\}$ all are examples of "abstract sheaves" (of complex vector spaces or $\mathbb{C}$-algebras) (see for instance the book[9]). The family $\{\mathrm{End}(\mathcal{O}_V), V \subset U \text{ open subset}\}$ is denoted by $\mathit{End}(\mathcal{O}_U)$, and we write $\mathit{End}(\mathcal{O}_U)(V) = \mathrm{End}(\mathcal{O}_V)$ for any open subset $V \subset U$.

Exercise 2.5. Let $L : \mathcal{O}_U \to \mathcal{O}_U$ be an endomorphism and let us consider the data $\ker L$ defined by $(\ker L)(V) = \ker L(V) \subset \mathcal{O}_U(V)$ for each open set $V \subset U$. Prove that $\ker L$ is a subsheaf of $\mathcal{O}_U$, that will be called the *kernel of L*.

Exercise 2.6. (1) Describe the kernel of the endomorphism $\frac{d}{dz} : \mathcal{O}_\mathbb{C} \to \mathcal{O}_\mathbb{C}$.
(2) Prove that $\ker\left(z\frac{d}{dz} + 1 : \mathcal{O}_\mathbb{C} \to \mathcal{O}_\mathbb{C}\right) = j_! \mathbb{C}_{\mathbb{C}-\{0\}}$.

Exercise 2.7. (1) Let $L : \mathcal{O}_U \to \mathcal{O}_U$ be an endomorphism and let us consider the data $\operatorname{im}_0 L$ defined by $(\operatorname{im}_0 L)(V) = \operatorname{im} L(V) \subset \mathcal{O}_U(V)$ for each open set $V \subset U$. Prove that, in general, $\operatorname{im}_0 L$ is not a subsheaf of $\mathcal{O}_U$. (Hint: Consider $L = \frac{d}{dz} : \mathcal{O}_\mathbb{C} \to \mathcal{O}_\mathbb{C}$. Is the function z^{-1} in $(\operatorname{im}_0 L)(\mathbb{C}^*)$? Nevertheless, for each simply connected open set $V \subset \mathbb{C}^*$, the function z^{-1} belongs to $(\operatorname{im}_0 L)(V)$.)

(2) Let us consider the data $\operatorname{im} L$ defined by

$$(\operatorname{im} L)(V) = \{g \in \mathcal{O}_U(V) \mid \forall p \in V, \exists W \subset V \text{ open neighborhood of } p, \\ \exists f \in \mathcal{O}_U(W) \text{ s.t. } L(W)(f) = g|_W\},$$

for each open set $V \subset U$. Prove that $\operatorname{im} L$ is a subsheaf of $\mathcal{O}_U$, that will be called the *image of L*. (Note that $\operatorname{im} L = (\operatorname{im}_0 L)^+$)

(3) Compute the image of the endomorphism $\frac{d}{dz} : \mathcal{O}_\mathbb{C} \to \mathcal{O}_\mathbb{C}$.

Definition 2.4. A (holomorphic) linear differential operator of order $\leq n$ on U is an endomorphism $L : \mathcal{O}_U \to \mathcal{O}_U$ such that there are $a_i \in \mathcal{O}_U(U)$, $0 \leq i \leq n$, such that for each open set $V \subset U$ and each $f \in \mathcal{O}_U(V)$ we have

$$L(V)(f) = (a_n|_V)\frac{d^n f}{dz^n} + \cdots + (a_1|_V)\frac{df}{dz} + (a_0|_V)f,$$

or equivalently, the equality $L = a_n \frac{d^n}{dz^n} + \cdots + a_1 \frac{d}{dz} + a_0$ holds in the ring $\operatorname{End}(\mathcal{O}_U)$.

Obviously, if $L : \mathcal{O}_U \to \mathcal{O}_U$ is a linear differential operator of order $\leq n$ and $U' \subset U$ is an open subset, the restriction $L|_{U'}$ is also a linear differential operator of order $\leq n$.

Exercise 2.8. In the above definition, prove that the a_i are unique.

Remark 2.2. In the above definition, the functions in $(\ker L)(V)$ are obviously the same as the solutions on V of the homogeneous linear differential equation

$$a_n \frac{d^n y}{dz^n} + \cdots + a_1 \frac{dy}{dz} + a_0 y = 0.$$

In this way, $\ker L$ is an object which simultaneously encodes the solutions of the above differential equation on each open subset of U.

Definition 2.5. A (holomorphic) linear differential operator on U is an endomorphism $L : \mathcal{O}_U \to \mathcal{O}_U$ for which there is an open covering $\{U_i\}_{i\in I}$ of U and a family of non-negative integers $\{n_i\}_{i\in I}$ such that the restriction $L|_{U_i}$ is a (holomorphic) linear differential operator of order $\leq n_i$ for each $i \in I$.

The set of (holomorphic) linear differential operators on U will be denoted by $\mathcal{D}(U)$. It is clear that for $V \subset U \subset \mathbb{C}$ open sets, the restriction to V of any linear differential operator on U is also a linear differential operator.

Exercise 2.9. (1) Prove that $\mathcal{D}(U)$ is a sub-$\mathbb{C}$-algebra of $\mathrm{End}(\mathcal{O}_U)$.

(2) Prove that if U is connected, then for any linear differential operator L on U there exist an integer $n \geq 0$ such that L is of order $\leq n$. What happens when U is not connected? Is any differential linear operator on U of finite order?

(3) Let $L : \mathcal{O}_U \to \mathcal{O}_U$ be an endomorphism and assume that there is an open covering $\{U_i\}_{i\in I}$ such that $L|_{U_i}$ is a (holomorphic) linear differential operator on U_i for each $i \in I$. Prove that L is also a (holomorphic) linear differential operator on U.

Remark 2.3. The family $\{\mathcal{D}(V), V \subset U \text{ open subset}\}$, as in remark 2.1, satisfies the same formal properties as subsheaves of $\mathcal{O}_U$ (see definition 2.2). It is the another instance of "abstract sheaf", that will be denoted by $\mathcal{D}_U$, and which is an "abstract subsheaf" of $\mathit{End}(\mathcal{O}_U)$ (see the book[9]).

Definition 2.6. If $\mathcal{F} \subset \mathcal{O}_U$ is a subsheaf and p is a point of U, we define the *stalk* of $\mathcal{F}$ at p, denoted by $\mathcal{F}_p$, as the quotient set $\mathcal{M}/\sim$, where

$$\mathcal{M} = \{(V, f) \mid V \subset U \text{ is an open neighborhood of } p, f \in \mathcal{F}(V)\}$$

and $\sim$ is the equivalence relation given by

$$(V, f) \sim (V', f') \overset{\text{def.}}{\Leftrightarrow} \exists W \subset V \cap V' \text{ open neighb. of } p \text{ s.t. } f|_W = f'|_W.$$

The stalk $\mathcal{F}_p$ is a complex vector space under the operations:

$$\lambda\overline{(V, f)} = \overline{(V, \lambda f)}, \quad \overline{(V, f)} + \overline{(V', f')} = \overline{(V \cap V', f|_{V\cap V'} + f'|_{V\cap V'})}.$$

If $V \subset U$ is an open subset and $f \in \mathcal{F}(V)$, the equivalence class of (V, f) in $\mathcal{F}_p$ will be called *the germ of f at p*, and will be denoted by f_p.

Remark 2.4. The stalk $\mathcal{F}_p$ can be described as the *inductive limit* (or *colimit*) of the system $\mathcal{F}(V)$ when V runs into the open neighborhoods of p contained in U, ordered by the reverse inclusion.

Exercise 2.10. (1) Prove that in the case $\mathcal{F} = \mathcal{O}_U$, the stalk $\mathcal{O}_{U,p}$ is a $\mathbb{C}$-algebra and that the Taylor expansion centered at p defines an isomorphism of $\mathbb{C}$-algebras

$$T_p : \mathcal{O}_{U,p} \xrightarrow{\sim} \mathbb{C}\{z\}, \quad T_p(f_p) = \sum_{i=0}^{\infty} \frac{i}{i!} \frac{d^i f}{dz^i}(p) z^i,$$

where $\mathbb{C}\{z\}$ is the $\mathbb{C}$-algebra of convergent power series in one variable z with complex coefficients.

(2) Prove that $\mathcal{O}_{U,p}$ is a local ring, with maximal ideal $\mathfrak{m}_{U,p} = \{\xi \in \mathcal{O}_{U,p} \mid \xi(p) = 0\}$, where $\xi(p) = f(p)$ whenever $\xi = \overline{(V, f)}$, $f \in \mathcal{O}_U(V)$.

(3) Prove that $\mathcal{O}_{U,p}$ is a discrete valuation ring (Cf. Atiyah-MacDonald's book[1] ch. 9), with valuation $\nu_p : \mathcal{O}_{U,p} \to \mathbb{N} \cup \{+\infty\}$ defined by $\nu_p(\xi) = r$ if $\xi \in \mathfrak{m}_{U,p}^r - \mathfrak{m}_{U,p}^{r+1}$, for any $\xi \neq 0$ and $\nu_p(0) = +\infty$. In other words, if $\xi = f_p$, then $\nu_p(\xi)$ is the vanishing order of f at p, i.e. $\nu_p(f_p) = r$ with $f(q) = (q-p)^r g(q)$ on a neighborhood of p, g holomorphic and $g(p) \neq 0$.

Exercise 2.11. Let $\mathcal{F} \subset \mathcal{O}_U$ be a subsheaf and $p \in U$. Prove that the stalk $\mathcal{F}_p$ can be considered as a vector subspace of $\mathcal{O}_{U,p}$. Prove also that $\mathcal{F} = \mathcal{O}_U$ if and only if $\mathcal{F}_p = \mathcal{O}_{U,p}$ for every $p \in U$.

The following proposition is a version of the analytic continuation principle.

Proposition 2.1. *Let $U \subset \mathbb{C}$ be a connected open set. Then the linear map $f \in \mathcal{O}_U(U) \mapsto f_p \in \mathcal{O}_{U,p}$ is injective for each point $p \in U$.*

Proof. Let us assume that $f_p = 0$ and consider the set

$$W = \{q \in U \mid f_q = 0 \text{ in } \mathcal{O}_{U,p}\} \subset U.$$

It is clear that W is open and $p \in W \neq \emptyset$.

Let us prove that $U - W$ is also open. If $q \in U - W$, then $f_q \neq 0$ and there is an open disc $D \subset U$ centered at q such that $f|_D \neq 0$. If $f(q) \neq 0$, then, for D small enough, $f(q') \neq 0$ for all $q' \in D$. If $f(q) = 0$, since zeros of holomorphic functions ($\neq 0$) in one variable are isolated, we deduce that, for D small enough, $f(q') \neq 0$ for all $q' \in D - \{q\}$. In any case we have that, for D small enough, $f_{q'} \neq 0$ for all $q' \in D - \{q\}$ and so $D \subset U - W$.

Since U is connected, we deduce that $W = U$ and $f = 0$. □

Corollary 2.1. *Let $U \subset \mathbb{C}$ be a connected open set and $V \subset U$ a non-empty open set. Then, the restriction map $f \in \mathcal{O}_U(U) \mapsto f|_V \in \mathcal{O}_U(V)$ is injective.*

Definition 2.7. Let $L : \mathcal{O}_U \to \mathcal{O}_U$ be an endomorphism and $p \in U$. The *stalk* of L at p, denoted by $L_p : \mathcal{O}_{U,p} \to \mathcal{O}_{U,p}$, is the linear map defined by

$$L_p(f_p) = L_p\left(\overline{(V,f)}\right) = \overline{(V, L(V)(f))} = (L(V)(f))_p$$

for every open neighborhood $V \subset U$ of p and every $f \in \mathcal{O}_U(V)$.

Exercise 2.12. (1) If $L, L' : \mathcal{O}_U \to \mathcal{O}_U$ are endomorphisms, prove that $(L + L')_p = L_p + L'_p$, $(L \circ L')_p = L_p \circ L'_p$.

(2) If $L : \mathcal{O}_U \to \mathcal{O}_U$ is an endomorphism, $L = 0$ if and only if $L_p = 0$ for all $p \in U$.

Exercise 2.13. In the situation of the above definition, prove that there are canonical isomorphisms $\ker L_p \simeq (\ker L)_p$, $\operatorname{im} L_p \simeq (\operatorname{im} L)_p$. Prove also that L is injectif, i.e. $\ker L = 0$ (resp. L is surjectif, i.e. $\operatorname{im} L = \mathcal{O}_U$) if and only if L_p is injectif (resp. L_p is surjectif) for all $p \in U$.

Example 2.2. Let $U \subset \mathbb{C}$ be an open set and $p \in U$. For simplicity, let us assume that $p = 0$. Let us consider the linear differential operator on U,

$$L = a_n \frac{d^n}{dz^n} + \cdots + a_1 \frac{d}{dz} + a_0,$$

with $a_i \in \mathcal{O}_U(U)$. Let us call $t_i \in \mathbb{C}\{z\}$ the Taylor expansion at 0 of a_i. Then, under the isomorphism of exercise 2.10, the stalk $L_0 : \mathcal{O}_{U,0} \to \mathcal{O}_{U,0}$ is identified with the linear endomorphism of $\mathbb{C}\{z\}$ given by[c]

$$s \in \mathbb{C}\{z\} \mapsto t_n \frac{d^n s}{dz^n} + \cdots + t_1 \frac{ds}{dz} + t_0 s \in \mathbb{C}\{z\}.$$

Exercise 2.14. Let $U \subset \mathbb{C}$ be a connected open set and $V \subset U$ a non-empty open set. Prove that the restriction map $\mathcal{D}_U(U) \to \mathcal{D}(V)$ is injective.

[c]In definition 7.1, we will study the ring of this kind of linear endomorphisms of $\mathbb{C}\{z\}$.

3. Sheaf Version of Cauchy Theorem

Definition 3.1. (1) Let $U \subset \mathbb{C}$ be a connected open set and $\mathcal{F} \subset \mathcal{O}_U$ a subsheaf. We say that $\mathcal{F}$ is *constant* if for any $p \in U$ the map $f \in \mathcal{F}(U) \mapsto f_p \in \mathcal{F}_p$ is an isomorphism.
(2) Let $U \subset \mathbb{C}$ be an open set and $\mathcal{F} \subset \mathcal{O}_U$ a subsheaf. We say that $\mathcal{F}$ is *locally constant*, or a *local system*, if there is an open covering of U, $U = \bigcup U_i$, by connected open sets, such that $\mathcal{F}|_{U_i}$ is constant for all i.

Exercise 3.1. Let $U \subset \mathbb{C}$ be a connected open set. Prove that:

(1) $\mathbb{C}_U$ is constant subsheaf of $\mathcal{O}_U$.

(2) If $\mathcal{F} \subset \mathcal{O}_U$ is a constant subsheaf and $U' \subset U$ is a connected open set, then the restriction $\mathcal{F}(U) \to \mathcal{F}(U')$ is an isomorphism. Conclude that $\mathcal{F}|_{U'}$ is also a constant subsheaf of $\mathcal{O}_{U'}$.

(3) Prove that any restriction of any locally constant subsheaf of $\mathcal{O}_U$ is locally constant.

(4) Prove that a subsheaf $\mathcal{F} \subset \mathcal{O}_U$ is locally constant if and only if there is an open covering $U = \bigcup U_i$ such that $\mathcal{F}|_{U_i}$ is locally constant for each i.

Exercise 3.2. (1) Prove that any constant subsheaf $\mathcal{F} \subset \mathcal{O}_U$ on a connected open set $U \subset \mathbb{C}$ is determined by the complex vector subspace $\mathcal{F}(U)$ of $\mathcal{O}_U(U)$. Namely, for any open set $V \subset U$, $\mathcal{F}(V)$ consists of functions which locally are restrictions of functions in $\mathcal{F}(U)$.

(2) Reciprocally, given a vector subspace $E \subset \mathcal{O}_U(U)$, prove that there is a unique constant subsheaf $\mathcal{F} \subset \mathcal{O}_U$ such that $\mathcal{F}(U) = E$.

Exercise 3.3. Let $\mathcal{F} \subset \mathcal{O}_U$ be a locally constant subsheaf. Prove that the function $p \in U \mapsto \dim_{\mathbb{C}} \mathcal{F}_p$ is locally constant.

If U is connected and $\mathcal{F} \subset \mathcal{O}_U$ is a locally constant subsheaf with $\mathcal{F}_p$ finite dimensional vector space for some $p \in U$, then $\dim_{\mathbb{C}} \mathcal{F}_q = \dim_{\mathbb{C}} \mathcal{F}_p = r$ for all $q \in U$ and we call $\mathcal{F}$ a locally constant subsheaf (or a local system) of (finite) rank r.

The proof of the following proposition is a standard argument of general Topology (see for instance prop. I.2.1 in the paper[22]).

Proposition 3.1. *Any locally constant subsheaf $\mathcal{F} \subset \mathcal{O}_U$ on a simply connected open set $U \subset \mathbb{C}$ is constant.*

Definition 3.2. Let $U \subset \mathbb{C}$ be a connected open set and

$$L = a_n \frac{d^n}{dz^n} + \cdots + a_1 \frac{d}{dz} + a_0 : \mathcal{O}_U \to \mathcal{O}_U$$

a linear differential operator of order n, i.e. the function a_n does not vanish identically on U. We say that $p \in U$ is a *regular point* of L if $a_n(p) \neq 0$. Otherwise, p will be called a *singular point* of L. The set of singular points of L will be denoted by $\Sigma(L)$.

The theorem 1.1 can be rephrased in the following way.

Theorem 3.1. *Let $U \subset \mathbb{C}$ be a connected open set and $L : \mathcal{O}_U \to \mathcal{O}_U$ a linear differential operator of order n. Then the following properties hold:*

(1) The restriction $(\ker L)\,|_{U-\Sigma(L)}$ is a local system of rank n.
(2) The restriction $L|_{U-\Sigma(L)} : \mathcal{O}_{U-\Sigma(L)} \to \mathcal{O}_{U-\Sigma(L)}$ is surjective.

Moreover, for any singular point $p \in \Sigma(L)$, $\ker L_p$ is a complex vector space of dimension $\leq n$.

Proof. (1) Let us call $\mathcal{L} = (\ker L)\,|_{U-\Sigma(L)}$, $U^0 = U - \Sigma(L)$ and let $V \subset U^0$ be a non-empty open disc. From Cauchy theorem 1.1 we know that for any non-empty open disc $W \subset V$ we have $\dim_{\mathbb{C}} \mathcal{L}(W) = n$. In particular, the restriction $\mathcal{L}(V) \to \mathcal{L}(W)$ is an isomorphism and so $\mathcal{L}|_V$ is a constant sheaf.

(2) Cauchy theorem 1.1 implies that for any non-empty open disc $V \subset U^0$, the map $L(V) : \mathcal{O}_{U^0}(V) \to \mathcal{O}_{U^0}(V)$ is surjective. Hence, for any $p \in U^0$ the map $L_p : \mathcal{O}_{U^0,p} \to \mathcal{O}_{U^0,p}$ is surjective.

For the last part, using proposition 2.1, it is clear that for any small open disc V centered at a singular point p, the dimension of $(\ker L)(V)$ is less or equal than the dimension of $(\ker L)(W)$, for any small open disc $W \subset V - \Sigma(L)$, but for a such W we know that $\dim_{\mathbb{C}}(\ker L)(W) = n$. $\square$

Corollary 3.1. *Let $U \subset \mathbb{C}$ be a connected and simply connected open set and $L : \mathcal{O}_U \to \mathcal{O}_U$ a linear differential operator of order n without singular points. Then, $L(U) : \mathcal{O}(U) \to \mathcal{O}(U)$ is surjective, i.e. the non-homogeneous equation $L(y) = g$ has always a holomorphic solution on U for any $g \in \mathcal{O}(U)$.*

Proof. The proof of this corollary needs to use a small (and motivating) argument of *sheaf cohomology* (see for instance[9]). Let us consider the exact sequence of sheaves

$$0 \to \ker L \to \mathcal{O}_U \xrightarrow{L} \mathcal{O}_U \to 0$$

and the associated long exact sequence of cohomology (cf. *loc. cit.*)

$$0 \to (\ker L)(U) \to \mathcal{O}_U(U) \xrightarrow{L(U)} \mathcal{O}_U(U) \to H^1(U, \ker L) \to \cdots$$

From proposition 3.1 we know that $\ker L$ is a constant sheaf, $\ker L \simeq \mathbb{C}^n_U$, and so $H^1(U, \ker L) \simeq H^1(U, \mathbb{C}^n) = 0$ since U is simply connected. □

4. Local Monodromy

The universal covering space of $\mathbb{C}^* = \mathbb{C} - \{0\}$, with base point 1, can be realized for instance by

$$q : (\mathbb{C}, 0) \to (\mathbb{C}^*, 1), \quad q(w) = e^{2\pi i w}.$$

Base points can be moved inside the set of positive real numbers $\mathbb{R}^*_+ \subset \mathbb{C}^*$ and inside the imaginary axis $\mathbb{R}i \subset \mathbb{C}$ without ambiguity, since both sets are contractible.

The group of automorphisms of q is infinite cyclic generated by the automorphism $M : w \in \mathbb{C} \mapsto w + 1 \in \mathbb{C}$.

For any open disk D centered at 0, we write $\widetilde{D^*} = q^{-1}D^*$ and we also choose $q : \widetilde{D^*} \to D^*$ as universal covering of D^* with base points in $\widetilde{D^*} \cap (\mathbb{R}i)$ and $D^* \cap \mathbb{R}^*_+$ respectively. Let us denote by D_R the open disk centered at 0 of radius $R \in]0, +\infty]$.

Definition 4.1. A *multivalued holomorphic function* on D^* is by definition a holomorphic function on $\widetilde{D^*}$.

The set of multivalued holomorphic functions on D^*_R is denoted by $\mathcal{A}^0_R$. It is clearly a conmmutative $\mathbb{C}$-algebra without zero divisors. For $0 < R' < R \leq +\infty$ we have restriction maps $\mathcal{A}^0_R \to \mathcal{A}^0_{R'}$ which are injective and $\mathbb{C}$-algebra homomorphisms.

Example 4.1. (1) The identity function $w \in \mathbb{C} \mapsto w \in \mathbb{C}$ is obviously an element of $\mathcal{A}^0_\infty$, which will be denoted by $\operatorname{Log} z$. We will also denote by $\operatorname{Log} z$ its restriction to any $\mathcal{A}^0_R$ with $R > 0$.

(2) Given a fixed complex number α, the function $w \in \mathbb{C} \mapsto e^{2\pi i \alpha w} \in \mathbb{C}$ is also an element of $\mathcal{A}^0_\infty$, wich will be denoted by z^α. We will also denote by z^α its restrictions to any $\mathcal{A}^0_R$.

The map $f \in \mathcal{O}(D_R^*) \mapsto f \circ q \in \mathcal{A}_R^0$ is injective and so we can think in $\mathcal{O}(D_R^*)$ as a sub-$\mathbb{C}$-algebra of $\mathcal{A}_R^0$. The automorphism M induces an automorphism of $\mathbb{C}$-algebras

$$T : g \in \mathcal{A}_R^0 \mapsto T(g) = g \circ M \in \mathcal{A}_R^0,$$

called *monodromy operator*. It is clear that T commutes with restrictions and that $T(g) = g$ for any $g \in \mathcal{O}(D_R^*)$.

Exercise 4.1. Prove that any multivalued holomorphic function $g \in \mathcal{A}_R^0$ which is "uniform", i.e. $T(g) = g$, belongs to $\mathcal{O}(D_R^*)$ and so

$$\mathcal{O}(D_R^*) = \{g \in \mathcal{A}_R^0 \mid T(g) = g\}.$$

Definition 4.2. Let g be a multivalued holomorphic function on D^* and $U \subset D^*$ a simply connected open set. A *determination* of g on U is a holomorphic function f on U which is obtained as $f = g \circ \sigma$, where $\sigma : U \to \widetilde{D^*}$ is a holomorphic section of q.

Let $f = g \circ \sigma$ a fixed determination of g on U. Since $q : \widetilde{D^*} \to D^*$ is a covering space, σ must be a biholomorphic map between U and the open set $\sigma(U)$. Any other holomorphic section of q on U must be of the form $M^k \circ \sigma$ and $q^{-1}U = \bigsqcup_{k\in\mathbb{Z}} M^k(\sigma(U))$. Hence, any determination of g on U is of the form $T^k(g) \circ \sigma$.

Definition 4.3. We say that a multivalued holomorphic function g on D^* is of *finite determination* if the vector space generated by $T^k(g)$, $k \in \mathbb{Z}$, is finite dimensional.

Proposition 4.1. *Let g be a multivalued holomorphic function on D^*. The following properties are equivalent:*

(a) g is of finite determination.
(b) The vector space generated by the determinations of g on any simply connected open set $U \subset D^*$ is finite dimensional.
(c) The vector space generated by the determinations of g on some simply connected open set $U \subset D^*$ is finite dimensional.

Proof. The key point is that if we take any simply connected open set $U \subset D^*$ and we fix a holomorphic section $\sigma : U \to \widetilde{D^*}$ of q, then σ must be a biholomorphic map between U and the open set $\sigma(U) \subset \widetilde{D^*}$, any other holomorphic section of q on U must be of the form $M^k \circ \sigma$ and

$q^{-1}U = \bigsqcup_{k\in\mathbb{Z}} M^k(\sigma(U))$. So, if $f = g\circ\sigma$ is a fixed determination of g on U, then the map

$$T^k(g) \mapsto T^k(g)\circ\sigma = g\circ M^k\circ\sigma$$

is a bijection between the set $\{T^k(g),\ k\in\mathbb{Z}\}$ and the set of determinations of g on U, which clearly preserves linear dependence. □

The set of all multivalued holomorphic function on D_R^* of finite determination is a sub-$\mathbb{C}$-algebra of $\mathcal{A}_R^0$, stable by T, and will be denoted by $\mathcal{A}_R$. It is clear that the restriction map $\mathcal{A}_R^0 \to \mathcal{A}_{R'}^0$ sends $\mathcal{A}_R$ into $\mathcal{A}_{R'}$.

Example 4.2. (1) Since $T(\operatorname{Log} z) = 1 + \operatorname{Log} z$, $\operatorname{Log} z$ is a multivalued holomorphic function of finite determination.

(2) Since $T(z^\alpha) = e^{2\pi i\alpha}z^\alpha$, z^α is a multivalued holomorphic function of finite determination.

Definition 4.4. Let $V \subset D_R^*$ be a convex open neighborhood of $\mathbb{R}_+^*\cap D_R^*$ and let $\widetilde{V}\subset\widetilde{D_R^*}$ be the unique connected component of $q^{-1}V$ which intersects the imaginary axis of $\mathbb{C}$. We say that a holomorphic function $f\in\mathcal{O}(V)$ *extends to a multivalued holomorphic function on* D_R^* if there is a (unique) $g\in\mathcal{A}_R^0$ such that $g|_{\widetilde{V}} = f\circ q|_{\widetilde{V}}$. In such a case we say that g is the multivalued extension of f.

Let us note that in the above definition, f extends to the multivalued holomorphic function g on D_R^* if and only if f is a determination of g on V.

Example 4.3. The restriction $q|_{\widetilde{V}} : \widetilde{V} \xrightarrow{\sim} V$ is biholomorphic. The inverse function $f = \left(q|_{\widetilde{V}}\right)^{-1} : V \to \widetilde{V}\subset\mathbb{C}$ extends, obviously by definition, to a multivalued function on D_R^*. In fact, its multivalued extension is the identity function of $\widetilde{D_R^*}$. We have $f(1) = 0$ and $e^{2\pi i f(z)} = (q\circ f)(z) = z$ for all $z\in V$, and so $dz = (2\pi i)e^{2\pi i f(z)}df = (2\pi i)z df$ and

$$f(z) = \frac{1}{2\pi i}\int_1^z \frac{d\zeta}{\zeta},\quad \forall z\in V,$$

where the integration path is taken inside the simply connected open set V. The function f coincides with the usual logarithm "ln" up to the scalar factor $(2\pi i)^{-1}$. This explains why we denote by "$\operatorname{Log} z$" the identity function on $\mathbb{C}$ considered as "multivalued function" on D_R^*.

We have then injective maps

$$\mathcal{O}(D_R) \hookrightarrow \mathcal{O}(D_R^*) \overset{\circ q}{\hookrightarrow} \mathcal{A}_R \hookrightarrow \mathcal{A}_R^0 \overset{\nabla}{\hookrightarrow} \mathcal{O}(V) \tag{4}$$

where the last one associates to any multivalued holomorphic function $g \in \mathcal{A}_R^0$ its "main determination" on V, $\nabla(g) = g \circ \left(q|_{\widetilde{V}}\right)^{-1}$. The compositions $\mathcal{O}(D_R) \to \mathcal{O}(V)$ and $\mathcal{O}(D_R^*) \to \mathcal{O}(V)$ are nothing but the restriction maps. For any radius $R' \in]0, R]$ we have a commutative diagram

$$\begin{tikzcd}
\mathcal{O}(D_R) \arrow[r, hook] \arrow[d, "\text{rest.}"] & \mathcal{O}(D_R^*) \arrow[r, hook, "\circ q"] \arrow[d, "\text{rest.}"] & \mathcal{A}_R \arrow[r, hook] \arrow[d, "\text{rest.}"] & \mathcal{A}_R^0 \arrow[r, hook] \arrow[d, "\text{rest.}"] & \mathcal{O}(V) \arrow[d, "\text{rest.}"] \\
\mathcal{O}(D_{R'}) \arrow[r, hook] & \mathcal{O}(D_{R'}^*) \arrow[r, hook, "\circ q"] & \mathcal{A}_{R'} \arrow[r, hook] & \mathcal{A}_{R'}^0 \arrow[r, hook] & \mathcal{O}(V \cap D_{R'}^*)
\end{tikzcd} \tag{5}$$

Exercise 4.2. (1) Prove that $\nabla(\mathcal{A}_R^0)$ is a subspace of $\mathcal{O}(V)$ stable under the action of the derivative $\frac{d}{dz}$. Conclude that $\mathcal{A}_R^0$ has a natural structure of left $\mathcal{D}(D_R^*)$-module in such a way that ∇ is $\mathcal{D}(D_R^*)$-linear. In particular, $\mathcal{A}_R^0$ is a left $\mathcal{D}(D_R)$-module.

(2) Prove that the monodromy $T : \mathcal{A}_R^0 \to \mathcal{A}_R^0$ is $\mathcal{D}(D_R^*)$-linear.

(3) Prove that $\mathcal{A}_R$ is a sub-$\mathcal{D}(D_R^*)$-module of $\mathcal{A}_R^0$.

Proposition 4.2. *In the situation of definition 4.4, for any holomorphic function $f \in \mathcal{O}(V)$, the following properties are equivalent:*

(a) f extends to a multivalued holomorphic function g on D_R^* of finite determination.
(b) There is a locally constant subsheaf $\mathcal{F} \subset \mathcal{O}_{D_R^*}$ of finite rank such that $f \in \mathcal{F}(V)$.

Proof. We can assume that $f \neq 0$.

(a) $\Rightarrow$ (b): Let us call $\widetilde{\mathcal{F}} \subset \mathcal{O}_{\widetilde{D_R^*}}$ the constant subsheaf determined by the finite dimensional vector subspace $E \subset \mathcal{O}_{\widetilde{D_R^*}}(\widetilde{D_R^*})$ generated by $T^k(g)$, $k \in \mathbb{Z}$ (see exercise 3.2).

For each open subset $W \subset D_R^*$, we define $\mathcal{F}(W) \subset \mathcal{O}_{D_R^*}(W)$ as the vector space of holomorphic functions h on W for which there is an open covering $W = \bigcup W_i$ such that $h|_{W_i} \circ q|_{q^{-1}W_i}$ belongs to $\widetilde{\mathcal{F}}(q^{-1}W_i)$ for all i. It is clear that $\mathcal{F}$ is a subsheaf of $\mathcal{O}_{D_R^*}$.

Let $U \subset D_R^*$ be a simply connected open subset and let us choose a simply connected open subset $U^0 \subset \widetilde{D_R^*}$ such that $q(U^0) = U$. One has $q^{-1}U = \bigsqcup_{k\in\mathbb{Z}} M^k(U^0)$ and $q : M^k(U^0) \xrightarrow{\sim} U$ for all $k \in \mathbb{Z}$. For each open set $W \subset U$, let us call $W^0 = U^0 \cap q^{-1}W$ and so $q^{-1}W = \bigsqcup_{k\in\mathbb{Z}} M^k(W^0)$ and $q : M^k(W^0) \xrightarrow{\sim} W$ for each $k \in \mathbb{Z}$. It is easy to see that for a holomorphic function h on W, the condition $h \circ q|_{q^{-1}W} \in \widetilde{\mathcal{F}}(q^{-1}W)$ is equivalent to the condition $h \circ q|_{W^0} \in \widetilde{\mathcal{F}}(W^0)$. In particular, one has that a holomorphic function h on W belongs to $\mathcal{F}(W)$ if and only if $h \circ q|_{W^0} \in \widetilde{\mathcal{F}}(W^0)$. Composition with q gives rise to a commutative diagram

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\sim"] \arrow[d, "\text{stalk}"'] & \widetilde{\mathcal{F}}(U^0) \arrow[d, "\simeq"', "\text{stalk}"] \\
\mathcal{F}_{q(x)} \arrow[r, "\sim"] & \mathcal{F}_x
\end{tikzcd}$$

for each $x \in U^0$, where the horizontal arrows are isomorphism because $q : U^0 \xrightarrow{\sim} U$ is biholomorphic and the right vertical arrow is an isomorphism because $\widetilde{\mathcal{F}}$ is a constant subsheaf of $\mathcal{O}_{\widetilde{D_R^*}}$. We deduce that the map $\mathcal{F}(U) \to \mathcal{F}_y$ is an isomorphism for each $y \in U$, and so $\mathcal{F}|_U$ is a constant subsheaf of $\mathcal{O}_U$ of finite rank. It is also clear that $f \in \mathcal{F}(V)$.

(b) $\Rightarrow$ (a): For each open set $G \subset \widetilde{D_R^*}$ let us define $\widetilde{\mathcal{F}}(G)$ as the vector space of holomorphic functions $\widetilde{h}$ on G for which there is an open covering $G = \bigsqcup_i G_i$, with $q : G_i \xrightarrow{\sim} q(G_i)$, and functions $h_i \in \mathcal{F}(q(G_i))$ such that $\widetilde{h}|_{G_i} = h_i \circ q|_{G_i}$ for all i. It is clear that $\widetilde{\mathcal{F}}$ is a subsheaf of $\mathcal{O}_{\widetilde{D_R^*}}$ and that $\widetilde{f} := f \circ q|_{\widetilde{V}} \in \widetilde{\mathcal{F}}(\widetilde{V})$.

It is not difficult to see that the restriction of $\widetilde{\mathcal{F}}$ to any open set $G \subset \widetilde{D_R^*}$ for which the restriction of q gives a biholomorhic map between G and $q(G)$ is a locally constant subsheaf of $\mathcal{O}_G$ of finite rank (the same one as the rank of $\mathcal{F}$). So, $\widetilde{\mathcal{F}}$ is locally constant of finite rank too, and from proposition 3.1 we deduce that $\widetilde{\mathcal{F}}$ is constant of finite rank. In particular, there is a (unique) $g \in \widetilde{\mathcal{F}}(\widetilde{D_R^*}) \subset \mathcal{A}_R^0$ such that $g|_{\widetilde{V}} = \widetilde{f}$ and f extends to the multivalued holomorphic function g. Finally, g is of finite determination because $T^k(g) \in \widetilde{\mathcal{F}}(\widetilde{D_R^*})$ for all $k \in \mathbb{Z}$ and this space is finite dimeinsional. $\square$

Let L be a linear differential operator on D_R of order n with $\Sigma(L) \subset \{0\}$:

$$L = a_n \frac{d^n}{dz^n} + \cdots + a_1 \frac{d}{dz} + a_0, \quad a_i \in \mathcal{O}(D_R), \quad a_n(z) \neq 0 \ \forall z \neq 0.$$

From Cauchy theorem (see 3.1) we know that $(\ker L)|_{D_R^*}$ is a locally constant sheaf of rank n.

Proposition 4.3. *Under the above hypothesis and with the notations of definition 4.4, the following properties hold:*

(1) Any multivalued holomorphic function $g \in \mathcal{A}_R^0$ annihilated by L is of finite determination and

$$\{g \in \mathcal{A}_R^0 \mid L(g) = 0\} = \{g \in \mathcal{A}_R \mid L(g) = 0\} \simeq (\ker L)(V).$$

In particular $\dim_{\mathbb{C}}(\{g \in \mathcal{A}_R \mid L(g) = 0\}) = n$.

(2) The maps $L : \mathcal{A}_R^0 \to \mathcal{A}_R^0$ and $L : \mathcal{A}_R \to \mathcal{A}_R$ are surjective.

Proof. (1) We have $\nabla(\{g \in \mathcal{A}_R^0 \mid L(g) = 0\}) \subset (\ker L)(V)$ and from proposition 4.2, we know that $(\ker L)(V) \subset \nabla(\mathcal{A}_R)$. We conclude that

$$\{g \in \mathcal{A}_R^0 \mid L(g) = 0\} = \{g \in \mathcal{A}_R \mid L(g) = 0\} \overset{\nabla}{\simeq} (\ker L)(V).$$

(2) For $g \in \mathcal{A}_R^0$, we have $\frac{d}{dz}(\nabla(g)) = \nabla(\delta(g))$, où $\delta = \frac{e^{-2\pi i w}}{2\pi i}\frac{d}{dw}$, and $L(\nabla(g)) = \nabla(\widetilde{L}(g))$ with

$$\widetilde{L} = a_n(e^{2\pi i w})\delta^n + \cdots + a_0(e^{2\pi i w}) =$$
$$a_n(e^{2\pi i w})\frac{e^{-2\pi n i w}}{(2\pi i)^n}\frac{d^n}{dw^n} + b_{n-1}(w)\frac{d^{n-1}}{dw^{n-1}} + \cdots + b_0(w) \in \mathcal{D}(\widetilde{D_R^*}).$$

Since $\Sigma(\widetilde{L}) = \emptyset$, we deduce from corollary 3.1 that $\widetilde{L} : \mathcal{O}(\widetilde{D_R^*}) \to \mathcal{O}(\widetilde{D_R^*})$ is surjective, and so $L : \mathcal{A}_R^0 \to \mathcal{A}_R^0$ is surjective.

If $g \in \mathcal{A}_R$, there is a non-vanishing polynomial $P(X)$ such that $P(T)(g) = 0$. We have proved that there is $h \in \mathcal{A}_R^0$ such that $L(h) = g$, but $L(P(T)(h)) = P(T)(g) = 0$. We deduce from (1) that $P(T)(h) \in \mathcal{A}_R$ and $h \in \mathcal{A}_R$. So, $L : \mathcal{A}_R \to \mathcal{A}_R$ is surjective. □

Example 4.4. (1) For $L = z\frac{d}{dz} - \alpha$, we have $\{g \in \mathcal{A}_R^0 \mid L(g) = 0\} = \langle z^\alpha \rangle$.
(2) For $L = z\frac{d^2}{dz^2} + \frac{d}{dz}$, we have $\{g \in \mathcal{A}_R^0 \mid L(g) = 0\} = \langle \operatorname{Log} z, 1 \rangle$.

Theorem 4.1. *Any multivalued holomorphic function $g \in \mathcal{A}_R$ of finite determination can be expressed as a finite sum*

$$g = \sum_{\alpha \in \mathbb{C}, k \geq 0} \phi_{\alpha,k} z^\alpha (\operatorname{Log} z)^k,$$

where the $\phi_{\alpha,k} \in \mathcal{O}(D_R^*)$ are uniform functions. Moreover, the $\phi_{\alpha,k}$ are uniquely determined if we impose that the difference $\alpha - \alpha'$ is not an integer whenever $\phi_{\alpha,k}, \phi_{\alpha',k} \neq 0$ for some k (this can be guaranteed for instance if we restrict ourselves to the set of complex numbers α with $-1 \leq \Re\alpha < 0$).

In fact we have a more precise statement. Let $E \subset \mathcal{A}_R$ be the finite dimensional vector subspace generated by the $T^k(g)$, $k \in \mathbb{Z}$, let $\prod(X - \lambda_j)^{r_j}$ be the minimal polynomial of the action of T on g (the λ_j are the eigenvalues of $T|_E$ with $\lambda_j \neq \lambda_l$ whenever $j \neq l$), let $d(g)$ be the degree of *this polynomial, and let us choose complex numbers $\alpha_j \in \mathbb{C}$ with $e^{2\pi i \alpha_j} = \lambda_j$. Then, there are unique $\phi_{j,k} \in \mathcal{O}(D^*)$ such that*

$$\xi = \sum_j \sum_{k=0}^{r_j-1} \phi_{j,k} z^{\alpha_j} (\operatorname{Log} z)^k.$$

Proof. The key point is that, for $\lambda = e^{2\pi i \alpha}$ and $k \geq 0$, we have

$$(T-\lambda)\left(z^\alpha (\operatorname{Log} z)^k\right) = \sum_{i=0}^{k-1} \lambda \binom{k}{i} z^\alpha (\operatorname{Log} z)^i$$

and, for any polynomial $P(X) \in \mathbb{C}[X]$,

$$P(T)\left(z^\alpha (\operatorname{Log} z)^k\right) = P(\lambda) z^\alpha (\operatorname{Log} z)^k + c_{k-1} z^\alpha (\operatorname{Log} z)^{k-1} + \cdots + c_0 z^\alpha,$$

where the c_i are complex numbers. As a consequence,

$$(T-\lambda)^k \left(z^\alpha (\operatorname{Log} z)^k\right) = k! \lambda^k z^\alpha, \quad (T-\lambda)^{k+1} \left(z^\alpha (\operatorname{Log} z)^k\right) = 0.$$

Let us start with uniqueness. Assume that $g = 0$. We proceed by induction on $r = \sum_j (r_j - 1)$. If $r = 0$, then $0 = g = \sum_j \phi_{j,0} z^{\alpha_j}$, and taking $P_l = \prod_{j \neq l} (X - \lambda_j)$ we obtain

$$0 = P_l(T)(g) = \sum_j P_l(T)(\phi_{j,0} z^{\alpha_j}) = P_l(\lambda_l) \phi_{l,0} z^{\alpha_l}$$

and so $\phi_{l,0} = 0$, for each l.

Let us suppose that we have the uniqueness of the coefficients $\phi_{j,k}$ every time $r \leq \nu$ and suppose that $g = 0$ with

$$g = \sum_j \sum_{k=0}^{r_j-1} \phi_{j,k} z^{\alpha_j} (\operatorname{Log} z)^k$$

and $\sum_j (r_j - 1) = \nu + 1$. Let us consider the polynomial $P(X) = (X - \lambda_1)^{r_1-1} \prod_{j \neq 1} (X - \lambda_j)^{r_j}$. We have

$$0 = P(T)(g) = \cdots = \phi_{1,r_1-1} (r_1 - 1)! \lambda_1^{r_1-1} \prod_{j \neq 1} (\lambda_1 - \lambda_j)^{r_j} z^{\alpha_1}$$

and so $\phi_{1,r_1-1} = 0$. To conclude, we apply the induction hypothesis to

$$0 = g = \sum_{k=0}^{r_1-2} \phi_{1,k} z^{\alpha_1} (\operatorname{Log} z)^k + \sum_{j \neq 1} \sum_{k=0}^{r_j-1} \phi_{j,k} z^{\alpha_j} (\operatorname{Log} z)^k.$$

Now, let us prove the existence of the $\phi_{j,k}$. We proceed by induction on the degree $d(g)$ of the minimal polynomial of the action of T on g.

If $d(g) = 1$ then there is a complex number $\lambda_1 \neq 0$ such that $(T - \lambda_1)(g) = 0$. Consequently, $T(z^{-\alpha_1}g) = z^{-\alpha_1}g$ and $\phi_{1,0} := z^{-\alpha_1}\xi$ is uniform: $g = \phi_{1,0}z^{\alpha_1}$.

Assume the result true for any multivalued function $h \in \mathcal{A}_R$ with $d(h) \leq d$.

Let $g \in \mathcal{A}_R$ be a multivalued holomorphic function of finite determination with $d(g) = d+1$ and let $P(X) = \prod_j (X - \lambda_j)^{r_j}$ be the minimal polynomial of the action of T on g. We have $d(g) = \sum_j r_j = d+1$. Let us write $Q(X) = \prod_{j\neq 1}(X-\lambda_j)^{r_j}$ and $P'(X) = P(X)/(X-\lambda_1) = (X-\lambda_1)^{r_1-1}Q(X)$. From the first step of the induction, we know that there exists a $\psi \in \mathcal{O}(D_R^*)$ such that $P'(T)(g) = \psi z^{\alpha_1}$.

We have $P'(T)\left(z^{\alpha_1}(\operatorname{Log} z)^{r_1-1}\right) = Q(T)(T-\lambda_1)^{r_1-1}\left(z^{\alpha_1}(\operatorname{Log} z)^{r_1-1}\right) = (r_1 - 1)!\lambda_1^{r_1-1}Q(\lambda_1)z^{\alpha_1}$ and so $P'(T)(\phi_{1,r_1-1}z^{\alpha_1}(\operatorname{Log} z)^{r_1-1}) = \psi z^{\alpha_1}$ with

$$\phi_{1,r_1-1} = \frac{\psi}{(r_1-1)!\lambda_1^{r_1-1}Q(\lambda_1)}.$$

We deduce that $P'(T)\left(g - \phi_{1,r_1-1}z^{\alpha_1}(\operatorname{Log} z)^{r_1-1}\right) = 0$ and we conclude by applying the induction hypothesis to $g - \phi_{1,r_1-1}z^{\alpha_1}(\operatorname{Log} z)^{r_1-1}$. $\square$

Remark 4.1. In the course of the proof of the above theorem, we have also proved that if $E \subset \mathcal{A}_R$ is the finite dimensional vector subspace generated by the determinations of g and

$$g = \sum_j \sum_{k=0}^{r_j-1} \phi_{j,k} z^{\alpha_j}(\operatorname{Log} z)^k$$

with $\phi_{j,k} \in \mathcal{O}(D_R^*)$ and $\phi_{j,r_j-1} \neq 0$ for all j, then each $\phi_{j,r_j-1}z^{\alpha_j}$ belongs to E and it is an eigenvector of $T|_E$ with respect to the eigenvalue λ_j.

Exercise 4.3. Prove that for any complex number λ, the map $T - \lambda : \mathcal{A}_R \to \mathcal{A}_R$ is surjective [d].

Remark 4.2. Let $\tau : \mathbb{C}/\mathbb{Z} \to \mathbb{C}$ be any section of the canonical projection $\mathbb{C} \to \mathbb{C}/\mathbb{Z}$. The above theorem says that $\{z^\alpha(\operatorname{Log} z)^k \mid \alpha \in \operatorname{im}\tau, k \geq 0\}$ is a basis of $\mathcal{A}_R$ as an $\mathcal{O}(D^*)$-module.

[d] Actually $T - \lambda : \mathcal{A}_R^0 \to \mathcal{A}_R^0$ is also surjective, but the proof needs a cohomological argument (cf. th. (4.1.2) in[20]).

5. Fuchs Theory

In this section we study the behavior of a linear differential equation, or of a linear differential operator, in the neighborhood of a singular point.

Definition 5.1. We say that a multivalued holomorphic function $g \in \mathcal{A}_R$ is *regular*, or of the *Nilsson class* (at 0), if in the expression

$$g = \sum_j \sum_{k=0}^{r_j - 1} \phi_{j,k} z^{\alpha_j} (\operatorname{Log} z)^k,$$

the $\phi_{j,k}$ are meromorphic functions at 0.

It is clear that a $g \in \mathcal{A}_R$ is regular at 0 if and only if its restriction to some (or to any) $\mathcal{A}_{R'}$, with $0 < R' < R$, is regular at 0

Let us denote by $\mathcal{N}_R$ the set of $g \in \mathcal{A}_R$ which are regular (at 0). It is clear that $\mathcal{N}_R$ is a sub-$\mathbb{C}$-algebra of $\mathcal{A}_R$.

Exercise 5.1. Prove that $\mathcal{N}_R$ is a sub-$\mathcal{D}(D)$-module of $\mathcal{A}_R$. Is $\mathcal{N}_R$ a sub-$\mathcal{D}(D^*)$-module of $\mathcal{A}_R$?.

Let $L = a_n \frac{d^n}{dz^n} + \cdots + a_1 \frac{d}{dz} + a_0$ be a linear differential operator on $D = D_R$ of order n ($a_n \neq 0$), and let us assume that 0 is the only singular point of L.

Definition 5.2. We say that 0 is a *regular singular* point of L if any $g \in \mathcal{A}_R$ such that $L(g) = 0$ is regular at 0.

Remark-Definition 5.1. It is clear that if $D' \subset D$ is an open disc centered at 0 and $L' = L|_{D'}$, then 0 is a regular singular point of L if and only if it is so of L'. In particular, if L is a linear differential operator on some open neighborhood of 0, and 0 is a singular point of L, we say that 0 is a *regular singular* point of L if it is so for the restriction of L to a small enough open disc centered at 0. More generally, if L is a linear differential operator on an open set $U \subset \mathbb{C}$ and $p \in U$ is a singular point of L, we say that p is a *regular singular* point of L if 0 is a regular singular point of the "translated" operator

$$L' = a'_n \frac{d^n}{dz^n} + \cdots + a'_1 \frac{d}{dz} + a'_0$$

with $a'_k(z) = a_k(z + p)$, which is defined on the open neighborhood of 0, $U' = \{z \in \mathbb{C} \mid z + p \in U\}$.

For a function $a \in \mathcal{O}(U)$ and a point $p \in U$, let us write $\nu_p(a)$ for the vanishing order of a at p. It only depends on the germ a_p (see exercise 2.10). If $a_p = 0$ then $\nu_p(0) = +\infty$.

Theorem 5.1. *(Fuchs) Let $U \subset \mathbb{C}$ be an open set, $L = a_n \frac{d^n}{dz^n} + \cdots + a_1 \frac{d}{dz} + a_0$ a linear differential operator on U of order $n \geq 1$ and $p \in U$ a singular point of L. Then, the following properties are equivalent:*

(a) p is a regular singular point of L.
(b) $\max\limits_{0 \leq k \leq n} \{k - \nu_p(a_k)\} = n - \nu_p(a_n)$.

Proof. The proof of this theorem can be found in the book,[8] 15.3. □

6. Index of Differential Operators at Singular Points

Let $U \subset \mathbb{C}$ be a connected open set and $L = a_n \frac{d^n}{dz^n} + \cdots + a_1 \frac{d}{dz} + a_0$ a linear differential operator on U of order n. Cauchy theorem 1.1 tells us that, for any non-singular point $p \in U$ of L ($a_n(p) \neq 0$), the stalk at p of L, $L_p : \mathcal{O}_{U,p} \to \mathcal{O}_{U,p}$, is a surjective map and $\dim_{\mathbb{C}} \ker L_p = n$. On the other hand, if $p \in \Sigma(L)$, we have $\dim_{\mathbb{C}} \ker L_p \leq n$ (see theorem 3.1), but what about $\dim_{\mathbb{C}} \operatorname{coker} L_p$?

We have the following important result, known as Komatsu-Malgrange *index theorem.*[11,17]

Theorem 6.1. *Under the above hypothesis, the following properties hold:*

(1) $\dim_{\mathbb{C}} \operatorname{coker} L_p < \infty$.
(2) $\chi(L_p) = \dim_{\mathbb{C}} \ker L_p - \dim_{\mathbb{C}} \operatorname{coker} L_p = n - \nu_p(a_n)$.

The proof of the above theorem consists of a reduction to the case where the differential operator is of the form $L^0 = a_n \frac{d^n}{dz^n}$, where an easy computation shows that $\chi(L^0_p) = \chi\left(\frac{d^n}{dz^n}\right) + \chi(a_n) = n - \nu_p(a_n)$. The reduction is based on the fact that L can be seen as a compact perturbation of L^0 on convenable Banach spaces.

Let us write $\mathcal{O} = \mathcal{O}_{U,p}$, $\mathfrak{m} = \mathfrak{m}_{U,p}$ for its maximal ideal and $P = L_p : \mathcal{O} \to \mathcal{O}$. We know that Taylor development at p establishes an isomorphism between $\mathcal{O}$ and the ring of convergent power series $\mathbb{C}\{z\}$, which sends the ideal $\mathfrak{m}$ to the ideal (z) (see exercise 2.10). It is easy to see that, for any integer $k \geq 0$, we have $P\left(\mathfrak{m}^{n+k}\right) \subset \mathfrak{m}^k$ and so P is continuous for the $\mathfrak{m}$-adic topology and induces a linear endomorphism $\widehat{P}$ of the $\mathfrak{m}$-adic completion

$\widehat{\mathcal{O}}$ of $\mathcal{O}$, which is isomorphic to the (z)-adic completion of $\mathbb{C}\{z\}$, i.e. to the formal power series ring $\mathbb{C}[[z]]$.

The proof of the following theorem is much easier than the proof of theorem 6.1 and can be found in the paper,[17] prop. 1.3 and th. 1.4.

Theorem 6.2. *In the above situation, the following properties hold:*

(1) The vector spaces $\ker \widehat{P}$ *and* $\operatorname{coker} \widehat{P}$ *are finite dimensional and*

$$\chi(\widehat{P}) = \dim_{\mathbb{C}} \ker \widehat{P} - \dim_{\mathbb{C}} \operatorname{coker} \widehat{P} = \max_{0 \le k \le n} \{k - \nu_p(a_k)\}.$$

(2) The induced map $\widetilde{P} = \widehat{\mathcal{O}}/\mathcal{O} \to \widehat{\mathcal{O}}/\mathcal{O}$ *is* <u>*always*</u> *surjective and* $\dim_{\mathbb{C}} \ker \widetilde{P} = \max_{0 \le k \le n} \{k - \nu_p(a_k)\} - (n - \nu_p(a_n))$.

Corollary 6.1. *In the above situation, the following properties are equivalent:*

(1) p is a regular singular point of L.
(2) $\chi(\widetilde{P}) = 0$.
(3) $\widetilde{P}$ *is an isomorphism.*
(4) $\ker \widetilde{P} = 0$.
(5) The canonical maps $\ker P \to \ker \widehat{P}$ *and* $\operatorname{coker} P \to \operatorname{coker} \widehat{P}$ *are isomorphisms.*
(6) $\dim_{\mathbb{C}} \ker P = \dim_{\mathbb{C}} \ker \widehat{P}$ *and* $\dim_{\mathbb{C}} \operatorname{coker} P = \dim_{\mathbb{C}} \operatorname{coker} \widehat{P}$.

Proof. From the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O} \arrow[r] \arrow[d, "P"'] & \widehat{\mathcal{O}} \arrow[r] \arrow[d, "\widehat{P}"'] & \widehat{\mathcal{O}}/\mathcal{O} \arrow[r] \arrow[d, "\widetilde{P}"'] & 0 \\
0 \arrow[r] & \mathcal{O} \arrow[r] & \widehat{\mathcal{O}} \arrow[r] & \widehat{\mathcal{O}}/\mathcal{O} \arrow[r] & 0
\end{tikzcd}$$

we obtain the exact sequence

$$0 \to \ker P \to \ker \widehat{P} \to \ker \widetilde{P} \xrightarrow{\delta} \operatorname{coker} P \to \operatorname{coker} \widehat{P} \to \operatorname{coker} \widetilde{P}(= 0) \to 0 \tag{6}$$

and so[e] $\chi(P) - \chi(\widehat{P}) + \chi(\widetilde{P}) = 0$. From theorems 5.1, 6.1 and 6.2 we have that (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4). On the other hand, from equation (6) we deduce that (4) $\Leftrightarrow$ (5) and (6) $\Rightarrow$ (4), and finally (5) $\Rightarrow$ (6) is obvious. $\square$

[e]In fact this is part of the proof of (b) in theorem 6.2.

Remark 6.1. The above corollary shows that the finite dimensional vector space $\ker \widetilde{L_p}$ is a measure of the non-regularity (or the *irregularity*) of the singular point p of L. This point of view is the first step of the notion of *irregularity complexes* of holonomic $\mathcal{D}$-modules in higher dimension (see the paper[21]).

7. Division Tools

The material of this section is taken from the papers.[2,13]

In this section we work over the ring of convergent power series in one variable $\mathcal{O} = \mathbb{C}\{z\}$, that we can think as the ring of germs at 0 of holomorphic functions defined on a open neighborhood of the origin. Let us denote by $\partial : \mathcal{O} \to \mathcal{O}$ the derivative with respect to z.

Definition 7.1. A $\mathbb{C}$-linear endomorphism $L : \mathcal{O} \to \mathcal{O}$ will be called a *linear differential operator* of $\mathcal{O}$ of order $\leq n$ if there exist $a_0, \dots, a_n \in \mathcal{O}$ such that, for any $g \in \mathcal{O}$ we have $L(g) = a_n \partial^n(g) + \cdots + a_1\partial(g) + a_0 g$. In such a case we will write, as usual, $L = a_n\partial^n + \cdots + a_1\partial + a_0$.

By example 2.2, linear differential operators of $\mathcal{O}$ are nothing but the stalk at the origin of linear differential operators defined on an open neighborhood of 0.

Let us denote by $F^n\mathcal{D} \subset \mathrm{End}_{\mathbb{C}}(\mathcal{O})$ the set of linear differential operators of order $\leq n$ and $\mathcal{D} = \bigcup_{n\geq 0} F^n\mathcal{D} \subset \mathrm{End}_{\mathbb{C}}(\mathcal{O})$. Let us note that the map

$$a \in \mathcal{O} \mapsto [g \in \mathcal{O} \mapsto ag \in \mathcal{O}] \in \mathrm{End}_{\mathbb{C}}(\mathcal{O})$$

is an injective homomorphism of $\mathbb{C}$-algebras and its image coincides with $F^0\mathcal{D}$. From now on, we will identify $\mathcal{O} = F^0\mathcal{D}$. We also set $F^{-1}\mathcal{D} = \{0\}$.

For a $P \in \mathcal{D}$, with $P \neq 0$, let us write $\operatorname{ord} P$ for its order, i.e $\operatorname{ord} P = n$ means that $P \in F^n\mathcal{D}$ but $P \notin F^{n-1}\mathcal{D}$. For $P = 0$ we write $\operatorname{ord} 0 = -\infty$.

Exercise 7.1. Prove the following recursive description of the $F^n\mathcal{D}$:

$$F^0\mathcal{D} = \{P \in \mathrm{End}_{\mathbb{C}}(\mathcal{O}) \mid [P, a] = Pa - aP = 0, \forall a \in \mathcal{O}\},$$
$$F^{n+1}\mathcal{D} = \{P \in \mathrm{End}_{\mathbb{C}}(\mathcal{O}) \mid [P, a] \in F^n\mathcal{D}, \forall a \in \mathcal{O}\}.$$

Exercise 7.2. (see the notes[3]) Prove that:
(1) $\mathcal{D}$ is a non-commutative sub-$\mathbb{C}$-algebra of $\mathrm{End}_{\mathbb{C}}(\mathcal{O})$.
(2) $(F^r\mathcal{D})(F^sD) \subset F^{r+s}\mathcal{D}$ (we say that the family $\{F^n\mathcal{D}\}_{n\geq 0}$ is a *filtration* of the ring $\mathcal{D}$, or that $(\mathcal{D}, F)$ is a *filtered ring*.)

(3) The vector space $\oplus_{n\geq 0} F^n\mathcal{D}/F^{n-1}\mathcal{D}$ has a natural structure of ring (in fact a $\mathbb{C}$-algebra), that we will call the *associated graded ring* of the filtered ring $(\mathcal{D}, F)$ and will be denoted by $\operatorname{gr}_F \mathcal{D}$.
(4) If $P, Q \in \mathcal{D}$ and $P, Q \neq 0$, then $PQ \neq 0$ and $\operatorname{ord} PQ = \operatorname{ord} P + \operatorname{ord} Q$.
(5) If $P, Q \in \mathcal{D}$, then $\operatorname{ord}(PQ - QP) \leq \operatorname{ord} P + \operatorname{ord} Q - 1$ and so $\operatorname{gr}_F \mathcal{D}$ is a commutative ring, and that it is isomorphic to the polynomial ring $\mathcal{O}[\xi]$.

Exercise 7.3. Prove that the ring $\mathcal{D}$ is *simple*, i.e. it has not any non trivial two-sided ideal.

Definition 7.2. If $P \in \mathcal{D}$ is a non-zero operator with $\operatorname{ord}(P) = n$, we define its *symbol* as

$$\sigma(P) = P + F^{n-1}\mathcal{D} \in F^n\mathcal{D}/F^{n-1}\mathcal{D} = \operatorname{gr}^n_F \mathcal{D}.$$

It is clear that if $P, Q \in \mathcal{D}$ are non-zero, then $\sigma(PQ) = \sigma(P)\sigma(Q)$.

Definition 7.3. Given a left ideal $I \subset \mathcal{D}$, we define $\sigma(I)$ as the ideal of $\operatorname{gr}_F \mathcal{D}$ generated by $\sigma(P)$, for all $P \in I$, $P \neq 0$.

Exercise 7.4. Prove that $\mathcal{D}$ is left and right noetherian.

Let P be a non-zero linear differential operator (of $\mathcal{O}$) of order $n \geq 0$, i.e. $P = \sum_{k=0}^n a_k \partial^k$, with $a_k \in \mathcal{O}$ and $a_n \neq 0$. Let us write $a_k = \sum_{l=0}^\infty a_{lk} z^l$ and so

$$P = \sum_{k=0}^{n} \sum_{l=0}^{\infty} a_{lk} x^l \partial^k.$$

We call the *Newton diagram* (or the *support*) of P the set

$$\operatorname{supp}(P) = \{(l, k) \in \mathbb{N}^2 \mid a_{lk} \neq 0\} \subset \mathbb{N}^2.$$

Definition 7.4. In the above situation, we define the *valuation* of P as $\nu(P) = \nu_0(a_n)$ and the *exponent* of P as $\exp(P) = (\nu(P), \operatorname{ord} P)$.

Exercise 7.5. Prove that if $P, Q \in \mathcal{D}$, $P, Q \neq 0$, then $\exp(PQ) = \exp(P) + \exp(Q)$.

Lemma 7.1 (Briançon-Maisonobe[2]). *Let $P \in \mathcal{D}$, $P \neq 0$ and $\exp(P) = (v, d)$. Then, for any $A \in \mathcal{D}$ there are unique $Q, R \in \mathcal{D}$ such that $A = QP + R$ with*

$$R = \sum_{k=d}^{\operatorname{ord}(A)} \sum_{l=0}^{v-1} r_{lk} x^l \partial^k + S, \quad \textit{with } \operatorname{ord}(S) < d.$$

The proof of the above lemma is easy, and in fact it is a particular case of the general division theorems in several variables (see the lectures by F. Castro). Let us note that the condition on the remainder R is equivalent to say that

$$\operatorname{supp}(R) \subset \mathbb{N}^2 \setminus \left(\exp(P) + \mathbb{N}^2\right).$$

Let us denote by $\mathcal{K}$ the field of fractions of the ring $\mathcal{O}$. Any element of $\mathcal{K}$ can be written as a/z^r, with $a \in \mathcal{O}$ and $r \geq 0$. We can think of elements of $\mathcal{K}$ as the germs at 0 of meromorphic functions defined on a neighborhood of 0 and with a pole eventually at 0. The derivative $\partial : \mathcal{O} \to \mathcal{O}$ extends obviously to $\mathcal{K}$.

Let $\mathcal{D}_{\mathcal{K}}$ be the ring of linear differential operators of $\mathcal{K}$, i.e. the subring of $\operatorname{End}_{\mathbb{C}}(\mathcal{K})$ with elements of the form

$$\sum_{k=0}^{n} a_k \partial^k, \quad a_k \in \mathcal{K}.$$

The ring is filtered in the obvious way and for any $P \in \mathcal{D}_{\mathcal{K}}$, $P \neq 0$, the definition of its order $\operatorname{ord}(P)$ is clear.

The proof of following lemma is easy.

Lemma 7.2. *Let $P \in \mathcal{D}_{\mathcal{K}}$, $P \neq 0$. Then, for any $A \in \mathcal{D}_{\mathcal{K}}$ there are unique $Q, R \in \mathcal{D}_{\mathcal{K}}$ such that $A = QP + R$ with* $\operatorname{ord}(R) < \operatorname{ord}(P)$.

Corollary 7.1. *Let $P \in \mathcal{D}$, $P \neq 0$. Then, for any $A \in \mathcal{D}$ there are $Q, R \in \mathcal{D}$ and an integer $r \geq 0$ such that $x^r A = QP + R$ with* $\operatorname{ord}(R) < \operatorname{ord}(P)$.

Definition 7.5. Let $I \subset \mathcal{D}$ be a non-zero left ideal. We define the set

$$\operatorname{Exp}(I) = \{\exp(P) \mid P \in I, P \neq 0\}.$$

It is clear that $\operatorname{Exp}(I)$ is an ideal of $\mathbb{N}^2$, i.e. $\operatorname{Exp}(I) + \mathbb{N}^2 \subset \operatorname{Exp}(I)$.

Given a non-zero left ideal $I \subset \mathcal{D}$ let us write

$$p = p(I) = \min\{\operatorname{ord}(P) \mid P \in I, P \neq 0\},$$

and for each $d \geq p$,

$$\alpha_d = \alpha_d(I) = \min\{\nu(P) \mid P \in I, P \neq 0, \operatorname{ord}(P) = d\}.$$

Since $\alpha_p \geq \alpha_{p+1} \geq \cdots$ we can define

$$q = q(I) = \min\{d \geq p \mid \alpha_d = \alpha_e, \forall e \geq d\}.$$

We also define

$$\nu(I) = \min\{\nu(P) \mid P \in I, P \neq 0\}.$$

It is clear that $\nu(I) = \alpha_{q(I)}(I)$.

Exercise 7.6. With the above notations, prove that

$$\mathrm{Exp}(I) = \bigcup_{d=p}^{q} \left((\alpha_d, d) + \mathbb{N}^2 \right).$$

Definition 7.6. With the above notations, a set of elements F_p, $F_{p+1}, \ldots, F_q \in I$ with $\exp(F_d) = (\alpha_d, d)$ for $p \leq d \leq q$, is called a *standard basis*, or a *Gröbner basis*, of I.

If $F_p, F_{p+1}, \ldots, F_q$ is a Gröbner basis of I, then $p(I) = \mathrm{ord}(F_p)$ and $\nu(I) = \nu(F_q)$.

For any $A \in \mathcal{D}$, and by successive division (lemma 7.1) by the elements $F_q, F_{q-1}, \ldots, F_p$ of I, we obtain a unique expression

$$A = Q_p F_p + \cdots + Q_{q-1} F_{q-1} + Q_q F_q + R$$

with $Q_p, \ldots, Q_{q-1} \in \mathcal{O}$, $Q_q \in \mathcal{D}$ and

$$R = \sum_{k=p}^{\mathrm{ord}(A)} \sum_{l=0}^{\alpha_k - 1} r_{lk} x^l \partial^k + S, \quad \text{with } \mathrm{ord}(S) < p,$$

or in other words

$$\mathrm{supp}(R) \subset \mathbb{N}^2 \setminus \mathrm{Exp}(I).$$

In particular, $A \in I \Leftrightarrow R = 0$ and so any Gröbner basis $F_p, F_{p+1}, \ldots, F_q$ of I is a system of generators I.

Exercise 7.7. Prove that if $F_p, F_{p+1}, \ldots, F_q$ is a Gröbner basis of I, then

$$\sigma(I) = (\sigma(F_p), \ldots, \sigma(F_q)).$$

Example 7.1. Let $I = \mathcal{D}$ be the total left ideal. It is clear that I is generated by ∂, z. However, $\sigma(I) = \sigma(\mathcal{D}) = \mathrm{gr}_F \mathcal{D}$ is not generated by $\sigma(\partial) = \xi, \sigma(z) = z$.

Given a left ideal $I \subset \mathcal{D}$ and a system of generators $P_1, \ldots, P_r$ of I, often we are interested in the module of *syzygies* (or relations) of the P_i

$$S(\underline{P}) = \{(Q_1, \ldots, Q_r) \in \mathcal{D}^r \mid \sum_i Q_i P_i = 0\}.$$

This module is a sub-$\mathcal{D}$-module of $\mathcal{D}^r$, and so it is finitely generated.

In general it is not clear how to exhibit a finite number of generators of $S(\underline{P})$, but the situation is simpler if the P_i form a Gröbner basis of I.

Let us keep the notations of definition 7.6, and let us assume that the F_d satisfy the following property:

$$F_d = z^{\alpha_d}\partial^d + \text{terms of lower order.}$$

We say in that case that our Gröbner basis is *normalized*.

For each $d = p+1,\ldots,q$, there are unique $Q_l^d \in \mathcal{O}$, $l = p,\ldots,d-1$ such that

$$\partial F_{d-1} - z^{\alpha_{d-1}-\alpha_d} F_d = Q_p^d F_p + \cdots + Q_{d-1}^d F_{d-1}.$$

We have then the following syzygies of $(F_p, F_{p+1},\ldots,F_q)$:

$$\mathcal{R}_d = (Q_p^d, Q_{p+1}^d, \ldots, \overbrace{-\partial + Q_{d-1}^d}^{d-1}, \overbrace{z^{\alpha_{d-1}-\alpha_d}}^{d}, 0, \ldots, 0)$$

for $d = p+1,\ldots,q$.

We have the following result (see prop. 3 in[2]). It is a particular case of a general result valid for Gröbner bases in several variables and in various settings (see the notes[3]).

Proposition 7.1. *The module of syzygies of $(F_p, F_{p+1},\ldots,F_q)$ is generated by $\mathcal{R}_{p+1},\ldots,\mathcal{R}_q$.*

Proposition 7.2. *(Cf. prop. 8.8 in[10] or lemme 10.3.1 in[27]) Let M be a left $\mathcal{D}$-module which is finitely generated as $\mathcal{O}$-module. Then it is free (of finite rank) as $\mathcal{O}$-module.*

Proof. We reproduce the proof of lemme 4 in.[2] Let $B = \{e_1,\ldots,e_p\}$ be a minimal system of generators of M as $\mathcal{O}$-module and let us write

$$\partial e_i = \sum_{j=1}^p v_{ij} e_j, \quad (v_{ij} \in \mathcal{O})\ \forall i = 1,\ldots,p.$$

Let $\mathbf{S}$ be the module of syzygies of B:

$$\mathbf{S} = \{\underline{u} = (u_1,\ldots,u_p) \in \mathcal{D}^p \mid \sum_i u_i e_i = 0\}.$$

If B is not a basis, then $\mathbf{S} \neq 0$ and we can define $\omega = \min\{\nu(\underline{u}) \mid \underline{u} \in \mathbf{S}, \underline{u} \neq 0\}$, where $\nu(\underline{u}) = \min\{\nu(u_i) \mid u_i \neq 0\}$. By Nakayama's lemma, the set of classes $\overline{B} = \{\overline{e}_1,\ldots,\overline{e}_p\}$ is a basis of the $(\mathcal{O}/\mathfrak{m} =)\mathbb{C}$-vector space $M/\mathfrak{m}M$ and so we have $\omega > 0$. Let $\underline{u} \in \mathbf{S}$ be a non-vanishing syzygy with $\nu(\underline{u}) = \nu(u_{j_0}) = \omega$. We have

$$0 = \partial \sum_{i=1}^p u_i e_i = \cdots = \sum_{j=1}^p w_j e_j, \quad \text{with} \quad w_j = \partial(u_j) + \sum_{i=1}^p u_i v_{ij},$$

but $\nu(\partial(u_{j_0})) = \nu(u_{j_0}) - 1$ and so $\nu(w_{j_0}) = \omega - 1$, which contradicts the minimality of ω. □

Proposition 7.3. *Let $I \subset \mathcal{D}$ a non-zero left ideal with*

$$\mathrm{Exp}(I) = \bigcup_{d=p}^{q} \left((\alpha_d, d) + \mathbb{N}^2\right)$$

(see exercise 7.6), and $F_p, \ldots, F_q \in I$ a Gröbner basis of I. Then, the following properties hold:

(1) For any $A \in I$, there is an integer $r \geq 0$ such that $x^r A \in \mathcal{D}F_p$.
(2) $I = \mathcal{D}(F_p, F_q)$.

Proof. We reproduce the proof of prop. 5 in.[2] Part (1) is a starightforward consequence of corollary 7.1. For part (2), let us consider the left $\mathcal{D}$-module $M = I/\mathcal{D}(F_p, F_q)$. For any $A \in I$, there are unique elements $Q_p, \ldots, Q_{q-1} \in \mathcal{O}$, $Q_q \in \mathcal{D}$ such that $A = Q_p F_p + \cdots + Q_{q-1}F_{q-1} + Q_q F_q$, and so M is generated as $\mathcal{O}$-module by $\{F_{p+1}, \ldots, F_{q-1}\}$. But part (a) implies that M is a torsion $\mathcal{O}$-module, and so, from proposition 7.2, we deduce that $M = 0$. □

Let us note that the ring $\mathcal{D}$ is the inductive limit $\varinjlim_{R\to 0} \mathcal{D}(D_R)$.

Example 7.2. Let us see some examples of left $\mathcal{D}$-modules:
(1) $\mathcal{O}$ is a left $\mathcal{D}$-module, since $\mathcal{D}$ is a subring of $\mathrm{End}_{\mathbb{C}}(\mathcal{O})$ and then any $P \in \mathcal{D}$ acts on any $a \in \mathcal{O}$ by $Pa = P(a)$.
(2) To any linear differential operator $P \in \mathcal{D}$ we associate the left $\mathcal{D}$-module $\mathcal{D}/\mathcal{D}P$.
(3) The field $\mathcal{K}$ of fractions of $\mathcal{O}$ is a left $\mathcal{D}$-module.
(4) The formal power series ring $\widehat{\mathcal{O}} = \mathbb{C}[[z]]$ is a left $\mathcal{D}$-module. In fact the action of any $P \in \mathcal{D}$ on $\mathcal{O}$ is continuous for the $\mathfrak{m} = (z)$-adic topology.
(5) Since each $\mathcal{A}_R^0$ is a left $\mathcal{D}(D_R)$-module, $\mathcal{A}^0 := \varinjlim_{R\to 0} \mathcal{A}_R^0$ is a left $\mathcal{D}$-module, and the monodromy operator $T : \mathcal{A}^0 \xrightarrow{\sim} \mathcal{A}^0$ is $\mathcal{D}$-linear.
(6) $\mathcal{A} := \varinjlim_{R\to 0} \mathcal{A}_R$ is a left sub-$\mathcal{D}$-module of $\mathcal{A}^0$. The elements in $\mathcal{A}$ can be written as finite sums

$$\sum_{\alpha,k} \phi_{\alpha,k} z^{\alpha} (\mathrm{Log}\, z)^k$$

where the $\phi_{\alpha,k}$ are germs at 0 of holomorphic functions with a possibly essential singularity at 0, i.e.

$$\phi_{\alpha,k} \in \varinjlim_{R\to 0} \mathcal{O}(D_R^*).$$

(7) Prove that $T : \mathcal{A} \xrightarrow{\sim} \mathcal{A}$ induces an automorphism on $\mathcal{A}/\mathcal{O}$. Prove also that for any $\lambda \in \mathbb{C}$, the map $T - \lambda : \mathcal{A} \to \mathcal{A}$ is surjective (see exercise 4.3).
(8) $\mathcal{N} := \lim\limits_{R \to 0} \mathcal{N}_R$ is a left sub-$\mathcal{D}$-module of $\mathcal{A}$. The elements in $\mathcal{N}$ can be written as finite sums

$$\sum_{\alpha,k} \phi_{\alpha,k} z^{\alpha} (\operatorname{Log} z)^k$$

where the $\phi_{\alpha,k} \in \mathcal{K}$.

Let us denote by $\operatorname{Mod}(\mathcal{D})$ the abelian category of left $\mathcal{D}$-modules.

Exercise 7.8. (1) Prove that the $\mathcal{D}$-linear map $P \in \mathcal{D} \mapsto P(1) \in \mathcal{O}$ is surjective and its kernel is the left ideal generated by ∂. In particular $\mathcal{O} \simeq \mathcal{D}/\mathcal{D}\partial$.
(2) Prove that the $\mathcal{D}$-linear map $P \in \mathcal{D} \mapsto P(z^{-1}) \in \mathcal{K}$ is surjective and its kernel is generated by $z\partial + 1$. In particular $\mathcal{K} \simeq \mathcal{D}/\mathcal{D}(z\partial + 1)$.
(3) Prove that the $\mathcal{D}$-linear map $P \in \mathcal{D} \mapsto P(\overline{z^{-1}}) \in \mathcal{K}/\mathcal{O}$ is surjective and its kernel is generated by z. In particular $\mathcal{K}/\mathcal{O} \simeq \mathcal{D}/\mathcal{D}z$.
(4) Let $a \in \mathcal{O}$ be any non-zero element. Prove that $\mathcal{O} = \mathcal{D}a$ and compute a Gröbner basis of the left ideal $\operatorname{ann}_{\mathcal{D}} a$.

Definition 7.7. Let us denote by $\mathcal{M}^0, \mathcal{M}$ the left $\mathcal{D}$-modules

$$\mathcal{M}^0 = \mathcal{A}^0/\mathcal{O}, \quad \mathcal{M} = \mathcal{A}/\mathcal{O}.$$

The following proposition is a straightforward consequence of Cauchy theorem and Komatsu-Malgrange index theorem.

Proposition 7.4. *For any non-zero $P \in \mathcal{D}$, the following properties hold:*

(1) $\ker(P : \mathcal{A}^0 \to \mathcal{A}^0) = \ker(P : \mathcal{A} \to \mathcal{A})$ *and* $\dim_{\mathbb{C}} \ker(P : \mathcal{A}^0 \to \mathcal{A}^0) = \operatorname{ord}(P)$.
(2) The maps $P : \mathcal{A}^0 \to \mathcal{A}^0$ and $P : \mathcal{A} \to \mathcal{A}$ are surjective.
(3) The maps $P : \mathcal{M}^0 \to \mathcal{M}^0$ and $P : \mathcal{M} \to \mathcal{M}$ are surjective.
(4) $\ker(P : \mathcal{M}^0 \to \mathcal{M}^0) = \ker(P : \mathcal{M} \to \mathcal{M})$ *and* $\dim_{\mathbb{C}} \ker(P : \mathcal{M}^0 \to \mathcal{M}^0) = \nu(P)$.

Proof. Properties (1) and (2) are a simple translation of proposition 4.3. Property (3) is a consequence of property (2). For property (4), let us consider the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O} & \longrightarrow & \mathcal{A}^0 & \longrightarrow & \mathcal{M}^0 & \longrightarrow & 0 \\ & & {\scriptstyle P}\big\downarrow & & {\scriptstyle P}\big\downarrow & & {\scriptstyle P}\big\downarrow & & \\ 0 & \longrightarrow & \mathcal{O} & \longrightarrow & \mathcal{A}^0 & \longrightarrow & \mathcal{M}^0 & \longrightarrow & 0. \end{array}$$

From theorem 6.1 we know that $\chi(P : \mathcal{O} \to \mathcal{O}) = \operatorname{ord}(P) - \nu(P)$, and from (2) and (3) we deduce that $\dim_{\mathbb{C}} \ker(P : \mathcal{M}^0 \to \mathcal{M}^0) = \cdots = \operatorname{ord}(P) - (\operatorname{ord}(P) - \nu(P)) = \nu(P)$. A similar argument works for $\mathcal{M}$ instead of $\mathcal{M}^0$. □

For a left ideal $I \subset \mathcal{D}$, let us denote $E(I) = \{f \in \mathcal{A} | Pf = 0,\ \forall P \in I\}$ and $F(I) = \{g \in \mathcal{M} | Pg = 0,\ \forall P \in I\}$. The following proposition is taken from prop. 6 in,[2] and gives a very precise information about the spaces of solutions $E(I)$ and $F(I)$.

Proposition 7.5. *Let $I \subset \mathcal{D}$ be a non-zero left ideal and $F_p, \ldots, F_q$ a Gröbner basis of I. Then the following properties hold:*

(1) $E(I) = \ker(F_p : \mathcal{A} \to \mathcal{A})(= E(\mathcal{D}F_p))$.
(2) $F(I) = \ker(F_q : \mathcal{M} \to \mathcal{M})(= E(\mathcal{D}F_q))$.
(3) $\dim_{\mathbb{C}} E(I) = p(I)(= p = \operatorname{ord}(F_p))$, $\dim_{\mathbb{C}} F(I) = \nu(I)(= \nu(F_q))$.
(4) $P \in I \Leftrightarrow Pf = 0, \forall f \in E(I)$ and $Pg = 0, \forall g \in F(I)$.

Proof. Property (1) is a consequence of proposition 7.3, (1) and the fact that $\mathcal{A}$ has no $\mathcal{O}$-torsion.

For property (2), we only need to prove that any $g \in \mathcal{M}$ annihilated by F_q is annihilated by $F_p, \ldots, F_q$. We can assume that our Gröbner basis is normalized. Then, the definition of the syzygies $\mathcal{R}_d$ (see proposition 7.1) can be written in the following compact form:

$$\begin{pmatrix} \partial F_p \\ \partial F_{p+1} \\ \vdots \\ \partial F_{q-2} \\ \partial F_{q-1} \end{pmatrix} = A \begin{pmatrix} F_p \\ F_{p+1} \\ \vdots \\ F_{q-2} \\ F_{q-1} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ z^{\alpha_{q-1}-\alpha_q} F_q \end{pmatrix} \tag{7}$$

with

$$A = \begin{pmatrix} Q_p^{p+1} & z^{\alpha_p - \alpha_{p+1}} & 0 & \cdots & 0 & 0 \\ Q_p^{p+2} & Q_{p+1}^{p+2} & z^{\alpha_{p+1}-\alpha_{p+2}} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ Q_p^{q-1} & Q_{p+1}^{q-1} & Q_{p+2}^{q-1} & \cdots & Q_{q-2}^{q-1} & z^{\alpha_{q-2}-\alpha_{q-1}} \\ Q_p^q & Q_{p+1}^q & Q_{p+2}^q & \cdots & Q_{q-2}^q & Q_{q-1}^q \end{pmatrix},$$

which is a matrix with entries in $\mathcal{O}$. If $g = \overline{a} \in \ker(F_q : \mathcal{M} \to \mathcal{M})$, $a \in \mathcal{A}$, then $F_q(a) = b \in \mathcal{O}$ and so, by evaluating the equation (7) at a we obtain

$$\frac{d}{dz}\begin{pmatrix} F_p(a) \\ F_{p+1}(a) \\ \vdots \\ F_{q-2}(a) \\ F_{q-1}(a) \end{pmatrix} = A \begin{pmatrix} F_p(a) \\ F_{p+1}(a) \\ \vdots \\ F_{q-2}(a) \\ F_{q-1}(a) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ z^{\alpha_{q-1}-\alpha_q} b \end{pmatrix}$$

and $\frac{d}{dz}(F_i(a)) \in \mathcal{O}$ for $i = p, \ldots, q-1$. By Cauchy's theorem we deduce that $F_i(a) \in \mathcal{O}$ for $i = p, \ldots, q-1$ and so $\overline{a} \in F(I)$.

Property (3) is a consequence of (2) and proposition 7.4.

For the last property, let us call $J \subset \mathcal{D}$ the left ideal $\{P \in \mathcal{D} \mid Pf = 0, \forall f \in E(I), Pg = 0, \forall g \in F(I)\}$. It is clear that $I \subset J$. Let A be any element in J. By division, there are unique $Q, T, S \in \mathcal{D}$ such that $A = QF_q + T + S$ with

$$T = \sum_{k=q}^{\mathrm{ord}(A)} \sum_{l=0}^{v-1} r_{lk} x^l \partial^k, \quad \mathrm{ord}(S) < q = \mathrm{ord}(F_q)$$

and $v = \nu(I) = \nu(F_q)$. So, $R = T + S \in J$ and $E(I) \subset E(\mathcal{D}R)$, $F(I) \subset F(\mathcal{D}R$. In particular, by property (3) applied to the ideal $\mathcal{D}R$, we have $\mathrm{ord}(R) \geq p$ and $\nu(R) \geq v$ and so $T = 0$. Consequently the classes $\overline{\partial^l}$, $0 \leq l \leq q-1$, form a (finite) system of generators of the $\mathcal{O}$-module J/I. On the other hand, for any $A \in J$ there are $Q, U \in \mathcal{D}$ and an integer $r \geq 0$ such that $x^r A = QF_p + U$ and $\mathrm{ord}(U) < \mathrm{ord}(F_p) = p$ (see corollary 7.1). We deduce that $U \in J$ and $E(I) \subset E(\mathcal{D}U)$. Property (3) again shows that, if $U \neq 0$, $\mathrm{ord}(U) \geq \dim_{\mathbb{C}} E(I) = p$. So, $U = 0$ and J/I is a torsion $\mathcal{O}$-module. To conclude we apply proposition 7.2. □

Remark 7.1. Proposition 7.5 remains true if we replace $\mathcal{A}$ and $\mathcal{M}$ by $\mathcal{A}^0$ and $\mathcal{A}^0$ respectively.

Corollary 7.2. *Let $I \subset I' \subset \mathcal{D}$ be non-zero left ideals. The following properties are equivalent:*

(a) $I = I'$.
(b) $E(I) = E(I')$ *and* $F(I) = F(I')$.
(b) $\mathrm{p}(I) = \mathrm{p}(I')$ *and* $\nu(I) = \nu(I')$.
(c) $\mathrm{p}(I) + \nu(I) = \mathrm{p}(I') + \nu(I')$.

Proof. The equivalence (a) $\Leftrightarrow$ (b) comes from property (4) in proposition 7.5. The equivalence (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) comes from property (3) in proposition 7.5 and the obvious inclusions $E(I') \subset E(I)$, $F(I') \subset F(I)$. $\square$

Corollary 7.3. *For any non-zero left ideal $I \subset \mathcal{D}$, we have $\lg(\mathcal{D}/I) \leq p(I) + \nu(I)$, and in particular the left $\mathcal{D}$-module $\mathcal{D}/I$ is of finite length.*

Exercise 7.9. Prove that for any non integer complex number α, the left $\mathcal{D}$-module $\mathcal{D}/\mathcal{D}(z\partial - \alpha)$ is simple, i.e. the left ideal $\mathcal{D}(z\partial - \alpha)$ is maximal.

Corollary 7.4. *For any non-zero left ideal $I \subset \mathcal{D}$, the left $\mathcal{D}$-module $\mathcal{D}/I$ is a torsion module.*

Proof. Let us take $A \in \mathcal{D}$, $A \notin I$, and consider the $\mathcal{D}$-linear map $\Phi : P \in \mathcal{D} \mapsto \Phi(P) = P\overline{A} \in \mathcal{D}/I$. Since $\mathcal{D}z \supset \mathcal{D}z^2 \supset \mathcal{D}z^3 \supset \cdots$ is an infinite strictly decreasing sequence of left ideals in $\mathcal{D}$, we have $\lg(\mathcal{D}) = +\infty$ and the map Φ cannot be injective. So, there is a $P \in \mathcal{D}$, $P \neq 0$, such that $P\overline{A} = \overline{0}$. $\square$

8. Generalized Solutions

If we start from a linear differential equation as (1), we may be interested in searching its solutions, not only holomorphic functions, but possibly distributions, hyperfunctions, etc.

In order to make sense the sentence "y is a solution" of (1) what we need is that y is an element of certain space $\mathcal{S}$, g is also an element of the same space $\mathcal{S}$, and it makes sense the action of any linear differential operator on elements of $\mathcal{S}$. Algebraically that corresponds to the fact that $\mathcal{S}$ is a (left) $\mathcal{D}$-module.

The solutions of the homogeneous equation associated with (1) in the space $\mathcal{S}$ can be expressed simply as

$$\ker(P : \mathcal{S} \to \mathcal{S}) = \{y \in \mathcal{S} \mid Py = 0\}$$

where $P = a_n \frac{d^n}{dz^n} + \cdots + a_1 \frac{d}{dz} + a_0$. But it is clear that

$$y \in \ker(P : \mathcal{S} \to \mathcal{S}) \mapsto [\overline{Q} \in \mathcal{D}/\mathcal{D}P \mapsto Qy \in \mathcal{S}] \in \operatorname{Hom}_{\mathcal{D}}(\mathcal{D}/\mathcal{D}P, \mathcal{S}) \quad (8)$$

is an isomorphism of vector spaces, and then the solutions of the homogeneous equation can be expressed in some way in terms of the $\mathcal{D}$-module $\mathcal{D}/\mathcal{D}P$. On the other hand, the fact that the equation (1) has solutions for any $g \in \mathcal{S}$ exactly means that $\operatorname{im}(P : \mathcal{S} \to \mathcal{S}) = \mathcal{S}$, i.e. that $P : \mathcal{S} \to \mathcal{S}$ is surjective. Algebraically, the obstruction to this surjectivity is measured by the cokernel $\operatorname{coker}(P : \mathcal{S} \to \mathcal{S}) = \mathcal{S}/\operatorname{im}(P : \mathcal{S} \to \mathcal{S})$.

If instead of having one linear differential equation, we have a system

$$\begin{array}{ccccccc} P_{11}y_1 & + \cdots + & P_{1r}y_r & = & g_1 \\ \vdots & \vdots\ \vdots\ \vdots & \vdots & \vdots & \vdots \\ P_{m1}y_1 & + \cdots + & P_{mr}y_r & = & g_m \end{array} \tag{9}$$

we can consider the left sub-$\mathcal{D}$-module $I \subset \mathcal{D}^r$ generated by $\underline{P}^i = (P_{i1}, \ldots, P_{ir})$, $i = 1, \ldots, m$, and we have as above an isomorphism (for the solutions of the associated homogeneous system)

$$\ker(\mathbf{P} : \mathcal{S}^r \to \mathcal{S}^m) \simeq \mathrm{Hom}_{\mathcal{D}}(\mathcal{D}^r/I, \mathcal{S}),$$

where $\mathbf{P}$ is the matrix of linear differential operators (P_{ij}).

But if we are interested in the non-homogeneous system (9), it is not reasonable to try to solve it for any choice of $g_1, \ldots, g_m$, since the existence of a solution would imply that any time we have a syzygy $Q_1\underline{P}^1 + \cdots + Q_m\underline{P}^m = \underline{0}$, with $Q_i \in \mathcal{D}$, then $Q_1g_1 + \cdots + Q_mg_m = 0$. So, to measure the obstruction to solve (9), we have to look not at $\mathrm{coker}(\mathbf{P} : \mathcal{S}^r \to \mathcal{S}^m)$, but at

$$\{(g_1, \ldots, g_m) \in \mathcal{S}^m \mid \sum_i Q_ig_i = 0, \forall \underline{Q} \in S(\mathbf{P})\}/\operatorname{im}\mathbf{P}, \tag{10}$$

where

$$S(\mathbf{P}) = \{\underline{Q} \in \mathcal{D}^m \mid \sum_i Q_i\underline{P}^i = \underline{0}\}.$$

In fact, due to the noetherianity of $\mathcal{D}$, $S(\mathbf{P})$ is a finitely generated sub-$\mathcal{D}$-module of $\mathcal{D}^m$ and then the apparently infinite number of conditions

$$\sum_i Q_ig_i = 0, \forall \underline{Q} \in S(\mathbf{P}) \tag{11}$$

reduce to a finite number of them.

The question now is if it is possible to get an isomorphism of type (8) for $\mathrm{coker}(P : \mathcal{S} \to \mathcal{S})$, in the one equation case, or for (10) in the general case of a system of several equations with several unknowns.

The answer is YES and is given by Homological Algebra:

$$\{\underline{g} \in \mathcal{S}^m \mid \sum_i Q_ig_i = 0, \forall \underline{Q} \in S(\mathbf{P})\}/\operatorname{im}\mathbf{P} \simeq \mathrm{Ext}^1_{\mathcal{D}}(\mathcal{D}^r/I, \mathcal{S}), \tag{12}$$

where the $\mathrm{Ext}^i_{\mathcal{D}}(M, N)$ are complex vector spaces conveniently defined for any left $\mathcal{D}$-modules M, N and $i \geq 0$. For $i = 0$ we have

$$\mathrm{Ext}^0_{\mathcal{D}}(M, N) = \mathrm{Hom}_{\mathcal{D}}(M, N)$$

(see the book[28]). In fact, the $\operatorname{Ext}^i_{\mathcal{D}}(M,N)$ appear as the cohomology of degree i of a certain complex of vector spaces

$$\boxed{\mathbb{R}\operatorname{Hom}_{\mathcal{D}}(M,N)}$$

which can be calculated by taking a projective resolution of M or an injective resolution of N (see for instance ch. II in[22] for a quick introduction to this subject and for a list of references).

Example 8.1. Let us see some examples.

(1) If $M=\mathcal{D}$, then it corresponds to the linear differential equation $0y=g$. Any element $y\in N$ is obviously a solution of the homogeneous equation, and the the compatibility conditions (11) mean that the g must be zero and the non-homogeneous equation must be actually homogenoeus, and then it always has solutions (the zero solution). In this case, since $\mathcal{D}$ is free as left $\mathcal{D}$-module, we have

$$\mathbb{R}\operatorname{Hom}_{\mathcal{D}}(\mathcal{D},N)=\operatorname{Hom}_{\mathcal{D}}(\mathcal{D},N)=\cdots\to 0\to \overset{0}{N}\to 0\to\cdots$$

and $\operatorname{Ext}^0_{\mathcal{D}}(\mathcal{D},N)=\operatorname{Hom}_{\mathcal{D}}(\mathcal{D},N)=N$ and $\operatorname{Ext}^i_{\mathcal{D}}(\mathcal{D},N)=0$ for $i\neq 0$.

(2) If $M=\mathcal{D}/\mathcal{D}P$, to describe $\mathbb{R}\operatorname{Hom}_{\mathcal{D}}(M,N)$ we take the free resolution of M

$$0\to \overset{-1}{\mathcal{D}}\xrightarrow{\cdot P}\overset{0}{\mathcal{D}}\to M=\mathcal{D}/\mathcal{D}P\to 0,$$

$$M^\bullet=\cdots\to 0\to \overset{-1}{\mathcal{D}}\xrightarrow{\cdot P}\overset{0}{\mathcal{D}}\to 0\to\cdots,$$

and

$$\begin{gathered}\mathbb{R}\operatorname{Hom}_{\mathcal{D}}(M,N)=\operatorname{Hom}_{\mathcal{D}}(M^\bullet,N)=\\ \cdots\to 0\to \overset{0}{\operatorname{Hom}_{\mathcal{D}}(\mathcal{D},N)}\xrightarrow{\operatorname{Hom}_{\mathcal{D}}(\cdot P,N)}\overset{1}{\operatorname{Hom}_{\mathcal{D}}(\mathcal{D},N)}\to 0\to\cdots=\\ \cdots\to 0\to \overset{0}{N}\xrightarrow{P}\overset{1}{N}\to 0\to\cdots.\end{gathered}$$

In particular,

$$\operatorname{Hom}_{\mathcal{D}}(M,N)=\operatorname{Ext}^0_{\mathcal{D}}(M,N)=h^0\mathbb{R}\operatorname{Hom}_{\mathcal{D}}(M,N)=\ker(P:N\to N),$$

$$\operatorname{Ext}^1_{\mathcal{D}}(M,N)=h^1\mathbb{R}\operatorname{Hom}_{\mathcal{D}}(M,N)=\operatorname{coker}(P:N\to N)$$

and $\operatorname{Ext}^i_{\mathcal{D}}(M,N)=0$ for all $i\neq 0,1$.

(3) If N is an injective (see for instance the book[28]) $\mathcal{D}$-module, we can solve any compatible system with unknowns in N and

$$\mathbb{R}\operatorname{Hom}_{\mathcal{D}}(M,N) = \operatorname{Hom}_{\mathcal{D}}(M,N),$$

i.e. $\operatorname{Ext}^i_{\mathcal{D}}(M,N) = 0$ for all $i \neq 0$.

Definition 8.1. If M is a finitely generated left $\mathcal{D}$-module, we define its *higher holomorphic solutions* as the complex of vector spaces

$$\operatorname{Sol} M = \mathbb{R}\operatorname{Hom}_{\mathcal{D}}(M,\mathcal{O}).$$

The proof of the following proposition is an interesting application of the division tools in the ring $\mathcal{D}$ and gives a "natural" injective resolution of the left $\mathcal{D}$-module $\mathcal{O}$.

Proposition 8.1. *The following exact sequence of left $\mathcal{D}$-modules*

$$0 \to \mathcal{O} \to \mathcal{A} \to \mathcal{M} \to 0$$

is an injective resolution of $\mathcal{O}$ as a left $\mathcal{D}$-module.

Proof. To prove that $\mathcal{A}$ is an injective $\mathcal{D}$-module, we have to check that for any left ideal $I \subset \mathcal{D}$ and any $\mathcal{D}$-linear map $\varphi : I \to \mathcal{A}$ there exists a $\mathcal{D}$-linear map $\widetilde{\varphi} : \mathcal{D} \to \mathcal{A}$ such that $\widetilde{\varphi}|_I = \varphi$ (see any book of Homological Algebra, for instance[28]). Let us take a Gröbner basis $F_p, \ldots, F_q$ of I. We know from the proposition 7.3, (2) that $I = \mathcal{D}(F_p, F_q)$. Let us write $\varphi(F_p) = f_p, \varphi(F_q) = f_q$. Finding $\widetilde{\varphi}$ is the same as finding $f = \widetilde{\varphi}(1)$, since $\widetilde{\varphi}(P) = P\widetilde{\varphi}(1)$ for all $P \in \mathcal{D}$. On the other hand, the condition $\widetilde{\varphi}|_I = \varphi$ exactly means that $F_p f = f_p, F_q f = f_q$.

From proposition 7.3, (1) there exists an integer $r \geq 0$ and an operator $Q \in \mathcal{D}$ such that $x^r F_q = QF_p$, and so $x^r f_q = Qf_p$. From proposition 7.4, (2) there exists $f \in \mathcal{A}$ such that $F_p f = f_p$. We have $x^r F_q f = QF_p f = Qf_p = x^r f_q$, and since $\mathcal{A}$ has no $\mathcal{O}$-torsion we deduce that $F_q f = f_q$.

Let us now prove the injectivity of $\mathcal{M}$. Assume that $I \subset \mathcal{D}$ is a left ideal and $\psi : I \to \mathcal{M}$ is a $\mathcal{D}$-linear map. Take a normalized Gröbner basis $F_p, \ldots, F_q$ of I and let us write $\psi(F_d) = g_d = \overline{f_d}$, $d = p, \ldots, q$. From proposition 7.4, there exists $f \in \mathcal{A}$ such that $F_q f = f_q$ (and so $F_q g = g_q$ for $g = \overline{f} \in \mathcal{M}$). The generating system of the syzygies of $F_p, \ldots, F_q$ 7.1

gives rise to the relation (see (7) in the proof of proposition 7.5)

$$\begin{pmatrix} \partial F_p \\ \partial F_{p+1} \\ \vdots \\ \partial F_{q-2} \\ \partial F_{q-1} \end{pmatrix} = A \begin{pmatrix} F_p \\ F_{p+1} \\ \vdots \\ F_{q-2} \\ F_{q-1} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ z^{\alpha_{q-1}-\alpha_q} F_q \end{pmatrix}$$

with A a matrix with entries in $\mathcal{O}$. By applying ψ we find

$$\begin{pmatrix} \partial g_p \\ \partial g_{p+1} \\ \vdots \\ \partial g_{q-2} \\ \partial g_{q-1} \end{pmatrix} = A \begin{pmatrix} g_p \\ g_{p+1} \\ \vdots \\ g_{q-2} \\ g_{q-1} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ z^{\alpha_{q-1}-\alpha_q} g_q \end{pmatrix}$$

or

$$\frac{d}{dz} \begin{pmatrix} f_p \\ f_{p+1} \\ \vdots \\ f_{q-2} \\ f_{q-1} \end{pmatrix} = A \begin{pmatrix} f_p \\ f_{p+1} \\ \vdots \\ f_{q-2} \\ f_{q-1} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ z^{\alpha_{q-1}-\alpha_q} f_q \end{pmatrix} + \begin{pmatrix} h_p \\ h_{p+1} \\ \vdots \\ h_{q-2} \\ h_{q-1} \end{pmatrix},$$

where $h_d \in \mathcal{O}$ for $d = p, \ldots, q-1$, and

$$\frac{d}{dz} \begin{pmatrix} f_p - F_p f \\ f_{p+1} - F_{p+1} f \\ \vdots \\ f_{q-2} - F_{q-2} f \\ f_{q-1} - F_{q-1} f \end{pmatrix} = A \begin{pmatrix} f_p - F_p f \\ f_{p+1} - F_{p+1} f \\ \vdots \\ f_{q-2} - F_{q-2} f \\ f_{q-1} - F_{q-1} f \end{pmatrix} + \begin{pmatrix} h_p \\ h_{p+1} \\ \vdots \\ h_{q-2} \\ h_{q-1} \end{pmatrix}.$$

By Cauchy theorem we deduce that $f_d - F_d f \in \mathcal{O}$ for $d = p, \ldots, q-1$ and so $F_d g = g_d$ for $d = p, \ldots, q-1$. The extension of ψ is given by $\widetilde{\psi} : P \in \mathcal{D} \mapsto Pg \in \mathcal{M}$. ☐

Example 8.2. We can use the injective resolution of proposition 8.1 to compute the higher holomorphic solutions of any left $\mathcal{D}$-module M:

$$\mathbb{R}\operatorname{Hom}_{\mathcal{D}}(M, \mathcal{O}) = \cdots \to 0 \to \overset{0}{\operatorname{Hom}_{\mathcal{D}}(M, \mathcal{A})} \to \overset{1}{\operatorname{Hom}_{\mathcal{D}}(M, \mathcal{M})} \to 0 \to \cdots.$$

Exercise 8.1. By taking the free resolutions in exercise 7.8 and the injective resolution of $\mathcal{O}$ given in proposition 8.1, compute in two different ways $\operatorname{Sol} M$ for: (1) $M = \mathcal{O}$. (2) $M = \mathcal{K}$. (3) $M = \mathcal{K}/\mathcal{O}$.

Exercise 8.2. Let $P = \partial^n + a_{n-1}\partial^{n-1} + \cdots + a_1\partial + a_0 \in \mathcal{D}$ and let us consider the left $\mathcal{D}$-module $M = \mathcal{D}/\mathcal{D}P$ associated with the (germ of) linear differential equation $Py = g$, where $g \in \mathcal{S}$. Let us also consider the system of (germs of) linear differential equations (see (3))

$$\mathbf{P}\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ g \end{pmatrix}$$

with

$$\mathbf{P} = \begin{pmatrix} \partial & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \partial & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \partial & -1 & 0 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-3} & -a_{n-2} & \partial - a_{n-1} \end{pmatrix}$$

and the associated $\mathcal{D}$-module $M' = \mathcal{D}^n/I$, where I is the left submodule of $\mathcal{D}^n$ generated by

$$\begin{array}{rcc} \underline{P}^1 & = & (\partial, -1, 0, \ldots, 0, 0, 0) \\ \underline{P}^2 & = & (0, \partial, -1, \ldots, 0, 0, 0) \\ \vdots & \vdots & \vdots \\ \underline{P}^{n-1} & = & (0, 0, 0, \ldots, \partial, -1, 0) \\ \underline{P}^n & = & (-a_0, -a_1, -a_2, \ldots, -a_{n-3}, -a_{n-2}, \partial - a_{n-1}). \end{array}$$

Prove that the map

$$\overline{(Q_0, \ldots, Q_{n-1})} \in M' = \mathcal{D}^n/I \mapsto \overline{\sum_{i=0}^{n-1} Q_i \partial^i} \in M = \mathcal{D}/\mathcal{D}P$$

is an isomorphism of left $\mathcal{D}$-modules, and so

$$\mathbb{R}\operatorname{Hom}_{\mathcal{D}}(M, \mathcal{S}) \simeq \mathbb{R}\operatorname{Hom}_{\mathcal{D}}(M', \mathcal{S}).$$

In the above exercise, the isomorphism $M \simeq M'$ is the algebraic counterpart of the classic reduction of an order n linear differential equation to an order 1 system of linear differential equations described in section 1.

9. Holonomic $\mathcal{D}$-Modules

In this section, all $\mathcal{D}$-modules considered will be left $\mathcal{D}$-modules.

Definition 9.1. We say[f] that a $\mathcal{D}$-module M is *holonomic* if it is finitely generated and a torsion module, i.e. for all $m \in M$ there is $P \in \mathcal{D}$, $P \neq 0$, such that $Pm = 0$.

It is clear that any submodule and any quotient of a holonomic $\mathcal{D}$-module is also holonomic, and that the direct sum of two holonomic $\mathcal{D}$-modules is again holonomic. In particular the category of holonomic $\mathcal{D}$-modules is abelian.

Let us denote by $\mathrm{Hol}(\mathcal{D})$ the (abelian) category of holonomic (left) $\mathcal{D}$-modules.

Example 9.1. Any $\mathcal{D}$-module of type $\mathcal{D}/I$, where $I \subset \mathcal{D}$ is a non-zero ideal, is holonomic after corollary 7.4.

In fact we have the following result.

Proposition 9.1. *Let M be a $\mathcal{D}$-module. The following properties are equivalent:*

(a) M is holonomic.
(b) M is of finite length.
(c) There is a non-zero ideal $I \subset \mathcal{D}$ such that $M \simeq \mathcal{D}/I$.

Proof. For (a) $\Rightarrow$ (b) we proceed by induction on the number of generators of M. If $M = \mathcal{D}m_1$ is cyclic, then $I = \mathrm{ann}_{\mathcal{D}}(m_1) \neq 0$ and $M \simeq \mathcal{D}/I$ is of finite length by corollary 7.3.

Assume that any holonomic $\mathcal{D}$-module generated by $n-1$ elements is of finite length and take a holonomic $\mathcal{D}$-module $M = \mathcal{D}(m_1, \ldots, m_n)$ generated by n elements. By induction hypothesis $M' = \mathcal{D}(m_2, \ldots, m_n)$ and $M'' = M/M' = \mathcal{D}\overline{m_1}$ are of finite length, and so M is also of finite length.

The implication (b) $\Rightarrow$ (c) follows from from a general result, which assures that any left module of finite length over a simple ring R of infinite length as left R-module is cyclic (cf. 5.7.3 in[18]).

[f]Holonomic $\mathcal{D}$-modules make sense in several variables, but their definition needs to work with filtrations (see the notes[3]). The present definition only works in one variable.

The implication (c) $\Rightarrow$ (a) is a consequence of corollary 7.4. $\square$

Theorem 9.1. *Let M be a holonomic $\mathcal{D}$-module. Then $\operatorname{Sol} M$ is a complex of vector space with finite dimensional cohomology. More precisely:*

$$\dim_{\mathbb{C}} h^0 \operatorname{Sol} M = \dim_{\mathbb{C}} \operatorname{Hom}_{\mathcal{D}}(M, \mathcal{O}) < +\infty,$$

$$\dim_{\mathbb{C}} h^1 \operatorname{Sol} M = \dim_{\mathbb{C}} \operatorname{Ext}^1_{\mathcal{D}}(M, \mathcal{O}) < +\infty,\ h^i \operatorname{Sol} M = 0\ \forall i \neq 0, 1.$$

Proof. From proposition 9.1, we know that $M \simeq \mathcal{D}/I$, where $I \subset \mathcal{D}$ is a non-zero left ideal. On the other hand, $\operatorname{Sol} M$ can be computed as (see example 8.2)

$$\cdots \to 0 \to \overset{0}{\operatorname{Hom}_{\mathcal{D}}(M, \mathcal{A})} \to \overset{1}{\operatorname{Hom}_{\mathcal{D}}(M, \mathcal{M})} \to 0 \to \cdots,$$

but $\operatorname{Hom}_{\mathcal{D}}(M, \mathcal{A}) \simeq \operatorname{Hom}_{\mathcal{D}}(\mathcal{D}/I, \mathcal{A}) \simeq E(I)$ and $F(I) \simeq \operatorname{Hom}_{\mathcal{D}}(M, \mathcal{M}) \simeq \operatorname{Hom}_{\mathcal{D}}(\mathcal{D}/I, \mathcal{M})$. So, the theorem is a consequence of proposition 7.5. $\square$

Remark 9.1. For a holonomic $\mathcal{D}$-module it is relatively easy to give a formula for

$$\chi(\mathbb{R} \operatorname{Hom}_{\mathcal{D}}(M, \mathcal{O})) = \dim_{\mathbb{C}} \operatorname{Hom}_{\mathcal{D}}(M, \mathcal{O}) - \dim_{\mathbb{C}} \operatorname{Ext}^1_{\mathcal{D}}(M, \mathcal{O})$$

in terms of two integers algebraically associated with M: the multiplicity e_0 of the "null section" and the multiplicity e_1 of the "conormal of 0 in the "characteristic variety" defined by means of filtrations and the theory of Hilbert polynomials (cf. ch. V in[6]). When $M = \mathcal{D}/I$, then $e_0 = p(I)$ and $e_1 = \nu(I)$

$$\chi(\mathbb{R} \operatorname{Hom}_{\mathcal{D}}(\mathcal{D}/I, \mathcal{O})) = \dim_{\mathbb{C}} \operatorname{Hom}_{\mathcal{D}}(\mathcal{D}/I, \mathcal{A}) - \dim_{\mathbb{C}} \operatorname{Hom}_{\mathcal{D}} /\mathcal{D}/I, \mathcal{M}) =$$
$$\dim_{\mathbb{C}} E(I) - \dim_{\mathbb{C}} F(I) = p(I) - \nu(I).$$

10. Regular $\mathcal{D}$-Modules

In this section, all $\mathcal{D}$-modules considered will be left $\mathcal{D}$-modules.

Definition 10.1. Let

$$P = \sum_{k=0}^{n} a_k \partial^k = \sum_{k=0}^{n} \sum_{l=0}^{\infty} a_{lk} x^l \partial^k$$

be a non-zero linear differential operator (of $\mathcal{O}$) of order $n \geq 0$.
(1) We say that P is *regular* if it satisfies property (b) of theorem 5.1, i.e.

$$\max\{k - \nu_0(a_k) \mid k = 0, \ldots, n\} = n - \nu_0(a_n).$$

(2) We define the *weight* of P as

$$\mathrm{w}(P) = \max\{k - l \mid (l,k) \in \mathrm{supp}(P)\} = \max\{k - \nu_0(a_k) \mid k = 0,\dots,n\}.$$

(3) The *initial form* of P is the operator

$$\mathrm{in}(P) = \sum_{k-l=\mathrm{w}(P)} a_{lk} x^l \partial^k.$$

Let us note that P is regular if and only if $\mathrm{w}(P) = \mathrm{ord}(P) - \nu(P)$.

Exercise 10.1. Prove that, if $P_1, P_2 \in \mathcal{D}$ are non-zero linear differential operators, then:
(a) $\mathrm{w}(P_1P_2) = \mathrm{w}(P_1) + \mathrm{w}(P_2)$.
(b) $\mathrm{in}(P_1P_2) = \mathrm{in}(P_1)\,\mathrm{in}(P_2)$.
(c) Prove that P_1P_2 is regular if and only if P_1 and P_2 are regular.

Theorem 5.1 can be rephrased in the following way: Let $L = a_n \frac{d^n}{dz^n} + \cdots + a_1 \frac{d}{dz} + a_0$ be a linear differential operator of order n on an open disc $D = D_R$ and let $P = L_0 \in \mathcal{D}$ be its stalk at the origin. The following properties are equivalent:

(a) 0 is a regular singular point of L.
(b) P is regular.

Theorem 10.1. *Let $I \subset \mathcal{D}$ be a non-zero left $\mathcal{D}$-ideal. The following properties are equivalent:*

(a) There is a regular element $P \in I$, $P \neq 0$.
(b) All the elements of a Gröbner basis of I are regular.
(c) $E(I) \subset \mathcal{N}$.
(d) $F(I) \subset \mathcal{N}/\mathcal{O}$.
(e) $\{\eta \in \widehat{\mathcal{O}}/\mathcal{O} \mid P\eta = \overline{0}, \forall P \in I\} = 0$.

Proof. (See II.3.1 in[13]) The equivalence of the first three properties comes from proposition 7.3, (1), exercise 10.1, proposition 7.5, (1) and theorem 5.1.

Let $\{F_p, \dots, F_q\}$ be a Gröbner basis of I. We know from proposition 7.5, (2) that $F(I) = F(\mathcal{D}F_q)$.
(b) $\Rightarrow$ (d): Let $g = \overline{f} \in \mathcal{M}$ be a class in $F(I)$, i.e. $F_q f \in \mathcal{O}$. We can find a non singular operator $P \in \mathcal{D}$ ($\nu(P) = 0$) such that $PF_q f = 0$, and so, by theorem 5.1 $f \in \mathcal{N}$ and $g \in \mathcal{N}/\mathcal{O}$.

(d) $\Rightarrow$ (a): If $f \in \mathcal{A}$ is annihilated by F_q, then $\overline{f} \in F(I)$ and so $f \in \mathcal{N}$. Hence F_q is regular by theorem 5.1.

Since $\widehat{\mathcal{O}}/\mathcal{O}$ has no $\mathcal{O}$-torsion, we can follow the proof of proposition 7.5, (1) to prove that

$$\{\eta \in \widehat{\mathcal{O}}/\mathcal{O} \mid P\eta = \overline{0}, \forall P \in I\} = \ker\left(\widetilde{F_p} : \widehat{\mathcal{O}}/\mathcal{O} \to \widehat{\mathcal{O}}/\mathcal{O}\right).$$

So, the equivalence (e) $\Leftrightarrow$ (a) is a consequence of corollary 6.1. $\square$

Remark 10.1. Let us note that for any holonomic $\mathcal{D}$-module M, the vector space $\operatorname{Hom}_{\mathcal{D}}(M, \widehat{\mathcal{O}}/\mathcal{O})$ is finite dimensional. For that, it is enough to consider the case where $M = \mathcal{D}/I$ with $I \subset \mathcal{D}$ a non-zero left ideal. In such a case we have an isomorphism

$$\operatorname{Hom}_{\mathcal{D}}(\mathcal{D}/I, \widehat{\mathcal{O}}/\mathcal{O}) \simeq \{\eta \in \widehat{\mathcal{O}}/\mathcal{O} \mid P\eta = 0, \forall P \in I\},$$

but the last space is finite dimensional by theorem 6.2, (1).

Definition 10.2. (1) Let M be a holonomic $\mathcal{D}$-module. We define its *irregularity* as the number $\operatorname{irr} M := \dim_{\mathbb{C}} \operatorname{Hom}_{\mathcal{D}}(M, \widehat{\mathcal{O}}/\mathcal{O}) \geq 0$.

(2) We say that a holonomic $\mathcal{D}$-module M is *regular* if $\operatorname{Hom}_{\mathcal{D}}(\mathcal{D}/I, \widehat{\mathcal{O}}/\mathcal{O}) = 0$, or equivalently, if $\operatorname{irr} M = 0$.

Proposition 10.1. *The $\mathcal{D}$-module $\widehat{\mathcal{O}}/\mathcal{O}$ is injective.*

Proof. The proof follows the same lines as the proof of the injectivity of $\mathcal{A}$ in proposition 8.1, since $\widehat{\mathcal{O}}/\mathcal{O}$ has no $\mathcal{O}$-torsion either and we can use theorem 6.2, (2) instead of proposition 7.4, (2). $\square$

The following theorem is a straightforward consequence of theorem 10.1 and proposition 10.1.

Theorem 10.2. *Let M be a holonomic left $\mathcal{D}$-module. The following properties are equivalent:*

(a) *M is regular.*
(b) $\mathbb{R}\operatorname{Hom}_{\mathcal{D}}(M, \widehat{\mathcal{O}}/\mathcal{O}) = 0$.
(c) *The map* $\operatorname{Hom}_{\mathcal{D}}(M, \mathcal{N}) \to \operatorname{Hom}_{\mathcal{D}}(M, \mathcal{A})$ *induced by the inclusion* $\mathcal{N} \subset \mathcal{A}$ *is an isomorphism.*
(d) *The map* $\operatorname{Hom}_{\mathcal{D}}(M, \mathcal{N}/\mathcal{O}) \to \operatorname{Hom}_{\mathcal{D}}(M, \mathcal{A}/\mathcal{O})$ *induced by the inclusion* $\mathcal{N}/\mathcal{O} \subset \mathcal{A}/\mathcal{O}$ *is an isomorphism.*

The proof of the following proposition is also a straightforward consequence of proposition 10.1.

Proposition 10.2. *The* irregularity irr *is an additive function on exact sequences of holonomic* $\mathcal{D}$*-modules.*

Corollary 10.1. *Given a short exact sequence of holonomic* $\mathcal{D}$*-modules*

$$0 \to M' \to M \to M'' \to 0,$$

M *is regular if and only if* M' *and* M'' *are regular. In particular the category of holonomic* $\mathcal{D}$*-modules is abelian.*

Let us denote by $\mathrm{RegHol}(\mathcal{D})$ the (abelian) category of regular holonomic (left) $\mathcal{D}$-modules.

Remark 10.2. The above results are the precursors of the *irregularity complexes* along a hypersurface and the notion of regular holonomic module in higher dimension (see the papers[20,21]).

Additional results and information about regular an irregular holonomic $\mathcal{D}$-modules in one variable can be found in the paper.[29]

11. A Local Version of the Riemann-Hilbert Correspondence in One Variable (in Collaboration with F. Gudiel Rodríguez)

In this section we explain proposition III.4.5 in[13] using the description of simple objects in the category $\mathcal{C}$ instead of the more involved description of indecomposable objects (see the master thesis[7]).

Definition 11.1. Let us call $\mathcal{C}^0$ the category defined in the following way:

(1) The objets of $\mathcal{C}^0$ are the diagrams (E, F, u, v) where E, F are complex vector spaces and $u : E \to F$ and $v : F \to E$ are linear maps such that $Id_E + v \circ u$ and $Id_F + u \circ v$ are automorphisms.
(2) If $O = (E, F, u, v)$, $O' = (E', F', u', v')$ are objets of $\mathcal{C}^0$, a morphism in $\mathcal{C}^0$ from O to O' is a pair (a, b) of linear maps $a : E \to E'$, $b : F \to F'$ such that $u' \circ a = b \circ u$, $v' \circ b = a \circ v$.

We also call $\mathcal{C}$ the full subcategory of $\mathcal{C}^0$ whose objects are those (E, F, u, v) with $\dim_{\mathbb{C}} E, \dim_{\mathbb{C}} F < +\infty$.

Definition 11.2. The functors $\mathbb{E}_0 : \mathrm{Hol}(\mathcal{D}) \to \mathcal{C}$, $\mathbb{E} : \mathrm{RegHol}(\mathcal{D}) \to \mathcal{C}$ are defined on objects by

$$\mathbb{E}_0(M) = (\mathrm{Hom}_{\mathcal{D}}(M, \mathcal{A}), \mathrm{Hom}_{\mathcal{D}}(M, \mathcal{M}), U_*, V_*),$$
$$\mathbb{E}(M) = (\mathrm{Hom}_{\mathcal{D}}(M, \mathcal{N}), \mathrm{Hom}_{\mathcal{D}}(M, \mathcal{N}/\mathcal{O}), U_*, V_*)$$

and on morphisms in the obvious way.

Proposition 11.1. *Functors* $\mathbb{E}_0 : \mathrm{Hol}(\mathcal{D}) \to \mathcal{C}$ *and* $\mathbb{E} : \mathrm{RegHol}(\mathcal{D}) \to \mathcal{C}$ *are exact.*

Proof. The exactness of $\mathbb{E}_0$ follows from proposition 8.1. The exactness of $\mathbb{E}$ comes from the fact that for a regular holonomic $\mathcal{D}$-module, the canonical inclusion $\mathbb{E}(M) \hookrightarrow \mathbb{E}_0(M)$ is an isomorphism (see theorem 10.2). $\square$

Exercise 11.1. (1) Prove that $\mathcal{C}^0$ is an abelian category. Which are the monomorphisms and the epimorphisms in $\mathcal{C}^0$?

(2) Prove that $\mathcal{C}$ is an abelian subcategory of $\mathcal{C}^0$. Prove that any object in $\mathcal{C}$ has finite length.

(3) Prove that the simple objects in $\mathcal{C}$ are isomorphic to one of the following types: (i) $(\mathbb{C}, 0, 0, 0)$; (ii) $(0, \mathbb{C}, 0, 0)$; (iii) $(\mathbb{C}, \mathbb{C}, 1, \lambda)$, $\lambda \neq -1, 0$.

Definition 11.3. We define an "universal" object $\mathcal{U}^0$ in $\mathcal{C}^0$ as $\mathcal{U}^0 = (\mathcal{A}, \mathcal{M}, U, V)$ with $U : \mathcal{A} \to \mathcal{M}$ the projection map and $V : \mathcal{M} \to \mathcal{A}$ the "variation" map defined as $V(\overline{f}) = T(f) - f$. This object contains another special object $\mathcal{U} = (\mathcal{N}, \mathcal{N}/\mathcal{O}, U, V)$.

The object $\mathcal{U}^0$ is enriched with a (left) $\mathcal{D}$-module structure, since $\mathcal{A}$ and $\mathcal{M}$ are left $\mathcal{D}$-modules and U, V are $\mathcal{D}$-linear. So, for any object $O = (E, F, u, v)$ in $\mathcal{C}$, the abelian group $\mathrm{Hom}_{\mathcal{C}}(O, \mathcal{U}^0)$ carries a natural structure of left $\mathcal{D}$-module given by the following operation: for $P \in \mathcal{D}$ and $(a, b) \in \mathrm{Hom}_{\mathcal{C}}(O, \mathcal{U}^0)$, $P(a, b)$ is defined as (Pa, Pb) where

$$Pa : x \in E \mapsto (Pa)(x) := P \cdot a(x) \in \mathcal{A},$$
$$Pb : y \in F \mapsto (Pb)(y) := P \cdot b(y) \in \mathcal{M}.$$

In that way we define a contravariant left exact additive functor

$$\mathbb{F}_0 = \mathrm{Hom}_{\mathcal{C}^0}(-, \mathcal{U}^0) : \mathcal{C} \to \mathrm{Mod}(\mathcal{D}).$$

Since $\mathcal{U}$ is also enriched with a (left) $\mathcal{D}$-module structure, we also have another contravariant left exact additive functor

$$\mathbb{F} = \mathrm{Hom}_{\mathcal{C}^0}(-, \mathcal{U}) : \mathcal{C} \to \mathrm{Mod}(\mathcal{D})$$

with $\mathbb{F} \subset \mathbb{F}_0$.

Exercise 11.2. Let O be a simple object in $\mathcal{C}$. Prove that:

(1) There is an injection $\iota : O \hookrightarrow \mathcal{U}$.

(2) For any injection $\iota : O \hookrightarrow \mathcal{U}$, the left $\mathcal{D}$-module $\mathbb{F}O$ is generated by ι.

(3) Let $\iota : O \hookrightarrow \mathcal{U}$ be an injection and $(E, F, U, V) = \operatorname{im} \iota \subset \mathcal{U}$. Let us define $I = \{P \in \mathcal{D} \mid Pf = 0, \forall f \in E, Pg = 0, \forall g \in F\}$. Prove that I is a left maximal ideal of $\mathcal{D}$ and $E(I) = E$, $F(I) = F$.
(4) $\mathbb{F}O$ is a simple regular holonomic left $\mathcal{D}$-module.
(Hint: Proceed following the three different types of simple objects in $\mathcal{C}$)

Proposition 11.2. *With the above notations, $\mathbb{F}O$ is a regular holonomic left $\mathcal{D}$-module for any object O of $\mathcal{C}$. Moreover,* $\lg \mathbb{F}O \leq \lg O$.

Proof. The proof goes easily by induction on the length of the object O. If O is simple, the result has been treated in exercise 11.2.

Assume that the proposition is true anytime that $\lg O < n$ and let O be an object in $\mathcal{C}$ with $\lg O = n$. We can find a short exact sequence $0 \to O' \xrightarrow{i} O \xrightarrow{p} O'' \to 0$ in $\mathcal{C}$ with $\lg O' = 1$ and $\lg O'' = n-1$. By applying $\mathbb{F}$ we obtain a left exact sequence of left $\mathcal{D}$-modules

$$0 \to \mathbb{F}O'' \xrightarrow{\mathbb{F}p} \mathbb{F}O \xrightarrow{\mathbb{F}i} \mathbb{F}O'.$$

By induction hypothesis $\mathbb{F}O'$ and $\mathbb{F}O''$ are regular holonomic with $\lg \mathbb{F}O' = 1$, $\lg \mathbb{F}O'' \leq n-1$, and so the image of $\mathbb{F}i$ is also regular holonomic and we conclude that $\mathbb{F}O$ is regular holonomic too (see corollary 10.1) with $\lg \mathbb{F}O = \lg \mathbb{F}O'' + \lg \mathbb{F}O' \leq n$. $\square$

As a consequence of the above proposition, we can consider the contravariant left exact additive functor

$$\mathbb{F} = \operatorname{Hom}_{\mathcal{C}^0}(-, \mathcal{U}) : \mathcal{C} \to \operatorname{RegHol}(\mathcal{D}).$$

Definition 11.4. For any regular holonomic $\mathcal{D}$-module, we define the map $\xi_M : M \to \mathbb{F}\mathbb{E}M = \operatorname{Hom}_{\mathcal{C}^0}(\mathbb{E}M, \mathcal{U})$ by $\xi_M(m) = (\xi^1_M(m), \xi^2_M(m))$ with

$$\xi^1_M(m) : \phi \in \operatorname{Hom}_{\mathcal{D}}(M, \mathcal{N}) \mapsto \phi(m) \in \mathcal{N},$$
$$\xi^2_M(m) : \psi \in \operatorname{Hom}_{\mathcal{D}}(M, \mathcal{N}/\mathcal{O}) \mapsto \psi(m) \in \mathcal{N}/\mathcal{O}.$$

Proposition 11.3. *The correspondence which associates to any regular holonomic $\mathcal{D}$-module M the map ξ_M is a morphism of functors $\xi : Id \to \mathbb{F}\mathbb{E}$. Moreover, ξ_M is injective for any regular holonomic $\mathcal{D}$-module M.*

Proof. The first part is clear. For the second part, we can restrict ourselves to the case $M = \mathcal{D}/I$, where $I \subset \mathcal{D}$ is a non-zero left ideal. In such a case, $\mathbb{E}M$ is canonically isomorphic to $O = (E(I), F(I), U, V) \subset \mathcal{U}$ and the map ξ_M can be seen as

$$\overline{P} \in \mathcal{D}/I \mapsto (P : E(I) \to \mathcal{N}, P : F(I) \to \mathcal{N}/\mathcal{O}) \in \mathbb{F}O.$$

The injectivity of ξ_M is so a consequence of proposition 7.5, (4). □

Lemma 11.1. *For any simple regular holonomic $\mathcal{D}$-module M, $\mathbb{E}M$ is a simple object in $\mathcal{C}$.*

Proof. We can assume that $M = \mathcal{D}/I$, with $I \subset \mathcal{D}$ a maximal left ideal and so $\mathbb{E}M$ is isomorphic to $O = (E(I), F(I), U, V)$. Let $O' = (E, F, U, V)$ be a simple sub-object of O and $J = \{P \in \mathcal{D} \mid Pf = 0, \forall f \in E, Pg = 0, \forall g \in F\}$. We know from exercise 11.2, (3) that $E(J) = E$, $F(J) = F$.

It is clear that J is a proper ideal containing I, and so $I = J$. We conclude that $O = O'$ and O is simple. □

Proposition 11.4. *For any regular holonomic $\mathcal{D}$-module M the map $\xi_M : M \to \mathbb{F}\mathbb{E}M$ is an isomorphism.*

Proof. We proceed by induction on the length of M. If M is simple, then $\mathbb{E}M$ is simple by lemma 11.1 and $\mathbb{F}\mathbb{E}M$ is simple by exercise 11.2, (4). So the injection $\xi_M : M \hookrightarrow \mathbb{F}\mathbb{E}M$ is an isomorphism.

Assume that ξ_M is an isomorphism anytime that $\lg M < n$ and let M be a regular holonomic $\mathcal{D}$-module of length n. Let us consider a short exact sequence of (regular holonomic left) $\mathcal{D}$-modules $0 \to M' \to M \to M'' \to 0$. We have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\
 & & \simeq \big\downarrow \xi_{M'} & & \big\downarrow \xi_M & & \simeq \big\downarrow \xi_{M''} & & \\
0 & \longrightarrow & \mathbb{F}\mathbb{E}M' & \longrightarrow & \mathbb{F}\mathbb{E}M & \longrightarrow & \mathbb{F}\mathbb{E}M'' & &
\end{array}$$

and so ξ_M is an isomorphism. □

Definition 11.5. For any object $O = (E, F, u, v)$ in $\mathcal{C}$, we define the map $\tau_O : O \to \mathbb{E}\mathbb{F}O = (\mathrm{Hom}_{\mathcal{D}}(\mathbb{F}O, \mathcal{N}), \mathrm{Hom}_{\mathcal{D}}(\mathbb{F}O, \mathcal{N}/\mathcal{O}), U_*, V_*)$ by $\tau_O = (\tau_O^1, \tau_O^2)$ with

$$\tau_O^1 : x \in E \mapsto \tau_O^1(x) \in \operatorname{Hom}_{\mathcal{D}}(\mathbb{F}O, \mathcal{N}),$$
$$\tau_O^1(x) : (a,b) \in \mathbb{F}O = \operatorname{Hom}_{\mathcal{C}^0}(O,\mathcal{U}) \mapsto a(x) \in \mathcal{N},$$
$$\tau_O^2 : y \in E \mapsto \tau_O^1(y) \in \operatorname{Hom}_{\mathcal{D}}(\mathbb{F}O, \mathcal{N}/\mathcal{O}),$$
$$\tau_O^2(y) : (a,b) \in \mathbb{F}O = \operatorname{Hom}_{\mathcal{C}^0}(O,\mathcal{U}) \mapsto b(y) \in \mathcal{N}/\mathcal{O}.$$

Exercise 11.3. Prove that the correspondence which associates to any object O of $\mathcal{C}$ the map τ_O is a morphism of functors $\tau : Id \to \mathbb{E}\mathbb{F}$.

Exercise 11.4. (1) Prove that for any objects $O \subset O'$ in $\mathcal{C}$ with O'/O simple, the induced map $\mathbb{F}O' \to \mathbb{F}O$ is surjective (Hint: Proceed following the three different types of simple objects in $\mathcal{C}$ and use exercise 7.2, (7)).

(2) Conclude that the functor $\mathbb{F} : \mathcal{C} \to \mathrm{RegHol}$ is exact.

Proposition 11.5. *For any object O in $\mathcal{C}$ the map $\tau_O : O \to \mathbb{E}\mathbb{F}O$ is an isomorphism.*

Proof. Thanks to the exactness of $\mathbb{F}$ (and $\mathbb{E}$), we can restrict ourselves to the case where O is simple, as in the proof of proposition 11.4.

Assume that O is simple. Let $\iota : O \hookrightarrow \mathcal{U}$ be an injection, $(E, F, U, V) = \operatorname{im} \iota \subset \mathcal{U}$, and $I = \{P \in \mathcal{D} \mid Pf = 0, \forall f \in E, Pg = 0, \forall g \in F\}$. It is easy to see that the map τ_O can be seen as the inclusion

$$(E, F, U, V) \hookrightarrow (E(I), F(I), U, V)$$

and so it is an isomorphism by exercise 11.2, (3). □

Proposition 11.4 and 11.5 can be summarized in the following theorem.

Theorem 11.1. *The functors*

$$\mathbb{E} : \mathrm{RegHol}(\mathcal{D}) \to \mathcal{C} \quad \textit{and} \quad \mathbb{F} : \mathcal{C} \to \mathrm{RegHol}(\mathcal{D})$$

are quasi-inverse contravariant equivalences of categories.

Remark 11.1. (1) Since $\mathcal{A}$ and $\mathcal{M}$ are in fact left modules over the ring $\mathcal{D}^\infty$ of germs at 0 of infinite order linear differential operators (cf.[23,26]), we can consider $\mathbb{F}_0$ as a functor from $\mathcal{C}$ to $\mathrm{Mod}(\mathcal{D}^\infty)$. One can prove that $\mathcal{A} = \mathcal{D}^\infty \otimes_{\mathcal{D}} \mathcal{N}$, $\mathcal{M} = \mathcal{D}^\infty \otimes_{\mathcal{D}} \mathcal{N}/\mathcal{O}$ and that $\mathcal{D}^\infty \otimes_{\mathcal{D}} \mathbb{F} \simeq \mathbb{F}_0$. Let us call $\mathrm{Hol}(\mathcal{D}^\infty)$ the full abelian subcategory[g] $\mathcal{D}^\infty \otimes_{\mathcal{D}} \mathrm{Hol}(\mathcal{D}) \subset \mathrm{Mod}(\mathcal{D}^\infty)$.

[g]One needs to use that the extension $\mathcal{D} \hookrightarrow \mathcal{D}^\infty$ is faithfully flat (cf. *loc. cit.*).

The functor $\mathbb{E}_0$ can be also extended to $\mathbb{E}_0 : \mathrm{Hol}(\mathcal{D}^\infty) \to \mathcal{C}$ and it is a quasi-inverse of $\mathbb{F}_0 : \mathcal{C} \to \mathrm{Hol}(\mathcal{D}^\infty)$.

(2) One can prove by elementary methods that the category $\mathcal{C}$ is equivalent to the category of germs at $0 \in \mathbb{C}$ of *perverse sheaves* (cf.[4,14,25]).

(1) and (2) are particular cases of the "full" Riemann-Hilbert correspondence in higher dimension (cf. 3.3 in[19] and the paper[21]).

12. D-Modules on a Riemann Surface

In this section, we briefly sketch some basic facts of the theory of D-modules on a Riemann surface. X will be a connected Riemann surface and $\mathcal{O}_X$ will denote its sheaf of holomorphic functions. It has the same properties as $\mathcal{O}_U$ in definition 2.1.

We also define the notion of subsheaf of $\mathcal{O}_X$ as in definition 2.2, and the notion of endomorphism of $\mathcal{O}_X$ as in definition 2.3.

We have a "generalized sheaf" in the sense that it is not a sheaf of functions, but a sheaf of rings, given in the following way: for any open set $U \subset X$ we define

$$\mathcal{H}om_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X)(U) := \mathrm{Hom}_{\mathbb{C}}(\mathcal{O}_U, \mathcal{O}_U).$$

The data $\mathcal{H}om_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X)$ satisfies the formal properties of the sheaves of holomorphic functions (see exercise 2.4). The reader can refer to the book[9] for the general notion of sheaf.

We have an injective morphism of sheaves of rings $\mathcal{O}_X \hookrightarrow \mathcal{H}om_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X)$.

To define the sheaf of (holomorphic linear) differential operators, we have to adapt definition 2.4, because on a Riemann surface we do not have global coordinates.

Definition 12.1. Let $U \subset X$ an open set. A linear differential operator on U is an endomorphism $L : \mathcal{O}_U \to \mathcal{O}_U$ which locally, on open sets U_i with local coordinate z_i there are holomorphic functions $a_0, \ldots, a_n$ on U_i (n may depends on i) such that

$$L|_{U_i} = a_n \frac{d^n}{dz_i^n} + \cdots + a_0.$$

The set of linear differential operators on U will be denoted by $\mathcal{D}_X(U)$.

Exercise 12.1. Prove that the data $\mathcal{D}_X$ is a subsheaf of non-commutative rings of $\mathcal{H}om_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X)$.

The stalk of $\mathcal{D}_X$ at a point $p \in X$, $\mathcal{D}_{X,p}$ is isomorphic to the ring $\mathcal{D}$ (by taking a local coordinate around p).

We have the filtration by the order at the level of sheaves $F^k\mathcal{D}_X$, $k \geq 0$. The graded sheaf $\operatorname{gr}_F \mathcal{D}_X$ is locally isomorphic to the sheaf of commutative rings $\mathcal{O}_X[\xi]$. The the sheaf $\mathcal{D}_X$ has an important property: it is a *coherent* sheaf of rings (cf. the paper[6]).

Definition 12.2. A left holonomic $\mathcal{D}_X$-module $\mathcal{M}$ is a left coherent $\mathcal{D}_X$-module such that $\mathcal{M}_p$ is a holonomic $\mathcal{D}_{X,p}$-module for each $p \in X$.

Alternatively, holonomicity can be defined by using local good filtrations at the sheaf level. In that way we define the characteristic variety $\operatorname{Ch}\mathcal{M}$ which is an analytic conical closed subset of the cotangent space T^*X, and a coherent left $\mathcal{D}_X$-module is holonomic if and only if $\dim \operatorname{Ch}\mathcal{M} = \dim X = 1$.

We can also define, for any left coherente $\mathcal{D}_X$-modules $\mathcal{M}, \mathcal{N}$, the sheaf of complex vector spaces $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{N})$. We also define

$$\mathcal{S}\mathrm{ol}(\mathcal{M}) = \mathbb{R}\,\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X),$$

in such a way that $\mathcal{S}\mathrm{ol}(\mathcal{M})_p = \operatorname{Sol}(\mathcal{M}_p)$ for each point $p \in X$.

Theorem 12.1. *Let $\mathcal{M}$ be a (left) holonomic $\mathcal{D}_X$-module. Then $\mathcal{S}\mathrm{ol}(M)$ is a* perverse *sheaf, i.e*
(1) $h^i\,\mathcal{S}\mathrm{ol}(M) = \mathcal{E}\mathrm{xt}^i_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) = 0$ for all $i \neq 0, 1$.
(2) There is a closed discrete set $\Sigma \subset X$ (the singular locus *of $\mathcal{M}$) such that:*

a) $h^0\,\mathcal{S}\mathrm{ol}(M)|_{X\setminus\Sigma} = \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)|_{X\setminus\Sigma}$ is a locally constant sheaf of finite rank.

b) $h^1\,\mathcal{S}\mathrm{ol}(M)|_{X\setminus\Sigma} = \mathcal{E}\mathrm{xt}^1_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)|_{X\setminus\Sigma} = 0$.

c) $h^i\,\mathcal{S}\mathrm{ol}(\mathcal{M})_p = h^i\,\operatorname{Sol}(\mathcal{M}_p)$ are finite dimensional spaces for $i = 0, 1$ and for each $p \in \Sigma$.

d) $h^0\,\mathcal{S}\mathrm{ol}(M)$ has no section supported by Σ (this is clear because locally we have $h^0\,\mathcal{S}\mathrm{ol}(M) \subset \mathcal{O}_X$, and there are no holomorphic functions supported by a discrete set).

The proof of the above theorem is a direct consequence of theorems 3.1 and 9.1. More details can be found, for instance, in the paper,[24] where it is given an elementary proof of the Riemann-Hilbert correspondence on a Riemann surface.

References

1. M.F. Atiyah and I.G. MacDonald. *Introduction to commutative algebra.* Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.
2. J. Briançon and Ph. Maisonobe. *Idéaux de germes d'opérateurs différentiels à une variable. Enseign. Math.*, 30 (1984), 7–38.
3. F. J. Castro Jiménez. *Modules over the Weyl Algebra.* This volume.
4. P. Deligne. Lettre à R. MacPherson. 1981.
5. E. Goursat. *Cours d'analyse mathématique, Tome II.* Gauthier-Villars, Paris, 1933.
6. J.M. Granger and Ph. Maisonobe. *A basic course on differential modules*, in[16] vol. I, 103–168.
7. F. Gudiel Rodríguez. *Sobre una variante local del problema de Riemann-Hilbert en una variable.* Tesis de Licenciatura, Facultad de Matemáticas, Universidad de Sevilla, October 1991.
8. O. Ince. *Ordinary Differential Equations.* Dover, New York, 1950.
9. B. Iversen. *Cohomology of Sheaves.* Universitext. Springer Verlag, Heidelberg, 1986.
10. N.M. Katz. *Nilpotent connections and the monodromy theorem: Applications of a result of Turritin. Inst. Hautes Études Sci. Publ. Math.*, 39 (1970), 175–232.
11. H. Komatsu. *On the index of ordinary differential operators. J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 18 (1971), 379–398.
12. Lê Dũng Tráng (editor). *Introduction à la théorie algébrique des systèmes différentiels.* Travaux en cours, 34. Hermann, Paris, 1988.
13. Ph. Maisonobe. *Germes de D-modules à une variable et leurs solutions*, in[12] 97–134.
14. Ph. Maisonobe. *Faisceaux pervers sur* $\mathbb{C}$ *relativement à* $\{0\}$ *et couple* $E \rightleftarrows F$, in[12] 135–146.
15. Ph. Maisonobe and L. Narváez Macarro (editors). *Éléments de la théorie des systèmes différentiels géométriques.* Cours du CIMPA, École d'été de Séville (1996). *Séminaires et Congrès*, vol. 8. Soc. Math. France, Paris, 2004.
16. Ph. Maisonobe and C. Sabbah (editors). *Eléments de la théorie des systèmes différentiels (vol. I, II).* Summer school at CIMPA, Nice, 1990. Travaux en cours, 45, 46. Hermann, Paris, 1993.
17. B. Malgrange. *Sur les points singuliers des équations différentielles. L'Enseig. Math.*, XX (1974), 147–176.
18. J.C. McConnell and J.C. Robson. *Noncommutative Noetherian Rings.* John Wiley & Sons, Chichester, 1987.
19. Z. Mebkhout. *Une autre équivalence de catégories. Compositio Math.*, 51 (1984), 63–88.
20. Z. Mebkhout. *Le théorème de comparaison entre cohomologies de De Rham d'une variété algébrique complexe et le théorème d'existence de Riemann. Inst. Hautes Études Sci. Publ. Math.*, 69 (1989), 47–89.
21. Z. Mebkhout. *Le théorème de positivité, le théorème de comparaison et le théorème d'existence de Riemann*, in[15] 165–308.

22. Z. Mebkhout and L. Narváez-Macarro. *Le théorème de constructibilité de Kashiwara*, in[16] vol. II, 47–98.
23. Z. Mebkhout and L. Narváez Macarro. *Le théorème de continuité de la division dans les anneaux d'opérateurs différentiels. J. Reine Angew. Math.*, 503 (1998), 193–236.
24. L. Narváez Macarro. *Systèmes différentiels linéaires sur une surface de Rieman*, in[12] 50–96.
25. L. Narváez-Macarro. Cycles évanescents et faisceaux pervers. II. Cas des courbes planes réductibles. In J.P. Brasselet, editor, *Singularities (Lille, 1991)*, volume 201 of *London Math. Soc. Lecture Note Ser.*, pages 285–323. Cambridge Univ. Press, Cambridge, 1994.
26. L. Narváez Macarro and A. Rojas León. *Continuous division of linear differential operators and faithful flatness of* $\mathcal{D}_X^\infty$ *over* $\mathcal{D}_X$, in[15] 129–148.
27. F. Pham. *Singularités des systèmes différentiels de Gauss-Manin*, volume 2 of *Progress in Math.* Birkhäuser Boston, Mass., 1979. With contributions by Lo Kam Chan, Philippe Maisonobe and Jean-Étienne Rombaldi.
28. J.J. Rotman. *Notes on Homological Algebra*. Math. Studies. Van Nostrand Reinhold Co., N.Y., 1970.
29. C. Sabbah. *Introduction to algebraic theory of linear systems of differential equations*, in[16] vol. I, 1–80.

MODULES OVER THE WEYL ALGEBRA

FRANCISCO J. CASTRO JIMÉNEZ

Department of Algebra
University of Seville
Campus Reina Mercedes
41012 Seville (Spain)
E-mail: castro@us.es

We develop some basic results on modules over the Weyl algebra including the existence of the Bernstein polynomial and the relationship between logarithmic modules and the so-called Logarithmic Comparison Theorem. We use computer algebra system `Macaulay 2` to explicitly compute some invariants arising in the whole subject.

Keywords: Weyl algebra; Linear Differential Operator; Characteristic variety; Holonomic module; Bernstein-Sato polynomial; Logarithmic derivation; Logarithmic differential form; Logarithmic A_n–module.

Introduction

These notes are an enlarged version of the lecture notes given at the *School on Algebraic Approach to Differential Equations* that took place at Bibliotheca Alexandrina (Alexandria, Egypt) from 12th to 24th November 2007.[a] The school was organized by the Mathematics Section of the ICTP (The Abdus Salam International Centre for Theoretical Physics, Trieste).

The content of these notes is the following. Section 1 is devoted to the definition of the complex Weyl algebra, the ring of linear differential operators with polynomial coefficients, and the study of different filtrations on the Weyl algebra and on left A_n–modules. The associated graded structures will be used in Section 2 to define the characteristic variety, the dimension and the multiplicity of a finitely generated A_n–module. In this section we prove the Bernstein's inequality and study the class of holonomic modules.

[a]Part of this material has been previously taught in the Aachen Summer School 2007 *Algorithmic D-module theory* that took place from 3rd to 7th September 2007, at the Söllerhaus in Kleinwalsertal (Austria).

One of the deepest result of this section, due to J. Bernstein, states that the A_n–module of rational functions, with poles along a hypersurface in $\mathbb{C}^n$, is holonomic. Following J. Bernstein we also introduce the Bernstein polynomial (also known as Bernstein-Sato polynomial) associated with a given polynomial $f \in \mathbb{C}[x] := \mathbb{C}[x_1, \ldots, x_n]$.

In Section 3 we consider the class of logarithmic A_n–modules. These modules are associated with a polynomial $f \in \mathbb{C}[x]$ and are defined by using differential operators of order 1 arising from the logarithmic derivations with respect to f.

Some parts of the lectures were devoted to algorithmic questions and explicit computations in the theory of finitely generated modules over the Weyl algebra. These explicit computations are possible because the theory of Groebner bases can be extended from the polynomial ring to the Weyl algebra. We have added in an Appendix some basic results on the Division Theorem for differential operators and the theory of Groebner bases in A_n and other rings of differential operators. Many of the used algorithms are due to T. Oaku and N. Takayama[37] and have been implemented in the `D-modules` package for the Computer Algebra system `Macaulay 2`.[28]

While giving the lectures and writing these notes we have supposed that the students and the readers have some familiarity with basic notions in Commutative Algebra and Algebraic Geometry. In particular they should have a good elementary knowledge of the theory of commutative rings and their modules, as contained for instance in the first three chapters of Atiyah-Macdonald.[4] They should also know the basic definitions and results in the theory of affine algebraic varieties at the level of the first chapter of R. Hartshorne.[30] This knowledge must include the Nullstellensatz and the theory of Hilbert functions and polynomials.

1. The Weyl Algebra

1.1. *Linear differential operators*

For the sake of simplicity we are going to consider the complex field $\mathbb{C}$ as base field. Nevertheless, in what follows many algebraic results also hold for any base field $\mathbb{K}$ of characteristic zero.

Let $n \geq 1$ be an integer number and $\mathbb{C}[x] = \mathbb{C}[x_1, \ldots, x_n]$ be the ring of polynomials in n variables with complex coefficients.

Let $\mathrm{End}_{\mathbb{C}}(\mathbb{C}[x])$ be the $\mathbb{C}$-algebra of endomorphisms of the $\mathbb{C}$-vector space $\mathbb{C}[x]$. As the product in this algebra is just the composition of endomorphisms then $\mathrm{End}_{\mathbb{C}}(\mathbb{C}[x])$ is a noncommutative ring with unit.

Let $f \in \mathbb{C}[x]$ be a polynomial. The multiplication by f

$$\mathbb{C}[x] \xrightarrow{\phi_f} \mathbb{C}[x]$$

defined by $\phi_f(g) = fg$ for all $g \in \mathbb{C}[x]$, is an endomorphism.

The unit of the ring $\mathrm{End}_{\mathbb{C}}(\mathbb{C}[x])$ is the identity map which coincides with ϕ_1. We will denote ϕ_1 just by 1 if no confusion is possible.

The partial derivative with respect to x_i

$$\mathbb{C}[x] \xrightarrow{\frac{\partial}{\partial x_i}} \mathbb{C}[x]$$

is also an endomorphism. We will denote ∂_i instead of $\frac{\partial}{\partial x_i}$ for $i = 1, \ldots, n$. So, for any $f \in \mathbb{C}[x]$ we have $\partial_i(f) = \frac{\partial f}{\partial x_i}$.

Definition 1.1. Let $n \geq 1$ be an integer. The *n-th complex Weyl algebra*, denoted by $A_n(\mathbb{C})$, is the subalgebra of $\mathrm{End}_{\mathbb{C}}(\mathbb{C}[x])$ generated by the endomorphisms

$$\phi_{x_1}, \ldots, \phi_{x_n}, \partial_1, \ldots, \partial_n.$$

We will adopt the convention $A_0(\mathbb{C}) = \mathbb{C}$ and we will simply write $A_n = A_n(\mathbb{C})$.

Remark 1.1. An element in A_n is nothing but a finite linear combination, with coefficients in $\mathbb{C}$, of words in the generators $\phi_{x_1}, \ldots, \phi_{x_n}, \partial_1, \ldots, \partial_n$. Each of these words must be identified with the corresponding endomorphism built up by composing the generators appearing in the word.

For any $f \in \mathbb{C}[x]$ we have

$$(\partial_i \circ \phi_{x_i})(f) = \partial_i(x_i f) = f + x_i \partial_i(f) = f + (\phi_{x_i} \circ \partial_i)(f).$$

This means that the equality $\partial_i \circ \phi_{x_i} = \phi_{x_i} \circ \partial_i + 1$ holds in $\mathrm{End}_{\mathbb{C}}(\mathbb{C}[x])$ and therefore in A_n. In particular A_n is a noncommutative ring (for $n \geq 1$).

More generally, for any $f, g \in \mathbb{C}[x]$ and $1 \leq i \leq n$ we have

$$(\partial_i \circ \phi_g)(f) = \partial_i(gf) = \partial_i(g)f + g\partial_i(f) = \phi_{\partial_i(g)}(f) + (\phi_g \circ \partial_i)(f).$$

That is: the equality $\partial_i \circ \phi_g = \phi_g \circ \partial_i + \phi_{\partial_i(g)}$ holds in $\mathrm{End}_{\mathbb{C}}(\mathbb{C}[x])$ and therefore in A_n. The last equality is known as Leibniz's rule.

Exercise 1.1. Prove that the following equalities hold in A_n:
$\partial_i \circ \phi_{x_j} = \phi_{x_j} \circ \partial_i$ for all $1 \leq i, j \leq n$ with $i \neq j$.
$\partial_i \circ \partial_j = \partial_j \circ \partial_i$ for all $1 \leq i \leq j \leq n$.
$\phi_{x_i} \circ \phi_{x_j} = \phi_{x_j} \circ \phi_{x_i}$ for all $1 \leq i \leq j \leq n$.

Proposition 1.1. *The map $\mathbb{C}[x] \longrightarrow A_n$ defined by $f \mapsto \phi_f$ is an injective morphism of rings (and of $\mathbb{C}$-algebras).*

Proof. The proof follows from the equalities

$$\phi_{f+g}(h) = (f+g)h = fh + gh = \phi_f(h) + \phi_g(h) = (\phi_f + \phi_g)(h)$$

and

$$\phi_{fg}(h) = (fg)h = f(gh) = \phi_f(\phi_g(h)) = (\phi_f \circ \phi_g)(h)$$

which hold for any $f, g, h \in \mathbb{C}[x]$. □

Notation 1.1. The notations above are not easy to use. Hence, we will simply write x_i instead of ϕ_{x_i}. This identification is justified by Proposition 1.1. We will also write PQ instead of $P \circ Q$ for the product in A_n.

For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ we will write $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ both for the monomial in $\mathbb{C}[x]$ and the corresponding element in A_n. We will also write $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} \in A_n$. By convention we have $x_i^0 = 1$ and $\partial_i^0 = 1$ for $i = 1, \ldots, n$.

An element $x^\alpha \partial^\beta \in A_n$ for $\alpha, \beta \in \mathbb{N}^n$ is called a monomial in A_n.

Proposition 1.2.

(1) Let $f \in \mathbb{C}[x]$ and $\beta \in \mathbb{N}^n$. The product $\partial^\beta f$ in A_n satisfies the equality

$$\partial^\beta f = \sum_{\sigma \ll \beta} \binom{\beta}{\sigma} \partial^\sigma(f) \partial^{\beta - \sigma}$$

where $\sigma \ll \beta$ stands for $\sigma_i \leq \beta_i$ for $i = 1, \ldots, n$, $\binom{\beta}{\sigma} = \frac{\beta!}{\sigma!(\beta-\sigma)!}$ and $\beta! = \beta_1! \cdots \beta_n!$

(2) If $\beta, \gamma \in \mathbb{N}^n$ then we have $\partial^\beta(x^\gamma) = \beta! \binom{\gamma}{\beta} x^{\gamma - \beta}$ where $\binom{\gamma}{\beta} = 0$ if the relation $\beta \ll \gamma$ doesn't hold.

(3) If $\alpha, \alpha', \beta, \beta' \in \mathbb{N}^n$ then we have

$$\begin{aligned} x^\alpha \partial^\beta x^{\alpha'} \partial^{\beta'} &= x^{\alpha + \alpha'} \partial^{\beta + \beta'} \\ &\quad + \sum_{\sigma \ll \beta, \sigma \ll \alpha', \sigma \neq 0} \sigma! \binom{\beta}{\sigma} \binom{\alpha'}{\sigma} x^{\alpha + \alpha' - \sigma} \partial^{\beta + \beta' - \sigma}. \end{aligned}$$

Proof. *(1)* It follows from the case $n = 1$ and the distributivity of the product with respect to the sum in A_n. For $n = 1$ (writing t and ∂_t instead of x_1 and ∂_1) the formula

$$\partial_t^j f = \sum_{k=0}^{j} \partial_t^k(f)\partial_t^{j-k}$$

can be proved by induction on j.

(2) The formula follows by induction on n. The case $n = 1$ can be proved by induction on β_1.

(3) It follows from *(1)* and *(2)*. □

Proposition 1.3. *The set of monomials $\mathcal{B} = \{x^\alpha \partial^\beta \mid \alpha, \beta \in \mathbb{N}^n\}$ is a basis of the $\mathbb{C}$-vector space A_n. Each nonzero element P in A_n can be written in an unique way as a finite sum*

$$P = \sum_{\alpha,\beta} p_{\alpha\beta} x^\alpha \partial^\beta$$

for some nonzero complex numbers $p_{\alpha\beta}$. Moreover, $P = \sum_\beta p_\beta(x)\partial^\beta$ with $p_\beta(x) = \sum_\alpha p_{\alpha\beta} x^\alpha$.

Proof. It is enough to prove the first statement since the second statement follows from it.

Any word in the generators $x_1, \ldots, x_n, \partial_1, \ldots, \partial_n$ is a product of monomials (see Remark 1.1). By Proposition 1.2 (3.) a product of monomials is a linear combination with coefficients in $\mathbb{C}$ of elements in $\mathcal{B}$. This proves that $\mathcal{B}$ is a generating system for the vector space A_n. Let us prove now that $\mathcal{B}$ is linearly independent. Let us consider

$$P = \sum p_{\alpha\beta} x^\alpha \partial^\beta$$

a non trivial linear combination of monomials in $\mathcal{B}$. Let $\beta' \in \mathbb{N}^n$ be the smallest element, with respect to the lexicographical order,[b] appearing as the exponent of ∂ in P. We have that $P(x^{\beta'}) = (\beta')!(\sum_\alpha p_{\alpha\beta'} x^\alpha)$ because if β' is strictly smaller than β in the lexicographic order then $\partial^\beta(x^{\beta'}) = 0$. Because of the choice of β' there exists $\alpha \in \mathbb{N}^n$ such that $p_{\alpha\beta'} \neq 0$ and then $P(x^{\beta'})$ is nonzero. In particular the endomorphism $P \in A_n$ is nonzero. □

Remark 1.2. There exists a natural action of A_n on the polynomial ring $\mathbb{C}[x]$ since each element $P \in A_n$ is an endomorphism of the $\mathbb{C}$-vector space $\mathbb{C}[x]$. This natural action induces on $\mathbb{C}[x]$ a structure of left A_n–module.

[b]Given $\beta, \beta' \in \mathbb{N}^n$ we say that β' is smaller than or equal to β with respect to the lexicographic order if there exists $1 \leq i \leq n-1$ such that $\beta'_j = \beta_j$ for $1 \leq j \leq i$ and $\beta'_{i+1} \leq \beta_{i+1}$.

Due to Proposition 1.3 the action of an element $P = \sum_\beta p_\beta(x)\partial^\beta \in A_n$ on an element $g \in \mathbb{C}[x]$ can be written as

$$P(g) = \sum_\beta p_\beta(x) \frac{\partial^{\beta_1+\cdots+\beta_n}(g)}{\partial x_1^{\beta_1}\cdots \partial x_n^{\beta_n}}$$

and this justifies the name of *Linear Differential Operators with polynomial coefficients* for the elements in A_n.

In fact $\mathbb{C}[x]$ is finitely generated as A_n-module. This can be proved just by considering the map

$$\phi : \mathbb{C}[x] \longrightarrow \frac{A_n}{A_n(\partial_1,\ldots,\partial_n)}$$

defined by $\phi(g) = \overline{g} = g + A_n(\partial_1,\ldots,\partial_n)$ where $A_n(\partial_1,\ldots,\partial_n)$ is the left ideal generated by $\partial_1,\ldots,\partial_n$.

This map is a morphism of left A_n–modules and it is injective by Proposition 1.3. Let's see that ϕ is also surjective. Consider $P \in A_n$ and write it as

$$P = \sum_\beta p_\beta(x)\partial^\beta.$$

It is clear that $\phi(p_0(x)) = \phi(P(1)) = \overline{P}$.

1.2. *Order and total order*

We will denote $|\beta| = \sum_i \beta_i$ for each $\beta \in \mathbb{N}^n$.

Definition 1.2. For a nonzero operator

$$P = \sum_{\alpha,\beta} p_{\alpha\beta} x^\alpha \partial^\beta = \sum_\beta p_\beta(x)\partial^\beta \in A_n,$$

the maximum of $|\beta|$ for $p_\beta(x) \neq 0$ is called the *order* of P. This nonnegative integer is denoted by $\mathrm{ord}(P)$. The maximum of $|\alpha| + |\beta|$ for $p_{\alpha\beta} \neq 0$ is called the *total order* of P and it is denoted by $\mathrm{ord}^T(P)$. We will write $\mathrm{ord}(0) = \mathrm{ord}^T(0) = -\infty$.

Definition 1.3. The *principal symbol* of the operator

$$P = \sum_{\alpha,\beta} p_{\alpha\beta} x^\alpha \partial^\beta = \sum_\beta p_\beta(x)\partial^\beta$$

is the polynomial

$$\sigma(P) = \sum_{|\beta| = \mathrm{ord}(P)} p_\beta(x)\xi^\beta \in \mathbb{C}[x,\xi] = \mathbb{C}[x_1,\ldots,x_n,\xi_1,\ldots,\xi_n]$$

where $\xi_1, \ldots, \xi_n$ are new variables. The *principal total symbol* of P is the polynomial

$$\sigma^T(P) = \sum_{|\alpha+\beta|=\mathrm{ord}^T(P)} p_{\alpha\beta} x^\alpha \xi^\beta \in \mathbb{C}[x, \xi].$$

Remark 1.3. Notice that $\sigma(P) \in \mathbb{C}[x, \xi]$ is a homogeneous polynomial of degree $\mathrm{ord}(P)$ in the ξ-variables while $\sigma^T(P) \in \mathbb{C}[x, \xi]$ is a homogeneous polynomial of degree $\mathrm{ord}^T(P)$. In general we have $\sigma(P) \neq \sigma^T(P)$. In general we do not have neither $\sigma(P+Q) = \sigma(P) + \sigma(Q)$ nor $\sigma^T(P+Q) = \sigma^T(P) + \sigma^T(Q)$. Nevertheless, we have the following Proposition.

Proposition 1.4. *For $P, Q \in A_n$ one has*

(1) $\mathrm{ord}(PQ) = \mathrm{ord}(P) + \mathrm{ord}(Q)$ *and* $\sigma(PQ) = \sigma(P)\sigma(Q)$.
(2) $\mathrm{ord}^T(PQ) = \mathrm{ord}^T(P) + \mathrm{ord}^T(Q)$ *and* $\sigma^T(PQ) = \sigma^T(P)\sigma^T(Q)$.
(3) $\mathrm{ord}(PQ-QP) \leq \mathrm{ord}(P) + \mathrm{ord}(Q) - 1$ *and* $\mathrm{ord}^T(PQ-QP) \leq \mathrm{ord}^T(P) + \mathrm{ord}^T(Q) - 2$.
(4) $\mathrm{ord}(P+Q) \leq \max\{\mathrm{ord}(P), \mathrm{ord}(Q)\}$ *(and similarly for* ord^T*).*
(5) If $\mathrm{ord}(P) = \mathrm{ord}(Q)$ *and* $\sigma(P) + \sigma(Q) \neq 0$ *then* $\sigma(P+Q) = \sigma(P) + \sigma(Q)$ *(and similarly for* ord^T *and* σ^T*).*

We are assuming $-\infty + k = k + (-\infty) = -\infty$ for $k \in \mathbb{Z} \cup \{-\infty\}$.

Proof. (1), (2) and (3) follow from Proposition 1.2. Parts (4) and (5) follow from the very Definitions 1.2 and 1.3. □

From (1) in Proposition 1.4 we have:

Corollary 1.1. *A_n is an integral domain.*

Definition 1.4. For each left (or right) ideal $I \subset A_n$ the *graded ideal* associated with I is the ideal $\mathrm{gr}(I)$ of $\mathbb{C}[x, \xi]$ generated by the family of principal symbols of elements in I

$$\mathrm{gr}(I) = \mathbb{C}[x, \xi]\{\sigma(P) \,|\, P \in I\}.$$

The *total graded ideal* associated with I is the ideal $\mathrm{gr}^T(I)$ of $\mathbb{C}[x, \xi]$ generated by the family of principal total symbols of elements in I

$$\mathrm{gr}^T(I) = \mathbb{C}[x, \xi]\{\sigma^T(P) \,|\, P \in I\}.$$

Remark 1.4. If $I = A_n P$ is the principal left ideal generated by an operator $P \in A_n$ then $\mathrm{gr}(I)$ is the principal ideal in $\mathbb{C}[x, \xi]$ generated by the principal symbol $\sigma(P)$, so we have $\mathrm{gr}(A_n P) = \mathbb{C}[x, \xi]\sigma(P)$. We have an analogous result for $\mathrm{gr}^T(I)$.

Proposition 1.5. *Let $I \subset A_n$ be a left ideal and $F(x,\xi)$ be a polynomial in $\mathrm{gr}(I)$ (resp. $\mathrm{gr}^T(I)$). If $F(x,\xi)$ is homogeneous with respect to the ξ-variables (resp. homogeneous) then it is the principal symbol $\sigma(P)$ (resp. the total principal symbol $\sigma^T(P)$) for some $P \in I$.*

Proof. Both cases being analogous we will only write the proof for the first one. Assume $F \in \mathrm{gr}(I)$ is not zero. As $\mathrm{gr}(I)$ is generated by $\{\sigma(P) \mid P \in I\}$ there are $P_1, \ldots, P_r$ in I such that

$$F = \sum_i H_i \sigma(P_i)$$

for some polynomials $H_i \in \mathbb{C}[x,\xi]$. Let's denote by d (resp. d_i) the ξ-degree of F (resp. $\sigma(P_i)$). Then we can assume H_i ξ–homogeneous of degree $d - d_i$ (in particular $H_i = 0$ if $d < d_i$). Let's denote by $Q_i(x,\partial)$ any element in A_n such that $\sigma(Q_i) = H_i$. As

$$F = \sum_i \sigma(Q_i)\sigma(P_i)$$

is a nonzero ξ–homogeneous polynomial of degree d then (see Proposition 1.4)

$$\sigma\left(\sum_i Q_i P_i\right) = \sum_i \sigma(Q_i)\sigma(P_i)$$

and $\sum_i Q_i P_i$ is the wanted operator in I. □

Proposition 1.6. *The ring A_n is left and right Noetherian domain.*

Proof. We will proof that any left ideal $I \subset A_n$ is finitely generated. A similar proof can be done for right ideals. We can assume $I \neq (0)$. By definition of $\mathrm{gr}^T(I)$ there are polynomials $\sigma^T(P_1), \ldots, \sigma^T(P_r)$ with $P_i \in I$ generating $\mathrm{gr}^T(I)$. Let us denote J the left ideal in A_n generated by $P_1, \ldots, P_r$. We will prove that $J = I$. Assume there exists $P \in I \setminus J$. We can also assume P is of minimal total order. As $\sigma^T(P) \in \mathrm{gr}^T(I)$ then there are homogeneous polynomials $H_1, \ldots, H_r \in \mathbb{C}[x,\xi]$ such that

$$\sigma^T(P) = \sum_i H_i \sigma^T(P_i).$$

We can also assume $\deg(H_i) + \mathrm{ord}^T(P_i) = \mathrm{ord}^T(P)$. Denote by Q_i any element in A_n with $\sigma^T(Q_i) = H_i$. Then the operator

$$P' := P - \sum_i Q_i P_i$$

has total order strictly smaller than $\mathrm{ord}^T(P)$ and then P' should be in J. This implies $P \in J$ which is a contradiction. This proves $J = I$. $\square$

Remark 1.5. Replacing in the above proof gr^T by gr, σ^T by σ, ord^T by ord and homogeneous polynomials by ξ-homogeneous polynomials we get another proof of Proposition 1.6.

Exercise 1.2. Prove that A_n is a simple ring.
Quick answer.- *Let J be a nonzero two-sided ideal in A_n. Let P be a nonzero element of minimal total order in J and write $d = \mathrm{ord}^T(P)$. If $d = 0$ then $P \in \mathbb{C}$ and as P is nonzero we have $J = A_n$. Assume $d > 0$. Let $(\alpha, \beta) \in \mathbb{N}^{2n}$ be such that $|\alpha + \beta| = d$ and the coefficient $p_{\alpha\beta}$ of $x^\alpha \partial^\beta$ in P is nonzero. Assume there is i such that $\beta_i > 0$.*
Claim.- *The operator $Q = [x_i, P] = x_i P - P x_i$ belongs to J and it is nonzero.*
Let us prove the claim. The first part follows because J is a two-sided ideal. Let us write $P = P' + P''$ where P' (resp. P'') is the sum of the monomials in P of total order equal to d (resp. less or equal to $d - 1$). We can write

$$P' = \sum_{j=0}^{\ell} P_j(x, \partial')\partial_i^j$$

where $P_j = P_j(x, \partial')$ doesn't depend on ∂_i, $P_\ell \neq 0$ and, moreover, all monomials in P_j have total order $d - j$. We also have $\ell \leq d$ and since $\beta_i > 0$ we have $\ell > 0$. From the equality $Q = [x_i, P'] + [x_i, P'']$ and by Proposition 1.4 we have $\mathrm{ord}^T([x_i, P'']) \leq d - 2$. Moreover, any monomial in $[x_i, P_j\partial_i^j] = jP_j\partial_i^{j-1}$ has total order less or equal than $d-1$ and $\ell P_\ell \partial_i^{\ell-1}$ is not zero. That proves the claim.
The claim contradicts the minimality of d. Then β should be zero as long as $p_{\alpha\beta} \neq 0$ and $|\alpha + \beta| = d$. Assume now there exists i with $\alpha_i > 0$ and $|\alpha| = d$. In a similar way, using $Q' = [\partial_i, P]$ we get a contradiction with the minimality of d. So P should be a nonzero element in $\mathbb{C}$ and then $J = A_n$.

Remark 1.6. The only invertible elements in A_n are the nonzero constants (i.e. the elements in $\mathbb{C} \setminus \{0\}$), since if $1 = PQ$ for some $P, Q \in A_n$ then $0 = \mathrm{ord}^T(1) = \mathrm{ord}^T(P) + \mathrm{ord}^T(Q)$ and this implies that both P, Q must be nonzero elements in $\mathbb{C}$.

Exercise 1.3. If $\phi : S \to S'$ is a ring morphism then $\ker(\phi)$ is a two-sided ideal in S.

Corollary 1.2. *Each ring morphism $\phi : A_n \to S$ (S a possibly noncommutative ring) is injective.*

1.3. *Filtrations on A_n*

Let us denote

$$F_k(A_n) = \{P \in A_n \,|\, \mathrm{ord}(P) \leq k\}$$

and

$$B_k(A_n) = \{P \in A_n \,|\, \mathrm{ord}^T(P) \leq k\}$$

for $k \in \mathbb{Z}$.

If no confusion arises we will simply write $F_k = F_k(A_n)$ and $B_k = B_k(A_n)$.

Proposition 1.7. *The following properties hold:*

(1) $B_k = F_k = \{0\}$ *for* $k \leq -1$.
(2) $B_k \subset F_k$ *for* $k \in \mathbb{Z}$.
(3) $B_k \subset B_{k+1}$, $F_k \subset F_{k+1}$ *for* $k \in \mathbb{Z}$.
(4) $B_k B_\ell \subset B_{k+\ell}$, $F_k F_\ell \subset F_{k+\ell}$ *for* $k, \ell \in \mathbb{Z}$.
(5) $A_n = \bigcup_k B_k = \bigcup_k F_k$.
(6) $1 \in B_0 = \mathbb{C}$, $1 \in F_0 = \mathbb{C}[x]$.
(7) Each B_k *is a* $\mathbb{C}$*–vector space with dimension* $\binom{2n+k}{k}$.
(8) Each F_k *is a free* $\mathbb{C}[x]$*-module with rank* $\binom{n+k}{k}$.

Definition 1.5. The family $(F_k)_k$ (resp. $(B_k)_k$) is called the *order filtration* (resp. the *total order filtration*) on A_n.

1.4. *The graded rings* $\mathrm{gr}^B(A_n)$ *and* $\mathrm{gr}^F(A_n)$

Exercise 1.4. Each quotient $\frac{B_k}{B_{k-1}}$ is a $\mathbb{C}$–vector space with dimension $\binom{2n+k-1}{2n-1}$.

Quick answer.- The residue classes

$$\overline{x^\alpha \partial^\beta} = x^\alpha \partial^\beta + B_{k-1}$$

with $|\alpha+\beta| = k$ generate the quotient vector space B_k/B_{k-1} and, moreover, they are linearly independent since a linear combination

$$\sum_{|\alpha+\beta|=k} \lambda_{\alpha\beta} x^\alpha \partial^\beta$$

belongs to B_{k-1} if and only if all $\lambda_{\alpha\beta}$ are 0.

Exercise 1.5. Each quotient F_k/F_{k-1} is a free $\mathbb{C}[x]$–module with rank $\binom{n+k-1}{n-1}$.

Quick answer.- The residue classes

$$\overline{\partial^\beta} = \partial^\beta + F_{k-1}$$

with $|\beta| = k$ generate the quotient $\mathbb{C}[x]$–module F_k/F_{k-1} and, moreover, they are linearly independent over $\mathbb{C}[x]$, since a $\mathbb{C}[x]$–linear combination

$$\sum_{|\beta|=k} \lambda_\beta(x) \partial^\beta$$

belongs to F_{k-1} if and only if each $\lambda_\beta(x)$ is 0.

Proposition 1.8. *The Abelian group*

$$\mathrm{gr}^B(A_n) := \bigoplus_{k\in\mathbb{Z}} \frac{B_k}{B_{k-1}}$$

$$\left(\textit{resp. } \mathrm{gr}^F(A_n) := \bigoplus_{k\in\mathbb{Z}} \frac{F_k}{F_{k-1}} \right)$$

has a natural structure of commutative ring with unit.

Proof. We will write down the case of the total order filtration $(B_k)_k$, the other one being analogous. Let us consider, for $k, \ell \in \mathbb{Z}$, the map

$$\mu_{k\ell} : \frac{B_k}{B_{k-1}} \times \frac{B_\ell}{B_{\ell-1}} \to \frac{B_{k+\ell}}{B_{\ell+k-1}}$$

defined by

$$\mu_{k\ell}(P + B_{k-1}, Q + B_{\ell-1}) = PQ + B_{k+\ell-1}$$

for $P \in B_k, Q \in B_\ell$.

The map $\mu_{k\ell}$ is well defined: $PQ + B_{k+\ell-1}$ does not depend on the chosen representatives $P \in B_k$ and $Q \in B_\ell$.

We will simply denote $\overline{P} = P + B_{k-1}, \overline{Q} = Q + B_{\ell-1}$ and $\overline{PQ}$ instead of $\mu_{k\ell}(\overline{P}, \overline{Q})$.

From Proposition 1.4 it follows that for $P \in B_k$ and $Q \in B_\ell$ we have $PQ - QP \in B_{k+\ell-1}$ (in fact in this case we also have $PQ-QP \in B_{k+\ell-2}$ but we do not need this stronger property here) and then $\overline{PQ} = \overline{QP}$. Moreover, we also have $\overline{P}(\overline{Q}\ \overline{R}) = \overline{P}(\overline{QR}) = \overline{P(QR)} = \overline{(PQ)R} = (\overline{PQ})\ \overline{R} = (\overline{P}\ \overline{Q})\overline{R}$.

We define a map

$$\mu' : \mathrm{gr}^B(A_n) \times \mathrm{gr}^B(A_n) \to \mathrm{gr}^B(A_n)$$

by bilinearity:

$$\mu'(\sum_k \overline{P_k}, \sum_\ell \overline{Q_\ell}) = \sum_{k,\ell} \overline{P_k Q_\ell}$$

where $P_k \in B_k$ and $Q_\ell \in B_\ell$ for all k, ℓ.

We will simply denote $(\sum_k \overline{P_k})(\sum_\ell \overline{Q_\ell})$ instead of $\mu'(\sum_k \overline{P_k}, \sum_\ell \overline{Q_\ell})$. The map μ' is well defined and defines a product on $\mathrm{gr}^B(A_n)$. To this end we can see that the binary operation defined by μ' is associative and commutative (since the corresponding properties hold for the maps $\mu_{k\ell}$). As μ' is defined by bilinearity it is distributive with respect to the sum.

Let us denote by $\overline{1}$ the residue class of 1 modulo $B_{-1} = \{0\}$. It is clear that $\overline{1}(\sum_k \overline{P_k}) = \sum_k \overline{P_k}$ and then it is the unit of the commutative ring $\mathrm{gr}^B(A_n)$. □

Remark 1.7. The family of Abelian groups $\left(\frac{B_k}{B_{k-1}}\right)_k$ (resp. $\left(\frac{F_k}{F_{k-1}}\right)_k$) is a *grading* on the ring $\mathrm{gr}^B(A_n)$ (resp. $\mathrm{gr}^F(A_n)$).

Recall that we have denoted $\mathbb{C}[x, \xi] = \mathbb{C}[x_1, \ldots, x_n, \xi_1, \ldots, \xi_n]$.

Proposition 1.9. *The graded ring* $\mathrm{gr}^B(A_n)$ *is isomorphic to the polynomial ring* $\mathbb{C}[x, \xi]$ *endowed with the grading defined by the degree of the polynomials.*

Proof. Let us consider, for $k \in \mathbb{N}$, the isomorphism (of vector spaces)

$$\eta_k : B_k/B_{k-1} \to \mathbb{C}[x, \xi]_k$$

defined

$$\eta_k\left(\left(\sum_{|\alpha+\beta|\le k} p_{\alpha\beta} x^\alpha \partial^\beta\right) + B_{k-1}\right) = \sum_{|\alpha+\beta|=k} p_{\alpha\beta} x^\alpha \xi^\beta.$$

Here $\mathbb{C}[x,\xi]_k$ denotes the $\mathbb{C}$–vector space of homogeneous polynomials of degree k. The family η_k yields by bilinearity a natural isomorphism

$$\eta : \mathrm{gr}^B(A_n) \to \mathbb{C}[x,\xi]$$

of graded rings. □

Remark 1.8. The polynomial ring $\mathbb{C}[x,\xi]$ can be also endowed with the grading defined by the degree in ξ:

$$\mathbb{C}[x,\xi] = \bigoplus_k \mathbb{C}[x,\xi]_{(k)}$$

with

$$\mathbb{C}[x,\xi]_{(k)} = \sum_{|\beta|=k} \mathbb{C}[x]\xi^\beta.$$

We will call this grading the ξ–grading on $\mathbb{C}[x,\xi]$.

Proposition 1.10. *The graded ring* $\mathrm{gr}^F(A_n)$ *is isomorphic to the polynomial ring* $\mathbb{C}[x,\xi]$ *endowed with the* ξ*–grading.*

Proof. Let us consider, for $k \in \mathbb{N}$, the isomorphism (of vector spaces)

$$\eta'_k : F_k/F_{k-1} \to \mathbb{C}[x,\xi]_{(k)}$$

defined by

$$\eta'_k\left(\sum_{|\beta|\le k} p_\beta(x)\partial^\beta\right) = \sum_{|\beta|=k} p_\beta(x)\xi^\beta.$$

Here $\mathbb{C}[x,\xi]_{(k)}$ denotes the free $\mathbb{C}[x]$–module of ξ–homogeneous polynomials of degree k (see Remark 1.8). The family η'_k yields by bilinearity a natural isomorphism

$$\eta' : \mathrm{gr}^F(A_n) \to \mathbb{C}[x,\xi]$$

of graded rings (when considering the ξ-grading on $\mathbb{C}[x,\xi]$). □

Remark 1.9. Notice that if $P \in B_k \backslash B_{k-1}$ (resp. $P \in F_k \backslash F_{k-1}$) then $\eta_k(P)$ (resp. $\eta'_k(P)$) is nothing but the principal total symbol (resp. the principal symbol) of P: $\eta_k(P) = \sigma^T(P)$ (resp. $\eta'_k(P) = \sigma(P)$) (see Definition 1.3).

Notation 1.2. From now on, we will identify the graded rings $\mathrm{gr}^B(A_n)$ and $\mathbb{C}[x,\xi]$ (endowed with the degree of the polynomials) (resp. $\mathrm{gr}^F(A_n)$ and $\mathbb{C}[x,\xi]$ (endowed with the ξ–degree)) by mean of the isomorphism η (resp. η').

1.5. *B–filtrations on A_n-modules*

From now on, all the A_n–modules (resp. ideals) will be left modules (resp. left ideals) unless otherwise specified.

Definition 1.6. Let M be an A_n-module. A B–filtration on M is a family $\Gamma = (M_k)_{k\in\mathbb{N}}$ of finitely dimensional $\mathbb{C}$–vector subspaces of M such that:

i) $M_k \subset M_{k+1}$ for all $k \in \mathbb{N}$.
ii) $\bigcup_k M_k = M$.
iii) $B_k M_\ell \subset M_{k+\ell}$ for all (k, ℓ).

That is not the more general definition of filtration on a module. There exist filtrations indexed by $\mathbb{Z}$ instead of $\mathbb{N}$ but we will not need them here. Nevertheless, if in these notes we write $\Gamma = (M_k)_{k\in\mathbb{Z}}$ for a B–filtration on M it is assumed to be $M_k = \{0\}$ for $k < 0$.

Remark 1.10.

(1) The total order filtration $(B_k(A_n))_k$ is a B–filtration on the left A_n–module A_n.
(2) Let us denote, for $k \in \mathbb{Z}$,
$$B_k(\mathbb{C}[x]) = \{f \in \mathbb{C}[x] \mid \deg(f) \le k\}.$$
The family $(B_k(\mathbb{C}[x]))_k$ is a B–filtration on the left A_n-module $\mathbb{C}[x]$.
(3) Let $I \subset A_n$ be an ideal and denote $B_k(I) = B_k(A_n) \cap I$ for $k \in \mathbb{Z}$. The family $(B_k(I))_k$ is a B–filtration on I considered as a left A_n–module (see also Subsection 1.9).
(4) Let $I \subset A_n$ be an ideal and define
$$B_k\left(\frac{A_n}{I}\right) = \frac{B_k(A_n) + I}{I}$$
for $k \in \mathbb{N}$. The family $(B_k(A_n/I))_k$ is a B–filtration on the left A_n–module A_n/I. It will be called the induced B–filtration on A_n/I (see also Subsection 1.9).

1.6. *F–filtrations on A_n-modules*

We can also define in a similar way as before F–filtrations on A_n–modules.

Definition 1.7. An F–filtration on a A_n–module M is a family $\Gamma = (M_k)_{k\in\mathbb{N}}$ of finitely generated $\mathbb{C}[x]$–submodules of M such that:

i) $M_k \subset M_{k+1}$ for all $k \in \mathbb{N}$.

ii) $\bigcup_k M_k = M$.
iii) $F_k M_\ell \subset M_{k+\ell}$ for all (k, ℓ).

As before, if in these notes we write $\Gamma = (M_k)_{k\in\mathbb{Z}}$ for an F–filtration on M it is assumed to be $M_k = \{0\}$ for $k < 0$.

Remark 1.11.

(1) The order filtration $(F_k(A_n))_k$ is an F–filtration on the left A_n–module A_n.
(2) Let us denote, for $k \in \mathbb{Z}$, $F_k(\mathbb{C}[x]) = \mathbb{C}[x]$ for $k \geq 0$ and $F_k(\mathbb{C}[x]) = 0$. The family $(F_k(\mathbb{C}[x]))_k$ is an F–filtration on the left A_n–module $\mathbb{C}[x]$.
(3) Let $I \subset A_n$ be an ideal and denote $F_k(I) = F_k(A_n) \cap I$ for $k \in \mathbb{Z}$. The family $(F_k(I))_k$ is an F–filtration on I considered as a left A_n–module (see also Subsection 1.9).
(4) Let $I \subset A_n$ be an ideal and denote
$$F_k\left(\frac{A_n}{I}\right) = \frac{F_k(A_n) + I}{I}$$
for $k \in \mathbb{Z}$. The family $(F_k(A_n/I))_k$ is an F–filtration on the left A_n–module A_n/I. It will be called the induced F–filtration on A_n/I (see also Subsection 1.9).

1.7. *The* $\boldsymbol{\Gamma}$*–order and the* $\boldsymbol{\Gamma}$*–symbol map*

Let $\Gamma = (M_k)_k$ be a filtration[c] on an A_n-module M. For each nonzero $m \in M$ we call the Γ–order of m and we denote by $\mathrm{ord}^\Gamma(m)$ the integer k such that $m \in M_k \setminus M_{k-1}$.

Let us denote
$$\sigma_k^\Gamma : M_k \to M_k/M_{k-1}$$
the canonical projection (which is a $\mathbb{C}$–linear map). We have $\sigma_k^\Gamma(m) = m + M_{k-1}$ for $m \in M_k$.

If Γ is an F–filtration then σ_k^Γ is also a morphism of $\mathbb{C}[x]$–modules.

The map σ_k^Γ is called the k-th Γ–symbol map associated with the filtration Γ.

If $M = A_n$ and $\Gamma = (B_k)_k$ is the total order filtration on A_n (also called the B–filtration on A_n), the corresponding k-th symbol map will be also denoted by σ_k^B.

[c]A filtration on an A_n–module M is either a B–filtration or an F–filtration.

If $\Gamma = (F_k)_k$ is the order filtration on A_n (also called the F–filtration on A_n), the corresponding k-th symbol map will be also denoted σ_k^F.

Remark 1.12. We will use here the notations of Propositions 1.9 and 1.10. If $P \in B_k \setminus B_{k-1}$ (resp. $P \in F_k \setminus F_{k-1}$) then

$$(\eta_k \circ \sigma_k^B)(P) = \sigma^T(P)$$

$$\big(\text{resp.} \quad (\eta_k' \circ \sigma_k^F)(P) = \sigma(P)\big)\,.$$

1.8. *Graded associated module*

Let M be an A_n–module and $\Gamma = (M_k)_k$ a B–filtration (resp. F–filtration) on M.

As each quotient M_k/M_{k-1} is an Abelian group (and a $\mathbb{C}$–vector space) the direct sum

$$\mathrm{gr}^\Gamma(M) := \bigoplus_k \frac{M_k}{M_{k-1}}$$

is also an Abelian group (and a $\mathbb{C}$–vector space).

For $m \in M_k$ the class

$$m + M_{k-1} \in \frac{M_k}{M_{k-1}}$$

will be simply denoted by $\overline{m}$ if no confusion arises.

An element in $\mathrm{gr}^\Gamma(M)$ is a finite sum $\sum_k \overline{m_k}$ where each m_k belongs to M_k.

Proposition 1.11. *Let M be an A_n–module and $\Gamma = (M_k)_k$ be a B–filtration (resp. F–filtration) on M. The Abelian group $\mathrm{gr}^\Gamma(M)$ has a natural structure of $\mathrm{gr}^B(A_n)$–module (resp. $\mathrm{gr}^F(A_n)$–module).*

Proof. We will only treat the case of the B–filtration the other one being analogous.

Let us consider the map

$$\nu : \mathrm{gr}^B(A_n) \times \mathrm{gr}^\Gamma(M) \to \mathrm{gr}^\Gamma(M)$$

defined by bilinearity from the maps

$$\nu_k : \frac{B_k}{B_{k-1}} \times \frac{M_\ell}{M_{\ell-1}} \to \frac{M_{k+\ell}}{M_{k+\ell-1}}$$

defined by

$$\nu_k(\overline{P_k}, \overline{m_\ell}) = \overline{P_k m_\ell}.$$

It is straightforward to show that the map ν defines on $\mathrm{gr}^{\Gamma}(M)$ a structure of $\mathrm{gr}^{B}(A_n)$–module. □

Definition 1.8. The graded module $\mathrm{gr}^{\Gamma}(M)$ will be called *the associated graded module* to the filtration $\Gamma = (M_k)_k$ on M.

1.9. *Induced filtrations*

Let us recall that a filtration on an A_n–module M will be either a B–filtration or an F–filtration on M.

Let M be an A_n–module, $N \subset M$ a submodule of M and $\Gamma = (M_k)_k$ a filtration on M. For each $k \in \mathbb{Z}$ let us denote $N_k := M_k \cap N$ and $(M/N)_k := (M_k + N)/N$.

The following proposition is easy to prove.

Proposition 1.12.

(1) The family $\Gamma' = (N_k)_k$ is a filtration on N. It will be called the induced *filtration by Γ on N.*
(2) The family $\Gamma'' = ((M/N)_k)_k$ is a filtration on M/N. It will be called the induced *filtration by Γ on M/N.*

Let I be an ideal in A_n. Using the notations in Remark 1.10 the family $(B_k(I))_k$ (resp. $(B_k(A_n/I))_k$) is the induced B–filtration on I (resp. on A_n/I). We also have the analogous statement for the F–filtration (using Remark 1.11).

Proposition 1.13. *Let M be an A_n–module, $N \subset M$ a submodule of M and $\Gamma = (M_k)_k$ a filtration on M. Then there exists a canonical exact sequence of graded modules*

$$0 \to \mathrm{gr}^{\Gamma'}(N) \to \mathrm{gr}^{\Gamma}(M) \to \mathrm{gr}^{\Gamma''}(M/N) \to 0.$$

Proof. For each $k \in \mathbb{Z}$ we have an exact sequence of $\mathbb{C}$-vector spaces

$$0 \to N_k \to M_k \to (M/N)_k \to 0$$

since $(M/N)_k = \frac{M_k+N}{N} \simeq \frac{M_k}{M_k \cap N} = \frac{M_k}{N_k}$. Then for each $k \in \mathbb{Z}$ there exists a canonical exact sequence of vector spaces

$$0 \to \frac{N_k}{N_{k-1}} \to \frac{M_k}{M_{k-1}} \to \frac{M_k + N}{M_{k-1} + N} \to 0.$$

□

Corollary 1.3. *Let I be an ideal in A_n. We have canonical isomorphisms*

$$\mathrm{gr}^B\left(\frac{A_n}{I}\right) \simeq \frac{\mathrm{gr}^B(A_n)}{\mathrm{gr}^B(I)} \qquad \textit{and} \qquad \mathrm{gr}^F\left(\frac{A_n}{I}\right) \simeq \frac{\mathrm{gr}^F(A_n)}{\mathrm{gr}^F(I)}.$$

Proof. It follows from Proposition 1.13 applied to the canonical exact sequence

$$0 \to I \longrightarrow A_n \longrightarrow \frac{A_n}{I} \to 0$$

where each term of the sequence is endowed with its corresponding B–filtration or F–filtration. □

By Corollary 1.3 there exists a canonical injective map $\mathrm{gr}^B(I) \to \mathrm{gr}^B(A_n)$ which is a morphism of graded modules and then $\mathrm{gr}^B(I)$ can be identified with a graded submodule of $\mathrm{gr}^B(A_n)$. This means that $\mathrm{gr}^B(I)$ is a graded ideal –modulo this identification– of the graded ring $\mathrm{gr}^B(A_n)$).

Let us recall (see Proposition 1.9) that there exists a natural isomorphism of graded rings

$$\eta : \mathrm{gr}^B(A_n) \to \mathbb{C}[x,\xi]$$

whose k-th homogeneous component

$$\eta_k : B_k/B_{k-1} \to \mathbb{C}[x,\xi]_k$$

is defined by

$$\eta_k\left(\left(\sum_{|\alpha+\beta|\leq k} p_{\alpha\beta}x^\alpha\partial^\beta\right) + B_{k-1}\right) = \sum_{|\alpha+\beta|=k} p_{\alpha\beta}x^\alpha\xi^\beta.$$

Here $\mathbb{C}[x,\xi]_k$ denotes the $\mathbb{C}$–vector space of homogeneous polynomials of degree k.

Proposition 1.14. *With the above notations, the ideal $\eta(\mathrm{gr}^B(I))$ is the homogeneous ideal of $\mathbb{C}[x,\xi]$ generated by the family $\{\sigma^T(P) \mid P \in I\}$.*

We have an analogous result for $\mathrm{gr}^F(I)$ and the family $\{\sigma(P) \mid P \in I\}$ (using the notation of Proposition 1.10).

Recall that we have identified $\mathrm{gr}^B(A_n)$ with $\mathbb{C}[x,\xi]$ (see Notation 1.2). Let I be an ideal of A_n and let us denote $M = A_n/I$ and $\Gamma'' = (B_k(M))_k$ the induced B–filtration on M (see Remark 1.10). By Corollary 1.3 $\mathrm{gr}^{\Gamma''}(M)$ is isomorphic as $\mathbb{C}[x,\xi]$–module to

$$\frac{\mathbb{C}[x,\xi]}{\mathrm{gr}^B(I)}.$$

We have an analogous result for the F–filtration.

1.10. *Good filtrations*

Proposition 1.15. *Let M be an A_n–module and $\Gamma = (M_k)_k$ a B–filtration on M. If $\mathrm{gr}^\Gamma(M)$ is a finitely generated $\mathbb{C}[x,\xi]$–module then M is Noetherian.*

Proof. Since A_n is a Noetherian ring it is enough to prove that M is finitely generated. Assume $\overline{m_1},\ldots,\overline{m_r}$ is a homogeneous generating system of $\mathrm{gr}^\Gamma(M)$. Assume $m_i \in M_{k_i}$ for some $k_i \in \mathbb{Z}$, $i = 1,\ldots,r$. We will prove that the set $\{m_1,\ldots,m_r\}$ generates M.
Let us denote M' the submodule of M generated by the m_i. We will prove that any $m \in M$ is in fact in M'. We use induction on $\mathrm{ord}^\Gamma(m)$ (see Subsection 1.7). There is nothing to prove if $\mathrm{ord}^\Gamma \leq 0$. Assume that any element $m' \in M$ such that $\mathrm{ord}^\Gamma(m') \leq k$ is in M' for some integer $k > 0$. Let $m \in M$ be such that $\mathrm{ord}^\Gamma(m) = k+1$. Let us write

$$\overline{m} = \sum_i f_i \overline{m_i}$$

for some homogeneous polynomials $f_i \in \mathbb{C}[x,\xi]$ where $\deg(f_i) = k+1-k_i$. Let us write

$$m' = m - \sum_i P_i(x,\partial) m_i$$

where $P_i = P_i(x,\partial)$ is a differential operator satisfying $\sigma^T(P_i) = f_i$. The residue class of m' modulo M_{k+1} is zero and then, by induction, m' is in M'. Then $m \in M'$. ☐

Definition 1.9. Let M be an A_n–module and $\Gamma = (M_k)_k$ a B–filtration . We say that Γ is a good filtration if $\mathrm{gr}^\Gamma(M)$ is a finitely generated $\mathrm{gr}^B(A_n)$–module.

Exercise 1.6. Let I be an ideal of A_n. Prove that:

i) The induced B–filtration on I is a good filtration.
ii) The induced B–filtration on A_n/I is a good filtration.
iii) Any finitely generated A_n–module M admits a good B–filtration.

Both i) and ii) have direct proofs. Hint for iii): If $m_1, \ldots, m_r$ is a generating system for M, define $M_k := \sum_j B_k m_j$ for $k \in \mathbb{N}$. The family $(M_k)_k$ is a good B–filtration on M.

Proposition 1.16. *Let M be an A_n–module and $\Gamma = (M_k)_k$ a B–filtration on M. The following conditions are equivalent:*

i) *Γ is a good B–filtration on M.*
ii) *There exists $k_0 \in \mathbb{N}$ such that $M_{k+\ell} = B_\ell M_k$ for all $\ell \geq 0$ and for all $k \geq k_0$.*

Proof. *ii) $\Rightarrow$ i).* It is enough to prove that $\mathrm{gr}^\Gamma(M)$ is generated by $M_0 \oplus M_1/M_0 \oplus \cdots \oplus M_{k_0}/M_{k_0-1}$ since each M_ℓ is a vector space of finite dimension. To see this, if $m \in M_k$ and $k > k_0$ then write $k = k_0 + i$ for $i = k - k_0 > 0$. Since $M_k = B_i M_{k_0}$ we have

$$m = \sum_{j=1}^{r} P_j m_j$$

where $P_j \in B_i$ and $m_1, \ldots, m_r$ is a basis (or simply a finite generating system) of the $\mathbb{C}$–vector space M_{k_0}. We can write

$$m + M_{k-1} = \overline{m} = \sum_j P_j m_j + M_{k-1} = \sum_j (P_j + B_{i-1})(m_j + M_{k_0-1}) = \sum_j \overline{P_j}\,\overline{m_j}.$$

i) $\Rightarrow$ ii). Let $\overline{m_1}, \ldots, \overline{m_r}$ be a homogeneous system of generators of $\mathrm{gr}^\Gamma(M)$. Suppose $m_j \in M_{k_j} \setminus M_{k_j-1}$ for $j = 1, \ldots, r$. Write $k_0 := \max\{k_j\}$. We will prove by induction on ℓ that $M_{k+\ell} = B_\ell M_k$ for all $\ell \geq 0$ and all $k \geq k_0$. There is nothing to prove for $\ell = 0$. Suppose the result is true for $\ell - 1$ for some $\ell > 0$. Let us consider $m \in M_{k+\ell}$ for $k \geq k_0$. We can write

$$\overline{m} = m + M_{k+\ell-1} = \sum_j f_j \overline{m_j}$$

for some homogeneous polynomial $f_j \in \mathbb{C}[x, \xi]$ of degree $k + \ell - k_j$. Let us write

$$m' = m - \sum_j P_j(x, \partial) m_j$$

for some $P_j \in B_{k+\ell-k_j}$ such that $\sigma^T(P_j) = f_j$. It is clear that $m' \in M_{k+\ell-1}$ and by induction $m' \in B_{\ell-1} M_k$. Since $k - k_j \geq 0$ we also have $B_{k+\ell-k_j} = B_\ell B_{k-k_j}$. Then

$$m = m' + \sum_j P_j(x, \partial) m_j.$$

Since $P_j(x,\partial) \in B_{k+\ell-k_j} = B_\ell B_{k-k_j}$ then $P_j(x,\partial)m_j \in B_\ell B_{k-k_j} M_{k_j} \subset B_\ell M_k$. $\square$

Exercise 1.7. Let us define $B'_k = B_{2k}$ for $k \in \mathbb{Z}$. Prove that $(B'_k)_k$ is a B–filtration on the A_n–module A_n. Prove that $(B'_k)_k$ is not a good filtration. (Hint: For $i > 0$ and $k > 0$ we have $B_i B'_k = B_i B_{2k} \subsetneqq B'_{i+k} = B_{2i+2k}$).

Proposition 1.17. *Let M be an A_n–module; $\Gamma = (M_k)_k$ and $\Gamma' = (M'_k)_k$ two B–filtrations on M. We have:*

i) If Γ is a good B–filtration then there exists $k_1 \in \mathbb{N}$ such that

$$M_k \subset M'_{k+k_1}$$

for all $k \in \mathbb{N}$.

ii) If Γ and Γ' are both good filtrations then there exists $k_2 \in \mathbb{N}$ such that

$$M'_{k-k_2} \subset M_k \subset M'_{k+k_2}$$

for all $k \in \mathbb{N}$.

Proof. Let us prove first that *ii)* follows from *i)*. So, let us assume *i)* is proved. Then there exists $j_1 \in \mathbb{N}$ such that $M_k \subset M'_{k+j_1}$ and there exists $j_2 \in \mathbb{N}$ such that $M'_k \subset M_{k+j_2}$ and both inclusions hold for all k. Let us define k_2 to be the maximum of j_1 and j_2. This k_2 satisfies *ii)*.
Let us prove *i)*. By Proposition 1.16 there exists $k_0 \geq 0$ such that $M_{k+\ell} = B_\ell M_k$ for all $\ell \geq 0$ and for all $k \geq k_0$. As M_{k_0} is a $\mathbb{C}$–vector space of finite dimension there exists $k_1 \in \mathbb{N}$ such that $M_{k_0} \subset M'_{k_1}$. If $k \geq k_0$ we have

$$M_k = M_{k-k_0+k_0} = B_{k-k_0} M_{k_0} \subset B_{k-k_0} M'_{k_1} \subset M'_{k-k_0+k_1} \subset M'_{k+k_1}.$$

If $0 \leq k \leq k_0$ then $M_k \subset M_{k_0} \subset M'_{k_1} \subset M'_{k+k_1}$. $\square$

Definition 1.10. Let M be an A_n–module and $\Gamma = (M_k)_k$ an F–filtration. We say that Γ is a good filtration if $\mathrm{gr}^\Gamma(M)$ is a finitely generated $\mathrm{gr}^F(A_n)$–module.

Exercise 1.8. If I is an ideal of A_n, prove that:

i) The induced F–filtration on I is a good filtration.
ii) The induced F–filtration on A_n/I is a good filtration.
iii) Any finitely generated A_n–module M admits a good F–filtration.

i) and ii) have direct proofs. Hint for iii). If $m_1, \dots, m_r$ is a generating system for M, define $M_k := \sum_j F_k m_j$ for $k \in \mathbb{N}$. The family $(M_k)_k$ is a good F–filtration on M.

Remark 1.13. Propositions 1.15, 1.16 and 1.17 remain true if one replaces B–filtrations by F–filtrations.

1.11. *Rational functions*

The ring $\mathbb{C}[x]$ has a natural structure of left A_n-module (see Remark 1.2) since each linear differential operator $P = \sum_\beta p_\beta(x)\partial^\beta \in A_n$ acts on any polynomial $g \in \mathbb{C}[x]$ just by

$$P(g) = \sum p_\beta(x) \frac{\partial^{\beta_1+\cdots+\beta_n}(g)}{\partial x_1^{\beta_1}\cdots \partial x_n^{\beta_n}} \in \mathbb{C}[x].$$

Let us consider $\mathbb{C}(x)$ the fraction field of the domain $\mathbb{C}[x]$. Elements in $\mathbb{C}(x)$ are rational functions, that is, quotients $\frac{g}{f}$ of polynomials $g, f \in \mathbb{C}[x]$ with $f \neq 0$.

For any of these rational functions $\frac{g}{f}$ its partial derivative $\partial_i(\frac{g}{f})$ is nothing but

$$\frac{\partial_i(g)f - g\partial_i(f)}{f^2}$$

and then it is also a rational function. This can be extended to an action of A_n on $\mathbb{C}(x)$ defining $\mathbb{C}(x)$ as a left A_n–module.

For any nonzero polynomial $f = f(x) \in \mathbb{C}[x]$ the ring

$$\mathbb{C}[x]_f = \{\frac{g}{f^m} \in \mathbb{C}(x) \,|\, g \in \mathbb{C}[x],\, m \in \mathbb{N}\}$$

is a $\mathbb{C}[x]$-module. In fact, $\mathbb{C}[x]_f$ has also a natural structure of left A_n-module since $\partial_i(\frac{g}{f^m}) \in \mathbb{C}[x]_f$ for all $g \in \mathbb{C}[x]$ and all $m \in \mathbb{N}$.

If $f \in \mathbb{C} \setminus \{0\}$ then $\mathbb{C}[x]_f$ is simply the ring $\mathbb{C}[x]$. If f is not a constant, then the elements in $\mathbb{C}[x]_f$ are called rational functions with poles along the affine hypersurface $\mathcal{V}(f) := \{a \in \mathbb{C}^n \,|\, f(a) = 0\}$.

If $f \in \mathbb{C}[x] \setminus \mathbb{C}$ then the $\mathbb{C}[x]$–module $\mathbb{C}[x]_f$ is not finitely generated. Nevertheless, we have

Theorem 1.1 (J. Bernstein[6]). *The A_n–module $\mathbb{C}[x]_f$ is finitely generated.*

This theorem was done by J. Bernstein.[6] It is deeply related to the existence of the global Bernstein (or global Bernstein-Sato) polynomial associated with f. We will give a stronger version of this result in Theorem 2.7.

Bibliographical note

Most of the material of this Section appears in the already cited article Bernstein[6] and in the books by J.E. Björk[7] and by S.C. Coutinho.[21]

2. Characteristic Variety. Holonomic A_n–Modules

2.1. *Classical characteristic vectors*

Let us consider a linear partial differential equation

$$P(x,\partial)(u) = \left(\sum_{\beta \in \mathbb{N}^n} p_\beta(x) \partial^\beta \right)(u) = v$$

with polynomial coefficients $p_\beta(x) \in \mathbb{R}[x] := \mathbb{R}[x_1, \ldots, x_n]$.

A vector $\xi_0 \in \mathbb{R}^n$ is called *characteristic* for P at a point $x_0 \in \mathbb{R}^n$ if $\sigma(P)(x_0, \xi_0) = 0$. Here $\sigma(P)$ is the principal symbol of P (see Definition 1.3). The set of all such ξ_0 is called the *characteristic variety* of the operator P (or of the equation $P(u) = v$) at $x_0 \in \mathbb{R}^n$ and is denoted by $\mathrm{Char}_{x_0}(P)$.

Notice that here, in contrast to some textbooks, the zero vector could be characteristic. More generally, the *classical characteristic variety* of the operator P is by definition the set

$$\mathrm{Char}(P) = \{(x_0, \xi_0) \in \mathbb{R}^n \times \mathbb{R}^n \mid \sigma(P)(x_0, \xi_0) = 0\}.$$

Assume $\mathrm{ord}(P) \geq 1$, then P is said to be *elliptic* at a point $x_0 \in \mathbb{R}^n$ if P has no nonzero characteristic vectors at x_0 (i.e. if $\mathrm{Char}_{x_0}(P) \subset \{0\}$) and it is said to be *elliptic* (on $\mathbb{R}^n$) if $\mathrm{Char}(P) \subset \mathbb{R}^n \times \{0\}$.

The *Laplace operator* $\sum_{i=1}^n \partial_i^2$ is elliptic on $\mathbb{R}^n$.

The characteristic variety of the *wave operator* $P = \partial_1^2 - \sum_{i=2}^n \partial_i^2$ is nothing but the hyperquadric defined in $\mathbb{R}^n \times \mathbb{R}^n$ by the equation $\xi_1^2 - \sum_{i=2}^n \xi_i^2 = 0$.

Characteristic vectors are important in the study of singularities of solutions as can be seen in any classical book on Differential Equations.

To define the characteristic vectors for a Linear Partial Differential System

$$\begin{array}{ccccc} P_{11}(u_1) & + \cdots + & P_{1m}(u_m) & = & v_1 \\ \vdots & \vdots & \vdots & & \vdots \\ P_{\ell 1}(u_1) & + \cdots + & P_{\ell m}(u_m) & = & v_\ell \end{array} \tag{1}$$

is more involved and in general the naive approach of simply considering the principal symbols of the equations turns out to be unsatisfactory (see Example 2.1).

We will use *graded ideals* and Groebner bases for Linear Differential Operators (see Appendix A and Subsection 2.2) to define and to compute the *characteristic variety* of a system as in (1). Here ℓ, m are nonzero integers, P_{ij} are Linear Differential Operators, u_j are unknown and v_i are given data (e.g. functions, distributions, hyperfunctions, ...).

2.2. *Characteristic variety*

Let us assume that the operators P_{ij} in the System (1) are in the Weyl algebra A_n and remember we are assuming A_n to be defined over the complex field $\mathbb{C}$ (see Subsection 1.1). With the System (1) we associate the quotient A_n–module

$$\frac{A_n^m}{A_n(P_1, \ldots, P_\ell)}$$

where $P_i = (P_{i1}, \ldots, P_{im}) \in A_n^m$ and $A_n(P_1, \ldots, P_\ell)$ denotes the submodule of A_n^m generated by $P_1, \ldots, P_\ell$.

Let us assume first that System (1) has only one unknown $u = u_1$ so that the associated A_n–module is nothing but $\frac{A_n}{A_n(P_1,\ldots,P_\ell)}$.

If $J \subset \mathbb{C}[x, \xi] = \mathbb{C}[x_1, \ldots, x_n, \xi_1, \ldots, \xi_n]$ is a polynomial ideal we denote by $\mathcal{V}_{\mathbb{C}}(J)$ (or simply by $\mathcal{V}(J)$) the *affine algebraic variety* defined in $\mathbb{C}^{2n}$ by J, that is

$$\mathcal{V}_{\mathbb{C}}(J) = \mathcal{V}(J) = \{(a, b) \in \mathbb{C}^{2n} \mid g(a, b) = 0, \forall g(x, \xi) \in J\}.$$

Recall that for any (left) ideal $I \subset A_n$ we denote by $\mathrm{gr}(I)$ the ideal of $\mathbb{C}[x, \xi]$ generated by the family of principal symbols of elements in I (see Definition 1.4).

Definition 2.1. Let $I \subset A_n$ be a left ideal. The *characteristic variety* of the left A_n–module A_n/I is defined as

$$\mathrm{Char}(A_n/I) = \mathcal{V}_{\mathbb{C}}(\mathrm{gr}(I)).$$

Remark 2.1. If $I = A_n P$ is a principal ideal then the characteristic variety of A_n/I coincides with the classical characteristic variety of the operator

P since in this case the graded ideal $\mathrm{gr}(I) \subset \mathbb{C}[x, \xi]$ is generated by the principal symbol $\sigma(P)$.

In general, if the ideal $I \subset A_n$ is generated by a family $P_1, \ldots, P_m$, the ideal $\mathrm{gr}^F(I)$ could be strictly bigger than the ideal generated by $\sigma(P_1), \ldots, \sigma(P_m)$. One example is given just bellow.

Example 2.1. If $I = A_2(P_1, P_2)$ with $P_1 = x_1\partial_1 + x_2\partial_2$ and $P_2 = x_1\partial_2 + x_2^2\partial_1$ then $\mathrm{gr}(I) = \langle \xi_1, \xi_2 \rangle$. This ideal strictly contains the ideal $\langle \sigma(P_1), \sigma(P_2) \rangle = \langle x_1\xi_1 + x_2\xi_2, x_1\xi_2 + x_2^2\xi_1 \rangle$.

The following `Macaulay 2` script can be used to compute generators of $\mathrm{gr}(I)$ by using the `Macaulay 2` command `charIdeal`. We need the `D-modules.m2` package to this end (see Ref. 28).

The input command lines are `i1` :, `i2` : ... while the output ones are `o4` =, `o5` =, ... (a semicolon `;` at the end of an input line prevents it from being printed).

Input `i1` defines the ring `R` as the polynomial ring in the variables x, y with coefficients in $\mathbb{Q}$ (we are using $x = x_1$, $y = x_2$).

Input `i2` loads the `D-modules.m2` package. Input `i3` defines `W` as the 2^{nd} Weyl algebra over the field $\mathbb{Q}$ (i.e. the algebra of linear differential operators with polynomial coefficients in `R`).

Inputs `i4` and `i5` define the operators P_1 and P_2 as above and the (left) ideal `I` in `W` generated by these two operators. In the Weyl algebra `W` the expressions `dx, dy` stand for ∂_1, ∂_2 respectively.

The `Macaulay 2` expression `charIdeal I` computes a generating system of the graded ideal $\mathrm{gr}(I)$. Notice the additional line of output labelled with `o6` : Output lines labelled with colons (:) provide information about the type of output. In this case, the symbol `QQ [x, y, dx, dy]` denotes the graded ring associated with the F–filtration on the Weyl algebra `W` (see Subsection 1.4). In particular the expressions `dx, dy`, when considered in the polynomial ring `QQ [x, y, dx, dy]`, stand for the commutative variables ξ_1, ξ_2.

Output `o6` = shows that $\mathrm{gr}(I) = \langle \xi_1, \xi_2 \rangle \subset \mathbb{Q}[x_1, x_2, \xi_1, \xi_2]$.

```
Macaulay 2, version 1.2 with packages: Elimination,
IntegralClosure, LLLBases, PrimaryDecomposition, ReesAlgebra,
SchurRings, TangentCone

i1 : load "D-modules.m2";

i2 : R=QQ[x,y];

i3 : W=makeWA R;

i4 : P1=x*dx+y*dy, P2=x*dy+y^2*dx
```

```
                          2
o4 = (x*dx + y*dy, y dx + x*dy)

o4 : Sequence

i5 : I=ideal(P1,P2)

                                2
o5 = ideal (x*dx + y*dy, y dx + x*dy)

o5 : Ideal of W

i6 : charIdeal I

o6 = ideal (dy, dx)

o6 : Ideal of QQ[x, y, dx, dy]
```

Let us add some extra computation in `Macaulay 2`. The following script gives a new insight into the previous ideal `I`.

Input `i7:` defines `J` as the (left) ideal in `W` generated by ∂_1, ∂_2. The output `o7 :` tells us that the (left) ideal `J` is considered in the ring `W`. Notice the similarity of the lines `o6 =` and `o7 =` although the first output represents an ideal in a polynomial ring and the second one an ideal in a Weyl algebra.

The string `==` stands for the binary operator testing equality of ideals. The last part of the script proves the equality `I = J` in `W` giving a new explanation for the equality $\mathrm{gr}(I) = \langle \xi_1, \xi_2 \rangle \subset \mathbb{C}[x_1, x_2, \xi_1, \xi_2]$.

```
i7 : J=ideal(dx,dy)

o7 = ideal (dx, dy)

o7 : Ideal of W

i8 : J==I

o8 = true
```

Remark 2.2. Let M be an A_n–module provided with an F–filtration (resp. a B–filtration) $\Gamma = (M_k)_k$. The annihilating ideal $Ann_{\mathrm{gr}^F(A_n)}(\mathrm{gr}^\Gamma(M))$ is a ξ–homogeneous ideal in $\mathrm{gr}^F(A_n)$ (resp. a homogeneous ideal in $\mathrm{gr}^B(A_n)$).

We will prove it for the F–filtration (the other case being analogous). To this end, let us consider a nonzero element $G = G(x, \xi) \in \mathrm{gr}^F(A_n) = \mathbb{C}[x, \xi]$ (see Proposition 1.10) annihilating $\mathrm{gr}^\Gamma(M)$. We can write $G = \sum_j G_j$ as sum of its ξ–homogeneous components $G_j \in \mathbb{C}[x, \xi]_{(j)}$ (see Remark 1.8). Let P_j be an element in $F_j(A_n)$ such that $\sigma(P_j) = P_j + F_{j-1}(A_n) = G_j$.

For each $k \in \mathbb{N}$ we have $G\frac{M_k}{M_{k-1}} = 0$ and then $(\sum_j (P_j + F_{j-1}(A_n))\frac{M_k}{M_{k-1}} = 0$. Then for each $j, k \in \mathbb{N}$ we have $P_j M_k \subset M_{j+k-1}$ and therefore $G_j = P_j + F_{j-1}(A_n)$ annihilates $\frac{M_k}{M_{k-1}}$ for all k.

Then $G_j \in Ann_{\mathrm{gr}^F(A_n)}(\mathrm{gr}^\Gamma(M))$ for each j. This proves that the ideal $Ann_{\mathrm{gr}^F(A_n)}(\mathrm{gr}^\Gamma(M))$ is ξ–homogeneous.

Proposition 2.1. *Let M be a finitely generated A_n–module provided with two good F–filtrations $\Gamma = (M_k)_k$ and $\Gamma' = (M'_k)_k$) (see Subsection 1.10). Then*

$$\sqrt{Ann_{\mathrm{gr}^F(A_n)}(\mathrm{gr}^\Gamma(M))} = \sqrt{Ann_{\mathrm{gr}^F(A_n)}(\mathrm{gr}^{\Gamma'}(M))}.$$

Proof. Let's write

$$J = Ann_{\mathrm{gr}^F(A_n)}(\mathrm{gr}^\Gamma(M))$$

and

$$J' = Ann_{\mathrm{gr}^F(A_n)}(\mathrm{gr}^{\Gamma'}(M)).$$

By symmetry it is enough to prove the inclusion $\sqrt{J} \subset \sqrt{J'}$. By Remark 2.2, the ideal J is homogeneous. Then also $\sqrt{J}$ is homogeneous. Let $G \in \sqrt{J}$ be a ξ–homogeneous element of ξ–degree $\nu \geq 0$. There exists an integer $\ell > 0$ such that $G^\ell \in J$. Let P be an element in $F_\nu(A_n)$ such that $\sigma(P) = G$. We have $P^\ell M_k \subset M_{\ell\nu+k-1}$ and then, for all $p \in \mathbb{N}$ we also have

$$P^{p\ell} M_k \subset M_{p\ell\nu+k-p}.$$

By Proposition 1.17 there exists $k_2 \in \mathbb{N}$ such that $M_{k-k_2} \subset M'_k \subset M_{k+k_2}$ for all $k \in \mathbb{N}$. In particular, for $p = 2k_2 + 1$ and for all $k \in \mathbb{N}$ we have

$$P^{(2k_2+1)\ell} M'_k \subset P^{(2k_2+1)\ell} M_{k+k_2} \subset M_{(2k_2+1)\ell\nu+k+k_2-2k_2-1} \subset M'_{(2k_2+1)\ell\nu+k-1}.$$

This proves that $\sigma(P^{(2k_2+1)\ell}) = G^{(2k_2+1)\ell}$ annihilates $\mathrm{gr}^{\Gamma'}(M)$ and then $G \in \sqrt{J'}$. □

The following Proposition can be proven similarly to the previous one.

Proposition 2.2. *Let M be a finitely generated A_n–module provided with two good B–filtrations $\Gamma = (M_k)_k$ and $\Gamma' = (M'_k)_k$) (see Subsection 1.10). Then*

$$\sqrt{Ann_{\mathrm{gr}^B(A_n)}(\mathrm{gr}^\Gamma(M))} = \sqrt{Ann_{\mathrm{gr}^F(B_n)}(\mathrm{gr}^{\Gamma'}(M))}.$$

Definition 2.2. Let M be a finitely generated A_n–module. The characteristic variety of M is defined as

$$\mathrm{Char}(M) = \mathcal{V}(Ann_{\mathrm{gr}^F(A_n)}(\mathrm{gr}^\Gamma(M)))$$

for a good F–filtration Γ on M.

By Proposition 2.1 the definition of $\mathrm{Char}(M)$ doesn't depend on the choice of the good F–filtration.

Remark 2.3. If $M = A_n/I$ for some ideal I in A_n then by Corollary 1.3 we have

$$\mathrm{gr}^F\left(\frac{A_n}{I}\right) \simeq \frac{\mathrm{gr}^F(A_n)}{\mathrm{gr}^F(I)}$$

and then both Definitions 2.1 and 2.2 coincide.

2.3. *Dimension of an A_n–module*

The Krull dimension of a *Zariski closed subset* Z in $\mathbb{C}^n$ (denoted by $\dim(Z)$) is by definition[d] the maximum of the lengths m of decreasing chains

$$Z \supseteq Z_0 \supset \cdots \supset Z_m$$

of irreducible Zariski closed subsets in Z.

By Hilbert's Nullstellensatz the Krull dimension of Z equals the Krull dimension of the $\mathbb{C}$–algebra $\mathbb{C}[x]/J$ if $J \subset \mathbb{C}[x]$ is any ideal verifying $\mathcal{V}(J) = Z$. The Krull dimension of a ring[e] is the maximum of the lengths of chains of prime ideals in the ring (see e.g. [30, Chapter I, Proposition 1.7]). By convention the Krull dimension of the empty set (and of the zero ring) is -1.

Krull dimension can be calculated by using Groebner basis computations in polynomial rings (see e.g. [24, Section 15.10.2]). For example, in `Macaulay 2` the string `dim I` computes the dimension of the quotient ring `R/I` if `I` is an ideal in the ambient polynomial ring `R`.

```
i1 : R=QQ[x,y,z];

i2 : f=x^2*y^2+x*z-y; g=x^2+y^2+z^2;

i4 : I=ideal(f,g)

                2 2              2    2    2
o4 = ideal (x y  + x*z - y, x  + y  + z )

o4 : Ideal of R

i5 : dim I

o5 = 1
```

[d]See e.g. [30, Ch.I, page 5]

[e]i.e. a commutative ring with unit

In the previous script, output `o5` tells us that the Krull dimension of the quotient ring $\frac{\mathbb{Q}[x,y,z]}{I}$ is 1, the ideal I being generated by polynomials $f = x^2y^2 + xz - y$ and $g = x^2 + y^2 + z^2$.

Definition 2.3. Let M be a finitely generated A_n–module. The dimension of M (denoted $\dim(M)$) is the Krull dimension of $\mathrm{Char}(M)$ the characteristic variety of M.

Example 2.2. Assume $I = A_nP$ for some $P \in A_n$. Then if P is a nonzero constant then $A_n/I = (0)$ and its dimension is -1. If $P = 0$ then $\mathrm{Char}(A_n/I) = \mathrm{Char}(A_n) = \mathbb{C}^n \times \mathbb{C}^n$ and then its dimension is $2n$. If $P \in A_n \setminus \mathbb{C}$ then $\sigma(P) \in \mathbb{C}[x, \xi]$ is a non-constant polynomial and the Krull dimension of $\mathrm{Char}(A_n/I) = \mathcal{V}_{\mathbb{C}}(\sigma(P))$ is $2n - 1$. Thus, in this case, $\dim(A_n/I) = 2n - 1$.

If `I` is an ideal in a Weyl algebra A_n, the `Macaulay 2` command `dim I` computes the dimension of the quotient module $A_n/$`I`.

The following `Macaulay 2` script shows that the dimension of the A_3–module A_3/I is 3, where $I \subset A_3$ is the ideal generated by the two operators $P = x^3ydxdy + dz$ and $Q = ydydz - zy$. We are using as usual x, y, z instead of x_1, x_2, x_3 and dx, dy, dz instead of $\partial_1, \partial_2, \partial_3$.

Input `i3` defines `W` as the Weyl algebra over the polynomial ring in the three variables `x,y,z` and with rational coefficients. Output `o5` shows that the dimension of `W/I` is 3. This last result is somehow unexpected because the ideal `I` is defined by 2 elements in a Weyl algebra over three variables.

```
i1 : load "D-modules.m2";

i2 : R=QQ[x,y,z];

i3 : W=makeWA R;

i4 : P=x^3*y*dx*dy+dz,Q=y*dy*dz-z*y;

i5 : I = ideal (P,Q);

i6 : dim I

o6 = 3

i7 : charIdeal I

                     3
o7 =  ideal (dz, y, x dx)

o7 : Ideal of QQ[x, y, z, dx, dy, dz]
```

Output `o7 =` gives a system of polynomials defining the characteristic variety of the module `W/I`. From this computation it is clear that the Krull dimension of this variety is 3.

2.4. *Hilbert polynomial*

Let M be a finitely generated A_n–module provided with a B–filtration $\Gamma = (M_k)_k$. Let us denote by

$$gHF_{\mathrm{gr}^\Gamma(M)} : \mathbb{N} \longrightarrow \mathbb{N}$$

the *Hilbert function*[f] of the $\mathrm{gr}^B(A_n)$–module $\mathrm{gr}^\Gamma(M)$. By definition we have

$$gHF_{\mathrm{gr}^\Gamma(M)}(\nu) = \dim_{\mathbb{C}}\left(\frac{M_\nu}{M_{\nu-1}}\right)$$

for all $\nu \in \mathbb{N}$. Let us notice that by Definition 1.6 each M_ν (and hence each quotient $M_\nu/M_{\nu-1}$) is a finite dimensional vector space.

Theorem 2.1 (Hilbert, Serre). *With the notation above, there exists a unique polynomial $gHP_{\mathrm{gr}^\Gamma(M)}(t) \in \mathbb{Q}[t]$ such that $gHF_{\mathrm{gr}^\Gamma(M)}(\nu) = gHP_{\mathrm{gr}^\Gamma(M)}(\nu)$ for $\nu \in \mathbb{N}$, ν big enough. Furthermore, the degree of $gHP_{\mathrm{gr}^\Gamma(M)}(t)$ equals $d-1$ where $d = \dim\left(\mathcal{V}(Ann_{\mathrm{gr}^B(A_n)}(\mathrm{gr}^\Gamma(M)))\right)$. Moreover the degree of $gHP_{\mathrm{gr}^\Gamma(M)}(t)$ is less than or equal to $2n-1$.*

Proof. See [44, Ch. VII, §12] or [30, Ch.1, Th. 7.5]. □

Definition 2.4. The polynomial $gHP_{\mathrm{gr}^\Gamma(M)}(t) \in \mathbb{Q}[t]$ is called the *Hilbert polynomial* of the graded module $\mathrm{gr}^\Gamma(M)$.

Remark 2.4. We will denote by

$$HF_{M,\Gamma} : \mathbb{N} \longrightarrow \mathbb{N}$$

the map defined by

$$HF_{M,\Gamma}(\nu) = \sum_{k=0}^{\nu} gHF_{\mathrm{gr}^\Gamma(M)}(k)$$

for all $\nu \in \mathbb{N}$.

By induction on ν and using the exact sequence

$$0 \to M_{\nu-1} \to M_\nu \to \frac{M_\nu}{M_{\nu-1}} \to 0$$

it is easy to prove that $HF_{M,\Gamma}(\nu) = \dim_{\mathbb{C}}(M_\nu)$.

[f]Also called *characteristic function* in [44, Ch. VII, §12]

If $\phi : \mathbb{Z} \to \mathbb{Z}$ is any map, we denote by $\Delta\phi : \mathbb{Z} \to \mathbb{Z}$ the map defined by $\Delta\phi(\nu) = \phi(\nu+1) - \phi(\nu)$.

By definition we have $\Delta HF_{M,\Gamma}(\nu) = gHF_{\mathrm{gr}^\Gamma(M)}(\nu+1)$.

Definition 2.5. A *numerical polynomial* is a polynomial $Q(t) \in \mathbb{Q}[t]$ (in one variable t) such that $Q(\nu) \in \mathbb{Z}$ for all $\nu \in \mathbb{Z}$, ν big enough.

Proposition 2.3.

i) *If $Q(t) \in \mathbb{Q}[t]$ is a numerical polynomial, then there are integers $c_0, \dots, c_d$ such that*

$$Q(t) = \sum_{k=0}^{d} c_k \binom{t}{k}$$

where

$$\binom{t}{k} = \frac{t(t-1)\cdots(t-k+1)}{k!}.$$

ii) *If $\phi : \mathbb{Z} \to \mathbb{Z}$ is any map, and if there exists a numerical polynomial $Q(t)$ such that $\Delta\phi(\nu) = Q(\nu)$ for $\nu \in \mathbb{N}$, ν big enough, then there exists a numerical polynomial $R(t)$ such that $\phi(\nu) = R(\nu)$ for $\nu \in \mathbb{N}$, ν big enough. Moreover, if the leading term of $Q(t)$ is $a_d t^d$ then the leading term of $R(t)$ is $\frac{a_d}{d+1} t^{d+1}$.*

Proof. See [44, Ch. VII, §12] or [30, Ch.1, Proposition 7.3]. □

Corollary 2.1. *Let M be a finitely generated A_n–module provided with a B–filtration $\Gamma = (M_k)_k$. There exists a unique polynomial $HP_{M,\Gamma}(t) \in \mathbb{Q}[t]$ such that $HF_{M,\Gamma}(\nu) = HP_{M,\Gamma}(\nu)$ for all $\nu \in \mathbb{N}$, ν big enough. Moreover, the degree of $HP_{M,\Gamma}(t)$ equals the Krull dimension of $\mathcal{V}_{\mathbb{C}}(Ann_{\mathrm{gr}^B(A_n)}(\mathrm{gr}^\Gamma(M)))$ and hence it is less than or equal to $2n$.*

Proof. It follows from Theorem 2.1, Proposition 2.3 and the fact that $\Delta HF_{M,\Gamma}(\nu) = gHF_{\mathrm{gr}^\Gamma(M)}(\nu+1)$ for all $\nu \in \mathbb{N}$. □

Definition 2.6. The polynomial $HP_{M,\Gamma}(t) \in \mathbb{Q}[t]$ is called the *Hilbert polynomial* of M with respect to the B–filtration Γ.

Proposition 2.4. *Let M be a finitely generated A_n–module provided with two good B–filtrations $\Gamma = (M_k)_k$ and $\Gamma' = (M'_k)_k$. Then the leading terms of $HP_{M,\Gamma}(t)$ and $HP_{M,\Gamma'}(t)$ coincide. In particular* $\deg(HP_{M,\Gamma}(t)) = \deg(HP_{M,\Gamma'}(t))$.

Proof. Assume that

$$HP_{M,\Gamma}(t) = a_d t^d + (\text{ lower terms in } t)$$

and

$$HP_{M,\Gamma'}(t) = a'_{d'} t^{d'} + (\text{ lower terms in } t)$$

with a_d and $a'_{d'}$ nonzero. By Proposition 1.17 there exists $k_2 \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ we have $M_{k-k_2} \subset M'_k \subset M_{k+k_2}$. Then for k big enough we have

$$HP_{M,\Gamma}(k-k_2) \leq HP_{M,\Gamma'}(k) \leq HP_{M,\Gamma}(k+k_2).$$

Dividing by k^d and taking the limit when $k \to \infty$ we get $d = d'$ and $a_d = a'_{d'}$. ◻

Definition 2.7. Let M be a finitely generated A_n–module. The *multiplicity* of M is $e(M) = a_d \cdot d!$ where $a_d t^d$ is the leading term of the polynomial $HP_{M,\Gamma}(t)$ for a (or any) good B–filtration Γ on M. [g]

Remark 2.5. If $M \neq (0)$ then the multiplicity $e(M)$ is a strictly positive integer. It follows from Proposition 2.3 and the fact that $HP_{M,\Gamma}(k) \in \mathbb{N}$ for k big enough.

An important result relating the dimension of a finitely generated A_n–module and the degree of its Hilbert polynomial, with respect to any good B–filtration, is the following

Theorem 2.2 (Th. 3.1., Bernstein[5]). *Let M be a finitely generated A_n–module provided with a good B–filtration Γ. Then* $\dim(M) = \deg(HP_{M,\Gamma}(t))$.

Concerning the statement of this Theorem let us remark that while the Hilbert polynomial $HP_{M,\Gamma}(t)$ and the multiplicity $e(M)$ are defined using a good B–filtration Γ of M, the dimension of M, $\dim(M)$, which is the Krull dimension of the characteristic variety $\mathrm{Char}(M)$, is defined using a good F–filtration on M.

To each good B–filtration $\Gamma = (M_k)_k$ on a finitely generated A_n–module M, we can also associate the algebraic variety defined in $\mathbb{C}^{2n}$ by the homogeneous ideal $Ann_{\mathrm{gr}^B(A_n)}(\mathrm{gr}^\Gamma(M))$ of the graded ring $\mathrm{gr}^B(A_n)$ (see Proposition 1.9).

[g] The notation $e(M)$ appears in [6, Def. 1.1].

Let us denote $\nabla(M) := \mathcal{V}_{\mathbb{C}}\left(Ann_{\mathrm{gr}^B(A_n)}(\mathrm{gr}^{\Gamma}(M))\right)$ which is an homogeneous affine algebraic variety in the affine space $\mathbb{C}^{2n}$.

By Proposition 2.2 the algebraic variety $\nabla(M)$ is independent of the choice of the good B–filtration Γ on M.

The variety $\nabla(M)$ can be different from the characteristic variety $\mathrm{Char}(M)$. The following is an example of this situation. Let's consider $P = x_1^2 + \partial_1^2$ in the Weyl algebra $A_1(\mathbb{C})$ and define $M = \frac{A_1}{A_1 P}$. Then the characteristic variety of M is the line $\xi_1 = 0$ in the plane $\mathbb{C}^2$ (with coordinates x_1, ξ_1) while $\nabla(M)$ is $\mathcal{V}(x_1^2 + \xi_1^2) \subset \mathbb{C}^2$.

Exercise 2.1.

i) Let $I = A_n P$ a proper principal ideal in A_n. Compute the dimension $\dim(A_n/I)$ and the multiplicity $e(A_n/I)$ (see Example 2.2).
ii) Prove that the dimension of the A_n–module $\mathbb{C}[x]$ is n and that its multiplicity is 1.
iii) Prove that $e(A_n) = 1$.

Quick answer.- i) We have $\dim(A_n/I) = \dim(\mathcal{V}(\sigma(P))) = 2n - 1$ (see Example 2.2). The leading term of the Hilbert polynomial of the graded $\mathrm{gr}^B(A_n)$–module

$$\mathrm{gr}^{\Gamma''}(A_n/I) \simeq \frac{\mathrm{gr}^B(A_n)}{\langle \sigma^T(P)\rangle}$$

is $\frac{d}{(2n-2)!}t^{2n-2}$ where $d = \mathrm{ord}^T(P)$. Here Γ'' stands for the induced B–filtration on A_n/I. Thus $e(A_n/I) = d$.
ii) We have an isomorphism $\mathbb{C}[x] \simeq \frac{A_n}{A_n(\partial_1,\ldots,\partial_n)}$ (see Remark 1.2). It is easy to prove that the ideal $\mathrm{gr}^B(A_n(\partial_1,\ldots,\partial_n)) \subset \mathbb{C}[x,\xi]$ is generated by $(\xi_1,\ldots,\xi_n)$. The Hilbert polynomial of the graded module $\mathbb{C}[x,\xi]/\langle \xi_1,\ldots,\xi_n\rangle \simeq \mathbb{C}[x]$ is $\begin{pmatrix} t+n-1 \\ n-1 \end{pmatrix}$ and then $e(\mathbb{C}[x]) = 1$.
iii) It follows from the fact that the Hilbert polynomial of the graded module $\mathbb{C}[x,\xi] \simeq \mathrm{gr}^B(A_n)$ is $\begin{pmatrix} t+2n-1 \\ 2n-1 \end{pmatrix}$.

Theorem 2.3. *Let M be a finitely generated A_n–module and $N \subset M$ a submodule. Then*

i) $\dim(M) = \max\{\dim(N), \dim(M/N)\}$.
ii) *If* $\dim(N) = \dim(M/N)$ *then* $e(M) = e(N) + e(M/N)$.
iii) $\dim(M) \leq 2n$.

Proof. Let us consider $\Gamma = (M_k)_k$ a good B–filtration on M and denote by Γ' and Γ'' the induced B–filtrations on N and M/N respectively (see Subsection 1.9). From the proof of Proposition 1.13 we have $HF_{M,\Gamma}(k) = HF_{N,\Gamma'}(k) + HF_{M/N,\Gamma''}(k)$ for all $k \in \mathbb{N}$. Then, for k big enough, we have $HP_{M,\Gamma}(k) = HP_{N,\Gamma'}(k) + HP_{M/N,\Gamma''}(k)$ and thus

$$HP_{M,\Gamma}(t) = HP_{N,\Gamma'}(t) + HP_{M/N,\Gamma''}(t).$$

The last equality proves *i)* and *ii)*. Part *iii)* follows from the very definition of $\dim(M)$ since the Krull dimension of any algebraic set in $\mathbb{C}^{2n}$ is less than or equal to $2n$. □

2.5. *Bernstein's inequality*

Theorem 2.4 (Bernstein's inequality). *Let M be a nonzero finitely generated A_n-module. Then* $\dim(M) \geq n$.

Proof. [A. Joseph's proof].
Claim.- Let M be a finitely generated A_n–module and let $(M_k)_k$ be a B–filtration with $M_0 \neq 0$. Then the $\mathbb{C}$-linear map

$$\phi_i : B_i \rightarrow Hom_{\mathbb{C}}(M_i, M_{2i})$$

defined by $\phi_i(P)(m) = Pm$ is injective for all $i \geq 0$.

Let's assume the claim. Let $m_1, \ldots, m_\ell$ be a finite system of generators of M and $\Gamma = (M_k)_k$ the good B-filtration defined by $M_k = \sum_j B_k m_j$. (see Exercise 1.6). Then $M_0 = \sum_j \mathbb{C}m_j \neq 0$. From the claim we have

$$\dim_{\mathbb{C}}(B_k) \leq \dim_{\mathbb{C}}(Hom_{\mathbb{C}}(M_k, M_{2k})) = \dim_{\mathbb{C}}(M_k)\dim_{\mathbb{C}}(M_{2k}).$$

For k a big enough integer we have

$$\binom{2n+k}{k} = \dim_{\mathbb{C}}(B_k)) \leq HP_{M,\Gamma}(k)HP_{M,\Gamma}(2k)$$

and so $2\deg(HP_{M,\Gamma}(t)) \geq 2n$ and $\dim(M) \geq n$. Here $HP_{M,\Gamma}(t)$ is the Hilbert polynomial of M with respect to the B–filtration Γ.

Let's prove the claim by induction on i. For $i = 0$ we have $B_0 = \mathbb{C}$ and for $\lambda \in \mathbb{C} \setminus \{0\}$ the $\mathbb{C}$-linear map $\phi_0(\lambda)$ is nonzero since $M_0 \neq (0)$ (by assumption). Assume $i > 0$ and the claim proved for $i-1$. Let $P \in B_i$ nonzero. Assume $PM_i = (0)$. Since $M_i \neq (0)$ then P is non-constant. Therefore, either there exists $j \in \{1, \ldots, n\}$ such that x_j appears in at least one monomial in P or there exists $k \in \{1, \ldots, n\}$ such that ∂_k appears in at least one monomial in P. In the first case we have $[P, \partial_j] \neq 0$ and in the second one we have $[P, x_k] \neq 0$.

By Proposition 1.4 we have that $[P, \partial_j]$ and $[P, x_k]$ belong to B_{i-1}. Moreover

$$[P, \partial_j]M_{i-1} \subset P\partial_j M_{i-1} + \partial_j P M_{i-1} = (0)$$

and analogously $[P, x_k]M_{i-1} = (0)$. Then by induction hypothesis $[P, \partial_j]$ and $[P, x_k]$ should be zero. Which is a contradiction. That proves the claim. $\square$

Exercise 2.2. Compute the dimension and the multiplicity of the quotient A_n–module $\frac{A_n}{A_n(\partial_{k+1},\ldots,\partial_n)}$ for each $k = 0, \ldots, n-1$.

Quick answer.- Let's write $I_k = A_n(\partial_{k+1}, \ldots, \partial_n)$ and $M^{(k)} = \frac{A_n}{I_k}$. It is easy to prove (e.g. using Buchberger's algorithm in A_n, see Appendix A and Remark 4.11) that $\mathrm{gr}^B(I_k) = \mathbb{C}[x, \xi](\xi_{k+1}, \ldots, x_n)$. The Hilbert polynomial of the graded quotient module

$$\frac{\mathrm{gr}^B(A_n)}{\mathrm{gr}^B(I_k)} \simeq \frac{\mathbb{C}[x, \xi]}{\mathbb{C}[x, \xi](\xi_{k+1}, \ldots, x_n)}$$

is $\binom{t+n+k-1}{n+k-1}$. Then the leading coefficient of the Hilbert polynomial $HP(t)$ of the A_n–module $\frac{A_n}{I_k}$ is $\frac{t^{n+k}}{(n+k)!}$. So, $\dim(M^{(k)}) = n+k$ and $e(M^{(k)}) = 1$.

2.6. *Holonomic A_n–modules*

Definition 2.8. A finitely generated A_n–module M is said to be *holonomic* if either $M = (0)$ or $\dim(M) = n$.

Remark 2.6. For $P \in A_n \setminus \mathbb{C}$ the quotient A_n/A_nP is holonomic if and only if $n = 1$ (see Example 2.2).

Example 2.3.

(1) Let I be a proper ideal in A_1 and P a nonzero element in I. Let us write $J = A_1P$. Let us consider the exact sequence of finitely generated A_1–modules

$$0 \longrightarrow I/J \longrightarrow A_1/J \longrightarrow A_1/I \longrightarrow 0.$$

The quotient A_1/J is holonomic (see Remark 2.6). By applying Theorems 2.3 and 2.4 we get that A_1/I is holonomic. However, the ideal I is not holonomic (considered as A_1–module): otherwise, using the exact sequence of A_n–modules

$$0 \longrightarrow I \longrightarrow A_1 \longrightarrow A_1/I \longrightarrow 0$$

one gets that A_1 is also holonomic and this is not true because $\dim A_1 = 2$.

(2) Assume that M is a finitely generated A_1–module, say $M = \sum_{\ell=1}^{r} A_1 m_\ell$, for some $m_1, \ldots, m_r \in M$. Then M is the sum of modules of type A_1/I_ℓ where $I_\ell = Ann_{A_1}(m_\ell)$. From Theorem 2.3 we have that M is holonomic if and only if all the I_ℓ are nonzero.

(3) The A_n–module $\mathbb{C}[x] = \mathbb{C}[x_1, \ldots, x_n]$ is isomorphic to $\frac{A_n}{A_n(\partial_1,\ldots,\partial_n)}$ and then it is holonomic (see Exercise 2.1).

Theorem 2.5.

(1) Let M be a finitely generated A_n–module and N a submodule of M. Then M is holonomic if and only if N and M/N are holonomic. If M is holonomic then $e(M) = e(N) + e(M/N)$.

(2) If M_ℓ, $\ell = 1, \ldots, r$ is a holonomic A_n–module then $\sum_\ell M_\ell$ is holonomic and

$$e(\oplus_{\ell=1}^{r} M_\ell) = \sum_{\ell=1}^{r} e(M_\ell).$$

Proof. *(1)* There is nothing to prove if $N = (0)$ or $N = M$. Assume N is a proper submodule of M and $M \neq (0)$. From Theorem 2.3 we get $\dim(M) = \max\{\dim(N), \dim(M/N)\}$ and therefore if N and M/N are holonomic then M is also holonomic.

From Theorem 2.4 we have $\dim(N) \geq n$ and $\dim(M/N) \geq n$.

Assume M is holonomic. Then we have $n = \dim(M) = dim(N) = \dim(M/N)$. Again applying Theorem 2.3 we get $e(M) = e(N) + e(M/N)$.

Part *(2)* follows from *(1)* by induction on r. □

Theorem 2.6. *Let M be a holonomic A_n–module. Then we have:*

(1) M is a torsion module, i.e. for each $m \in M$ there exists $P \in A_n$, $P \neq 0$, such that $Pm = 0$.

(2) M is an Artinian module of finite length. Moreover, the length of M is less or equal than $e(M)$.

Proof. *(1)* We can assume $M \neq (0)$. Take $m \in M$, $m \neq 0$. Let us consider the morphism of A_n–modules

$$\phi : A_n \longrightarrow M$$

defined by $\phi(P) = Pm$. The image of ϕ, $\mathrm{Im}(\phi)$, is a nonzero A_n–module (since $m \in \mathrm{Im}(\phi) \subset M$) and, moreover, it is holonomic (see Theorem 2.5). Since A_n is non-holonomic the kernel $\ker(\phi)$ is nonzero.

(2) Let $M = M_0 \supset M_1 \supset M_2 \supset \cdots$ a decreasing chain of A_n–submodules of M. By Theorem 2.5 each M_i is holonomic. If there exists i such that $M_i = (0)$ then the chain is stationary. Assume the chain non-stationary and that each M_i is nonzero. From the exact sequence

$$0 \longrightarrow M_{i+1} \longrightarrow M_i \longrightarrow M_i/M_{i+1} \longrightarrow 0$$

we get that $e(M_i) = e(M_{i+1}) + e(M_i/M_{i+1})$ (see Theorem 2.5). Then we have $e(M) = e(M_{r+1}) + \sum_{i=0}^{r} e(M_i/M_{i+1}) \geq r+1$, for each $r \geq 0$. This is a contradiction. Moreover, the length of M should be less or equal than $e(M)$. □

Remark 2.7. There are finitely generated A_n–modules of finite length –and even irreducible– which are non-holonomic.

An example of that –due to J.T. Stafford– is the following. Consider $M = A_2(\mathbb{C})/A_2(\mathbb{C})P$ with $P = x_2\partial_1\partial_2 - \partial_2 + x_1 + x_2$. We have that $\dim(M) = 3$ and then M is non-holonomic (see Example 2.2).

J. T. Stafford proved that M is irreducible as A_2–module [41, Th. 1.1].

Exercise 2.3. Prove that if a holonomic A_n–module has multiplicity 1 then it is irreducible. Prove that $\mathbb{C}[x]$ is irreducible.

Quick answer.- Assume M is a holonomic A_n–module with $e(M) = 1$ and consider $N \subset M$ a nonzero submodule of M. By Theorem 2.5 we have $1 = e(M) = e(N) + e(M/N)$. Since the integer $e(N)$ is strictly positive (see Remark 2.5) we have $e(N) = 1$ and $e(M/N) = 0$. The last equality implies $M = N$. So M is irreducible. The last part of the exercise follows since $\mathbb{C}[x]$ is holonomic and $e(\mathbb{C}[x]) = 1$ (see Exercise 2.1).

Proposition 2.5 (Cor. 1.4, Bernstein[6]). *Let M be an A_n–module endowed with a B–filtration $\Gamma = (M_k)_k$ such that there are two rational numbers c_1, c_2 satisfying*

$$\dim_{\mathbb{C}} M_j \leq \frac{c_1}{n!} j^n + c_2(j+1)^{n-1}$$

for $j \in \mathbb{N}$, j big enough. Then M is finitely generated. Moreover, it is holonomic and $e(M) \leq c_1$.

Proof. First of all we will prove that any nonzero finitely generated submodule N of M is holonomic and $e(N) \leq c_1$. Since N is finitely generated it admits a good B–filtration say $\Gamma_N = (N_k)_k$. Consider the B–filtration

$\Gamma' = (M_k \cap N)_k$. By Proposition 1.17 there exists a positive integer r such that $N_j \subset M_{j+r} \cap N$ for all j. We have

$$\dim_{\mathbb{C}}(N_j) \leq \dim_{\mathbb{C}}(M_{j+r}) \leq \frac{c_1}{n}(j+r)^n + c_2(j+r+1)^{n-1}$$

and then the degree of the Hilbert polynomial of N with respect to Γ_N is less than or equal to n. Applying 2.4 we get $\dim(N) = n$ and then N is holonomic. Moreover, from the previous inequality we also get $e(N) \leq c_1$. We will prove now that M is finitely generated. If M is nonzero let us consider a nonzero element $m_1 \in M$. Denote $M_1 = A_n m_1$. If $M \neq M_1$ let us consider a nonzero $m_2 \in M \setminus M_1$ and denote $M_2 = A_n m_1 + A_n m_2$. Assume we can construct an infinite increasing chain

$$M_1 \subset M_2 \subset M_2 \subset \cdots \subset M_i \subset \cdots$$

of finitely generated submodules of M. From the first part of the proof we deduce that each M_i is holonomic and $e(M_i) \leq c_1$. We also have that $e(M_i) \geq i$ for each i, which is a contradiction. So there exists a finite generating set $m_1, \ldots, m_r$ of M. □

2.7. $\mathbb{C}[x]_f$ *is holonomic*

Let f be a nonzero polynomial in $\mathbb{C}[x]$.

Theorem 2.7 (§2, Bernstein[6]). *The A_n–module $\mathbb{C}[x]_f$ is holonomic.*

Proof. Put $N = \mathbb{C}[x]_f$ and $\deg(f) = d \geq 0$.

For each $k \in \mathbb{N}$ define

$$N_k = \{g/f^k \in \mathbb{N} \mid \deg(g) \leq (d+1)k\}.$$

Let's prove first that the family $\Gamma = (N_k)_k$ is a B–filtration on N.

It's clear that $N_k \subset N_\ell$ for $k \leq \ell$.

Assume $g/f^k \in N_k$. We have $\deg(x_i g) = \deg(g) + 1 \leq (d+1)k + 1 \leq (d+1)(k+1)$. That proves the inclusion $x_i N_k \subset N_{k+1}$. We also have

$$\partial_i \left(\frac{g}{f^k} \right) = \frac{\partial_i(g)f - kg\partial_i(f)}{f^{k+1}}$$

and $\deg(\partial_i(g)f - kg\partial_i(f)) \leq d + \deg(g) - 1 \leq d - 1 + (d+1)k \leq (d+1)(k+1)$. That proves $\partial_i N_k \subset N_{k+1}$. Then $B_1 N_k \subset N_{k+1}$. Since $B_\ell = (B_1)^\ell$ we have $B_\ell N_k \subset N_{k+\ell}$.

We will now prove $N = \cup_k N_k$. To this end take $g/f^k \in N$ and assume $\deg(g) = m$. We have

$$\frac{g}{f^k} = \frac{gf^m}{f^{k+m}}$$

and $\deg(gf^m) = m + dm \leq (d+1)(k+m)$. That proves $g/f^k \in N_{m+k}$.

We have proved that $(N_k)_k$ is a B–filtration on $N = \mathbb{C}[x]_f$.

We will now prove that this filtration satisfies the hypothesis of Proposition 2.5 for adequate c_1, c_2.

The dimension of the $\mathbb{C}$–vector space N_k is bounded by the number of monomials x^α in $\mathbb{C}[x]$ with degree $|\alpha| \leq (d+1)k$. This number is

$$\binom{(d+1)k+n}{n} = \frac{1}{n!}(d+1)^n k^n + p(k)$$

where $p(t)$ is polynomial in t with rational coefficients and degree less than or equal to $n-1$. Then there exists an integer number $c_2 > 0$ such that

$$\dim_{\mathbb{C}}(N_k) \leq \binom{(d+1)k+n}{n} = \frac{1}{n!}(d+1)^n k^n + p(k) \leq \frac{(d+1)^n k^n}{n!} + c_2(k+1)^{n-1}$$

for $k \gg 0$. Then by Proposition 2.5 $\mathbb{C}[x]_f$ is holonomic. □

Remark 2.8. From the above proof we can also deduce, applying Proposition 2.5, that the multiplicity of $\mathbb{C}[x]_f$ is bounded by $(d+1)^n$. This bound is far to be sharp. See Exercise 2.4.

Exercise 2.4. Let us write $f = x_1$. Prove that:

i) $\mathbb{C}[x]_f = A_n \frac{1}{f}$.
ii) $Ann_{A_n}(1/f) = A_n(x_1\partial_1 + 1, \partial_2, \ldots, \partial_n)$.
iii) $\dim(\mathbb{C}[x]_f) = n$, $e(\mathbb{C}[x]_f) = 2$.

Answer.- Recall that the annihilating ideal $Ann_{A_n}(1/f)$ is by definition the ideal $\{P \in A_n \mid P(1/f) = 0\}$.

i) By definition we have that $A_n \frac{1}{f} \subset \mathbb{C}[x]_f$. The equality

$$g\partial_1\left(\frac{1}{x_1^m}\right) = \frac{(-m)g}{x_1^{m+1}}$$

holds for any $g \in \mathbb{C}[x]$ and any $m \in \mathbb{N}$. This equality proves that any rational function of type $\frac{g}{x_1^{m+1}}$ belongs to $A_n \frac{1}{x_1}$ and then it proves the equality $\mathbb{C}[x]_f = A_n \frac{1}{f}$.

ii) The inclusion $A_n(x_1\partial_1 + 1, \partial_2, \ldots, \partial_n) \subset Ann_{A_n}(1/f)$ is obvious. Let us consider an operator $P = P(x, \partial) \in A_n$ annihilating $\frac{1}{x_1}$. We can write

$$P = Q_2\partial_2 + \cdots + Q_n\partial_n + P_1$$

for some $Q_2, \dots, Q_n, P_1 \in A_n$ and $P_1 = \sum_\ell a_\ell(x)\partial_1^\ell$ for some $a_\ell(x) \in \mathbb{C}[x]$. The operator P_1 annihilates $\frac{1}{x_1}$ since P also does.

We can write

$$P_1 = Q(x_1\partial_1 + 1) + S(x', \partial_1) + r(x)$$

for some $Q, S(x', \partial_1) \in A_n$, $r(x) \in \mathbb{C}[x]$ and $S := S(x', \partial_1) = \sum_{k>0} b_k(x')\partial_1^k$ for some $b_k(x') \in \mathbb{C}[x'] := \mathbb{C}[x_2, \dots, x_n]$.

We have $0 = P_1(\frac{1}{x_1}) = S(\frac{1}{x_1}) + \frac{r(x)}{x_1}$. Assuming that S is nonzero, let us write $d > 0$ the degree of S with respect to ∂_1. The order of the pole of $S(\frac{1}{x_1})$ at $x_1 = 0$ is $d + 1$ while $\frac{r(x)}{x_1}$ has a pole of order at most 1. This implies that $d = 0$ which is a contradiction. Then we have $S = 0$ and $r(x) = 0$ since $\frac{r(x)}{x_1} = 0$. This proves that $P = Q_2\partial_2 + \cdots Q_n\partial_n + Q(x_1\partial_1 + 1) \in A_n(x_1\partial_1 + 1, \partial_2, \dots, \partial_n)$.

iii) Let us denote $I = A_n(x_1\partial_1 + 1, \partial_2, \dots, \partial_n)$ and let us write $J \subset \mathbb{C}[x, \xi]$ the ideal generated by $(x_1\xi_1, \xi_2, \dots, \xi_n)$. We have the inclusion $J \subset \mathrm{gr}^F(I)$ and then the inclusion $\mathcal{V}(\mathrm{gr}^F(I)) \subset \mathcal{V}(J)$. The last affine algebraic set has Krull dimension n. Then the Krull dimension of $\mathcal{V}(\mathrm{gr}^F(I))$ is less than or equal to n. So, $\dim(A_n/I) = \dim(\mathcal{V}(\mathrm{gr}^F(I))) \le n$ and then, from Theorem 2.4, we get $\dim(A_n/I) = \dim(\mathbb{C}[x]_f) = n$.

Let us now compute the multiplicity of $\mathbb{C}[x]_f$. We will prove first the equality $\mathrm{gr}^B(I) = J$. It is easy to prove that each nonzero $P \in I$ can be written as

$$P = Q_2\partial_2 + \cdots + Q_n\partial_n + Q(x_1\partial_1 + 1)$$

with $\mathrm{ord}^T(Q) = \mathrm{ord}^T(P) - 2$ and $\mathrm{ord}^T(Q_i) = \mathrm{ord}^T(P) - 1$ for $i = 2, \dots, n$ (see, e.g. Theorem A.1).

Then, by Proposition 1.4, we have

$$\sigma^T(P) = \sum_{i=2}^n \sigma^T(Q_i)\xi_i + \sigma^T(Q)x_1\xi_1$$

and then $\sigma^T(P) \in J$. This proves the equality $\mathrm{gr}^B(I) = J$. The leading term of the Hilbert polynomial $gHP_{\mathrm{gr}^\Gamma(\mathbb{C}[x]_f)}(t)$ of the quotient graded module

$$\frac{\mathbb{C}[x, \xi]}{J} \simeq \mathrm{gr}^\Gamma(\mathbb{C}[x]_f)$$

equals $\frac{2}{(n-1)!}t^{n-1}$ (here Γ denotes the induced B–filtration on $A_n/I \simeq \mathbb{C}[x]_f$). This proves that $e(\mathbb{C}[x]_f) = 2$.

2.8. *The Bernstein polynomial*

The Bernstein (or the Bernstein-Sato) polynomial associated with a given polynomial $f \in \mathbb{C}[x]$ has been introduced in its general form in Ref. 6 (and independently by M. Sato in Ref. 38).

Let f be a nonzero polynomial in $\mathbb{C}[x]$. Let s be a new variable and $\mathbb{C}(s)$ the field of rational functions on s. We denote by $A_n(s)$ the Weyl algebra over the field $\mathbb{C}(s)$ and $A_n[s] := A_n \otimes_{\mathbb{C}} \mathbb{C}[s]$. Denote by $\mathbb{C}(s)[x]_f f^s$ the free $\mathbb{C}(s)[x]_f$–module of rank 1 with basis the formal symbol f^s. This free module admits a natural structure of left $A_n(s)$–module by defining

$$\partial_i f^s = s f^{-1} \partial_i(f) f^s$$

for $i = 1, \ldots, n$ (the action of $\mathbb{C}(s)[x]$ being the natural one).

Proposition 2.6 (§2, Bernstein[6]). *The $A_n(s)$–module $\mathbb{C}(s)[x]_f f^s$ is holonomic.*

Proof. Put $N = \mathbb{C}(s)[x]_f f^s$ and $\deg(f) = d \geq 0$.

For each $k \in \mathbb{N}$ define

$$N_k = \left\{ \frac{g(s,x)}{f^k} \in N \mid \deg(g) \leq (d+1)k \right\}.$$

It can be proved, in a similar way to the proof of Proposition 2.7, that the family $\Gamma = (N_k)_k$ is a B–filtration on N.

We will now prove that this filtration satisfies the hypothesis of Proposition 2.5 for adequate c_1, c_2.

The dimension of the $\mathbb{C}(s)$–vector space N_k is bounded by the number of monomials x^α in $\mathbb{C}(s)[x]$ with degree $|\alpha| \leq (d+1)k$. This number is

$$\binom{(d+1)k+n}{n} = \frac{1}{n!}(d+1)^n k^n + p(k)$$

where $p(t)$ is a polynomial in t with rational coefficients and degree less than or equal to $n-1$. Then there exists a integer $c_2 > 0$ such that

$$\begin{aligned} \dim_{\mathbb{C}(s)}(N_k) \leq \binom{(d+1)k+n}{n} &= \frac{1}{n!}(d+1)^n k^n + p(k) \\ &\leq \frac{(d+1)^n k^n}{n!} + c_2(k+1)^{n-1} \end{aligned}$$

for $k \in \mathbb{N}$, k big enough. Then by Proposition 2.5 the $A_n(s)$–module $N = \mathbb{C}(s)[x]_f f^s$ is holonomic. □

Theorem 2.8. *Let f be a nonzero polynomial in $\mathbb{C}[x]$. There exists a nonzero polynomial $b(s) \in \mathbb{C}[s]$ and a differential operator $P(s) \in A_n[s]$ such that the equality*

$$P(s)ff^s = b(s)f^s$$

holds in $\mathbb{C}(s)[x]_f f^s$.

Proof. The module $A_n(s)f^s$ is an $A_n(s)$–submodule of $\mathbb{C}(s)[x]_f f^s$ and then, by Proposition 2.6, it is holonomic and furthermore of finite length (see Theorems 2.5 and 2.6). Then the descending sequence

$$A_n(s)f^s \supseteq A_n(s)ff^s \supseteq \cdots \supseteq A_n(s)f^\ell f^s \supseteq \cdots$$

is stationary. Thus there exists $\ell \in \mathbb{N}$ such that

$$f^\ell f^s \in A_n(s)f^{\ell+1}f^s.$$

So, there exists $Q(s) \in A_n(s)$ such that $f^\ell f^s = Q(s)f^{\ell+1}f^s$. Then $f^s = Q(s-\ell)ff^s$. Let $b(s) \in \mathbb{C}[s]$ a nonzero polynomial such that $P(s) := b(s)Q(s-\ell) \in A_n[s]$. Thus we have $b(s)f^s = P(s)ff^s$. □

For a given nonzero polynomial f in $\mathbb{C}[x]$ the set of polynomials $c(s) \in \mathbb{C}[s]$ such that there exists an operator $P(s) \in A_n[s]$ such that $P(s)ff^s = c(s)f^s$ is an ideal in $\mathbb{C}[s]$. We will denote this ideal by $\mathcal{B}_f$.

Definition 2.9. Let f be a nonzero polynomial in $\mathbb{C}[x]$. The monic generator of the ideal $\mathcal{B}_f$ is denoted by $b_f(s)$ and it is called the *Bernstein polynomial* (or the *Bernstein-Sato polynomial*) of f.

The computation of $b_f(s)$ is difficult although there exists an algorithm computing the Bernstein polynomial $b_f(s)$ for a given polynomial $f \in \mathbb{C}[x]$ (see T. Oaku;[36] see also M. Noro[34]).

A variant of this algorithm has been implemented in the `D-modules` package for `Macaulay 2`. We can use this implementation to make some experiments.

```
Macaulay 2, version 1.2 with packages: Elimination, IntegralClosure,
LLLBases, PrimaryDecomposition, ReesAlgebra, SchurRings, TangentCone

i1 : load "D-modules.m2"

i2 : R=QQ[x,y,z];

i3 : W=makeWA R;

i4 : f=x^2+y^2+z^2;

i5 : globalBFunction f
```

```
       2
o5 = 2s  + 5s + 3

o5 : QQ[s]
```

The previous command `globalBFunction f` computes the Bernstein polynomial of the given polynomial `f`. We have to notice here that, in order to simplify the output, `Macaulay 2` clears the denominators of the Bernstein polynomial (and so, the output of `globalBFunction` is not necessarily monic).

The previous script tells us that the Bernstein polynomial of $x_1^2+x_2^2+x_3^2$ is just the polynomial $s^2+(5/2)s+3/2$.

Let us continue with the following computation

```
i6 : f=x^3+y^3+z^3;

i7 : globalBFunction f

       5      4       3       2
o7 = 9s  + 63s  + 173s  + 233s  + 154s + 40

o7 : QQ[s]

i8 : factor o7

             2
o8 = (s + 1) (s + 2)(3s + 4)(3s + 5)
```

The command `factor` factorizes the given polynomial. Let us make some other experiments

```
i9 : g=x^2-y^3;

i10 : globalBFunction g

          3       2
o10 = 36s  + 108s  + 107s + 35

o10 : QQ[s]

i11 : factor o13

o11 = (s + 1)(6s + 5)(6s + 7)

i12 : f=x^2-y^3+1;

i13 : globalBFunction f

o13 = s + 1

o13 : QQ[s]
```

The last two examples show that a little change on the expression of the polynomial f can produce completely different Bernstein polynomials.

The expansion of the polynomial g is very close to the one of f although the corresponding Bernstein polynomials are very different.

For each polynomial $f \in R$ the Bernstein polynomial $b_f(s)$ depends on the singularities of the hypersurface $\mathcal{V}(f) = \{a \in \mathbb{C}^n \mid f(a) = 0\}$. Bernstein polynomial is a very useful invariant in singularity theory. Two main references in this topic are Malgrange[33] and Kashiwara.[31]

Bibliographical note

Most of the material of this Section appears in the articles Bernstein[6] and Ehlers[23] and in the books Björk[7] and Coutinho.[21]

3. Logarithmic A_n–Modules

In this Section[h] unless otherwise stated, we will denote $R = \mathbb{C}[x] = \mathbb{C}[x_1, \ldots, x_n]$.

3.1. *Logarithmic derivations*

Let us denote by $\mathrm{Der}_{\mathbb{C}}(R)$ the R–module of $\mathbb{C}$-derivations of the ring R. An element $\delta \in \mathrm{Der}_{\mathbb{C}}(R)$ can be written as

$$\delta = \sum_{i=1}^{n} a_i(x)\partial_i$$

for some $a_i(x) \in R$. Moreover, $\mathrm{Der}_{\mathbb{C}}(R)$ is a free R–module of rank n the set $\{\partial_1, \ldots, \partial_n\}$ being one of its bases. Elements in $\mathrm{Der}_{\mathbb{C}}(R)$ are also called *vector fields* on $\mathbb{C}^n$ with polynomial coefficients.

If $f \in R$ is not a constant we can define the notion of (global) logarithmic derivation (or logarithmic vector field) with respect to the hypersurface $D = \mathcal{V}(f) = \{a \in \mathbb{C}^n \mid f(a) = 0\} \subset \mathbb{C}^n$, as follows:

Definition 3.1 (K. Saito[39]). *A vector field $\delta = \sum_{i=1}^{n} a_i(x)\partial_i$ with coefficients $a_i(x) \in R$ is said to be* logarithmic with respect to D *if $\delta(f) \in Rf$.*

The R–module of logarithmic vector fields (or logarithmic derivations) with respect to $D \subset \mathbb{C}^n$ is denoted by $\mathrm{Der}_R(-\log D)$. The R–module of logarithmic vector fields with respect to $D \subset \mathbb{C}^n$ has been also denoted in

[h]Part of this Section follows the talk *Computational methods for testing the range of validity of the Logarithmic Comparison Theorem* given by the author at the Workshop *Geometry and analysis on complex algebraic varieties* held at RIMS, Kyoto University, from 11th to 15th December 2006.

the literature as $\mathrm{Der}_R(\log D)$. The sign before $\log D$ in our notation will be explained latter (see Remark 3.4).

For each polynomial $g \in R$, we can also denote

$$\mathrm{Der}_R(-\log g) = \{\delta = \sum_{i=1}^{n} a_i(x)\partial_i \in \mathrm{Der}_{\mathbb{C}}(R) \mid \delta(g) \in Rg\}.$$

Exercise 3.1.

(1) Prove the equality $\mathrm{Der}_R(-\log gh) = \mathrm{Der}_R(-\log g) \cap \mathrm{Der}_R(-\log h)$ for $g, h \in R$.
(2) Prove that if $f, g \in R$ and $\mathcal{V}(f) = \mathcal{V}(g)$ then $\mathrm{Der}_R(-\log f) = \mathrm{Der}_R(-\log g)$. (Hint: Use Nullstellensatz).

Exercise 3.1 justifies the notation $\mathrm{Der}_R(-\log D)$ for the R–module of logarithmic vector fields with respect to the hypersurface $D = \mathcal{V}(f)$.

Exercise 3.2. Prove that $\mathrm{Der}_R(-\log D)$ is a Lie algebra (i.e. prove that it is closed under the Lie bracket $[-,-]$).

Let us write $f_i = \partial_i(f)$ for $i = 1, \ldots, n$. To each logarithmic derivation $\delta = \sum_i a_i(x)\partial_i$ one can associate the syzygy $(-\frac{\delta(f)}{f}, a_1(x), \ldots, a_n(x))$ of the polynomials $(f, f_1, \ldots, f_n)$. This defines a map

$$\epsilon : \mathrm{Der}_R(-\log D) \longrightarrow Syz_R(f, f_1, \ldots, f_n)$$

where $Syz_R(f, f_1, \ldots, f_n)$ is the R–module of syzygies of $(f, f_1, \ldots, f_n)$.

Exercise 3.3. Prove that the previous map ϵ is an isomorphism of R–modules.

Remark 3.1. As R is a Noetherian ring then $Syz_R(f, f_1, \ldots, f_n)$ and $\mathrm{Der}_R(-\log D)$ are finitely generated R-modules.

Moreover, for each $f \in R$ one can compute, by using Groebner bases in R, a system of generators of the syzygy module $Syz_R(f, f_1, \ldots, f_n)$ (and then of the R–module $\mathrm{Der}_R(-\log D)$ by using the isomorphism ϵ) (see e.g. [1, Section 3.4]).

Let's treat the case $f = x^2 - y^3$ in `Macaulay 2`. In the following script, `R` denotes the polynomial ring with variables `x, y` and with coefficients if the field of rational numbers (which is denoted as `QQ` in `Macaulay 2`).

Input `i5 :` computes a system of generators of the syzygy module of $(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$. Notice that output `o5 :` gives a submodule of the free module $\mathtt{W}^3$ and that this submodule is generated by vectors in $\mathtt{R}^3$. Thus, these two vectors also generate the R–module of syzygies of $(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$.

Input `i6 :` computes a (1×2)–matrix which entries are the logarithmic vector fields with respect to `f` associate to each previously computed syzygy. Input `i7 :` gives the ideal in the Weyl algebra `W` generated by these two logarithmic vector fields.

```
Macaulay 2, version 1.2 with packages: Elimination,
IntegralClosure, LLLBases, PrimaryDecomposition, ReesAlgebra,
SchurRings, TangentCone

i1 : load "D-modules.m2";

i2 : R=QQ[x,y];

i3 : W=makeWA R;

i4 : f=x^2-y^3

       3    2
o4 = - y  + x

o4 : W

i5 : kernel matrix({{f,diff(x,f),diff(y,f)}})

o5 = image {3} | 6   0    |
           {1} | -3x -3y2 |
           {2} | -2y -2x  |

                             3
o5 : W-module, submodule of W

i6 : matrix({{0,dx,dy}}) * gens o5

o6 = | -3xdx-2ydy -3y2dx-2xdy |

             1       2
o6 : Matrix W  <--- W

i7 : ideal o6
                                   2
o7 = ideal (- 3x*dx - 2y*dy, - 3y dx - 2x*dy)

o7 : Ideal of W
```

Exercise 3.4. Prove that one has an exact sequence of R–modules

$$0 \to f\mathrm{Der}_{\mathbb{C}}(R) \to \mathrm{Der}_R(-\log D) \to \frac{\mathrm{Der}_R(-\log D)}{f\mathrm{Der}_{\mathbb{C}}(R)} \to 0$$

where the morphism $f\mathrm{Der}_{\mathbb{C}}(R) \to \mathrm{Der}_R(-\log D)$ is an inclusion.

Exercise 3.5.

(1) Assume $f = x_1 \in R$ and $D = \mathcal{V}(x_1) \subset \mathbb{C}^n$. Prove the equality $\mathrm{Der}_R(-\log D) = Rx_1\partial_1 \oplus R\partial_2 \oplus \cdots \oplus R\partial_n$.

(2) Assume $f = x_1x_2\cdots x_r$ for some $1 \leq r \leq n$, and $D = \mathcal{V}(f) \subset \mathbb{C}^n$. Prove the equality $\mathrm{Der}_R(-\log D) = Rx_1\partial_1 \oplus Rx_2\partial_2 \oplus \cdots \oplus Rx_r\partial_r \oplus R\partial_{r+1} \oplus \cdots \oplus R\partial_n$.

To each nonzero polynomial $f \in \mathbb{C}[x]$ we associate the quotient A_n–module defined as

$$M^{\log f} := \frac{A_n}{A_n\mathrm{Der}(-\log f)}.$$

For example, if $f = x_1x_2\cdots x_r$ then

$$M^{\log f} = \frac{A_n}{A_n(x_1\partial_1, x_2\partial_2, \ldots, x_r\partial_r, \partial_{r+1}, \ldots, \partial_n)}.$$

3.2. *The ideal* $Ann^{(1)}_{A_n}(\frac{1}{f})$

Let f be a nonzero polynomial in R. We denote

$$\widetilde{\mathrm{Der}}(-\log f) := \{\delta + \frac{\delta(f)}{f} \mid \delta \in \mathrm{Der}(-\log f)\}.$$

For each $\delta \in \mathrm{Der}(-\log f)$, the operator $\delta + \frac{\delta(f)}{f}$ annihilates the rational function $\frac{1}{f}$. We notice here that the operator $\delta + \frac{\delta(f)}{f}$ has order 1 (see Definition 1.2).

Reciprocally, if an operator $P \in A_n$ annihilates $\frac{1}{f}$ and $\mathrm{ord}(P) = 1$ then we can write

$$P = \eta + a_0(x)$$

for some polynomial $a_0(x) \in R$ and some derivation $\eta = \sum_{i=1}^n a_i(x)\partial_i$ (with $a_i(x)$ in R for $i = 1, \ldots, n$). Then, from $P(1/f) = 0$ we get $fa_0(x) = \eta(f)$. So, $\eta \in \mathrm{Der}(-\log f)$ and $P = \eta + \frac{\eta(f)}{f}$.

We denote by $Ann^{(1)}_{A_n}(\frac{1}{f})$ (or simply $Ann^{(1)}(\frac{1}{f})$) the left ideal in A_n generated by the operators of the form $\delta + \frac{\delta(f)}{f}$ for some δ in $\mathrm{Der}(-\log f)$. That is:

$$Ann^{(1)}\left(\frac{1}{f}\right) = A_n\widetilde{\mathrm{Der}}(-\log f).$$

To each nonzero polynomial $f \in \mathbb{C}[x]$ we have associated (see Subsection 3.1) the quotient A_n–module

$$M^{\log f} := \frac{A_n}{A_n\mathrm{Der}(-\log f)}.$$

Moreover, to the polynomial $f \in R$ we can also associate a new quotient A_n–module:

$$\widetilde{M}^{\log f} := \frac{A_n}{A_n \widetilde{\mathrm{Der}}(-\log f)} = \frac{A_n}{Ann^{(1)}(1/f)}.$$

Both modules $M^{\log f}$ and $\widetilde{M}^{\log f}$ will be called the *logarithmic* A_n–modules associated with the polynomial $f \in R$. These modules encode information about the *singularities* of the hypersurface $\mathcal{V}(f) \subset \mathbb{C}^n$ defined by $f \in R$.

Since $Ann^{(1)}(\frac{1}{f})$ is included in the annihilating ideal $Ann(\frac{1}{f})$ we have a natural surjective morphism of A_n–modules

$$\widetilde{M}^{\log f} = \frac{A_n}{Ann^{(1)}(1/f)} \xrightarrow{\phi_f} \frac{A_n}{Ann(1/f)}$$

defined by

$$\phi_f \left(P + Ann^{(1)} \left(\frac{1}{f} \right) \right) = P + Ann(\frac{1}{f}).$$

The morphism ϕ_f is an isomorphism if and only

$$Ann^{(1)} \left(\frac{1}{f} \right) = Ann \left(\frac{1}{f} \right)$$

(we say in this case that the annihilating ideal of $1/f$ is generated by operators of order 1).

It is an open question to characterize the class of polynomials $f \in R$ such that the $Ann(1/f)$ is generated by operators of order 1.

Annihilating ideals can be computed using Groebner bases in A_n (see Oaku and Takayama[37]).

We will use `Macaulay 2` to compute some examples.

The next script computes a system of generators for the annihilating ideal $Ann(1/f) \subset A_2$ for $f = x_1^4 + x_2^5$. The computation gives two generators for $Ann(1/f)$, namely $P = 5x_1\partial_1 + 4x_2\partial_2 + 20$ and $Q = 5x_2^4\partial_1 - 4x_1^3\partial_2$. So, the annihilating ideal of $1/f$ is generated by operators of order 1 and then, in this case, the morphism

$$\phi_f : \frac{A_2}{Ann^{(1)}(1/f)} \longrightarrow \frac{A_2}{Ann(1/f)}$$

is an isomorphism.

In the next script the command `RatAnn f` computes a generating system for the annihilating ideal $Ann(1/f)$.

```
Macaulay 2, version 1.2 with packages: Elimination,
IntegralClosure, LLLBases, PrimaryDecomposition, ReesAlgebra,
SchurRings, TangentCone

i1 : load "D-modules.m2"

i2 : R=QQ[x,y];

i3 : W=makeWA R;

i4 : f=x^4+y^5

      5    4
o4 = y  + x

o4 : W

i5 : RatAnn f

                                          4       3
o5 = ideal (5x*dx + 4y*dy + 20, 5y dx - 4x dy)

o5 : Ideal of W
```

We will change slightly the polynomial f just by adding the monomial $x_1x_2^4$. So, we get a new polynomial $g = x_1^4 + x_2^5 + x_1x_2^4$.

We are going to use `Macaulay 2` to compute, using the following script, a system of generators of the rational function $1/g$.

This computation is performed using the command `Ann=RatAnn g`. The value of the string `Ann` is then a system of generators of the annihilating ideal $Ann(1/g)$.

In order to easily manage the output `o7` we use the command `toString Ann`. Output `o8` gives a string representation of the previous system of generators of the annihilating ideal $Ann(1/g)$. This system does not have to be minimal. The fourth generator has order 2. To prove that $Ann(1/g)$ is not generated by operators of order 1 we will compute generators of $Ann^{(1)}(1/g)$. This is performed from input `i9` until input `i11`.

The value of the name `Ann1`, in the next script, is then a system of generators of the ideal $Ann^{(1)}(1/g)$. Then we compare the ideals $Ann^{(1)}(1/g)$ and $Ann(1/g)$, by using the command `Ann==Ann1`. The corresponding output being `false` means that the inclusion $Ann^{(1)}(1/g) \subset Ann(1/g)$ is strict and then the morphism

$$\phi_g : \frac{A_2}{Ann^{(1)}(1/g)} \longrightarrow \frac{A_2}{Ann(1/g)}$$

is not an isomorphism.

```
i6 : g=f+x*y^4

           4    5    4
o6 = x*y  + y  + x

o6 : W

i7 : Ann=RatAnn g;

i8 : toString Ann

o8 = ideal(4*x^2*dx+5*x*y*dx+3*x*y*dy+4*y^2*dy+16*x+20*y,
16*x*y^2*dx+4*y^3*dx+12*y^3*dy-125*x*y*dx-4*x^2*dy+5*x*y*dy-100*y^2*dy+64*y^2-500*y,
4*x*y^3*dx+5*y^4*dx-y^4*dy-4*x^3*dy,
-64*x^2*y*dx^2+36*y^3*dx^2-96*x*y^2*dx*dy-32*y^3*dx*dy-36*y^3*dy^2+500*x^2*dx^2
+125*x*y*dx^2-36*x^2*dx*dy+720*x*y*dx*dy+100*y^2*dx*dy+24*x^2*dy^2-29*x*y*dy^2
+260*y^2*dy^2-368*x*y*dx-72*y^2*dx-264*y^2*dy+2425*x*dx+625*y*dx-105*x*dy
+1495*y*dy-192*y-300)


i9 : kernel matrix({{g,diff(x,g),diff(y,g)}})

o9 = image {5} | -16x-20y -16y2-100x   |
           {4} | 4x2+5xy  4xy2+y3+25x2 |
           {4} | 3xy+4y2  3y3-x2+20xy  |

                              3
o9 : W-module, submodule of W

i10 : matrix({{-1,dx,dy}}) * gens o9

o10 = | 4x2dx+5xydx+3xydy+4y2dy+16x+20y 4xy2dx+y3dx+3y3dy+25x2dx-x2dy+20xydy+16y2+100x |

              1       2
o10 : Matrix W  <--- W

i11 : Ann1=ideal o10

              2                            2        2      3       3        2      2
o11 = ideal (4x dx + 5x*y*dx + 3x*y*dy + 4y dy, 4x*y dx + y dx + 3y dy + 25x dx - x dy + 20x*y*dy)

o11 : Ideal of W

i12 : Ann==Ann1

o12 = false
```

Notice that the input `i9` computes a generating system of the syzygy module $Syz_R(g, \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y})$ while the input `i10` computes a generating system of the ideal $Ann^{(1)}(1/g)$.

It is not easy to determine when the modules $M^{\log f}$ and $\widetilde{M}^{\log f}$ are holonomic.

For any non constant polynomial $f \in \mathbb{C}[x_1, x_2]$ both A_2–modules $M^{\log f}$ and $\widetilde{M}^{\log f}$ are holonomic (see Calderón[11]).

The A_3–modules $M^{\log f}$ and $\widetilde{M}^{\log f}$ are not holonomic for $f = (xz + y)(x^4 + y^5 + xy^4)$ (see [20, Example 6.4.]). Let us compute the dimension of these two A_3–modules using `Macaulay 2`.

```
i1 : load "D-modules.m2"

i2 : R=QQ[x,y,z]; W=makeWA R;

i3 : f=(x*z+y)*(x^4+y^5+x*y^4)

      2 4       5       5    6    5     4
o3 = x y z + x*y z + x*y  + y  + x z + x y

o3 : W

i4 : kernel matrix({{f,diff(x,f),diff(y,f),diff(z,f)}})

o4 = image {7} | -76x-96y  x     -xz-19x-24y -2y2z-38y2-150x+120y |
           {6} | 16x2+20xy 0     4x2+5xy     8xy2+2y3+30x2-25xy   |
           {6} | 12xy+16y2 0     3xy+4y2     6y3-2x2+25xy-20y2    |
           {6} | -4xz-4yz  -xz-y xz2-xz      2y2z2-2y2z+5yz+2x+5y |

                             4
o4 : W-module, submodule of W

i5 : matrix({{0,dx,dy,dz}})* gens o4;

o5 = | 16x2dx+20xydx+12xydy+16y2dy-4xzdz-4yzdz -xzdz-ydz xz2dz+4x2dx+5xydx+3xydy+4y2dy-xzdz
     ---------------------------------------------------------------------------------------
     2y2z2dz+8xy2dx+2y3dx+6y3dy-2y2zdz+30x2dx-25xydx-2x2dy+25xydy-20y2dy+5yzdz+2xdz+5ydz |

             1       4
o5 : Matrix W  <--- W

i6 : Ilog=ideal o5;

i7 : toString o6

o7 = matrix {{16*x^2*dx+20*x*y*dx+12*x*y*dy+16*y^2*dy-4*x*z*dz-4*y*z*dz, -x*z*dz-y*dz,
      x*z^2*dz+4*x^2*dx+5*x*y*dx+3*x*y*dy+4*y^2*dy-x*z*dz,
      2*y^2*z^2*dz+8*x*y^2*dx+2*y^3*dx+6*y^3*dy-2*y^2*z*dz+30*x^2*dx-25*x*y*dx-2*x^2*dy
      +25*x*y*dy-20*y^2*dy+5*y*z*dz+2*x*dz+5*y*dz}}

i8 : dim Ilog

o8 = 4
```

In the previous script, input `i4 :` computes a generating system of the syzygy module $Syz_R(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$ and inputs `i5 :` to `i7 :` compute a system of generators of the ideal $A_3\mathrm{Der}(-\log f)$, which is denoted here as `Ilog`. Notice that `W` stands for the Weyl algebra A_3. As `dim Ilog` is 4 that means that the A_3–module

$$M^{\log f} = \frac{A_3}{A_3\mathrm{Der}(-\log f)}$$

is not holonomic.

In the next script we will first compute a system of generators of the ideal $Ann^{(1)}(1/f)$ for the same $f = (xz+y)(x^4+y^5+xy^4)$. The value of the string `Ann1` is a generating system of the ideal $Ann^{(1)}(1/f)$.

```
i9 : matrix({{-1,dx,dy,dz}})* gens o4

o9 =
| 16x2dx+20xydx+12xydy+16y2dy-4xzdz-4yzdz+76x+96y -xzdz-ydz-x xz2dz+4x2dx+5xydx+3xydy+4y2dy-xzdz+xz+19x+24y
------------------------------------------------------------------------------------------------------------
2y2z2dz+8xy2dx+2y3dx+6y3dy-2y2zdz+2y2z+30x2dx-25xydx-2x2dy+25xydy-20y2dy+5yzdz+38y2+2xdz+5ydz+150x-120y |

             1       4
o9 : Matrix W  <--- W

i10 : Ann1=ideal o9;

i11 : dim Ann1

o11 = 4
```

As the dimension of the ideal `Ann1` is 4 then the A_3–module

$$\widetilde{M}^{\log f} = \frac{A_3}{Ann^{(1)}(1/f)}$$

is not holonomic.

3.3. *Logarithmic differential forms*

For each $q \in \mathbb{N}$, let us denote by Ω^q_R the R–module of differential q-forms with coefficients in R and let us write $\Omega^q_{R,f} = R_f \otimes_R \Omega^q_R$ the R–module of rational differential q–forms with poles along $D = \mathcal{V}(f)$.

Both $\Omega^\bullet_R$ and $\Omega^\bullet_{R,f}$ are complexes of $\mathbb{C}$–vector spaces once endowed with the exterior derivative and in fact $\Omega^\bullet_R$ is a subcomplex of $\Omega^\bullet_{R,f}$.

If $D = \mathcal{V}(f) = \mathcal{V}(g)$ then $\Omega^\bullet_{R,f} = \Omega^\bullet_{R,g}$ and this complex (endowed with the exterior derivative) is called the complex of (global) rational differential forms with respect to D and it is denoted by $\Omega^\bullet_R(*D)$ (or simply $\Omega^\bullet(*D)$).

Remark 3.2. The natural map

$$\mathrm{Der}_{\mathbb{C}}(R) \times \Omega^1(*D) \longrightarrow R_f$$

defined by $(\delta = \sum_i a_i(x)\partial_i, \omega = \sum_i g_i dx_i) \mapsto \sum a_i g_i \in R_f$ is R–bilinear and non-degenerate.

Definition 3.2 (K. Saito[39]). *A rational differential form $\omega \in \Omega^q(*D)$ is said to be* logarithmic with respect to D *if ω and $d\omega$ have at most a first-order pole along D.*

The R–module of logarithmic q-forms with respect to D is denoted by $\Omega^q_R(\log D)$.

If $f \in R$ is a *reduced* equation of D then $\omega \in \Omega^q(*D)$ is logarithmic with respect to D if and only if $f\omega \in \Omega^q_R$ and $fd\omega \in \Omega^{q+1}_R$.

Exercise 3.6. Prove that $\omega \in \Omega^q(*D)$ is logarithmic with respect to D if and only if $f\omega \in \Omega^q_R$ and $df \wedge \omega \in \Omega^{q+1}_R$. In particular, prove that $\omega = \sum_i \frac{b_i}{f} dx_i \in \Omega^1_R(\log D)$ (with $b_i \in R$) if and only if

$$f_i \frac{b_j}{f} - f_j \frac{b_i}{f} \in R$$

for all $1 \leq i, j \leq n$. Here $f_i := \frac{\partial f}{\partial x_i}$. Prove that $\frac{df}{f} \in \Omega^1(\log D)$.

Remark 3.3. We have a natural exact sequence

$$0 \to \Omega^1_R(\log D) \to \frac{1}{f}\Omega^1_R \to \frac{\frac{1}{f}\Omega^1_R}{\Omega^1_R(\log D)} \to 0$$

and the last quotient is a torsion R–module.

$\Omega^\bullet_R(\log D)$ is a sub-complex of $\Omega^\bullet_R(*D)$. The corresponding inclusion, which is a morphism of complexes of vector spaces, is denoted by

$$\iota_D : \Omega^\bullet_R(\log D) \to \Omega^\bullet_R(*D).$$

Proposition 3.1. $\Omega^q_R(\log D)$ *is a finitely generated R–module for all q.*

Proof. Recall that $f \in R$ is a reduced equation for the hypersurface $D = \mathcal{V}(f) \subset \mathbb{C}^n$. We have the inclusion

$$\Omega^q_R(\log D) \subset \frac{1}{f}\Omega^q_R$$

and $\frac{1}{f}\Omega^q_R$ is a free R–module with basis $\{\frac{1}{f}dx_{i_1} \wedge \cdots \wedge dx_{i_q} \mid 1 \leq i_1 < \cdots < i_q \leq n\}$. As R is a Noetherian ring, $\frac{1}{f}\Omega^q_R$ is a Noetherian module and its submodules are finitely generated. □

If A is a commutative ring and M is an A–module, the dual of M is $M^* := Hom_A(M, A)$. An A–module M is called *reflexive* if $M \simeq M^*$.

Exercise 3.7. Prove that the dual of $\mathrm{Der}_{\mathbb{C}}(R)$ is naturally isomorphic to Ω^1_R.

Proposition 3.2. *The R–modules* $\mathrm{Der}_R(-\log D)$ *and* $\Omega^1_R(\log D)$ *are dual to each other. Both modules are then reflexive.*

Remark 3.4. Because of the result in Proposition 3.2, V. Goryunov and D. Mond said (see [26, page 207]) that Kyoji Saito suggested, following the conventions of algebro-geometric notation, that the module of logarithmic derivations should have $-\log D$ in parentheses rather than $\log D$.

Proof. [of Proposition 3.2] We consider the R-bilinear map

$$\langle -, - \rangle : \mathrm{Der}_R(-\log D) \times \Omega^1_R(\log D) \longrightarrow R_f$$

defined by

$$\left\langle \sum_i a_i \partial_i, \sum_i \frac{b_i}{f} dx_i \right\rangle = \frac{\sum_i a_i b_i}{f}.$$

This bilinear form is just the restriction of the one of Remark 3.2. First of all, we will prove that if $\delta = \sum_i a_i \partial_i \in \mathrm{Der}_R(-\log D)$ and $\omega = \sum_i \frac{b_i}{f} dx_i \in \Omega^1(\log D)$ then $\langle \delta, \omega \rangle \in R$, i.e.

$$\sum_i a_i b_i \in Rf.$$

Let us write $g = \sum_i a_i b_i$. Recall (see Exercise 3.6) that if $\omega = \sum_i \frac{b_i}{f} dx_i \in \Omega^1(\log D)$ then

$$\frac{f_i b_j - f_j b_i}{f} \in R$$

for all i, j. Let us write $f_i b_j - f_j b_i = h_{ij} f$ for some $h_{ij} \in R$. Then, for any $i = 1, \ldots, n$,

$$f_i g = f_i (\sum_j a_j b_j) = \sum_j a_j f_i b_j = \sum_j a_j b_i f_j = b_i \delta(f) \in Rf.$$

So we have

$$J \cdot Rg \subset Rf$$

and $\mathcal{V}(f) \subset \mathcal{V}(J) \cup \mathcal{V}(g)$ where $J = R(f_1, \ldots, f_n)$ is the Jacobian ideal associated with f.

We can write $\mathcal{V}(f) = \mathcal{V}(J, f) \cup \mathcal{V}(g, f)$. Recall that $\mathcal{V}(J, f)$ is the set of singular points of the hypersurface $\mathcal{V}(f)$ and that it is a proper Zariski closed subset of $\mathcal{V}(f)$. So any irreducible component of $\mathcal{V}(f)$ should be included in $\mathcal{V}(g, f) \subset \mathcal{V}(g)$. By Hilbert's Nullstellensatz $g \in Rf$.

There is a natural injective R-module morphism

$$\Omega^1(\log D) \to (\mathrm{Der}_R(-\log D))^*$$

which associates to any logarithmic 1-form ω the R-module morphism

$$\phi_\omega : \mathrm{Der}_R(-\log D) \to R$$

defined by $\phi_\omega(\delta) = \langle \delta, \omega \rangle$. We will prove that this natural injective morphism is also surjective.

Assume $\varphi \in (\mathrm{Der}_R(-\log D))^*$. We will see that the rational form

$$\omega = \sum_i \frac{\varphi(f\partial_i)}{f} dx_i$$

belongs to $\in \Omega^1_R(\log D)$ and, moreover, $\varphi = \phi_\omega$.

First of all, we will prove that $\omega = \sum_i \frac{\varphi(f\partial_i)}{f} dx_i$ belongs to $\Omega^1_R(\log D)$. By, Exercise 3.6 it is enough to prove that

$$f_i\varphi(f\partial_j) - f_j\varphi(f\partial_i) \in Rf$$

for all i, j. To this end, $f_i\varphi(f\partial_j) - f_j\varphi(f\partial_i) = \varphi(f_i f\partial_j) - \varphi(f_j f\partial_i) = f\varphi(f_i\partial_j - f_j\partial_i) \in Rf$.

By definition of ω we have $\varphi(f\partial_i) = \phi_\omega(f\partial_i)$ for all i. Moreover, for $\delta = \sum_i a_i\partial_i \in \mathrm{Der}_R(-\log D)$ we have

$$f\varphi(\delta) = \varphi(f\delta) = \sum_i a_i\varphi(f\delta_i) = \sum_i a_i\phi_\omega(f\partial_i) = f\phi_\omega(\delta)$$

and then $\varphi(\delta) = \phi_\omega(\delta)$ for all $\delta \in \mathrm{Der}_R(-\log D)$. That proves $\varphi = \phi_\omega$ and, moreover, the natural morphism

$$\Omega^1_R(\log D) \to (\mathrm{Der}_R(-\log D))^*$$

is an isomorphism of R–modules.

We will prove in a similar way that the natural injective R-module morphism

$$\mathrm{Der}_R(-\log D) \to (\Omega^1_R(\log D))^*$$

which associates to any logarithmic vector field δ the R-module morphism

$$\phi_\delta : \Omega^1_R(\log D) \to R$$

defined by $\phi_\delta(\omega) = \langle\delta, \omega\rangle$ is an isomorphism.

Assume $\varphi \in (\Omega^1_R(\log D))^*$. We will see that the vector field

$$\delta = \sum_i \varphi(dx_i)\partial_i$$

belongs to $\mathrm{Der}_R(-\log D)$ and, moreover, $\varphi = \phi_\delta$.

We have $\varphi(\frac{df}{f}) \in R$ and then $f\varphi(\frac{df}{f}) = \varphi(df) \in Rf$. We also have

$$\delta(f) = \sum_i \varphi(dx_i)\frac{\partial f}{\partial x_i} = \varphi(\sum_i \frac{\partial f}{\partial x_i} dx_i) = \varphi(df) \in Rf.$$

Let's prove that $\varphi = \phi_\delta$. By definition of δ we have $\varphi(dx_i) = \phi_\delta(dx_i)$.

Assume $\omega = \sum_i \frac{g_i}{f} dx_i \in \Omega^1_R(\log D)$ for some $g_i \in R$. We have

$$f\varphi(\omega) = \varphi(f\omega) = \varphi(\sum_i g_i dx_i) = \sum_i g_i \varphi(dx_i)$$

$$= \sum_i g_i \phi_\delta(dx_i) = \phi_\delta(f\omega) = f\phi_\delta(\omega)$$

and then $\varphi(\omega) = \phi_\delta(\omega)$. □

Recall that the complex of (global) logarithmic differential forms $\Omega^\bullet_R(\log D)$ is a sub-complex of $\Omega^\bullet_R(*D)$ (both endowed with the exterior derivative) and that we denoted by $\iota_D : \Omega^\bullet_R(\log D) \to \Omega^\bullet_R(*D)$ the corresponding inclusion.

Definition 3.3 (Calderón et al.[12] Granger et al.[27]). *The hypersurface $D \subset \mathbb{C}^n$ satisfies the Global Logarithmic Comparison Theorem if ι_D is a quasi-isomorphism. In this case we will say that D satisfies GLCT or that GLCT holds for D.*

An open question is to characterize the class of hypersurfaces $D = \mathcal{V}(f) \subset \mathbb{C}^n$ such that D satisfies GLCT. This question is related to the characterization of polynomials $f \in R$ such that $Ann(1/f)$ is generated by operators of order 1. These questions (and the extension of the previous notions and results to the local analytic case) are treated in many classical as well as many recent research papers. The interested reader can consult the references 39, 17, 11, 12, 18, 19, 43, 13, 27, 2, 42.

Appendix A. Division Theorems and Groebner Bases in Rings of Differential Operators

In this Appendix we first recall the definition of the rings of germs of linear differential operators with holomorphic and formal coefficients. Then we recall the main results on the division theorems and the theory of Groebner bases in these rings of differential operators and in the Weyl algebra.

More rings of linear differential operators

Let $\mathbb{C}\{x\} = \mathbb{C}\{x_1, \ldots, x_n\}$ (resp. $\mathbb{C}[[x]] = \mathbb{C}[[x_1, \ldots, x_n]]$) be the ring of convergent power series (resp. of formal power series) in n variables with complex coefficients. The ring $\mathbb{C}\{x\}$ can also be viewed as the ring of germs of holomorphic functions at the origin in $\mathbb{C}^n$.

We will denote by $\mathcal{D}$ (resp. $\widehat{\mathcal{D}}$) the ring of linear differential operators with coefficients in $\mathbb{C}\{x\}$ (resp. $\mathbb{C}[[x]]$).

By definition, any nonzero element P in $\mathcal{D}$ (resp. in $\widehat{\mathcal{D}}$) can be written in a unique way as a finite sum

$$P = \sum_{\beta \in \mathbb{N}^n} p_\beta(x) \partial^\beta$$

where $p_\beta(x) \in \mathbb{C}\{x\}$ (resp. $\mathbb{C}[[x]]$) for all $\beta \in \mathbb{N}^n$ and, as in the case of the Weyl algebra, ∂^β stands for $\partial_1^{\beta_1} \cdots \partial_n^{\beta_n}$.

As any power series $p_\beta(x)$ can be written as

$$p_\beta(x) = \sum_{\alpha \in \mathbb{N}^n} p_{\alpha\beta} x^\alpha$$

with $p_{\alpha\beta} \in \mathbb{C}$, then each element $P \in \mathcal{D}$, unlike in the case of the Weyl algebra, can be written as a, possibly infinite, sum

$$P = \sum_{\alpha,\beta} p_{\alpha\beta} x^\alpha \partial^\beta$$

and we have a similar result for $\widehat{\mathcal{D}}$.

Some notions in Section 1 can be easily extended from the Weyl algebra to $\mathcal{D}$ and $\widehat{\mathcal{D}}$. Among these notions we have the *order* $\mathrm{ord}(P)$ and the *principal symbol* $\sigma(P)$ of an operator P in $\mathcal{D}$ or $\widehat{\mathcal{D}}$ (see Definitions 1.2 and 1.3).

The ring $\mathcal{D}$ is then *filtered* by the order of its elements. We will denote this filtration by

$$(F_k(\mathcal{D}))_{k \in \mathbb{Z}}.$$

The associated graded ring $\mathrm{gr}^F(\mathcal{D})$ is naturally isomorphic to the ring

$$\mathbb{C}\{x\}[\xi] = \mathbb{C}\{x\}[\xi_1, \ldots, \xi_n]$$

which is a polynomial ring with coefficients in $\mathbb{C}\{x\}$. With any (left) ideal I in $\mathcal{D}$ we can associate (see Definition 1.4) its graded ideal

$$\mathrm{gr}^F(\mathcal{D}) = \mathbb{C}\{x\}[\xi] \cdot \{\sigma(P), \mid P \in I\}$$

which is homogeneous with respect to the ξ–variables. We also have analogous results for $\widehat{\mathcal{D}}$.

Alike A_n the ring $\mathcal{D}$ is left and right Noetherian. The proof of the noetherianity of A_n uses the total order in A_n (see Proposition 1.6). By Remark 1.5 we can also give a proof of the same result by using the order in A_n instead of the total order. This last proof is also valid in $\mathcal{D}$ and in $\widehat{\mathcal{D}}$. For another proof of the noetherianity of A_n (also valid for $\mathcal{D}$ and $\widehat{\mathcal{D}}$) see Remark 4.10.

Division theorem in A_n

Definition A.1. A well ordering $\prec$ on $\mathbb{N}^n$ is said to be a *monomial order* if it is compatible with the sum: $\alpha \prec \beta$ implies $\alpha+\gamma \prec \beta+\gamma$ for all $\gamma \in \mathbb{N}^n$.

Remark 4.5. For any monomial order $\prec$ on $\mathbb{N}^n$ one has $0 = (0,\ldots,0) \prec \alpha$ for all $\alpha \in \mathbb{N}^n$. Moreover, for $\alpha, \beta \in \mathbb{N}^n$ such that $\alpha_i \leq \beta_i$ for all i one has $\alpha \prec \beta$. In other words, any monomial order refines the componentwise order on $\mathbb{N}^n$.

We usually translate any order $\prec$ on $\mathbb{N}^n$ to an order —also denoted by $\prec$— on the set of monomial $\{x^\alpha \,|\, \alpha \in \mathbb{N}^n\}$ just by writing $x^\alpha \prec x^\beta$ if and only if $\alpha \prec \beta$.

Example 4.1. (1) The *lexicographical* or *lexicographic order* (denoted by $<_{\text{lex}}$) on $\mathbb{N}^n$ is defined as follows:

$$(\alpha_1,\ldots,\alpha_n) <_{\text{lex}} (\beta_1,\ldots,\beta_n)$$

if and only if the first nonzero component of

$$(\alpha_1-\beta_1,\ldots,\alpha_n-\beta_n)$$

is negative. The well ordering $<_{\text{lex}}$ is a monomial order.
(2) Let $\prec$ be a monomial order on $\mathbb{N}^n$. Let $L : \mathbb{Q}^n \to \mathbb{Q}$ be a linear form with non negative coefficients. The binary relation $\prec_L$ defined on $\mathbb{N}^n$ by:

$$\alpha \prec_L \beta \text{ if and only if } \begin{cases} L(\alpha) < L(\beta) \\ \text{or } L(\alpha) = L(\beta) \text{ and } \alpha \prec \beta \end{cases}$$

is a monomial order on $\mathbb{N}^n$.

Let $P = P(x,\partial) = \sum_{\beta\in\mathbb{N}^n} p_\beta(x)\partial^\beta$ be a differential operator in A_n. The operator P can be rewritten as

$$P = \sum_{\alpha\beta} p_{\alpha\beta}x^\alpha\partial^\beta$$

just by considering the polynomial $p_\beta(x)$ as $p_\beta(x) = \sum_\alpha p_{\alpha\beta}x^\alpha$, with $p_{\alpha\beta} \in \mathbb{C}$.

The *Newton diagram* of P is the set

$$\mathcal{N}(P) := \{(\alpha,\beta) \in \mathbb{N}^{2n} \,|\, p_{\alpha\beta} \neq 0\}.$$

Let us fix a monomial order $\prec$ on $\mathbb{N}^{2n}$.

Definition A.2. We call *privileged exponent*[i] with respect to $\prec$ of a nonzero operator P —and we denote it by $\exp_{\prec}(P)$— the maximum $(\alpha, \beta) \in \mathbb{N}^{2n}$, with respect to $\prec$, such that $p_{\alpha\beta} \neq 0$. In other words

$$\exp_{\prec}(P) = \max_{\prec} \mathcal{N}(P).$$

We will simply write $\exp(P)$ if no confusion is possible.

Proposition 4.3. *Let $P, Q \in A_n$. We have:*
1) $\exp(PQ) = \exp(P) + \exp(Q)$.
2) If $\exp(P) \neq \exp(Q)$ *then* $\exp(P + Q) = \max_{\prec}\{\exp(P), \exp(Q)\}$*. More generally, for any family* $P_1, \ldots, P_m \in A_n$ *such that* $\exp(P_i) \neq \exp(P_j)$ *for all* i, j, $i \neq j$ *we have* $\exp(\sum_i P_i) = \max_{\prec}\{\exp(P_i) \,|\, i = 1, \ldots, m\}$.

With each m–tuple $((\alpha^1, \beta^1), \ldots, (\alpha^m, \beta^m))$ of elements in $\mathbb{N}^{2n}$, we associate a *partition*[j]

$$\{\Delta^0, \Delta^1, \ldots, \Delta^m\}$$

of $\mathbb{N}^{2n}$ in the following way. We set:

$$\Delta^1 = (\alpha^1, \beta^1) + \mathbb{N}^{2n}$$

$$\Delta^{i+1} = ((\alpha^{i+1}, \beta^{i+1}) + \mathbb{N}^{2n}) \setminus (\Delta^1 \cup \cdots \cup \Delta^i) \text{ if } i \geq 1$$

$$\Delta^0 = \mathbb{N}^{2n} \setminus (\cup_{i=1}^m \Delta^i).$$

The following theorem generalizes the division theorem for polynomials in $\mathbb{C}[x]$ (see e.g. [22, p. 9] or [1, Th. 1.5.9]).

Theorem A.1 (Division in A_n). *Let $(P_1, \ldots, P_m)$ be an m–tuple of nonzero elements of A_n and let $\{\Delta^0, \Delta^1, \ldots, \Delta^m\}$ be the partition of $\mathbb{N}^{2n}$ associated with $(\exp(P_1), \ldots, \exp(P_m))$. Then, for any P in A_n, there exists a unique $(m+1)$–tuple $(Q_1, \ldots, Q_m, R)$ of elements in A_n, such that:*

(1) $P = Q_1 P_1 + \cdots + Q_m P_m + R$.
(2) $\exp(P_i) + \mathcal{N}(Q_i) \subset \Delta^i$, $i = 1, \ldots, m$.
(3) $\mathcal{N}(R) \subset \Delta^0$.

---

[i]This notion generalizes the one of privileged exponent of a power series, due to H. Hironaka. It was introduced in Lejeune and Teissier[32] (see also Aroca et al.[3]).
[j]The word *partition* is used here in a broad sense, which means that an element of the family may be empty.

Proof. Let us prove uniqueness first. Assume that two $(m+1)$–tuples, $(Q_1, \ldots, Q_m, R)$ and $(Q'_1, \ldots, Q'_m, R')$, satisfy the conditions of the theorem. We have:

$$\sum_{i=1}^{m} (Q_i - Q'_i)P_i + R - R' = 0 \qquad (1)$$

If $Q_i \neq Q'_i$ then $\exp((Q_i - Q'_i)P_i) \in \Delta^i$. If $R \neq R'$ then $\exp(R - R') \in \Delta^0$. Since $\Delta^0, \Delta^1, \ldots, \Delta^m$ is a partition of $\mathbb{N}^{2n}$, the equality (1) is only possible if $Q_i = Q'_i$ for any i and if $R = R'$ (see Proposition 4.3). That proves the uniqueness.

It is clear that it is enough to prove the existence for the monomials $x^\alpha \partial^\beta \in A_n$. We will use induction on (α, β). If $x^\alpha \partial^\beta = 1$ (i.e. if $\alpha = \beta = (0, \ldots, 0)$), then either $\exp(P_i) \neq 0 \in \mathbb{N}^{2n}$ for all i and in this case it is enough to write $1 = \sum_{i=1}^{m} 0 P_i + 1$, or there exists an integer j such that $\exp(P_j) = 0 \in \mathbb{N}^{2n}$. In this case P_j is a nonzero constant because 0 is the first element in $\mathbb{N}^{2n}$ with respect to the well ordering $\prec$ and, moreover, $\Delta^0 = \emptyset$. Assume that j is minimal. We write

$$1 = \sum_{i \neq j} 0 \cdot P_i + (1/P_j)P_j + 0.$$

This proves the existence at the first step of the induction. Assume that the result is proved for any (α', β') strictly less than some $(\alpha, \beta) \neq 0 \in \mathbb{N}^{2n}$. Let $j \in \{0, 1, \ldots, m\}$ be such that $(\alpha, \beta) \in \Delta^j$. If $j = 0$ we write

$$x^\alpha \partial^\beta = \sum_{i=1}^{m} 0 \cdot P_i + x^\alpha \partial^\beta.$$

If $j \geq 1$ let $(\gamma, \delta) = (\alpha, \beta) - \exp(P_j) \in \mathbb{N}^{2n}$.

We can write

$$x^\alpha \partial^\beta = \frac{1}{c_j} x^\gamma \partial^\delta P_j + G_j$$

where c_j is the coefficient of the privileged monomial of P_j and all the monomials in G_j are smaller (with respect to $\prec$) than (α, β) (see Exercise 1.2). By the induction hypothesis there exists $(Q'_1, \ldots, Q'_m, R')$ satisfying the conditions of the theorem for $P = G_j$. In particular we have:

$$x^\alpha \partial^\beta = \sum_{i \neq j} Q'_i P_i + (\frac{1}{c_j} x^\gamma \partial^\delta + Q'_j)P_j + R'.$$

This proves the result for (α, β). Thus, existence is proved for any $P \in A_n$. □

Remark 4.6. The linear differential operator Q_i in the theorem is called the i–th *quotient* and R is called the *remainder* of the division of P by $(P_1, \ldots, P_m)$. The remainder will be denoted by $R(P; P_1, \ldots, P_m)$.

Remark 4.7. It follows from the proof of Theorem A.1 that for any division $P = Q_1P_1 + \cdots + Q_mP_m + R$ as in the theorem we have $\max\{\max_i\{\exp_\prec(Q_iP_i)\}, \exp_\prec(R)\} = \exp_\prec(P)$.

Groebner bases in A_n

The theory of Groebner bases in the polynomial ring $\mathbb{C}[x]$ (see Buchberger[9,10]) can be extended to the Weyl algebra A_n and also to the rings $\mathcal{D}$ and $\widehat{\mathcal{D}}$ (see Castro[14,15] and Briançon and Maisonobe[8]). The book M. Saito et al.[40] contains many applications of Groebner basis theory to the study of modules over the Weyl algebra A_n.

Let us fix a monomial order $\prec$ on $\mathbb{N}^{2n}$. For each non zero ideal I of A_n let $\mathrm{Exp}_\prec(I)$ denote the set

$$\{\exp_\prec(P) | P \in I \setminus \{0\}\}.$$

If no confusion is possible we write $\mathrm{Exp}(I)$ instead of $\mathrm{Exp}_\prec(I)$. From Proposition 4.3 we have: $\mathrm{Exp}(I) + \mathbb{N}^{2n} = \mathrm{Exp}(I)$.

Proposition 4.4. *Let $m > 0$ be an integer and $E \subset \mathbb{N}^m$ such that $E + \mathbb{N}^m = E$. Then there exists a finite subset $F \subset E$ such that*

$$E = \bigcup_{\alpha \in F} (\alpha + \mathbb{N}^m)$$

Proof. This is a version of Dickson's lemma. The proof is by induction on m. For $m = 1$ a (finite) family of generators of E is given by the smallest element of E (for the usual ordering in $\mathbb{N}$). Assume that $m > 1$ and that the result is true for $m - 1$. Let $E \subset \mathbb{N}^m$ be such that $E + \mathbb{N}^m = E$. We can assume that E is non empty. Let $\alpha \in E$. For any $i = 1, \ldots, m$ and $j = 0, \ldots, \alpha_i$ we consider the bijective mapping

$$\begin{array}{cccc} \phi_{i,j} : & \mathbb{N}^{i-1} \times \{j\} \times \mathbb{N}^{m-i} & \longrightarrow & \mathbb{N}^{m-1} \\ & (\beta_1, \ldots, \beta_{i-1}, j, \beta_{i+1}, \ldots, \beta_m) & \mapsto & (\beta_1, \ldots, \beta_{i-1}, \beta_{i+1}, \ldots, \beta_m) \end{array}$$

and we denote

$$E_{i,j} = \phi_{i,j}(E \cap (\mathbb{N}^{i-1} \times \{j\} \times \mathbb{N}^{m-i})).$$

It is clear that $E_{i,j} + \mathbb{N}^{n-1} = E_{i,j}$ and by the induction hypothesis there is a finite subset $F_{i,j} \subset E_{i,j}$ generating $E_{i,j}$. The finite set

$$F = \{\alpha\} \bigcup \left(\bigcup_{i,j} (\phi_{i,j})^{-1}(F_{i,j}) \right)$$

generates E. This proof is taken from Galligo.[25] □

Remark 4.8. The previous proposition can be rephrased as follows: *Any monomial ideal in* $\mathbb{C}[x]$ *(Or more generally in a polynomial ring* $\mathbb{K}[x_1, \ldots, x_n]$ *with coefficients in a field* $\mathbb{K}$*) is finitely generated.* This is a particular case of the Hilbert basis theorem.

In the same way, we can see that any increasing sequence $(E_k)_k$ of subsets of $\mathbb{N}^m$, stable under the action of $\mathbb{N}^m$, is stationary. We shall often use this property called the *noetherian property for* $\mathbb{N}^m$.

We can adapt the proof above to show that, given $E \subset \mathbb{N}^n$ as in Proposition 4.4, we can find in any set of generators, a finite subset of generators of E. This proves in particular that in any system of generators made of monomials of a monomial ideal in the polynomial ring $\mathbb{C}[x]$, we can find a finite subset of generators. This is Dickson's lemma.

Definition A.3. Let I be a nonzero ideal of A_n. We call any family $P_1, \ldots, P_m$ of elements in I such that

$$\operatorname{Exp}_{\prec}(I) = \bigcup_{i=1}^{m} (\exp_{\prec}(P_i) + \mathbb{N}^{2n})$$

a *Groebner basis* of I, relative to $\prec$ (or a $\prec$–Groebner basis of I).

Remark 4.9. There always exists a Groebner basis of I by definition of $\operatorname{Exp}(I)$ and by Proposition 4.4.

We have the following two corollaries of Theorem A.1.

Corollary A.1. *Let I be a nonzero ideal of A_n and let $P_1, \ldots, P_m$ be a family of elements of I. The following conditions are equivalents:*
1) $P_1, \ldots, P_m$ is a Groebner basis of I.
2) For any P in A_n, we have: $P \in I$ if and only if $R(P; P_1, \ldots, P_m) = 0$ (see Remark 4.6).

Corollary A.2. *Let I be a nonzero (left) ideal of A_n and let $P_1, \ldots, P_m$ be a Groebner basis of I. Then $P_1, \ldots, P_m$ is a system of generators of I.*

Remark 4.10. Corollary A.2 proves in particular that A_n is a left-Noetherian ring (see also Proposition 1.6).

Remark 4.11. Assume the monomial order $\prec$ satisfies the following property: for any $P \in A_n$ we have $\exp_\prec(P) = \exp_\prec(\sigma(P))$. Then, for any ideal I in A_n and any finite subset $\{P_1, \ldots, P_m\} \subset I$, the family $\{P_1, \ldots, P_m\}$ is a Groebner basis of I if and only if the family $\{\sigma(P_1), \ldots, \sigma(P_m)\}$ is a Groebner basis of $\mathrm{gr}^F(I) \subset \mathbb{C}[x, \xi]$.

If $L : \mathbb{Q}^n \times \mathbb{Q}^n \longrightarrow \mathbb{Q}$ is the linear form defined by $L(a_1, \ldots, a_n, b_1, \ldots, b_n) = \sum_i b_i$ and $\prec$ is any monomial order on $\mathbb{N}^{2n}$ then the monomial order $\prec_L$ (see Example 4.1) satisfies the desired condition $\exp_\prec(P) = \exp_\prec(\sigma(P))$ for all $P \in A_n$.

Buchberger's algorithm in A_n

Buchberger's algorithm for polynomials (see Buchberger[10]) can be easily adapted to the Weyl algebra (see Briançon and Maisonobe[8] and Castro[14] (see also Saito et al.[40])).

Considering as input a monomial order $\prec$ in $\mathbb{N}^{2n}$ and a finite set $\mathcal{F} = \{P_1, \ldots, P_m\}$ of differential operators in A_n, one can compute a Groebner basis, with respect to $\prec$, of the ideal $I \subset A_n$ generated by $\mathcal{F}$. So, one can also compute a finite set of generators of the subset $\mathrm{Exp}_\prec(I) \subset \mathbb{N}^{2n}$.

The Division Theorem A.1 and the theory of Groebner bases can be extended for vectors and for submodules of free modules A_n^r for any integer $r > 0$ (see Castro[14]).

Moreover, the Division Theorem and the Groebner basis notion can be also considered, in a straightforward way, for *right ideals* (or more generally for *right submodules* of a free module A_n^r).

Similarly to the commutative polynomial case, Groebner bases in A_n are used to compute, in an explicit way, some invariants in A_n-module theory. Most of the algorithms in this subject appears in Oaku and Takayama.[37] In particular, Groebner bases in A_n are used:

a) to compute a generating system of $Syz_{A_n}(P_1, \ldots, P_m)$, the A_n–module of syzygies of a given family $P_1, \ldots, P_m$ in a free module A_n^r $(r \geq 1)$.
b) to solve the membership problem (i.e. to decide if a given vector $P \in A_n^r$ belongs to the submodule generated by $P_1, \ldots, P_m$) and to decide if two submodules of A_n^r are equal.
c) to compute the graded ideal and the total graded ideal associated with a (left) ideal I in A_n (see Definition 1.4) and to compute the dimension of a quotient module A_n/I.

d) to decide if a finitely presented A_n–module is holonomic (i.e. to decide if its characteristic variety has dimension n. See Definition 2.8).
e) to construct a finite free resolution of a given finitely presented A_n–module.
f) to decide if a finite complex of free A_n–modules is exact.

Many computer algebra systems can handle this kind of computations. Among the most used should be mentioned `Macaulay`[28] `Risa/Asir`[35] and `Singular`.[29]

Remark 4.12 (Division theorem and Groebner bases in $\mathcal{D}$ and $\widehat{\mathcal{D}}$). *A Division Theorem (analogous to Theorem A.1) can be proved for elements in $\mathcal{D}$ or in $\widehat{\mathcal{D}}$ (see Briançon and Maisonobe[8] and Castro[14]). This is not straightforward from the Weyl algebra case because Definition A.2 of* privileged exponent *doesn't work for general operators in $\mathcal{D}$ or in $\widehat{\mathcal{D}}$. Nevertheless, Groebner bases also exist for left (or right) ideals in $\mathcal{D}$ (and in $\widehat{\mathcal{D}}$) and the analogous of Corollaries A.1 and A.2 also hold in $\mathcal{D}$ and $\widehat{\mathcal{D}}$. This proves in particular that $\mathcal{D}$ and $\widehat{\mathcal{D}}$ are Noetherian rings. We will not give here the details and refer the interested reader to the references above.*

Bibliographical note

Most of the material of this Appendix appears in Castro,[14,15] Briançon and Maisonobe,[8] Saito et al.[40] and Castro and Granger.[16]

References

1. Adams W.W. and Loustaunau Ph. *An introduction to Gröbner bases*. Graduate Studies in Mathematics, 3. American Mathematical Society, 1994.
2. Álvarez-Montaner J., Castro-Jiménez F.J., Ucha-Enríquez J.M. *Localization at hyperplane arrangements: combinatorics and D-modules*, J. Algebra, available on-line 28 Dec 2006; `http://dx.doi.org/10.1016/j.jalgebra.2006.12.006`
3. Aroca J. M., Hironaka H. and Vicente J.L. *The theory of maximal contact.* Volume 29 of *Memorias del Instituto "Jorge Juan"*, Madrid, 1975.
4. Atiyah M.F. and Macdonald I.G. *Introduction to Commutative Algebra.* Addison-Wesley, 1969.
5. Bernstein J. *Modules over the ring of differential operators. Study of the fundamental solutions of equations with constant coefficients.* Funkcional. Anal. i Priložen. 5 (1971), no. 2, 1–16.
6. Bernstein J. *Analytic continuation of generalized functions with respect to a parameter.* Funkcional. Anal. i Priložen. 6 (1972), no. 4, 26–40.
7. Björk J-E. *Rings of Differential Operators.* North-Holland, Amsterdam 1979.

8. Briançon J. and Maisonobe Ph. *Idéaux de germes d'opérateurs différentiels à une variable*, L'Enseignement Math. **30**, (2004), 7–38.
9. Buchberger B. *An Algorithm for Finding the Bases Elements of the Residue Class Ring Modulo a Zero Dimensional Polynomial Ideal (German)*, PhD thesis, (1965), Univ. of Innsbruck (Austria).
10. Buchberger B. *An Algorithmical Criterion for the Solvability of Algebraic Systems of Equations (German)*, Aequationes Mathematicae **4**(3),(1970), 374–383.
11. Calderón-Moreno F.J. *Logarithmic Differential Operators and Logarithmic De Rham Complexes relative to a Free Divisor.* Ann. Sci. E.N.S., 4^e série, t. 32, 1999, 701-714.
12. Calderón-Moreno F.J., Mond D., Narváez-Macarro L. and Castro-Jiménez F.J. *Logarithmic Cohomology of the Complement of a Plane Curve.* Comment. Math. Helv. 77 (2002), no. 1, 24–38.
13. Calderón-Moreno F.J. and Narváez-Macarro L. *Dualité et comparaison sur les complexes de de Rham logarithmiques par rapport aux diviseurs libres.* Ann. Inst. Fourier (Grenoble) 55 (2005), no. 1, 47–75.
14. Castro F. *Théorème de division pour les opérateurs différentiels et calcul des multiplicités*, Thèse de 3^{eme} cycle, Univ. of Paris VII, (1984).
15. Castro-Jiménez F. *Calculs effectifs pour les idéaux d'opérateurs différentiels*, in Actas de la II Conferencia Internacional de Geometría Algebraica. La Rábida, Travaux en Cours 24, (1987), Hermann, Paris.
16. Castro Jiménez F. J. and Granger, M. *Explicit calculations in rings of differential operators.* Éléments de la théorie des systèmes différentiels géométriques, 89–128, Sémin. Congr., 8, Soc. Math. France, Paris, 2004.
17. Castro-Jiménez F.J., Mond, D. and Narváez-Macarro L. *Cohomology of the complement of a free divisor.* Trans. Amer. Math. Soc. 348 (1996), no. 8, 3037–3049.
18. Castro-Jiménez, F.J. and Ucha-Enríquez J.M. *Explicit comparison theorems for $\mathcal{D}$-modules.* J. Symbolic Comput., Special Issue on Effective Methods in Rings of Differential Operators, 32 (2001) no. 6, 677–685.
19. Castro Jiménez F. J. and Ucha Enríquez J. M. *Testing the Logarithmic Comparison Theorem for free divisors.* Experimental Math. 13:4, 441-449, 2004.
20. Castro-Jiménez, F.J. and Ucha-Enríquez J.M. *Gröbner bases and logarithmic $\mathcal{D}$-modules.* J. Symbolic Comput., 41 (2006), 317–335.
21. Coutinho, S.C. *A primer on $\mathcal{D}$-modules.* Student Texts 33, London Mathematical Society. Cambridge University Press 1995.
22. Cox D., Little J. and O'Shea D. *Using algebraic geometry.* Graduate Texts in Mathematics, 185, Springer-Verlag, New York, 2005.
23. Ehlers F. *The Weyl algebra* in Borel A. et al. Algebraic D-modules. Perspectives in Mathematics, 2. Academic Press, Inc., Boston, MA, 1987. xii+355 pp.
24. Eisenbud, D. *Commutative algebra. With a view toward algebraic geometry.* Graduate Texts in Mathematics, 150. Springer-Verlag, New York, 1995.
25. Galligo A. Théorème de division et stabilité en Géométrie Analytique locale. *Ann. Inst. Fourier*, 29,2:107–184, 1979.

26. Goryunov V. and Mond D. *Tjurina and Milnor numbers of matrix singularities*. J. London Math. Soc. (2) 72 (2005) 205-224.
27. Granger M., Mond D., Nieto A. and Schulze M. *Linear free divisors.* arXiv :math/0607045.
28. Grayson D. and Stillman M. Macaulay2: *a software system for research in algebraic geometry.* Available at http://www.math.uiuc.edu/Macaulay2 Leykin, A. and Tsai, H. *D-modules package for Macaulay 2. Available at http://www.math.cornell.edu/~htsai*
29. Greuel G.-M., Pfister G. and Schönemann H. Singular a computer algebra system for polynomial computations, Centre for Computer Algebra, University of Kaiserslautern, http://www.singular.uni-kl.de.
30. Hartshorne R. Algebraic Geometry. Graduate Texts in Mathematics. Springer-Verlag. 1977.
31. Kashiwara M. *B-functions and holonomic systems. Rationality of roots of B-functions.* Invent. Math. 38, 1 (1976) 33–53.
32. M. Lejeune-Jalabert and B. Teissier. Quelques calculs utiles pour la résolution des singularités, Volume of *Prépub. Centre de Mathématiques, Ecole Polytechnique*, 1971.
33. Malgrange, B. *Le polynome de Bernstein d'une singularite isolée*. Lecture Notes in Math. 459, (1976), Springer-Verlag, 98-119.
34. Noro M. *An efficient modular algorithm for computing the global b-function.* Mathematical software (Beijing, 2002), 147–157. World Sci. Publishing, River Edge, NJ, (2002).
35. Noro M., Shimoyama T. and Takeshima T. *A Computer Algebra System Risa/Asir*. Available at http://www.math.kobe-u.ac.jp/Asir/index.html.
36. Oaku T. *An algorithm of computing b-functions*. Duke Math. J. 87 (1997), no. 1, 115–132
37. Oaku T. and Takayama N. *Algorithms for D-modules – restriction, tensor product, localization, and local cohomology groups.* Journal of Pure and Applied Algebra, 156 (2001) pp. 495-518.
38. M. Sato. *Theory of prehomogeneous vector spaces (algebraic part)*—the English translation of Sato's lecture from Shintani's note. Notes by T. Shintani. Translated from the Japanese by M. Muro. Nagoya Math. J. 120 (1990), 1–34.
39. Saito K. *Theory of logarithmic differential forms and logarithmic vector fields.* J. Fac Sci. Univ. Tokyo 27:256-291, 1980.
40. Saito M., Sturmfels B. and Takayama N., *Gröbner deformations of hypergeometric differential equations.* Algorithms and Computation in Mathematics, 6. Springer-Verlag, Berlin, (2000).
41. Stafford, J.T. *Nonholonomic modules over Weyl algebras and enveloping algebras.* Invent. Math., 79, (1985), no. 3, 619-638.
42. Terao H. and Yuzvinsky S. *Logarithmic forms on affine arrangements,* Nagoya Math. J. **139** (1995), 129–149.
43. Torrelli, T. *On meromorphic functions defined by a differential system of order 1.* Bull. Soc. Math. France 132 (2004), no. 4, 591–612.

44. Zariski, O. and Samuel, P. *Commutative algebra. Vol. I and II.* Springer Verlag, 1975

GEOMETRY OF CHARACTERISTIC VARIETIES

LÊ DŨNG TRÁNG

The Abdus Salam International Centre for Theoretical Physics,
Trieste, Italy
E-mail: ledt@ictp.it

BERNARD TEISSIER

Inst. Math. de Jussieu, 175 Rue Chevalleret, 75013 Paris, France
E-mail: teissier@math.jussieu.fr

0. Introduction

In this paper we give a quick presentation of the characteristic variety of a complex analytic linear holonomic differential system. The fact that we can view a complex analytic linear holonomic differential system on a complex manifold as a holonomic $\mathcal{D}_X$-module, allows us to present the characteristic variety in an algebro-geometric way as we do here.

We do not define what is a sheaf; for this we refer to the famous book of R. Godement.[2] We also do not define properly $\mathcal{D}_X$-modules although one can find in the lectures of F. Castro the definition of left modules on the Weyl algebra of operators on $\mathbb{C}^n$ with polynomial coefficients which give a local version of $\mathcal{D}_X$-modules.

Also we consider categories, complexes in abelian categories, derived functors and hypercohomologies[1] which are the natural language to be used here. Of course, we cannot define all these notions. One should view these notes as a provocation rather than a self-contained exposition. We hope they will encourage the reader to learn more in the subject.

1. Whitney Conditions

In Ref. 19, §19 p. 540, H. Whitney introduced Whitney conditions. The general idea is to find conditions for the attachment of a non singular analytic space having an analytic closure along a non singular part of its boundary which ensure that the closure is "locally topologically trivial" along the boundary, that is, locally topologically a product of the boundary by a "transverse slice". Whitney's approach can be deemed to be based on the fundamental fact, which he discovered, that any analytic set is, at each of its points, "asymptotically a cone". More precisely, given any closed complex analytic subspace $\mathcal{X} \subset U \subset \mathbb{C}^N$ and a point of X which we may take as the origin 0, given any sequence of non singular points $x_n \in X$ we may consider the tangent spaces $T_{\mathcal{X},x_n}$ of $\mathcal{X}$ and the secant lines $0x_n$. They respectively define points in the Grassmanian $\mathrm{G}(N,d)$ and the projective space $\mathbb{P}^{N-1}$, which are compact. Therefore, possibly after taking a subsequence we may assume the limits $T = \lim_n T_{\mathcal{X},x_n}$ and $\ell = \lim_n 0x_n$ exist. Then we have $\ell \subset T$, which ensures that $\mathcal{X}$ is locally "cone-like" with respect to the "vertex" 0.

Whitney's idea may have been that the topological triviality of the closed analytic space $\mathcal{X}$ along a part $\mathcal{Y}$ would be ensured by the condition that $\mathcal{X}$ should be "cone-like" along the "vertex" $\mathcal{Y}$, which is a way to ensure that locally $\mathcal{X}$ is transversal to the boundaries of small tubes around $\mathcal{Y}$. It gives this:
Let $\mathcal{X}$ be a complex analytic subset of $\mathbb{C}^N$ and $\mathcal{Y}$ be a complex analytic subset of $\mathcal{X}$. One says that $\mathcal{X}$ satisfies the Whitney condition along $\mathcal{Y}$ at the point $y \in \mathcal{Y}$, if:

(1) the point y is non-singular in $\mathcal{Y}$;
(2) for any sequence $(x_n)_{n\in\mathbb{N}}$ of non-singular points of $\mathcal{X}$ which converges to y such that the tangent spaces $T_{x_n}(\mathcal{X})$ have a limit T and, for any sequence $(y_n)_{n\in\mathbb{N}}$ of points of $\mathcal{Y}$ which converges to y, such that the lines $y_n x_n$ have a limit ℓ, we have $\ell \subset T$.

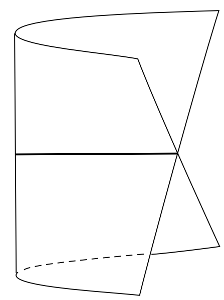
Examples. Consider the complex algebraic subset $\mathcal{X}$ of $\mathbb{C}^3$ defined by the equation:

$$X^2 - YZ^2 = 0.$$

The line $L = \{X = Y = 0\}$ is contained in $\mathcal{X}$. The surface $\mathcal{X}$ satisfies the Whitney condition along L at any point $y \in L \setminus \{0\}$, but not at the point $\{0\}$, because, for the sequence $(0, y_n, 0)$ of non-singular points of $\mathcal{X}$, the limit of tangent spaces is the plane $T = \{Z = 0\}$, and for the points $(0, 0, y_n)$ the limit ℓ is the line $\{X = 0, Y = -Z\}$ which is not contained in T.

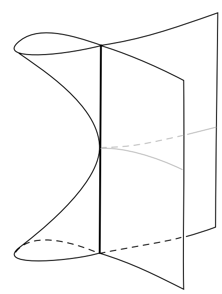
Consider the complex algebraic subset $\mathcal{X}$ of $\mathbb{C}^3$ defined by the equation (see Fig. 2):

$$X^2 - Y^3 - Z^2Y^2 = 0.$$



$X^2 - YZ^2 = 0$

Fig. 1.



$X^2 - Y^3 - Z^2Y^2 = 0$

Fig. 2.

The surface $\mathcal{X}$ satisfies Whitney condition along $L = \{X = Y = 0\}$ at any point $y \in L \setminus \{0\}$, but not at $\{0\}$, because, one may consider the sequence of non-singular points $(0, z_n^2, z_n)$ of $\mathcal{X}$ and $(0, 0, z_n)$ of L as z_n tends to 0. The limit ℓ is the line which contains $(0, 0, 1)$, the limit T is the plane orthogonal to $(0, 0, 1)$.

2. Stratifications

Let $\mathcal{X}$ be a complex analytic subset of $\mathbb{C}^N$. A partition $\mathcal{S} = (X_\alpha)_{\alpha \in A}$ is a stratification of $\mathcal{X}$, if:

(1) The closure $\overline{X}_\alpha$ of X_α in $\mathcal{X}$ and $\overline{X}_\alpha \setminus X_\alpha$ are complex analytic subspaces of $\mathcal{X}$;
(2) The family (X_α) is locally finite;
(3) Each X_α is a complex analytic manifold;

(4) If, for a pair (α, β), we have $X_\alpha \cap \overline{X}_\beta \neq \emptyset$, then $X_\alpha \subset \overline{X}_\beta$.

The subsets X_α are called the strata of the stratification $\mathcal{S}$. The condition (4) is called the frontier condition (see e.g. Ref. 13, Définition (1.2.3)).

Examples. Consider the complex algebraic subset of $\mathbb{C}^3$ defined by the equation:

$$X^2 - Y^3 - Z^2Y^2 = 0.$$

One can consider this surface as a deformation of complex plane algebraic curves parametrized by Z. The singular points of these curves are on the Z-axis which if given by $X = Y = 0$. For $Z = 0$, one has a cusp $X^2 - Y^3 = 0$. For $Z \neq 0$, one has a strophoıd with a singular point at the origin.

One has a stratification by considering the strata given by the non-singular points, which are the points of the surface outside of the line $X = Y = 0$ on the surface, and by the singular points which are the points of the line $X = Y = 0$.

Now consider the complex algebraic subset of $\mathbb{C}^3$ defined by the equation (see Fig. 3):

$$X^2 - Y^2Z^3 = 0.$$

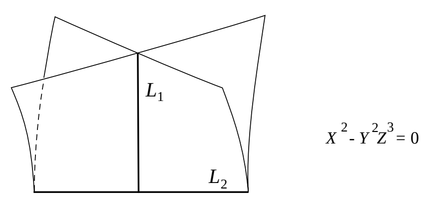


Fig. 3.

It is a surface whose singular points lie on two lines $L_1 = \{X = Y = 0\}$ and $L_2 = \{X = Z = 0\}$. We can define a partition of this surface by considering $\mathcal{S}_0 := \mathcal{X}^0$, the subset of non-singular points of the surface, the punctured line $\mathcal{S}_1 = L_1 \setminus \{0\}$ and the line $\mathcal{S}_2 = L_2$.

This partition does not define a stratification, because it does not satisfy the frontier condition. However, one can consider instead the partition given by $\mathcal{S}_0$, $\mathcal{S}_1$, the punctured line $\mathcal{S}_2' = L_2 \setminus \{0\}$ and the origin $\{0\}$. This is a stratification.

More generally one can prove that, by choosing a "refinement" of a partition satisfying (1), (2) and (3), one obtains the frontier condition (see Ref. 19).

A *complex analytic partition* $\mathcal{S} = (X_\alpha)_{\alpha \in A}$ of a complex analytic space $\mathcal{X}$ is a partition of $\mathcal{X}$ which satisfies (1), (2) and (3) above.

Then, a complex analytic partition $S' = (X'_\alpha)_{\alpha \in A'}$ of $\mathcal{X}$ is finer than the partition $\mathcal{S} = (X_\alpha)_{\alpha \in A}$ of $\mathcal{X}$, if any stratum X_α of $\mathcal{S}$ is the union of strata of $\mathcal{S}'$. One may quote a proposition of H. Whitney in the following way:

Proposition 2.1. *For any complex analytic partition $\mathcal{S} = (X_\alpha)_{\alpha \in A}$ of $\mathcal{X}$, there is a finer complex analytic partition which satisfies the frontier condition.*

Remark also that there is a coarsest finer partition with connected strata; this is obvious by taking the connected components of strata. In stratification theory one often assumes that the strata are connected.

3. Constructible Sheaves

All the sheaves that we consider are sheaves of complex vector spaces.

First, let us define local systems.

Definition 3.1. Let A be a topological space. A sheaf $\mathcal{F}$ on A is called a local system on A if it is locally isomorphic to a constant sheaf.

For instance, a constant sheaf is a local system. In fact, when A is an arcwise connected space, a local system $\mathcal{F}$ on A defines a homomorphism $\pi_1(A, a) \stackrel{\rho_\mathcal{F}}{\to} Aut_\mathbb{C}\mathcal{F}_a$. This homomorphism is defined in the following way: let γ be a loop at a; one can extend a section of $\mathcal{F}_a$ by continuity along γ, since there is a neighbourhood of a on which $\mathcal{F}$ is constant; because we can cover the image of γ by a finite number of open sets over which $\mathcal{F}$ is constant we define a map of $\mathcal{F}_a$ into itself determined by γ; one can show that this map depends only on the homotopy class of γ and is a complex linear automorphism of $\mathcal{F}_a$.

The correspondence $\mathcal{F} \mapsto \rho_\mathcal{F}$ defines an equivalence of category between the category of local systems on A and the category of representations of the fundamental group $\pi_1(A, a)$ in finite dimensional vector spaces.

Example: In the case A is the circle $\mathbb{S}^1$, the fundamental group $\pi_1(A,a)$ is the group of relative integers $\mathbb{Z}$. Local systems of rank one are given by maps $\mathbb{C} \to \mathbb{C}$ given by $z \mapsto z^k$, with $k \in \mathbb{Z}$.

Now, we have an important concept which, in some sense, generalises complex analytic partitions:

Definition 3.2. A sheaf L over a complex analytic set $\mathcal{X}$ is constructible if there is a complex analytic partition $\mathcal{S} = (X_\alpha)_{\alpha\in A}$ of $\mathcal{X}$, such that the restriction of L over each X_α is a local system.

Examples: a) Let $f : \mathcal{X} \to \mathcal{Y}$ be a proper algebraic map between two algebraic varieties, the sheaf on $\mathcal{Y}$ whose stalks are the k-th cohomology $\mathrm{H}^k(f^{-1}(y), \mathbb{C})$ of the fibers $f^{-1}(y)$ of f is a constructible sheaf.

b) The cohomologies of the solutions $\mathrm{RHom}_D(M,O)$ of a holonomic D-module M over a complex space $\mathbb{C}^N$ are constructible sheaves (Kashiwara's constructibility Theorem).

c) Let $\mathcal{Z}$ be an algebraic subvariety of $\mathcal{X}$ and $i : \mathcal{Z} \to \mathcal{X}$ be the inclusion. Let $z \in \mathcal{Z}$. The local cohomology $(\mathrm{R}^k i^!)(\mathbb{C})_z \simeq \mathrm{H}^k_{\mathcal{Z}\cap B_z}(\mathcal{X} \cap B_z, \mathbb{C})$, where B_z is a good neighbourhood of z in $\mathcal{X}$, defines a constructible sheaf on $\mathcal{Z}$.

These examples are not easy to prove. The first and third ones are difficult theorems on the topology of algebraic maps, the second one is a basic theorem of the theory of $\mathcal{D}$-modules, also difficult to prove.

4. Whitney Stratifications

Let $\mathcal{S} = (X_\alpha)_{\alpha\in A}$ be a stratification of a complex analytic set $\mathcal{X}$. We say that $\mathcal{S}$ is a Whitney stratification of $\mathcal{X}$ if, for any pair (X_α, X_β) of strata, such that $X_\alpha \subset \overline{X}_\beta$, the complex analytic set $\overline{X}_\beta$ satisfies Whitney condition along X_α at any point of X_α.

As we announced above, the interest for Whitney stratifications comes from the fact that they imply local topological triviality. Namely, a theorem of J. Mather[14] and R. Thom[18] gives:

Theorem 4.1. *Let $\mathcal{S} = (X_\alpha)_{\alpha\in A}$ be a Whitney stratification of a complex analytic subset $\mathcal{X}$ of $\mathbb{C}^N$. For any point $x \in X_\alpha$ there is an open neighbourhood U of x in $\mathbb{C}^N$ such that $\mathcal{X} \cap U$ is homeomorphic to $(X_\alpha \cap U) \times (N_\alpha \cap U)$ where N_α is a slice of X_α at x in $\mathcal{X}$ by a transversal affine space in $\mathbb{C}^N$.*

This theorem shows that locally on $\mathcal{X}$, along any strata, the analytic set is topologically a product.

A theorem of H. Whitney[19] gives that:

Theorem 4.2. *Let $\mathcal{S} = (X_\alpha)_{\alpha\in A}$ be a stratification of a complex analytic set $\mathcal{X}$. There is a stratification of $\mathcal{X}$ which is finer than $\mathcal{S}$ and is a Whitney stratification for $\mathcal{X}$.*

As a consequence on a singular compact space one can observe only finitely many different topological types of embedded germs of complex spaces, while in general there are continua of different analytic types.

Examples. In the examples given before the stratification of:

$$X^2 - YZ^2 = 0$$

given by the non-singular part $\mathcal{X}\setminus\mathcal{Y}$ of $\mathcal{X}$, $\mathcal{Y}\setminus\{0\}$ and $\{0\}$ gives a Whitney stratification of $\mathcal{X}$ which is finer than $\mathcal{X}\setminus\mathcal{Y}$ and $\mathcal{Y}$.

We have the same for:

$$X^2 - Y^3 - Z^2Y^2 = 0.$$

For the example given by:

$$X^2 - Y^2Z^3 = 0,$$

a Whitney stratification is given by $\mathcal{X}\setminus L_1\cup L_2$, $L_1\setminus\{0\}$, $L_2\setminus\{0\}$ and $\{0\}$.

A theorem of B. Teissier shows that a Whitney stratification can be characterized algebraically and is very useful to know if a stratification is a Whitney stratification. In order to give this criterion, we need to define Polar Varieties.

Let $\mathcal{X}$ be a complex analytic subset of $\mathbb{C}^N$. Let x be a point of $\mathcal{X}$. Assume for simplicity that $\mathcal{X}$ is equidimensional at x. Consider affine projections of $\mathcal{X}$ into $\mathbb{C}^{k+1}$, for $1\le k\le dim_x(\mathcal{X})$. Then, one can prove:

Theorem 4.3. *There is a non-empty Zariski open set Ω_k in the space of projections of $\mathbb{C}^N$ into $\mathbb{C}^{k+1}$, such that for any $p\in\Omega$, there is an open neighbourhood U, such that either the critical locus of the the restriction of p to the non-singular part of $\mathcal{X}\cap U$ is empty or the closure of the critical locus of the restriction of p to the non-singular part of $\mathcal{X}\cap U$ is reduced and has dimension k at x and its multiplicity at x is an integer which is independent of $p\in\Omega_k$.*

In the case where the critical locus of the restriction of p to the non-singular part of $\mathcal{X}\cap U$ is not empty for $p\in\Omega_k$, the closure of the critical locus in $\mathcal{X}\cap U$ is called "the" polar variety $P_k(\mathcal{X},x,p)$ of $\mathcal{X}$ at x of dimension k, defined by p and the multiplicity $m(P_k(\mathcal{X},x,p))$ is called the k-th polar multiplicity of $\mathcal{X}$ at the point x. For $P_k(\mathcal{X},x,p)$ this is an abuse of language since only the "equisingularity type" of $P_k(\mathcal{X}, x, p)$ is well-defined for $p \in \Omega_k$, but the theorem just stated shows that there is no abuse as far as the multiplicity is concerned. When the critical locus of the restriction of p is empty for $p \in \Omega_k$, we say that $m_k(\mathcal{X}, x) = 0$.

Then, the criterion of B. Teissier is the following (Ref. 17):

Theorem 4.4. *Let $\mathcal{X}$ be a complex analytic set. Let $\mathcal{S} = (X_\alpha)_{\alpha \in A}$ be a stratification of $\mathcal{X}$ with connected strata. Suppose that for any pair (X_α, X_β) such that $X_\alpha \subset \overline{X}_\beta$, the multiplicities $m(P_k(\overline{X}_\beta), y)$ are constant for $y \in X_\alpha$, for $1 \leq k \leq dim_y(\overline{X}_\beta)$, then, the stratification $\mathcal{S}$ is a Whitney stratification. Conversely, any Whitney stratification with connected strata has this property.*

Beware that this theorem is true with stratifications: for instance the frontier condition is important. One can consider the case of the surface defined by:

$$X^2 - Y^2 Z^3 = 0.$$

Examples. Let $\mathcal{X}$ be a surface, i.e. a complex analytic set of dimension 2. If, $x \in \mathcal{X}$ is a non-singular point, the 2-nd polar variety at x is $\mathcal{X}$ itself and, by definition, its multiplicy is 1. At x the 1-st polar variety is empty, so $m_1 = 0$. If $x \in \mathcal{X}$ is singular, again the 2-nd polar variety at x is $\mathcal{X}$ itself and $m_2 = m_x(X) > 1$. For almost all singular points, except a finite number locally, the 1-st polar curve is empty, so $m_1 = 0$. So, a Whitney stratification of $\mathcal{X}$ is given by the non-singular part $\mathcal{X}^0$ of $\mathcal{X}$, the nonsingular part of the singular locus minus the points where the polar curve is not empty, the points where the polar curve is not empty and finally the singular points of the singular locus.

For the surface given by:

$$X^2 - Y^3 - Z^2 Y^2 = 0$$

the stratification given by $\mathcal{X} \setminus \mathcal{Y}$, $\mathcal{Y} \setminus \{0\}$ and $\{0\}$ is a Whitney stratification.

A consequence of this theorem is the existence of a minimal Whitney stratification refining a given one (see Ref. 17):

Theorem 4.5. *Let $\mathcal{X}$ be a complex analytic set. Let $\mathcal{S} = (X_\alpha)_{\alpha \in A}$ be a complex analytic partition of $\mathcal{X}$. Then there is a unique coarsest refinement of $\mathcal{S}$ which is a Whitney stratification of $\mathcal{X}$ with connected strata.*

If one takes as partition the non-singular part of $\mathcal{X}$, the non-singular part of the singular locus, and so on, one sees that every complex-analytic space has a unique "minimal" Whitney stratification in the sense that any other Whitney stratification is a refinement of it.

5. Milnor Fibrations

Let $f : U \subset \mathbb{C}^N \to \mathbb{C}$ be a complex analytic function defined on a neighbourhood of the origin 0 in $\mathbb{C}^N$ such that the image $f(0)$ of the origin 0 by f is 0.

One can prove:

Theorem 5.1. *There is $\varepsilon_0 > 0$, such that, for any ε such that $0 < \varepsilon < \varepsilon_0$, there is $\eta(\epsilon) > 0$, such that for any η such that $0 < \eta < \eta(\varepsilon)$, the map*

$$\varphi_{\varepsilon,\eta} : \mathbb{B}_\varepsilon \cap f^{-1}(\mathbb{S})_\eta \to \mathbb{S}_\eta,$$

induced by f over the circle of radius η centered at the origin 0 in the complex plane $\mathbb{C}$, is a locally trivial smooth fibration.

See Ref. 16 for the case where f has an isolated critical point at 0 and Ref. 5 for the general case. We call the fibration given by the theorem the Milnor fibration of f at 0.

When f has a critical point at 0, the fibers of $\varphi_{\varepsilon,\eta}$ have a non-trivial homotopy. Since the "fiber" at 0, i.e., $\mathbb{B}_\varepsilon \cap f^{-1}(0)$ is contractible by Ref. 16, one usually calls *vanishing cycles* the cycles of a fiber of $\varphi_{\varepsilon,\eta}$, i.e. the elements of the homology $\mathrm{H}_*(\mathbb{B}_\varepsilon \cap f^{-1}(t), \mathbb{C})$ for $t \in \mathbb{S}_\eta$.

One calls *neighbouring cycles* the elements of the cohomology $\mathrm{H}^*(\mathbb{B}_\varepsilon \cap f^{-1}(t), \mathbb{C})$. The theorem above shows that these definitions do not depend on $t \in \mathbb{S}_\eta$. One can prove that it does not depend on ε, η chosen conveniently.[13]

One may observe that the complex cohomology $\mathrm{H}^*(\mathbb{B}_\varepsilon \cap f^{-1}(t), \mathbb{C})$ is the sheaf cohomology of $\mathbb{B}_\varepsilon \cap f^{-1}(t)$ with coefficients in the constant sheaf $\mathbb{C}$.

We also have on any analytic set a theorem similar to the one above (see Ref. 9):

Theorem 5.2. *Let $f : \mathcal{X} \to \mathbb{C}$ be a complex analytic function on a complex analytic subset of $\mathbb{C}^N$. Suppose that $0 \in \mathcal{X}$ and $f(0) = 0$. There is $\varepsilon_0 > 0$, such that, for any ε such that $0 < \varepsilon < \varepsilon_0$, there is $\eta(\epsilon) > 0$, such that for any η such that $0 < \eta < \eta(\varepsilon)$, the map*

$$\varphi_{\varepsilon,\eta} : \mathbb{B}_\varepsilon \cap \mathcal{X} \cap f^{-1}(\mathbb{S})_\eta \to \mathbb{S}_\eta,$$

induced by f over the circle of radius η centered at the origin 0 in the complex plane $\mathbb{C}$, is a locally trivial topological fibration.

In particular, this theorem shows that the homology

$$\mathrm{H}_*(\mathbb{B}_\varepsilon \cap \mathcal{X} \cap f^{-1}(t))$$

or the cohomology

$$\mathrm{H}^*(\mathbb{B}_\varepsilon \cap \mathcal{X} \cap f^{-1}(t), \mathbb{C})$$

does not depend on $t \in \mathbb{S}_\eta$ and on ε, η chosen appropriately.

6. Local Constructible Sheaves and Whitney Conditions

Since the constant sheaf $\mathbb{C}$ is a constructible sheaf, one may also consider the neighbouring cycles of a constructible sheaf $\mathcal{L}$ along the function $f : \mathcal{X} \to \mathbb{C}$ as the sheaf cohomology $\mathrm{H}^*(\mathbb{B}_\varepsilon \cap X \cap f^{-1}(t), \mathcal{L})$.

Let p be a projection $\mathbb{C}^N$ into $\mathbb{C}^{k+1}$ which defines a k-th polar variety $P_k(\mathcal{X}, x, p)$ of X at a point $x \in \mathcal{X}$. Then, one has neighbourhood U and V of x and $p(x)$ in $\mathcal{X}$ and $\mathbb{C}^{k+1}$, such that p induces a map $\pi : U \to V$. One can show that $\pi(P_k(\mathcal{X}, x, p) \cap U)$ is a complex analytic subset of V and π induces a locally trivial topological fibration of $\pi^{-1}(V \setminus \pi(P_k(\mathcal{X}, x, p) \cap U))$ over $V \setminus \pi(P_k(\mathcal{X}, x, p) \cap U)$.

The sheaf $(\mathrm{R}^\ell \pi_*)(\mathbb{C}_U)$, whose fiber at $y \in V$ is the ℓ-th cohomology of $\pi^{-1}(y)$, is a constructible sheaf. The Euler characteristic $\chi_k(\mathcal{X}, x)$ of the general fiber of π is called the k-th vanishing Euler characteristic of $\mathcal{X}$ at x. At x, one has $\dim_x(\mathcal{X})$ Euler characteristics $\chi(\mathcal{X}, x) := (\chi_1(\mathcal{X}, x), \ldots, \chi_{dim\mathcal{X}}(\mathcal{X}, x))$.

For simplicity assume that $\mathcal{X}$ is equidimensional. Then, one has a characterization of a Whitney stratification by a result of Lê and Teissier (Ref. 13, Théorème (5.3.1)) similar to the one of Teissier given above:

Theorem 6.1. *Let $\mathcal{X}$ be an equidimensional complex analytic set. Let $\mathcal{S} = (X_\alpha)_{\alpha \in A}$ be a stratification of $\mathcal{X}$. Suppose that, for any pair (X_α, X_β), such that $X_\alpha \subset \overline{X}_\beta$, the Euler characteristics $(\chi_1(X_\beta, y), \ldots, \chi_{\dim_y X_\beta}(X_\beta, y))$ are constant for $y \in X_\alpha$, then, the stratification $\mathcal{S}$ is a Whitney stratification.*

7. Neighbouring Cycles

We saw in Section 5 shows that any analytic function on an open set defines locally a locally trivial fibration on a circle $\mathbb{S}$. The last theorem of Section 5 shows that this extends to functions on any complex analytic sets.

Let $f : \mathcal{X} \to \mathbb{C}$ be a complex analytic function on a complex analytic subset of $\mathbb{C}^N$. On the fiber $f^{-1}(f(x))$ of the function f through any point

x of $\mathcal{X}$ one can define a sheaf $\mathrm{R}^k(\psi_{f-f(x)})(\mathbb{C})$, with $\mathrm{R}^k(\psi_{f-f(x)})(\mathbb{C})_y \simeq \mathrm{H}^k(F_y, \mathbb{C})$ where F_y is a fiber of the Milnor fibration defined above at $y \in f^{-1}(f(x))$ by f.

This sheaf $\mathrm{R}^k(\psi_{f-f(x)})(\mathbb{C})$ is a constructible sheaf on the fiber of f over $f(x)$. One calls it the *sheaf of k-th neighbouring cycles of f at x*.

Similarly, for any constructible sheaf $\mathcal{L}$, one can define the sheaf of neighbouring cycles $\mathrm{R}^k(\psi_{f-f(x)})(\mathcal{L})$ of f at x, where $\mathrm{R}^k(\psi_{f-f(x)})(\mathcal{L})_y \simeq \mathrm{H}^k(F_y, \mathcal{L})$.

When f is defined on $\mathbb{C}^{n+1}$ and has an isolated singular point at x, the sheaf $\mathrm{R}^k(\psi_{f-f(x)})(\mathbb{C})$ is non-zero when $k = n$ or 0. In this case, $\mathrm{R}^0(\psi_{f-f(x)})(\mathbb{C})$ is the constant sheaf on $f^{-1}(x)$ and $\mathrm{R}^n(\psi_{f-f(x)})(\mathbb{C})$ is a sheaf whose value at x is $\mathbb{C}^\mu$ and which is zero on a neighbourhood of x outside $\{x\}$. In this special case, when $n \geq 1$, one also call $\mathrm{R}^n(\psi_{f-f(x)})(\mathbb{C})$ the sheaf of n-neighbouring cycles of f at x.

When the complex analytic space $\mathcal{X}$ is a Milnor space (see Ref. 10) and the function $f : \mathcal{X} \to \mathbb{C}$ has isolated singularities, i.e. there is a Whitney stratification of $\mathcal{X}$, such that the restriction of f to the strata has maximal rank except at isolated points, one can define the same type of sheaf over the fiber above the image of a singularity. For instance, complete intersection spaces, e.g., hypersurfaces, are Milnor spaces.

Because of the fibration theorem, neighbouring cycles and vanishing cycles at a point y of $f^{-1}(f(x))$ are endowed with the monodromy of the fibration. We have the important theorem (see e.g. Ref. 8, Theorem I p. 89, or Ref. 4):

Theorem 7.1. *The monodromy automorphism of neighbouring (or vanishing) cycles is a quasi-unipotent automorphism, i.e. its eigenvalues are roots of unity.*

8. Constructible Complexes

Constructible sheaves over a complex analytic set $\mathcal{X}$ make a category where objects are constructible sheaves over $\mathcal{X}$, morphisms are morphisms of sheaves. Unit morphisms are identities of constructible sheaves and the composition is the composition of sheaf morphisms.

One can notice that, for each morphism of constructible sheaves over $\mathcal{X}$, one can define the kernel, the image and the cokernel of the morphism.

A complex of sheaves of complex vector spaces over $\mathcal{X}$ is a sequence of morphisms $(\varphi_n)_{n\in\mathbb{Z}}$, say $\phi_n \colon E_n \to E_{n+1}$, such that, for every $n \in \mathbb{Z}$, $\varphi_n \circ \varphi_{n-1} = 0$.

Complexes of sheaves of complex vector spaces over $\mathcal{X}$ make a category whose objects are complexes of sheaves of complex vector spaces over $\mathcal{X}$ and morphisms from $(\varphi_n)_{n\in\mathbb{Z}}$ into $(\psi_n)_{n\in\mathbb{Z}}$ are families $(h_n)_{n\in\mathbb{Z}}$ of sheaf morphisms such that, for any $n \in \mathbb{Z}$,

$$h_{n+1} \circ \varphi_n = \psi_n \circ h_n.$$

In practice we are interested in bounded complexes $(\varphi_n)_{n\in\mathbb{Z}}$ of sheaves of complex vector spaces over $\mathcal{X}$ such that for all $n \in \mathbb{Z}$ except a finite number φ_n is the trivial morphism from the null-sheaf 0 into itself. These complexes make also a full subcategory $D^b(\mathcal{X})$ of the preceding one. Now let $(\varphi_n)_{n\in\mathbb{Z}} \in D^b(\mathcal{X})$, since $\varphi_n \circ \varphi_{n-1} = 0$, the image of φ_{n-1} is a subsheaf of the kernel of φ_n. The quotient of the kernel sheaf of φ_n by the image sheaf of φ_{n-1} is by definition the n-th cohomology of $(\varphi_n)_{n\in\mathbb{Z}}$.

We say that the complex $(\varphi_n)_{n\in\mathbb{Z}}$ is constructible over $\mathcal{X}$, if it is bounded and, for any $n \in \mathbb{Z}$, the n-th cohomology of $(\varphi_n)_{n\in\mathbb{Z}}$ is a constructible sheaf over $\mathcal{X}$.

Constructible complexes over make a full subcategory $D^b_c(\mathcal{X})$ of $D^b(\mathcal{X})$.

Given a constructible complex $\mathbb{K}$ over $\mathcal{X}$, one can consider the hypercohomology $\mathbb{H}^*(\mathcal{X}, \mathbb{K})$ of $\mathbb{K}$ (see Ref. 1, Chap. XVII).

Using the notations of 7, one can consider the neighbouring cycles $\mathrm{R}^k(\psi_{f-f(x)})(\mathbb{K})$ of $\mathbb{K}$ along f at x, where

$$\mathrm{R}^k(\psi_{f-f(x)})(\mathbb{K})_y \simeq \mathbb{H}^k(F_y, \mathbb{K}).$$

9. Vanishing Cycles

Let X be a complex analytic space and $\mathbb{K}$ a constructible complex over $\mathcal{X}$. Let x be a point of $\mathcal{X}$ and $f : U \to \mathbb{C}$ be a complex analytic function defined on an open neighbourhood U of x in $\mathcal{X}$. We have defined the k-th neighbouring cycles of $\mathbb{K}$ along $f - f(x)$ at x as the sheaf $\mathrm{R}^k(\psi_{f-f(x)})(\mathbb{K})$, where $\mathrm{R}^k(\psi_{f-f(x)})(\mathbb{K})_y \simeq \mathbb{H}^k(F_y, \mathbb{K})$, over the space $f^{-1}(f(x))$.

It is convenient to define the sheaf $\mathrm{R}^k(\psi_{f-f(x)})(\mathbb{K})$ as the k-th cohomology of a bounded complex $\mathbf{R}(\psi_{f-f(x)})(\mathbb{K})$. The definition of the complex $\mathbf{R}(\psi_{f-f(x)})(\mathbb{K})$ uses derived categories and is rather abstract. Then, there is a natural morphism from the restriction $\mathbb{K}|f^{-1}(f(x))$ to $\mathbf{R}(\psi_{f-f(x)})(\mathbb{K})$. There is a natural triangle in the appropriate derived category:

$$\begin{tikzcd} \mathbb{K}|f^{-1}(f(x)) \arrow[rr] & & \mathbf{R}(\psi_{f-f(x)})(\mathbb{K}) \arrow[dl] \\ & \mathbf{R}(\phi_{f-f(x)})(\mathbb{K}) \arrow[ul, "+1"] & \end{tikzcd}$$

The complex $\mathbf{R}(\phi_{f-f(x)})(\mathbb{K})$ is called the *complex of vanishing cycles of* $\mathbb{K}$ *along* $f - f(x)$. The cohomology gives a long exact sequence:

$$\begin{aligned}\to \mathrm{H}^k(\mathbb{K}|f^{-1}(f(x))) \to \mathrm{R}^k(\psi_{f-f(x)})(\mathbb{K}) \to \mathrm{R}^k(\phi_{f-f(x)})(\mathbb{K})\\ \to \mathrm{H}^{k+1}(\mathbb{K}|f^{-1}(f(x))) \to\end{aligned}$$

The sheaf $\mathrm{R}^k(\phi_{f-f(x)})(\mathbb{K})$ is the sheaf of k-th vanishing cycles of $\mathbb{K}$ along f.

In special cases these sheaves are easy to interpret. Let $\mathcal{X} = \mathbb{C}^{n+1}$ and $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be complex analytic function with an isolated critical point at 0. Let U be an open neighbourhood of 0 in $\mathbb{C}^{n+1}$ where f has the only critical point 0. Let the complex $\mathbb{K}$ be the complex having one term in degree 0 equal to the constant sheaf $\mathbb{C}$ over $\mathbb{C}^{n+1}$ and the sheaf 0 in other degrees. The results of J. Milnor in Ref. 16 show that:

$$\mathrm{R}^k(\psi_{f-f(0)})(\mathbb{K}|U) = \begin{cases}\mathbb{C} \ \text{ if k=0}\\ 0 \text{ if } k \neq 0, n\\ \mathbb{C}^\mu \text{ at } 0 \text{ if } k = n \text{ and } 0 \text{ at } x \neq 0\end{cases}$$

Therefore if $n \geq 2$, the complex $\mathbf{R}(\psi_{f-f(0)})(\mathbb{K}|U)$ is the complex with the constant sheaf $\mathbb{C}|U$ in degree 0, the sheaf with one non-trivial stalk $\mathbb{C}^\mu$ at 0 in degree 2, and all morphisms are zero.

The complex $\mathbf{R}(\phi_{f-f(0)})(\mathbb{K}|U)$ has only one term in degree 2 which is the the sheaf with one non-trivial stalk $\mathbb{C}^\mu$ at 0, all the other terms in degree $\neq 2$ being 0.

One can observe that in this special case of isolated critical point the complex of vanishing cycles of the complex $\mathbb{K}|U$ along f consists of a sheaf non-trivial in the degree equal to the dimension of $f^{-1}(0)$ which has only one non-trivial stalk over the isolated critical point 0.

It can be proved that it is true for any space $\mathcal{X}$ satisfying the Milnor condition (see Ref. 10, §5) for functions having isolated singularities in the general sense of (Ref. 10, §1) and for any complex $\mathbb{K}_X$ equal to the constant sheaf $\mathbb{C}_\mathcal{X}$ over $\mathcal{X}$ in degree 0 and to the trivial sheaf 0 in other degrees.

The support of a constructible sheaf $\mathcal{L}$ on the complex analytic space $\mathcal{X}$ is the complex analytic subspace $\mathcal{Y}$ closure of the set of points x where $\mathcal{L}_x \neq 0$.

One says that a constructible complex $\mathbb{K}$ on $\mathbb{C}^N$ satisfies the support condition if the codimension of the support of its i-th cohomology $\mathrm{H}^i(\mathbb{K})$ is $\geq i$.

The Verdier dual of a constructible complex $\mathbb{K}$ on $\mathbb{C}^N$ is the derived complex $\mathrm{RHom}_{\mathbb{C}_{\mathbb{C}^N}}(\mathbb{K}, \mathbb{C}_{\mathbb{C}^N})$. Beware that the Verdier dual of a complex on

$\mathbb{C}^N$ is not the complex of duals. On the other hand, there is also a notion of duality for a constructible complex on a complex analytic space. However in this general case the construction is not as "simple" as the Verdier dual. The construction of the Verdier dual is usually difficult. Properties of the Verdier dual can be found in Ref. 3. One can prove that the Verdier dual of a constructible complex is also a constructible complex.

One says that the constructible complex $\mathbb{K}$ on $\mathbb{C}^N$ satisfies the cosupport condition, if the Verdier dual $\check{\mathbb{K}}$ of $\mathbb{K}$ satisfies the support condition.

A constructible complex $\mathbb{K}$ on $\mathbb{C}^N$ is *perverse*, if it satisfies the support condition and the cosupport condition. We shall call perverse sheaf a perverse constructible complex.

One can prove that more generally, if $\mathbb{K}$ is a perverse sheaf on $\mathbb{C}^N$, the complex of vanishing cycles along a function f which has an isolated critical point in the general sense of Ref. 10, §1 at 0 for all the restrictions of f to the closures of the strata on which all the cohomologies $\mathrm{H}^k(\mathbb{K})$ are locally constant, in an open neighbourhood U of 0, consists in the degree equal to the dimension n of $f^{-1}(0)$, i.e. for $\mathrm{R}^n(\psi_{f-f(0)})(\mathbb{K})|U$, of a sheaf which has only one non-trivial stalk over the isolated critical point 0 and 0 in the other degrees:

$$\mathrm{R}^n(\phi_{f-f(0)})(\mathbb{K})_0 = \mathbb{C}^k$$

$$\mathrm{R}^n(\phi_{f-f(0)})(\mathbb{K})_y = 0, \text{ for } y \neq 0$$

One has the following result due to P. Deligne (see e.g. Ref. 11):

Theorem 9.1. *Let $\mathbb{K}$ be a perverse sheaf on $\mathbb{C}^N$. Let $x \in \mathbb{C}^N$. For almost all linear fonction ℓ of $\mathbb{C}^N$, the sheaf of vanishing cycles of $\mathbb{K}$ along ℓ in an open neighbourhood of x is either zero or a complex which is zero in all degrees except in degree equal to the dimension of $f^{-1}(f(x)$, where it has a non-trivial stalk only at x.*

We can define:

Definition 9.1. Let $\mathbb{K}$ be a perverse sheaf on $\mathbb{C}^N$. The subvariety $\mathcal{V}(\mathbb{K})$ of the cotangent bundle $T^*(\mathbb{C}^N)$ of $\mathbb{C}^N$ is the characteristic variety of $\mathbb{K}$ if, a point $(x, \ell) \in \mathcal{V}(\mathbb{K})$ if and only if it belongs to the closure of the points (y, l) for which there is a neighbourhood U where $\mathrm{R}^k(\phi_{l-l(y)})(\mathbb{K})|U = 0$ for $k \neq n = \dim l^{-1}(0)$ and $\mathrm{R}^n(\phi_{l-l(y)})(\mathbb{K})|U$ is a non-trivial skyscraper sheaf with a non-zero fiber at y.

10. Holonomic $\mathcal{D}$-Modules

Let $\mathcal{X}$ a complex analytic manifold of complex dimension n. On $\mathcal{X}$ consider the sheaf $\mathcal{D}$ of complex analytic differential operators. Notice that this sheaf is filtered by the sheaves $\mathcal{D}(k)$ of differential operators of order $\leq k$.

We consider sheaves on $\mathcal{X}$ which are coherent left $\mathcal{D}$-modules. We shall call these sheaves simply $\mathcal{D}$-modules.

For instance, if $\mathcal{X}$ is a Riemann surface, in this meeting L. Narvaez considered $\mathcal{D}$-modules over a complex analytic manifold of dimension 1.

Since the sheaf $\mathcal{D}$ itself is coherent, a $\mathcal{D}$-module $\mathcal{M}$ is locally of finite presentation, i.e. for any $x \in \mathcal{X}$, there is an open neighbourhood U_x over which one has an exact sequence:

$$(\mathcal{D})|U_x{}^p \xrightarrow{\psi} (\mathcal{D}|U_x)^q \xrightarrow{\varphi} \mathcal{M}|U_x \to 0.$$

One can notice that $\varphi((\mathcal{D}(k)|U_x)^q) = \mathcal{M}(k)$ defines a *good filtration* of $\mathcal{M}$ (see the lectures of F. Castro).

The complex analytic space Specan $\oplus_{k\geq 0}(\mathcal{D}(k)|U_x)/(\mathcal{D}(k-1)|U_x)$ corresponding to the commutative graded ring $\mathrm{gr}(\mathcal{D}|U_x)$associated to the filtration of $\mathcal{D}|U_x$ by the $\mathcal{D}(k)|U_x$ is the cotangent space of U_x. The support of the $\mathrm{gr}(\mathcal{D}|U_x)$-module $\oplus_{k\geq 0}\mathcal{M}|U_x(k)/\mathcal{M}|U_x(k-1)$ is the characteristic variety $Ch(\mathcal{M}|U_x)$ of the $\mathcal{D}|U_x$-modules $\mathcal{M}|U_x$. One can show that this definition does not depend on the local finite presentation of $\mathcal{M}$.

We shall say that the $\mathcal{D}$-module $\mathcal{M}$ is *holonomic* if the dimension of the characteristic variety $Ch(\mathcal{M})$ is equal to the dimension n of the manifold $\mathcal{X}$.

As it is done in Ref. 7, we obtain the following relation between the characteristic variety of a holonomic $\mathcal{D}$-module and the topology (see Proposition 10.6.5 of Ref. 7):

Theorem 10.1. *Let $\mathcal{M}$ be a holonomic $\mathcal{D}$-module on a complex analytic manifold X. There is a Whitney stratification $(X_\alpha)_{\alpha\in A}$ of X, such that the characterisitc variety of $\mathcal{M}$ is contained in the union of the conormal bundles $T^*_{X_\alpha}X$ of the strata X_α in X.*

Recall that if S is a submanifold of X, the conormal $T^*_S X$ of S in X is the bundle of $(s,\ell) \in T^*X$, such that ℓ is a linear form which vanishes on $T_s(S)$. It can be seen that, when X is a manifold and S is submanifold, the conormal bundle $T^*_S X$ of S in X is a Lagrangean submanifold of T^*X.

In fact we can obtain a more precise result.

As it is done for $n = 1$ in the lectures of L. Narvaez, we define the analytic solutions of $\mathcal{D}$-module $\mathcal{M}$ to be the complex $\mathrm{RHom}_{\mathcal{D}}(\mathcal{M}, \mathcal{O})$. This is the stable (derived) version of the "naive" definition $\mathrm{Hom}_{\mathcal{D}}(\mathcal{M}, \mathcal{O})$.

M. Kashiwara proved in Ref. 6 that:

Theorem 10.2. *If the $\mathcal{D}$-module $\mathcal{M}$ is holonomic, the analytic solutions $\mathrm{RHom}_{\mathcal{D}}(\mathcal{M}, \mathcal{O})$ of $\mathcal{M}$ is a constructible complex which satisfies the support condition.*

A theorem of Z. Mebkhout (see Ref. 15) shows that:

Theorem 10.3. *The Verdier dual of the analytic solutions $\mathrm{RHom}_{\mathcal{D}}(\mathcal{M}, \mathcal{O})$ of the holonomic $\mathcal{D}$-module $\mathcal{M}$ is the analytic solutions of a holonomic $\mathcal{D}$-module, dual of $\mathcal{M}$.*

It follows that:

Corollary 10.1. *The analytic solutions $\mathrm{RHom}_{\mathcal{D}}(\mathcal{M}, \mathcal{O})$ of the holonomic $\mathcal{D}$-module $\mathcal{M}$ is a perverse sheaf.*

From this result and the ones of the preceding section we can state a result of D. T. Lê and Z. Mebkhout:[12]

Theorem 10.4. *The characteristic variety of an holonomic $\mathcal{D}$-module $\mathcal{M}$ is the characteristic variety of the perverse sheaf of its analytic solutions.*

References

1. H. Cartan and S. Eilenberg, *Homological Algebra* (Princeton University Press, 1956).
2. R. Godement, Topologie algébrique et théorie des faisceaux, Actualités Sci. Ind. No. 1252. Publ. Math. Univ. Strasbourg No. 13 (Hermann, Paris, 1958).
3. M. Goresky and R. MacPherson, Intersection Homology II, *Invent. Math.* **72**, 77 (1983).
4. *Periods of Integrals on Algebraic Manifolds*, Mimeographed notes (U.C. Berkeley, 1965-1966).
5. H. Hamm and Lê Dũng Tráng, *Ann. Sci. École Norm. Sup.* **6**, 317 (1973).
6. M. Kashiwara, *Pub. Res. Inst. Math. Sci.* **10**, 563 (1975).
7. M. Kashiwara and P. Schapira, *Acta Math.* **142**, 1 (1979).
8. A. Landman, *Trans. Amer. Math. Soc.* **181**, 89 (1973).
9. Lê Dũng Tráng, Some remarks on relative monodromy, in real and complex singularities (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976), pp.397-403, Sijthoff and Noordhoff, Alphen aan den Rijn (1977).
10. Lê Dũng Tráng, *J. Algebraic Geom.* **1**, 83 (1992).
11. Lê Dũng Tráng, *Bol. Soc. Mat. Mex.* **4**, 229 (1998).

12. Lê Dũng Tráng and Z. Mebkhout, *C. R. Acad. Sci. Paris Sér. I Math.* **296**, 129 (1983).
13. Lê Dũng Tráng and B. Teissier, Cycles évanescents, sections planes et conditions de Whitney II, in Singularities, Part 2 (Arcata, Calif., 1981), pp.65-103, Proc. Sympos. Pure Math. **40**, Amer. Math. Soc., Providence, RI (1983).
14. J. Mather, Notes on topological stability, Harvard booklet (1971).
15. Z. Mebkhout, Théorème de dualité pour les $\mathcal{D}_X$-modules cohérents, C. R. Acad. Sci. Paris Sér. A-B 285 (1977), A785 - A787/ Math. Scand. **50**, 25 (1982).
16. J. Milnor, *Singular Points of Complex Hypersurfaces*, Annals of Mathematics Studies, No. 61 (Princeton University Press, Princeton, N.J., 1968).
17. B. Teissier, Variétés polaires II. Multiplicités polaires, sections planes, et conditions de Whitney, in Algebraic geometry (La Rábida, 1981), pp.314-491, Lecture Notes in Math. **961** (Springer, Berlin, 1982).
18. R. Thom, *Bull. Amer. Math. Soc.* **75**, 240 (1969).
19. H. Whitney, *Ann. of Math.* **81**, 496 (1965).

SINGULAR INTEGRALS AND THE STATIONARY PHASE METHODS

E. DELABAERE

Université Nantes, Angers, Le Mans
Université d'Angers, UMR CNRS 6093
2 Boulevard Lavoisier, 49045 Angers Cedex 01, France
**E-mail: eric.delabaere@univ-angers.fr*

The paper is based on a course given in 2007 at an ICTP school in Alexandria, Egypt. It aims at introducing young scientists to methods to calculate asymptotic developpments of singular integrals.

Keywords: Singular integrals, Asymptotic in the complex domain, Steepest descent, Singularities. AMS class. : 30E15, 30E20, 32Cxx, 32S30, 34E05, 34Mxx, 41A60

1. Introduction

Classically a singular integral is an integral whose integrand reaches an infinite value at one or more points in the domain of integration. Even so, such integrals can converge as shown in the following example.

Example : Hilbert transform For $f \in L^2(\mathbb{R})$, we would like to define the integral $\dfrac{1}{\pi}\displaystyle\int_{\mathbb{R}} \frac{f(x-y)}{y}\,dy$. Since $y = 0$ is a singular point for the integrand, we define instead

$$\mathcal{H}[f](x) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|y|>\varepsilon} \frac{f(x-y)}{y}\,dy.$$

It can be shown that $\mathcal{H}[f]$ is well defined in $L^2(\mathbb{R})$. In fact one gets an automorphism of $L^2(\mathbb{R})$,

$$\mathcal{H} : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \qquad \mathcal{H}^2 = -Id,$$

which is called the Hilbert transform.[29]

In this lecture, we shall be interested in analytic functions defined by integrals. One more example is provided by the Gauss hypergeometric functions.

Example : Gauss hypergeometric function We briefly discuss here the Gauss hypergeometric functions (see also Le Dung Trang, M. Jambu, M. Yoshida in this volume). For $a \in \mathbb{C}$ we write as usual

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)\cdots(a+n-1), \qquad (a)_0 = 1.$$

(Γ is the Gamma function). We consider for $c \neq 0, -1, -2, \cdots$,

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n \tag{1}$$

which converges absolutely for $z \in \mathbb{C}$, $|z| < 1$. We have also, for $\Re c > \Re b > 0$ and $|z| < 1$, the Euler's formula :

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\, dt \tag{2}$$

Indeed, since for $|tz| < 1$

$$(1-tz)^{-a} = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} t^n z^n$$

one has, for $|z| < 1$ (normal convergence)

$$\begin{aligned} &\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\, dt \\ &\quad = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} z^n \int_0^1 t^{n+b-1}(1-t)^{c-b-1}\, dt \end{aligned}$$

and we recognize the Euler's Beta function. Therefore, for $\Re c > \Re b > 0, \quad |z| < 1$,

$$\begin{aligned} &\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\, dt \\ &\qquad = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} \frac{\Gamma(n+b)\Gamma(c-b)}{\Gamma(n+c)} z^n \\ &\qquad = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n. \end{aligned}$$

As a consequence of the Lebesgue dominated convergence theorem, we note that the integral representation (2) allows to extend analytically $F(a,b;c;z)$ in z to $|\arg(1-z)| < \pi$ (the cut plane $\mathbb{C} \setminus]1, +\infty[$). This raises the following questions :

- what it the behaviour of $F(a,b;c;z)$ at $z=1$?
- what it the behaviour of $F(a,b;c;z)$ at $z=\infty$?

Here part of the answers comes from the fact that the Gauss hypergeometric function $F(a,b;c;z)$ is solution of the following Gauss hypergeometric equation,

$$\Big[z(1-z)\frac{d^2}{dz^2}+\big[c-(a+b+1)z\big]\frac{d}{dz}-ab\Big]f=0. \tag{3}$$

Now (cf. L. Narváez, this volume):

- From the Cauchy existence theorem for linear differential equations, any solution f of (3) extends analytically in z as a multivalued holomophic function on $\mathbb{C}\backslash\{0,1\}$ (or on $\mathbb{CP}^1\backslash\{0,1,\infty\}$).
- Actually equation (3) is an example of a linear differential equation of Fuchsian type : the singular points $0,1,\infty$ are regular singular points (to see what happens at ∞, make the change of variable $z=\dfrac{1}{Z}$).

  This ensures that for any solution f of (3), one can write locally f as a finite sum,

  $$f(z)=\sum_{\alpha\in\mathbb{C},k\in\mathbb{N}}\phi_{\alpha,k}(X)X^{\alpha}(\mathrm{Log}\ X)^k,\qquad \phi_{\alpha,k}\in\mathbb{C}\{X\}.$$

  (Here X stands for $X=z$, $X=z-1$ and $X=\dfrac{1}{z}$ respectively depending on where we localize.)

Remark : to calculate the above decomposition at ∞ explicitely for the Gauss hypergeometric function $F(a,b;c;z)$ it is convenient to use its Mellin-Barnes integral representation[18] (see also M. Granger, this volume, for Mellin integrals).

Roughly speaking, in this course we shall be mainly concerned by analytic functions defined as Laplace transforms of solutions of linear differential equations with regular singular points, for instance the following function,

$$I(a,b;c;k)=\int_0^{\infty}e^{-kz}F(a,b;c;z)\,dz$$

where we integrate along a path on $\mathbb{C}\backslash\{1\}$. For such a function, we shall be interested in the (Poincaré) asymptotics at infinity in k.

The paper is base on a course given at the ICTP "School on Algebraic approach to differential equations", Alexandria - Egypt, November 12-24, 2007. It aims at introducing the reader into a subject where analysis and geometry are linked. It is written so as to be self-contained up to some other courses given during the school and written in this volume.

The structure of the paper is as follows.

In section 2 we consider some gentle examples of analytic functions defined as Laplace transforms in dimension 1 so as to introduce the reader into the subject. In section 3 we concentrate on properties of integrals of holomorphic differential forms along cycles so as to derive, in section 4 the asymptotics of Laplace-type integrals. Two appendices remind some results in homology theory and some properties of analytic spaces which are required in the paper.

2. Laplace-Type Integrals and Asymptotics: Some Examples

2.1. *A basic example*

We start this section with the following Laplace-type integral

$$\begin{cases} I(k) = \displaystyle\int_{\mathbb{R}} e^{-kf}\sigma \\ \\ f(z) = z^2, \qquad \sigma = dz \end{cases} \tag{4}$$

which defines a holomorphic function in $\Re(k) > 0$ (Lebesgue dominated convergence theorem). We are interested in the asymptotics of $I(k)$ when $|k| \to +\infty$.

2.1.1. *First method*

By a direct calculation, one gets

$$I(k) \underset{(s=k^{1/2}z)}{=} \int_{\mathbb{R}} e^{-s^2}\frac{ds}{k^{1/2}} = \frac{\sqrt{\pi}}{k^{1/2}}$$

Of course, this exact result provides the asymptotics of $I(k)$.

2.1.2. *Second method*

By symmetry, we reduce $I(k)$ into a Laplace transform :

$$I(k) \underset{\text{(symmetry)}}{=} 2\int_0^{+\infty} e^{-kf}\sigma \underset{(f=t)}{=} \int_0^{+\infty} e^{-kt}\frac{dt}{t^{1/2}} = \frac{\Gamma(1/2)}{k^{1/2}}$$

For the last equality, we have used the following formula:

$$\int_0^{+\infty} e^{-kt}\, t^{\alpha}(\ln\, t)^l\, dt = \left(\frac{d}{d\alpha}\right)^l \left(\frac{\Gamma(\alpha+1)}{k^{\alpha+1}}\right) \qquad (5)$$
$$l \in \mathbb{N}, \quad \Re(\alpha) > -1.$$

2.1.3. *Third method : the stationary phase method*

The first two methods rely on exact calculations. The third method develops another viewpoint and prepares the reader for some generalisations.

First step : localisation We recall that were are interested in the asymptotics of $I(k)$ when $|k| \to +\infty$, $\Re(k) > 0$. When $|k| \gg 1$, we note that the integrand $z \in \mathbb{R} \mapsto e^{-kf(z)}$ has a support which is essentially concentrated near $z = 0$ where $df = 0$, approaching the Dirac δ-distribution, see Fig. 1. This is why we localize near that point. This can be done in the following way : we introduce a $t_0 > 0$ and we note that $f^{-1}([0, t_0]) = [-\sqrt{t_0}, \sqrt{t_0}]$.

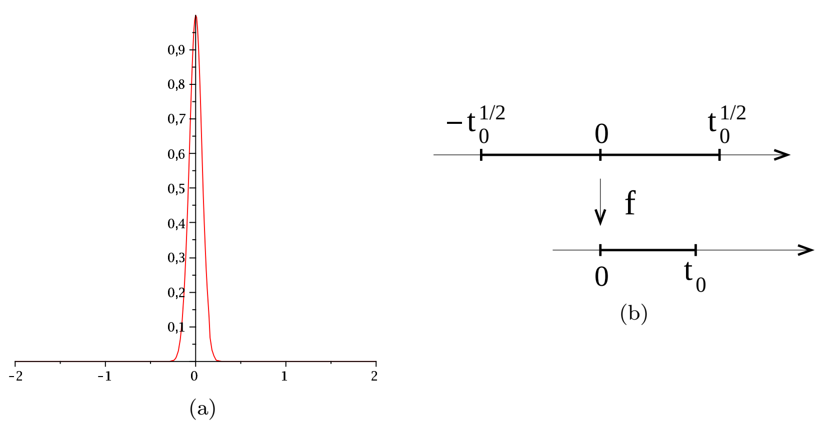


Fig. 1. (a) The graph of $z \in \mathbb{R} \mapsto |e^{-kf(z)}|$ for $k = 100$. (b) The map f.

We thus write

$$\begin{cases} I(k) = I(k,t_0) + R(k,t_0) \\ I(k,t_0) = \displaystyle\int_{0\le f\le t_0} e^{-kf}\sigma, \qquad R(k,t_0) = \int_{f>t_0} e^{-kf}\sigma \end{cases} \tag{6}$$

We introduce also the following open **sectorial neighbourdhood of infinity** of aperture $\pi - 2\delta$,

$$\Sigma_{r,\delta} = \{k \in \mathbb{C},\ |k| > r,\ |\arg k| < \frac{\pi}{2} - \delta\}, \qquad r > 0,\ 0 < \delta < \frac{\pi}{2}. \tag{7}$$

(see Fig. 2) so that,

$$\forall\, k \in \Sigma_{r,\delta}, \qquad \Re(k) > |k|\sin(\delta) > r\sin(\delta).$$

Therefore,

$$\exists\, C > 0\,,\, \exists\, M > 0,\, \forall\, k \in \Sigma_{r,\delta},\ |R(k,t_0)| \le M.e^{-C|k|} \tag{8}$$

($C = C(t_0,\delta) = t_0\sin(\delta)$), and in particular (since for all $N \in \mathbb{N}$, $|k|^N e^{-C|k|} \to 0$ when $|k| \to +\infty$) :

$$\forall N \in \mathbb{N},\, \exists C > 0,\, \forall\, k \in \Sigma_{r,\delta},\ |R(k,t_0)| \le \frac{C}{|k|^{N+1}}. \tag{9}$$

Remark : In fact, (8) implies the following more precise result (use the Stirling formula[21]):

$$\exists C > 0,\, \forall N \in \mathbb{N},\, \forall\, k \in \Sigma_{r,\delta},\ |R(k,t_0)| \le C^{N+1}\frac{\Gamma(N+2)}{|k|^{N+1}}. \tag{10}$$

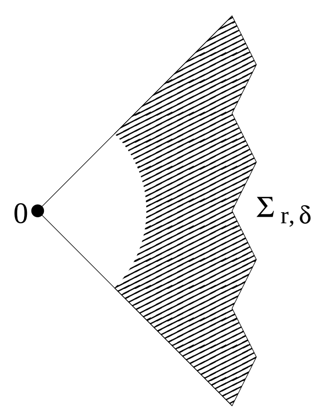


Fig. 2. The sectorial neighbourdhood of infinity $\Sigma_{r,\delta}$.

Second step : reduction to an incomplete Laplace transform We now concentrate on

$$I(k,t_0)=\int_{\Gamma_{t_0}} e^{-kf}\sigma, \qquad \Gamma_{t_0}=[-\sqrt{t_0},\sqrt{t_0}].$$

For k fixed, we notice that $e^{-kf}\sigma \in \Omega^1(\mathbb{C})$, where $\Omega^1(\mathbb{C})$ is the space of holomorphic differential 1-forms on $\mathbb{C}$.

[**Recall :** if $z=x+iy$ and $\bar{z}=x-iy$, $(x,y)\in\mathbb{R}^2$, one introduces the differential 1-forms $dz=dx+idy$ and $d\bar{z}=dx-idy$. If $g\in\mathcal{C}^1(U,\mathbb{C})$ with $U\subset\mathbb{C}$ an open set, then

$$dg=\frac{\partial g}{\partial z}dz+\frac{\partial g}{\partial \bar{z}}d\bar{z}.$$

Now

$$g\in\mathcal{O}(U) \underset{\text{Cauchy-Riemann}}{\Leftrightarrow} \frac{\partial g}{\partial \bar{z}}=0 \Leftrightarrow dg=\frac{\partial g}{\partial z}dz$$

and by definition,

$$\omega\in\Omega^1(U)\Leftrightarrow \exists\, h\in\mathcal{O}(U),\ \omega=hdz.$$

We recall also that if U is a *simply connected* open subset of $\mathbb{C}$, then every closed differential 1-form is exact (Poincaré lemma). We note also that if $\omega\in\Omega^1(U)$, then ω is closed.]

From the fact that $e^{-kf}\sigma\in\Omega^1(\mathbb{C})$, there exists $\varphi\in\Omega^0(\mathbb{C})=\mathcal{O}(\mathbb{C})$ such that $d\varphi=e^{-kf}\sigma$ (we are just saying that $e^{-kf(z)}$ has a primitive φ). Thus,

$$I(k,t_0)=\int_{\Gamma_{t_0}} e^{-kf}\sigma=\int_{\Gamma_{t_0}} d\varphi \underset{\text{Stokes formula}}{=} \int_{\gamma(t_0)}\varphi \tag{11}$$

where $\gamma(t_0)$ is the boundary of Γ_{t_0}, that is the formal difference of two points,

$$\gamma(t_0)=[\sqrt{t_0}]-[-\sqrt{t_0}].$$

In other words (11) is just a pedantic way of writing:

$$I(k,t_0)=\varphi(\sqrt{t_0})-\varphi(-\sqrt{t_0}). \tag{12}$$

We note $D_{t_0}=D(t_0,\eta)\subset\mathbb{C}$ the open disc centred at t_0 with radius small enough so that $0\notin D_{t_0}$ $(0<\eta<t_0)$. We note also by $S^1=\partial D_{t_0}$ its boundary with its natural orientation, Fig. 3, and we consider for $t\in D_{t_0}$:

$$I_1(k,t)=\varphi(\sqrt{t}).$$

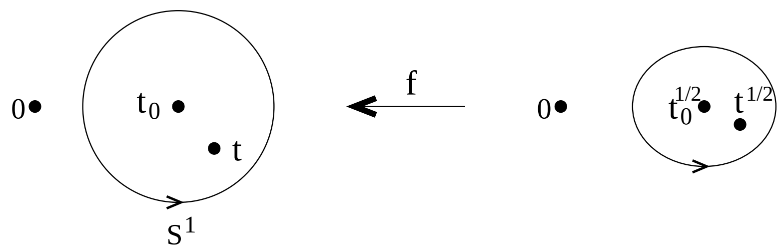


Fig. 3.

Obviously $I_1(k,t) \in \mathcal{O}(D_{t_0}) \cap \mathcal{C}^0(\overline{D_{t_0}})$ so that, by the Cauchy formula,

$$\forall\, t \in D_{t_0},\;\; I_1(k,t) = \frac{1}{2i\pi}\int_{S^1} \frac{\varphi(\sqrt{s})}{s-t}\, ds.$$

By the inverse function theorem, f realizes a biholomorphic mapping between a neighbourhood of $\sqrt{t_0}$ and D_{t_0}. In other words, one can take f as a coordinate : taking $s = f(z) = z^2$, we get

$$\forall\, t \in D_{t_0},\;\; I_1(k,t) = \frac{1}{2i\pi}\oint \frac{\varphi(z)}{z^2-t}\, 2zdz = \frac{1}{2i\pi}\oint \frac{\varphi\, df}{f-t}$$

where we integrate along an oriented loop surrounding $\sqrt{t_0}$ and $\sqrt{t}$, see Fig. 3.

This integral representation has the following consequence : since the path of integration does not depend on t, we obtain: for $t \in D_{t_0}$,

$$\frac{d}{dt} I_1(k,t) = \frac{1}{2i\pi}\oint \frac{\varphi\, df}{(f-t)^2}$$

Also,

$$d\left(\frac{\varphi}{f-t}\right) = \frac{d\varphi}{f-t} - \frac{\varphi\, df}{(f-t)^2}$$

and by Stokes

$$\frac{1}{2i\pi}\oint d\left(\frac{\varphi}{f-t}\right) = 0.$$

Therefore

$$\forall\, t \in D_{t_0},\;\; \frac{d}{dt} I_1(k,t) = \frac{1}{2i\pi}\oint \frac{d\varphi}{f-t}.$$

We remark now that, as far as f can be chosen as a coordinate, one can write:

$$d\varphi = df.\Psi$$

where Ψ is a holomorphic 0-form, usually denoted as a quotient form

$$\Psi = \frac{d\varphi}{df} = e^{-kf}\frac{\sigma}{df},$$

which is the meromorphic function here

$$\Psi(z) = \frac{e^{-kf(z)}}{2z}.$$

What we get is finally:

$$\forall\, t \in D_{t_0},\ \frac{d}{dt}I_1(k,t) = \frac{1}{2i\pi}\oint \frac{df.\Psi}{f-t} \underset{\text{Cauchy formula}}{=} \frac{e^{-kf}}{2z}|_{f=t, z=\sqrt{t}}$$

What have been done for $I_1(k,t)$ can be done as well for $I_2(k,t) = \varphi(-\sqrt{t})$, $t \in D_{t_0}$, so that :

$$\forall\, t \in D_{t_0},\ \frac{d}{dt}I(k,t) = \frac{d}{dt}\left(\int_{\gamma(t)} \varphi\right) = \int_{\gamma(t)} \frac{d\varphi}{df}|_{f=t} = e^{-kt}\int_{\gamma(t)} \frac{\sigma}{df}|_{f=t} \tag{13}$$

which is nothing but writing

$$\forall\, t \in D_{t_0},\ \frac{d}{dt}I(k,t) = e^{-kt}\left(\frac{1}{2\sqrt{t}} - \frac{1}{-2\sqrt{t}}\right) = \frac{e^{-kt}}{\sqrt{t}}.$$

To write $I(k,t_0)$ as an incomplete Laplace transform, what remains to do now is to notice that

$$\lim_{t>0, t\to 0} I(k,t) = 0$$

(see (12), we recall that $\varphi \in \mathcal{O}(\mathbb{C})$), so that $I(k,t_0)$ can we written as the following incomplete Laplace transform :

$$I(k,t_0) = \int_0^{t_0} e^{-kt}\left(\int_{\gamma(t)} \frac{\sigma}{df}|_{f=t}\right)dt = \int_0^{t_0} e^{-kt}\frac{dt}{\sqrt{t}}. \tag{14}$$

Final step From (14) one has

$$I(k,t_0) = \int_0^{+\infty} e^{-kt}\frac{dt}{\sqrt{t}} + r(k,t_0) = \frac{\Gamma(1/2)}{k^{1/2}} + r(k,t_0) \tag{15}$$

where

$$r(k,t_0) = -\int_{t_0}^{+\infty} e^{-kt}\frac{dt}{\sqrt{t}}$$

is an exponentially decreasing function in $k \in \Sigma_{r,\delta}$: the reasoning is quite similar to what have been done for $R(k,t_0)$ (in fact here $r(k,t_0) = -R(k,t_0)$) so that

$$\forall N \in \mathbb{N},\, \exists C > 0,\, \forall\, k \in \Sigma_{r,\delta},\;\; |r(k,t_0)| \leq \frac{C}{|k|^{N+1}}. \tag{16}$$

Finally, (6), (15), (9) and (16) imply :

$$\forall N \in \mathbb{N},\, \exists C > 0,\, \forall\, k \in \Sigma_{r,\delta},\;\; |I(k) - \frac{\Gamma(1/2)}{k^{1/2}}| \leq \frac{C}{|k|^{N+1}}.$$

2.2. *Airy and the steepest-descent method in dimension 1*

We turn to the Airy equation

$$\frac{d^2y}{dx^2} - xy = 0 \tag{17}$$

which has ∞ as an irregular singular point (set $y(x) = Y(X)$ in (17) with $X = 1/x$, then $X^5\dfrac{d^2Y}{dX^2} + 2X^4\dfrac{dY}{dX} - Y = 0$).
By Fourier transformation, one easily obtains the following particular solution of (17),

$$Ai(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i(xs+\frac{s^3}{3})}\, ds, \tag{18}$$

known as the Airy function. Applying the Cauchy existence theorem for linear differential equations, we know that $Ai(x)$ extends as a holomorphic function on $\mathbb{C}$.

Our aim in this subsection is to analyse the asymptotic behaviour of the Airy function when $x \to +\infty$. More precisely we shall demonstrate the following result (see also Ref. 5 for an hyperasymptotic viewpoint):

Proposition 2.0.1. If $\Sigma_{r,\delta}$ is the sectorial neighbourdhood of infinity defined by (7), then

$$\forall N \in \mathbb{N},\, \exists C > 0,\, \forall\, k \in \Sigma_{r,\delta},\;\; \left| 2\sqrt{\pi}k^{1/6}e^{2k/3}Ai(k^{2/3}) - \sum_{n=0}^{N}\frac{a_n}{k^n}\right| \leq \frac{C}{|k|^{N+1}}.$$

where the a_n belong to $\mathbb{C}$ (explicitely, $a_n = (-\dfrac{3}{4})^n\dfrac{\Gamma(n+1/6)\Gamma(n+5/6)}{2\pi\Gamma(n+1)}$).

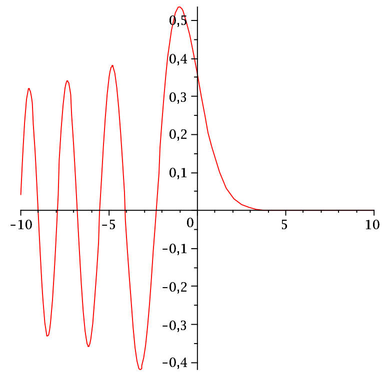


Fig. 4. Trace on the real of the Airy function.

2.2.1. *Prepared form*

We assume for the moment that $x > 0$. In (18) we first make the change of variable

$$s = iu$$

then we set

$$k = x^{3/2}, \qquad \zeta = uk^{-1/3}.$$

We thus get :

$$\text{for } k > 0, \qquad Ai(k^{2/3}) = \frac{k^{1/3}}{2i\pi}\int_C e^{-kg(\zeta)}\,d\zeta, \qquad g(\zeta) = \zeta - \frac{\zeta^3}{3} \tag{19}$$

where the path C is drawn on Fig. 5. It is easy to see that (19) is equivalent to writing:

$$\text{for } k > 0, \qquad Ai(k^{2/3}) = \frac{k^{1/3}}{2i\pi}\int_{C_1} e^{-kg(\zeta)}\,d\zeta, \qquad g(\zeta) = \zeta - \frac{\zeta^3}{3} \tag{20}$$

where the path C_1 is drawn on Fig. 5. (By integration by part, first write $\displaystyle\int_C e^{-kg(\zeta)} = \frac{1}{k}\int_C e^{-kg(\zeta)}\frac{2\zeta}{(1-\zeta^2)^2}\,d\zeta$ then check that this last integral is equal to the integral along C_1).

2.2.2. *Integrability and space of allowed paths of integration*

We pause a moment. Let us consider the integral

$$\mathcal{I}(k) = \int_c e^{-kg(\zeta)}\,d\zeta = \int_{\mathbb{R}} c^\star(e^{-kg(\zeta)}\,d\zeta) = \int_{\mathbb{R}} e^{-kg(c(s))}\,c'(s)ds \tag{21}$$

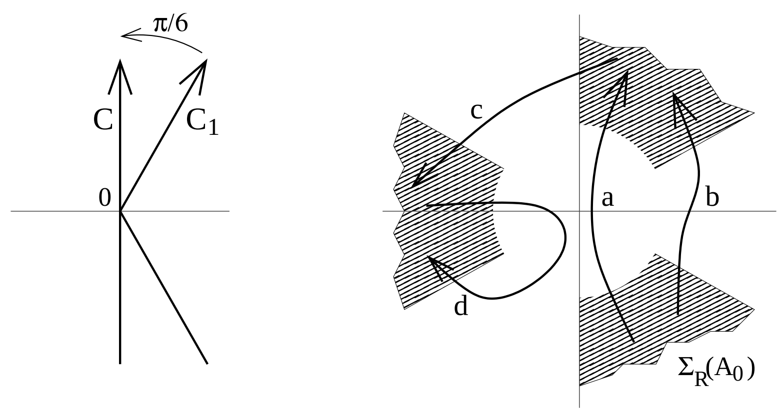


Fig. 5. On the left the paths C and C_1. On the right the open set $\Sigma_R(A_0)$. On this right picture, a, b and c are relative 1-cycles of $\mathbb{C}$ with respect to $\Sigma_R(A_0)$, and d is a relative 1-boundary. The relative 1-cycles a and b are homologous and thus represent the same class in $H_1(\mathbb{C}, \Sigma_R(A_0))$. This 1-homology group is a free $\mathbb{Z}$-module generated by the classes of a and c.

where $c : s \in \mathbb{R} \mapsto c(s) \in \mathbb{C}$ is assumed to be smooth-piecewise, $\lim_{s\to\pm\infty} |c(s)| = +\infty$ and with moderate variations ($|c'(s)|$ has moderate growth). (C and C_1 are such paths).
To make this integral $\mathcal{I}(k)$ absolutely convergent, we would like that $\Re(-kg(\zeta)) \to -\infty$ when $\zeta \to \infty$ along the endless path of integration c. This translates into the condition that $\Re(k\zeta^3) \to -\infty$ at infinity along c. We write this in another way. If

$$\theta = \arg(k) \in \mathbb{S} = \mathbb{R}/2\pi\mathbb{Z} \tag{22}$$

we introduce the following open subsets of $\mathbb{S}$,

$$\begin{aligned} A_{\theta,\delta} &= \bigcup_{j=0}^{2} \left]\frac{\pi}{6} + \frac{2\pi}{3}j - \frac{\theta}{3} + \delta, \frac{\pi}{2} + \frac{2\pi}{3}j - \frac{\theta}{3} - \delta\right[, \qquad \delta > 0 \text{ small enough} \\ A_{\theta} &= \bigcup_{j=0}^{2} \left]\frac{\pi}{6} + \frac{2\pi}{3}j - \frac{\theta}{3}, \frac{\pi}{2} + \frac{2\pi}{3}j - \frac{\theta}{3}\right[, \end{aligned} \tag{23}$$

and the associated family of sectorial neighbourdhoods of infinity

$$\begin{aligned} \Sigma_R(A_{\theta,\delta}) &= \{\zeta \in \mathbb{C}, |\zeta| > R, \ \arg(\zeta) \in A_{\theta,\delta}\}, \qquad R > 0. \\ \Sigma_R(A_{\theta}) &= \{\zeta \in \mathbb{C}, |\zeta| > R, \ \arg(\zeta) \in A_{\theta}\}, \qquad R > 0. \end{aligned} \tag{24}$$

Then, the integral $\mathcal{I}(k)$ will be absolutely convergent if apart from a compact set the (support of the) path c belongs to $\Sigma_R(A_{\theta,\delta})$ for some $R > 0$ and $\delta > 0$.

To precise the space of such allowed paths c, we now introduce the 1-homology group of the pair $(\mathbb{C}, \Sigma_R(A_{\theta,\delta}))$,

$$H_1(\mathbb{C}, \Sigma_R(A_{\theta,\delta})) = H_1(\mathbb{C}, \Sigma_R(A_{\theta,\delta}); \mathbb{Z}).$$

(see Appendix A for some notions on homology theory, see also Fig. 5).

We see that $H_1(\mathbb{C}, \Sigma_R(A_{\theta,\delta})) \cong \mathbb{Z}^2$. Also, if $R' \geq R$, so that $\Sigma_{R'}(A_{\theta,\delta}) \subseteq \Sigma_R(A_{\theta,\delta})$ one has a natural isomorphism

$$\mathcal{R}_{RR'} : H_1(\mathbb{C}, \Sigma_{R'}(A_{\theta,\delta})) \to H_1(\mathbb{C}, \Sigma_R(A_{\theta,\delta}))$$

since the pairs $(\mathbb{C}, \Sigma_{R'}(A_{\theta,\delta}))$ and $(\mathbb{C}, \Sigma_R(A_{\theta,\delta}))$ are homotopic. This allows to define the following 1-homology group by inverse limit:

$$H_1^{A_{\theta,\delta}}(\mathbb{C}) = \varprojlim H_1(\mathbb{C}, \Sigma_R(A_{\theta,\delta}))$$

("we make $R \to +\infty$") which is still a free $\mathbb{Z}$-module of rank 2. Finally we define the 1-homology group we have in mind by inductive limit

$$H_1^{A_\theta}(\mathbb{C}) = \varinjlim H_1^{A_{\theta,\delta}}(\mathbb{C}) \tag{25}$$

("we make $\delta \to 0$"), a free $\mathbb{Z}$-module of rank 2 which deserves to be the space of equivalent classes of our allowed paths of integration.
Note indeed that if c and c' are two endless paths (smooth-piecewise, with moderate variations) which represent the same element in $H_1^{A_\theta}(\mathbb{C})$, then (by integrability and by Stokes since $e^{-kg(\zeta)}\, d\zeta \in \Omega^1(\mathbb{C})$ is closed)

$$\int_c e^{-kg(\zeta)}\, d\zeta = \int_{c'} e^{-kg(\zeta)}\, d\zeta.$$

2.2.3. *The steepest-descent method*

We return to our integral (20) and we recall that we assume $k > 0$. From what have been said previously, in (20) we can replace C_1 by any other path of integration (smooth-piecewise, with moderate variations) belonging to the same class of homology $[C_1] \in H_1^{A_\theta}(\mathbb{C})$.
This is that freedom that we are going to use in the following stationary phase method[8,13] known as the Riemann-Debye "steepest-descent" or "saddle-point" method[a].

[a]It is amazing how Riemann introduced this method to obtain what is now known as the Riemann-Siegel formula for the zeta function, see Ref. 4.

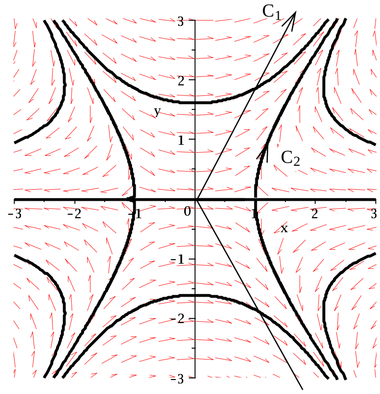


Fig. 6. The vector field $\nabla\big(\Re(kg)\big)$, some steepest-descent curves of $\Re(-kg)$ for $k > 0$, and the path C_2.

In the homology class of C_1 we look for a path C_2 (or a linear combination of paths) which follows *a level curve of* $\Im(kg)$: if $kg(z) = P(x,y) + iQ(x,y)$, such a curve is an integral curve of the vector field $\dfrac{\partial Q}{\partial y}\partial x - \dfrac{\partial Q}{\partial x}\partial y$. By the Cauchy-Riemann equalities, this vector field is also the gradient field $\nabla\big(\Re(kg)\big) = \dfrac{\partial P}{\partial x}\partial x + \dfrac{\partial P}{\partial y}\partial y$ whose (oriented) level curves are the *steepest-ascent curves* of $\Re(kg)$, or equivalently the *steepest-descent curves* of $\Re(-kg)$. See Fig. (6). Note that g has two critical points $\zeta = \pm 1$ where $dg = 0$. These are nondegenerate saddle points (since g is holomorphic, g as well as $\Im(g)$ and $\Re(g)$ are harmonic functions and therefore they satisfy the maximum principle : any critical point is necessarily a saddle point).

Consequence : In the homology class $[C_1] \in H_1^{A_\theta}(\mathbb{C})$ of C_1, we can choose a path C_2 (smooth-piecewise, with moderate variations) such that:

- C_2 is a level curve of $\Im(kg)$,
- the support of C_2 (i.e., its image) contains the saddle point $\zeta = +1$.

See Fig. (6). In practice to obtain C_2, one can consider the deformation of C_1 under the flow Φ of the gradient field $\Re(kg)$: if $C_1 : s \in \mathbb{R} \mapsto c(s) \in \mathbb{C}$, consider the homotopy map

$$h \ : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{C}, \quad h(\tau, s) := \Phi\left(\tau, C_1(s)\right).$$

Write $C_2 : s \in \mathbb{R} \mapsto C_2(s) \in \mathbb{C}$ and assume that $C_2(0) = 1$. Then for $k > 0$ the map $s \in \mathbb{R} \mapsto -k\big[g\big(C(s)\big) - g(1)\big]$ defines a real function with the following variations :

s	$-\infty$		0		$+\infty$
			0		
$-k\big[g\big(C(s)\big) - g(1)\big]$		$\nearrow$		$\searrow$	
	$-\infty$				$-\infty$

In (20) thus replacing C_1 by C_2, one gets

$$Ai(k^{2/3}) = \frac{k^{1/3}}{2i\pi} \int_{C_2} e^{-kg(\zeta)}\, d\zeta, \qquad g(\zeta) = \zeta - \frac{\zeta^3}{3} \tag{26}$$

and we note that the integral defines a holomorphic function in $k \in \Sigma_{r,\delta}$ ($\Sigma_{r,\delta}$ is the sectorial neighbourdhood of infinity defined by (7)).

We are now in position to copy what we done in 2.1.

2.2.4. *Localisation*

In (26) we set $\zeta = 1 + z$ so as to be centred on the critical point:

$$\text{for } k \in \Sigma_{r,\delta}, \quad \begin{aligned} Ai(k^{2/3}) &= \frac{k^{1/3} e^{-2k/3}}{2i\pi} I(k) \\ I(k) &= \int_\Gamma e^{-kf(z)}\, dz, \qquad f(z) = -z^2 - \frac{z^3}{3} \end{aligned} \tag{27}$$

where Γ is the translated of C_2.

We now fix a $t_0 > 0$ and we note Γ_{t_0} the restriction of Γ (precisely its support) to $f^{-1}([0, t_0])$, see Fig. 7.

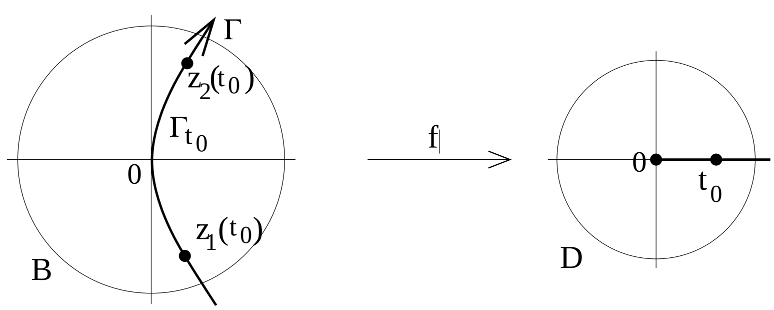


Fig. 7. The path Γ and its restriction Γ_{t_0}. The path Γ is mapped twice on $\mathbb{R}^+$, *resp.* Γ_{t_0} twice on $[0, t_0]$, by f.

We write

$$I(k) = I(k,t_0) + R(k,t_0), \qquad I(k,t_0) = \int_{\Gamma_{t_0}} e^{-kf(z)}\,dz. \tag{28}$$

Concerning $R(k,t_0)$ one easily shows that

$$\exists\, C>0\,,\ \exists\, M>0,\ \forall\, k\in\Sigma_{r,\delta},\ \ |R(k,t_0)| \le M.e^{-C|k|}$$

and in particular

$$\forall N\in\mathbb{N},\ \exists C>0,\ \forall\, k\in\Sigma_{r,\delta},\ \ |R(k,t_0)| \le \frac{C}{|k|^{N+1}} \tag{29}$$

or more precisely

$$\exists C>0,\ \forall N\in\mathbb{N},\ \forall\, k\in\Sigma_{r,\delta},\ \ |R(k,t_0)| \le C^{N+1}\frac{\Gamma(N+2)}{|k|^{N+1}}. \tag{30}$$

This allows to concentrate on $I(k,t_0)$.

2.2.5. *First method*

We briefly mention this first method just for completeness. Since the origin is a nondegenerate critical point for f, by the complex Morse lemma one can find a local coordinate Z such that

$$f(z) = Z^2$$

Explicitely,

$$z = iZ + \frac{1}{6}Z^2 - \frac{5}{72}iZ^3 - \frac{1}{27}Z^4 + \cdots \in \mathbb{C}\{Z\}.$$

Therefore, for $t_0>0$ small enough,

$$\begin{aligned} I(k,t_0) &= \int_{-\sqrt{t_0}}^{\sqrt{t_0}} e^{-kZ^2} h(Z)\,dZ, \\ h(Z) &= \sum_{n\ge 0} h_n Z^n = i + \frac{1}{3}Z - \frac{5}{24}iZ^2 - \frac{4}{27}Z^3 + \cdots \in \mathbb{C}\{Z\}. \end{aligned}$$

As we shall see later (see 2.2.8), to get the asymptotics of $I(k,t_0)$ and $I(k)$ when $|k|\to+\infty$ in $\Sigma_{r,\delta}$ reduces in exchanging $\sum$ and $\displaystyle\int$ and in integrating on $\mathbb{R}$:

$$I(k) \sim i\frac{\sqrt{\pi}}{k}^{-1/2} - i\frac{5\sqrt{\pi}}{48}k^{-3/2} + \cdots$$

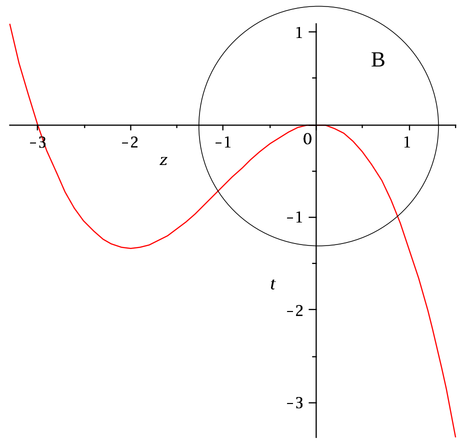


Fig. 8. Trace on the real of $\mathcal{L}$.

2.2.6. *Second method : reduction to an incomplete Laplace transform (1)*

We have here in mind to represent $I(k,t_0) = \displaystyle\int_{\Gamma_{t_0}} e^{-kf(z)}\,dz$ as an incomplete Laplace transform, that is in a way to make the change of variable $t = f(z)$. For doing that we may introduce the complex algebraic curve

$$\mathcal{L} = \{(z,t) \in \mathbb{C}^2,\, t = f(z)\}.$$

This is an analytic submanifold of $\mathbb{C}^2$, $\dim_{\mathbb{C}} \mathcal{L} = 1$, see Fig. 8.

If one considers $\pi_1 : \begin{array}{c} (z,t) \in \mathcal{L} \\ \downarrow \\ z \in \mathbb{C} \end{array}$ then $(\mathcal{L}, \pi_1)$ is nothing but the graph of f. If one considers instead

$$\pi_2| : \begin{array}{c} (z,t) \in \mathcal{L}\backslash\{(0,0),(-2,-4/3)\} \\ \downarrow \\ t \in \mathbb{C}\backslash\{0,-4/3\} \end{array} \qquad \pi_2 : \begin{array}{c} (z,t) \in \mathcal{L} \\ \downarrow \\ t \in \mathbb{C} \end{array}$$

then $(\mathcal{L}\backslash\{(0,0),(-2,-4/3)\},\pi_2|)$ is a 3-sheeted covering of $\mathbb{C}\backslash\{0,-4/3\}$ which is the Riemann surface of the inverse function $z(t)$, 0 and $-4/3$ being algebraic branch points of order 2. ($(\mathcal{L},\pi_2)$ is the so-called ramified Riemann surface).

However for our purpose we do not need such a global information, since the support of Γ_{t_0} is localized near $z = 0$.

We thus introduce an open disc $D = D(0, \eta)$, *resp.* an open ball $B = B(0, \varepsilon)$ with $0 < \varepsilon < 2$, see Fig.7 and Fig. 8. We can choose $0 < \eta$ small enough ($\eta < \frac{4}{3}$) so that:

- for any $t_0 \in D^\star = D \backslash \{0\}$, the set $B \cap f^{-1}(t_0)$ consists in two distinct points $z_1(t_0)$ and $z_2(t_0)$.
- for t near t_0, each of these two functions $z_1(t)$ and $z_2(t)$ represents a germ of holomorphic functions at t_0 (this is a consequence of the implicit function theorem since $t_0 \neq 0$ is not a critical value of f).

In other words, $z_1(t)$ (say) extends to a multivalued function $z(t) \in \widetilde{\mathcal{O}_{D^\star, t_0}}$ with two determinations $z_1(t)$ and $z_2(t)$ (cf. L. Narváez, this volume).

It is not hard to calculate $z(t)$ explicitely:

$$z(t) = \left(\frac{1}{6}t - \frac{1}{27}t^2 + \cdots\right) + t^{1/2}\left(-i + \frac{5}{72}it + \cdots\right)$$

where the series expansions belong to $\mathbb{C}\{t\}$ with $4/3$ for their radius of convergence.

We now go back to $I(k, t_0)$ as defined by (28), where we assume that $t_0 > 0$, $t_0 \in D^\star$. Making the change of variable $z = z(t)$ one has

$$I(k, t_0) = \int_{\widetilde{\Gamma_{t_0}}} e^{-kt} \overset{\vee}{J}(t)\, dt, \qquad \overset{\vee}{J}(t) = -\frac{1}{z(t)\big(2 + z(t)\big)}$$

where the path $\widetilde{\Gamma_{t_0}}$ drawn on Fig. 9 should be thought of as a path on $(\mathcal{L}, \pi_2)$.

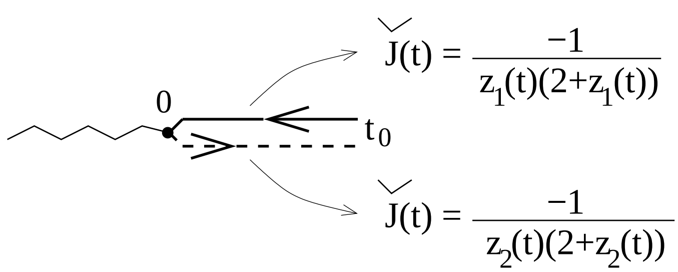


Fig. 9. The path $\widetilde{\Gamma_{t_0}}$.

This reduces into

$$I(k,t_0)=\int_0^{t_0} e^{-kt}J(t)\,dt,$$
$$J(t)=\sum_{n=0}^{\infty} b_n t^{n-1/2} = t^{-1/2}\left(i-\frac{5}{24}it+\frac{385}{3456}it^2+\cdots\right) \tag{31}$$

where the series expansion belongs to $\mathbb{C}\{t\}$ with $4/3$ for its radius of convergence.

2.2.7. *Third method : reduction to an incomplete Laplace transform (2)*

Rather than the previous method, we can repeat what has been done in 2.1.3. With the notations and hypotheses of the previous subsection, we have

$$I(k,t_0)=\int_{\Gamma_{t_0}} e^{-kf}\sigma, \quad \text{with} \quad \sigma=dz,\ \ f(z)=-z^2-\frac{z^3}{3}$$

so that $e^{-kf}\sigma \in \Omega^1(B)$ and thus $e^{-kf}\sigma$ is closed. Since B is simply connected we deduce that $e^{-kf}\sigma$ is exact : there exists $\varphi\in\Omega^0(B)=\mathcal{O}(B)$ such that $d\varphi=e^{-kf}\sigma$. By Stokes,

$$I(k,t_0)=\int_{\Gamma_{t_0}} d\varphi=\int_{\gamma(t_0)}\varphi \tag{32}$$

where $\gamma(t_0)$ is the boundary of Γ_{t_0},

$$\gamma(t_0)=[z_2(t_0)]-[z_1(t_0)]. \tag{33}$$

For t near t_0 we introduce a small circle S^1 of radius small enough ($S^1\subset D^\star$) which surrounds t_0 and t in the positive oriented way, Fig. 10.

To this circle S^1 and to $\gamma(t)$ we associate $\delta_t\gamma(t)=\delta_2-\delta_1$ which consists in the formal difference of the two closed paths drawn of Fig. 10, where δ_2 surrounds $z_2(t_0)$, $z_2(t)$, and δ_1 surrounds $z_1(t_0)$, $z_1(t)$.
By Cauchy and taking f as a coordinate, one gets:

$$I(k,t)=\int_{\gamma(t)}\varphi=\frac{1}{2i\pi}\int_{\delta_t\gamma(t)}\frac{\varphi df}{f-t}.$$

This implies that (see 2.1.3):

$$\frac{d}{dt}I(k,t)=\frac{1}{2i\pi}\int_{\delta_t\gamma(t)}\frac{d\varphi}{f-t}=\frac{1}{2i\pi}\int_{\delta_t\gamma(t)}\frac{df}{f-t}\frac{d\varphi}{df}$$

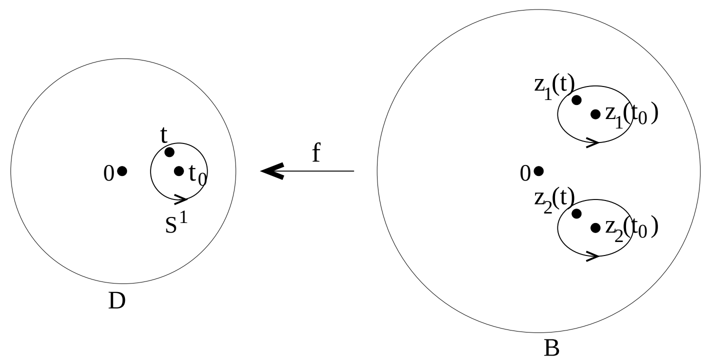


Fig. 10.

where the quotient form is the following meromorphic function:

$$\frac{d\varphi}{df} = -\frac{e^{-kf(z)}}{z(2+z)}.$$

By Cauchy, this means that

$$\frac{d}{dt}I(k,t) = \int_{\gamma(t)} \frac{d\varphi}{df}|_{f=t} = e^{-kt}\int_{\gamma(t)} \frac{\sigma}{df}|_{f=t}.$$

Using the fact that

$$\lim_{t>0, t\to 0} I(k,t) = 0$$

one finally concludes that :

$$I(k,t_0) = \int_0^{t_0} e^{-kt} J(t)\, dt, \qquad J(t) = \int_{\gamma(t)} \frac{\sigma}{df}|_{f=t}. \tag{34}$$

2.2.8. *The asymptotics*

We first analyse the asymptotics of $I(k,t_0)$ for $k \in \Sigma_{r,\delta}$. From (31),

$$I(k,t_0) = \int_0^{t_0} e^{-kt}\left(\sum_{n=0}^{\infty} b_n t^{n-1/2}\right) dt$$

so that for any $N \in \mathbb{N}$,

$$\begin{aligned} I(k,t_0) = &\int_0^{+\infty} e^{-kt}\sum_{n=0}^{N} b_n t^{n-1/2}\, dt - \int_{t_0}^{+\infty} e^{-kt}\sum_{n=0}^{N} b_n t^{n-1/2}\, dt \\ &+ \int_0^{t_0} e^{-kt}\sum_{n=N+1}^{\infty} b_n t^{n-1/2}\, dt \end{aligned}$$

Using (5), this means that

$$I(k,t_0) - \sum_{n=0}^{N} b_n \frac{\Gamma(n+1/2)}{k^{n+1/2}}$$
$$= \sum_{n=N+1}^{\infty} b_n \int_0^{t_0} e^{-kt} t^{n-1/2}\,dt - \sum_{n=0}^{N} b_n \int_{t_0}^{+\infty} e^{-kt} t^{n-1/2}\,dt.$$

Making the change of variable $t = t_0 s$ one obtains:

$$I(k,t_0) - \sum_{n=0}^{N} b_n \frac{\Gamma(n+1/2)}{k^{n+1/2}} =$$
$$\sum_{n=N+1}^{\infty} b_n {t_0}^{n+1/2} \int_0^{1} e^{-kt_0 s} s^{n-1/2}\,ds - \sum_{n=0}^{N} b_n {t_0}^{n+1/2} \int_1^{+\infty} e^{-kt_0 s} t^{n-1/2}\,ds.$$

Since $s^{n-1/2} \le s^{N+1/2}$ in the integrals, it follows that for $k \in \Sigma_{r,\delta}$,

$$\left| I(k,t_0) - \sum_{n=0}^{N} b_n \frac{\Gamma(n+1/2)}{k^{n+1/2}} \right| \le \left(\sum_{n=0}^{\infty} |b_n| t_0^{n+1/2} \right) \frac{\Gamma(N+3/2)}{\Big(t_0 \sin(\delta)|k|\Big)^{N+3/2}}. \tag{35}$$

Using (28) and (30), we can conclude for the asymptotics that:

$$\exists C > 0,\, \forall N \in \mathbb{N},\, \forall\, k \in \Sigma_{r,\delta},\quad \left| I(k) - \sum_{n=0}^{N} b_n \frac{\Gamma(n+1/2)}{k^{n+1/2}} \right|$$
$$\le C^{N+3/2} \frac{\Gamma(N+3/2)}{|k|^{N+3/2}}. \tag{36}$$

Note this result (36) is stronger than the usual Poincaré asymptotics (see Malgrange[21]).
With this result and (27) we get the proposition 2.0.1.

2.2.9. *Remark 1 : geometric monodromy*

Let us go back to the function $J(t)$ defined by (34), namely

$$J(t) = \int_{\gamma(t)} \frac{\sigma}{df}|_{f=t}, \qquad \gamma(t) = [z_2(t)] - [z_1(t)].$$

This function $J(t)$ can be viewed as defining a germ of holomorphic functions at t_0 which extends to a multivalued function on $D^\star$. So we now think of J as a germ in $\widetilde{\mathcal{O}}_{D^\star,t_0}$.

Let λ be the following loop in $D^\star$,

$$\lambda : [0,1] \to D^\star, \qquad \lambda : s \mapsto t = t_0 e^{2i\pi s}.$$

This loop λ generates the first homotopy group $\pi_1(D^\star, t_0)$. We note $T = \rho(\lambda)$ the associated monodromy operator, where

$$\rho : \pi_1(D^\star, t_0) \to \mathrm{Aut}_{\mathcal{O}(D^\star)-\mathrm{alg.}}(\widetilde{\mathcal{O}_{D^\star, t_0}})$$

(cf. L. Narváez, this volume). Then

$$T(J) = -J$$

because $z_1(t)$ and $z_2(t)$ are exchanged when t follows the loop λ, so that $\gamma(t_0)$ is transformed into $-\gamma(t_0)$.

Thus J is of finite determination and one infers from the geometry that

$$T(t^{1/2} J) = t^{1/2} J,$$

that is $t^{1/2} J$ is uniform.

Since we know by other means that J is a multivalued function with moderate growth at 0, we deduce that J belongs to the Nilsson class at $t = 0$ (cf. L. Narváez, this volume). This implies that $J(t)$ is solution of a holomorphic linear differential equation $L(J) = 0$, $L \in \mathrm{End}_{\mathbb{C}}(\mathcal{O}_D)$ with a regular singular point at 0. By inverse Laplace transformation, there is a relationship between this linear differential equation and the Airy equation (17) we started with. It can be shown that

$$J(t) = it^{-1/2} F\left(\frac{1}{6}, \frac{5}{6}, \frac{1}{2}; -\frac{3}{4}t\right)$$

where $F(a, b; c; z)$ is the Gauss hypergeometric function.

2.2.10. *Remark 2 : an example of local system*

In 2.2.2 we introduced the family of homology groups

$$\mathcal{F}_\theta = H_1^{A_\theta}(\mathbb{C}), \qquad \theta \in \mathbb{S} = \mathbb{R}/2\pi\mathbb{Z},$$

each of them being a free $\mathbb{Z}$-module of rank 2.

More generally, for any $U \subset \mathbb{S}$, a connected open arc of length $< \pi$, one can defined

$$A_U = \bigcap_{\theta \in U} A_\theta \subset \mathbb{S}$$

and the group

$$\mathcal{F}(U) = H_1^{A_U}(\mathbb{C}) \cong \mathbb{Z}^2.$$

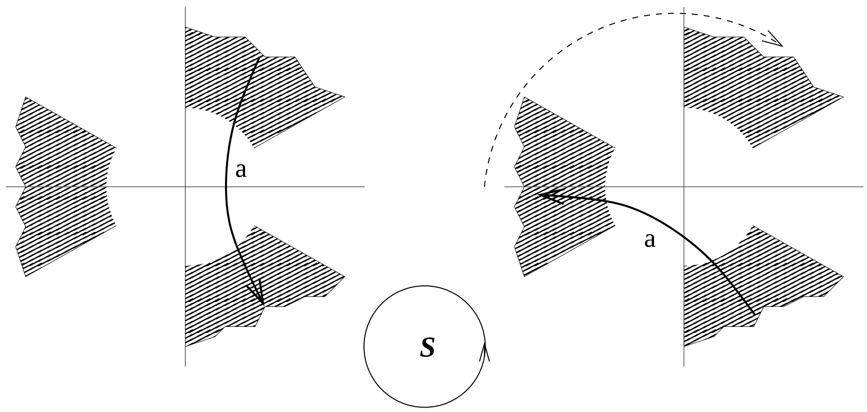


Fig. 11. Continous deformation of a relative 1-cycle a when θ moves along $\mathbb{S}$.

The data of all these groups $(\mathcal{F}(U))$ with the obvious isomorphisms

$$\rho_{V,U} \,:\, \mathcal{F}(U) \to \mathcal{F}(V), \qquad V \subset U$$

makes a sheaf[b] of groups $\mathcal{F}$ on the topological space $\mathbb{S}$ which is locally constant ($\mathcal{F}$ is a local system). Note that $\mathcal{F}$ is not a constant sheaf, see Fig. 11.

2.2.11. *Exercise*

One considers $f(z) = \dfrac{z^3}{3} + \dfrac{z^4}{4}$. Note that $z = 0$ is a degenerate singular point of order 2 for f, whereas $z = -1$ is a nondegenerate singular point, see Fig. 12.

For $k \in \Sigma_{r,\delta}$ (for some $r > 0$ and $0 < \delta < \pi/2$) we define

$$I_1(k) = \int_{-\infty}^{+\infty} e^{-kf(z)}\,dz, \qquad I_2(k) = \int_{-i\infty}^{+i\infty} e^{-kf(z)}\,dz.$$

Show that for $k \in \Sigma_{r,\delta}$:

[b]For $U \subset \mathbb{S}$, one constructs a section $\Gamma \in \mathcal{F}(U)$ by considering a covering $(U_i)_{i\in I}$ of U made of open connected arcs of length $< \pi$ and a family $(\Gamma_i)_{i\in I}$ of elements of $\mathcal{F}(U_i)$ such that for all $i, j \in I$ one has

$$\rho_{U_i\cap U_j, A_i}(\Gamma_i) = \rho_{U_i\cap U_j, U_j}(\Gamma_j).$$

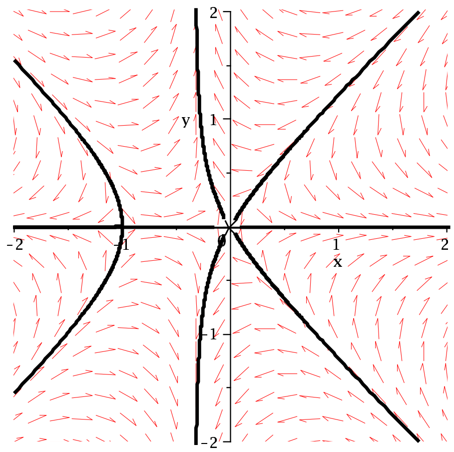


Fig. 12. The vector field $\nabla\big(\Re(kf)\big)$, and some steepest-descent curves of $\Re(-kf)$ for $k>0$ and $f(z)=\dfrac{z^3}{3}+\dfrac{z^4}{4}$.

(1) $I_1(k)$ is asymptotic to

$$e^{k/12}\sqrt{2\pi}\left(\frac{1}{k^{1/2}}+\frac{31}{12}\frac{1}{k^{3/2}}+\cdots\right).$$

(2) $I_2(k)$ is asymptotic to

$$-\frac{2\pi\sqrt{3}}{9\Gamma(2/3)}\frac{1}{k^{1/3}}+\frac{3^{2/3}\Gamma(2/3)}{6}\frac{1}{k^{2/3}}+\frac{9}{8}\frac{1}{k}+\cdots$$

With the methods described in this subsection, we know that the asymptotics of $I_2(k)$ is governed by those of

$$I(k,t_0)=\int_0^{t_0}e^{-kt}J(t)\,dt,\qquad J(t)=\int_{\gamma(t)}\frac{dz}{df}|_{f=t}.$$

with $t_0>0$ small enough, and a convenient $\gamma(t)$. With the notations of 2.2.9, show that $T^3(J)=J$ where T is the monodromy operator.

2.3. *An example in higher dimension*

2.3.1. *Two division lemmas*

We start with two lemmas (see Refs. 19 and 25).

Lemma 2.0.1 (Local division of forms). *We assume that f is a $\mathcal{C}^\infty$ function on the open set $U\subset\mathbb{R}^n$* (resp. *a holomorphic function on $U\subset\mathbb{C}^n$).*

Let ω *be a* $\mathcal{C}^\infty$ *differential* p*-form (*resp. *a holomorphic differential* p*-form,* $\omega \in \Omega^p(U)$*) such that*

$$df \wedge \omega = 0$$

on U*. If* $z_0 \in U$ *is not a critical point of* f *(*$df(z_0) \neq 0$*) then there exists a neighbourhood* $U_{z_0} \subset U$ *of* z_0 *and a* $\mathcal{C}^\infty$ *differential* $(p-1)$*-form* Ψ *on* U_{z_0} *(*resp. *a holomorphic differential* $(p-1)$*-form,* $\Psi \in \Omega^{p-1}(U_{z_0})$*) such that*

$$\omega = df \wedge \Psi.$$

Moreover, the restriction of Ψ *to any hypersurface* $U_{z_0} \cap f^{-1}(t)$ *is a uniquely defined* $\mathcal{C}^\infty$ $(p-1)$*-form (*resp. *holomorphic* $(p-1)$*-form).*
One usually notes Ψ *as a quotient form*

$$\Psi = \frac{\omega}{df}$$

which is called a Leray-Gelfand quotient form.

Proof. We just consider the case $p = n$ so that $\omega = g(z) dz_1 \wedge \cdots \wedge dz_n$.
• From the implicit function theorem, one can choose a local system of coordinates $s = (s_1, \cdots, s_n)$ such that $f = s_1$ is a coordinate while $\omega = h(s) ds_1 \wedge \cdots \wedge ds_n$. Defining $\Psi = h(s) ds_2 \wedge \cdots \wedge ds_n$ gives the result. To make things explicit, there exists a $\mathcal{C}^\infty$ (*resp.* holomorphic) diffeomorphism $\phi : s \in (V, 0) \subset \mathbb{R}^n \mapsto z \in (U_{z_0}, z_0)$ such that

$$\phi^\star \circ f(s) = f \circ \phi(s) = s_1.$$

Meanwhile,

$$\phi^\star \circ \omega(s) = \Big(\phi^\star \circ g(s)\Big) \det \Big(d\phi(s)\Big) ds_1 \wedge \cdots \wedge ds_n.$$

Defining

$$\phi^\star \circ \Psi(s) = \Big(\phi^\star \circ g(s)\Big) \det \Big(d\phi(s)\Big) ds_2 \wedge \cdots \wedge ds_n$$

one gets

$$\phi^\star \circ \omega = d\Big(\phi^\star \circ f\Big) \wedge \Big(\phi^\star \circ \Psi\Big) = \Big(\phi^\star \circ df\Big) \wedge \Big(\phi^\star \circ \Psi\Big) = \phi^\star \circ \Big(df \wedge \Psi\Big).$$

• Assume that $\omega = df \wedge \Psi_1 = df \wedge \Psi_2$, then using the above local system of coordinates one has $ds_1 \wedge (\Psi_1 - \Psi_2) = 0$ where $f = s_1$, so that $\Psi_1 - \Psi_2 = ds_1 \wedge (\cdots)$ and finally $i_t^\star \circ (\Psi_1 - \Psi_2) = 0$, where $i_t : U_{z_0} \cap f^{-1}(t) \to U_{z_0}$ is the canonical injection. □

Lemma 2.0.2 (Global division of forms). *With the notations of the previous lemma, we assume that* $df \wedge \omega = 0$ *on* U *and furthermore*

$$\forall\, z \in U,\ df(z) \neq 0.$$

Then there exists a $\mathcal{C}^\infty$ $(p-1)$*-form* $\dfrac{\omega}{df}$ *on* U *such that*

$$\omega = df \wedge \frac{\omega}{df}.$$

Moreover, the restriction of $\dfrac{\omega}{df}$ *to any hypersurface* $U \cap f^{-1}(t)$ *is a uniquely defined* $\mathcal{C}^\infty$ $(p-1)$*-form (*resp. *holomorphic* $(p-1)$*-form).*

Proof. We recall that on a $\mathcal{C}^\infty$-manifold (separated with countable basis), every open covering has a $\mathcal{C}^\infty$ partition of 1 subordinate to it. Lemma 2.0.2 is thus a consequence of Lemma 2.0.1.
Note that in the holomorphic case $\dfrac{\omega}{df}$ is only a $\mathcal{C}^\infty$ $(p-1)$-form on U as a rule. However its restriction to a level hypersurface $U \cap f^{-1}(t)$ is holomorphic thanks to its uniqueness and Lemma 2.0.1. □

2.3.2. *An application*

We consider here the integral

$$I(k) = \int_{\mathbb{R}^n} \omega, \qquad \omega = e^{-kf}\sigma$$

where $z = (z_1, \cdots, z_n)$, $\sigma = g(z)dz_1 \wedge \cdots \wedge dz_n$ ($\mathbb{R}^n$ with its canonical orientation). We assume that $f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ and $g \in \mathcal{C}_c^\infty(\mathbb{R}^n, \mathbb{R})$.

Assuming that the support of the differential form ω does not meet the singular locus of f (the set of critical points) and using Lemma 2.0.2, one obtains :

$$\int_{\mathbb{R}^n} \omega = \int_{\mathbb{R}^n} df \wedge \frac{\omega}{df} \underset{\text{(Fubini)}}{=} \int_{\mathbb{R}} \left(\int_{\gamma(t)} \frac{\omega}{df} \right) dt = \int_{\mathbb{R}} e^{-kt} \left(\int_{\gamma(t)} \frac{\sigma}{df} \right) dt$$

where $\gamma(t)$ is level hypersurface $f = t$ (naturally oriented as the boundary of $f \leq t$).
As a matter of fact, the above equality extends to the case where the support of ω meets some singular fibres $f = t$ since:

- the set of critical values of the smooth function f is a Lebesgue null set (Sard's theorem).
- each fibre $f = t$ is a Lebesgue null set for the Lebesgue measure in $\mathbb{R}^n$.

2.3.3. *An example*

By way of a simple example we consider for $k > 0$ (or even $k \in \Sigma_{r,\delta}$ for some $r > 0$ and $0 < \delta < \pi/2$),

$$I(k) = \int_{\mathbb{R}^2} e^{-kf}\sigma, \qquad f(z_1, z_2) = z_1^a + z_2^b,\ (a, b) \in (2\mathbb{N}^\star)^2$$

and $\sigma = g(z_1, z_2)dz_1 \wedge dz_2$, $g \in \mathbb{C}[z_1, z_2]$. Writing

$$g(z_1, z_2) = \sum_{p,q} g_{p,q} z_1^p z_2^q$$

with $(p, q) \in \mathbb{N}^2$, it is straighforward (by Fubini) to calculate $I(k)$,

$$I(k) = \sum_{p,q} 4\delta_{p,q} \frac{g_{p,q}}{ab} \frac{\Gamma(\frac{p+1}{a})\Gamma(\frac{q+1}{b})}{k^{\frac{p+1}{a}+\frac{q+1}{b}}}, \qquad \begin{array}{l} \delta_{p,q} = 0 \text{ if } p \text{ or } q \text{ is odd} \\ \delta_{p,q} = 0 \text{ otherwise} \end{array} \tag{37}$$

Also from what precedes (since f is a positive function),

$$I(k) = \int_0^{+\infty} e^{-kt} J(t)\, dt, \qquad J(t) = \int_{\gamma(t)} \frac{\sigma}{df}$$

with $\gamma(t)$ the closed curve $\gamma(t) = \{(z_1, z_2) \in \mathbb{R}^2,\ f(z_1, z_2) = t\}$ and from (37) by inverse Laplace transformation,

$$J(t) = \sum_{p,q} 4\delta_{p,q} \frac{g_{p,q}}{ab} \mathcal{B}(\frac{p+1}{a}, \frac{q+1}{b}) t^{\frac{p+1}{a}+\frac{q+1}{b}-1}$$

with $\mathcal{B}$ the Euler Beta function. See also M. Granger, this volume.

3. Integrals of Holomorphic Differential Forms along Cycles

We would like to extend the constructions seen in §2 to multidimensional integrals defined in the complex field. Here we shall define and analyse the functions

$$J(t) = \int_{\gamma(t)} \omega$$

where:

- ω is a holomorphic differential $(n-1)$-form on $U \subset \mathbb{C}^n$,
- $\gamma(t)$ is a $(n-1)$-cycle on the level hypersurface $f = t$,
- $f : U \subset \mathbb{C}^n \to \mathbb{C}$ is holomorphic.

The notions of homology theory which are used in the sequel are recalled in the appendix A, see also Refs. 12, 16 and 28.

3.1. *Integrals along cycles on fibres*

3.1.1. *Definition and first property*

We assume here that $f : \mathbb{C}^n \to \mathbb{C}$ is a (non constant) holomorphic function ($f \in \mathcal{O}(\mathbb{C}^n)$) and that $\omega \in \Omega^{n-1}(\mathbb{C}^n)$ is a holomorphic differential $(n-1)$-form : $\omega = \sum_{|I|=n-1} g_I(z)dz_I$ where $z = (z_1, \cdots, z_n)$, $g_I \in \mathcal{O}(\mathbb{C}^n)$ and if $I = (i_1, \cdots, i_p)$ then $dz_I = dz_{i_1} \wedge \cdots \wedge dz_{i_p}$.

We assume that $t_0 \in \mathbb{C}$ is not a critical value of f so that $f^{-1}(t_0)$ is an holomorphic submanifold of $\mathbb{C}^n$ of dimension $\dim_{\mathbb{C}} f^{-1}(t_0) = n-1$.

Let $\gamma(t_0) \in Z_{n-1}(f^{-1}(t_0))$ be a $(n-1)$-cycle of $f^{-1}(t_0)$.

Proposition 3.0.2. With the above hypotheses, the integral $\int_{\gamma(t_0)} \omega$ depends only on the homology class $[\gamma(t_0)] \in H_{n-1}(f^{-1}(t_0))$.

Proof. We first mention that $\int_{\gamma(t_0)} \omega$ is a short way of writing $\int_{i_\star\gamma(t_0)} \omega$ where $i : f^{-1}(t_0) \hookrightarrow \mathbb{C}^n$ is the canonical injection.
We have $\int_{i_\star\gamma(t_0)} \omega = \int_{\gamma(t_0)} i^\star\omega$ where $i^\star\omega \in \Omega^{n-1}(f^{-1}(t_0))$ is a holomorphic differential $(n-1)$-form on $f^{-1}(t_0)$. Since $\dim_{\mathbb{C}} f^{-1}(t_0) = n-1$, this means that $i^\star\omega$ is of maximal order, so that $i^\star\omega$ is closed. We conclude with the Corollary A.4.1. □

3.1.2. *The Ehresmann fibration theorem*

We would like to think of the integral $\int_{\gamma(t)} \omega$ with $[\gamma(t)] \in H_{n-1}(f^{-1}(t))$ as a continuous function of t. This requires to being able to deform continuously $\gamma(t_0)$ to a nearby $\gamma(t)$ for t near t_0. This relies on some fibration theorems.

We first mention the following result, see Ref 12:

Theorem 3.1 (Ehresmann fibration theorem). *If M and N are differentiable* (resp. $\mathcal{C}^\infty$) *manifolds and if $f : M \to N$ is a proper submersion, then f is a locally trivial differentiable* (resp. $\mathcal{C}^\infty$) *fibration.*
If M is a manifold with boundary ∂M, and if both $f : M \to N$ and $f| : \partial M \to N$ are proper submersions, then both f and $f|$ are locally trivial differentiable (resp. $\mathcal{C}^\infty$) *fibrations.*

So when f is a submersion (for any $m \in M$, rank $T_m f = \dim N$) and proper (f^{-1}(a compact) is compact), then for every $t_0 \in N$, there exist a neighbourhood U of t_0 and a differentiable (*resp.* $\mathcal{C}^\infty$) diffeomorphism $\Phi : f^{-1}(U) \to U \times f^{-1}(t_0)$ (a so-called local trivialization map) such that the following diagram commutes:

$$\begin{array}{ccc} f^{-1}(U) & \xrightarrow{\Phi} & U \times f^{-1}(t_0) \\ f| \searrow & & \swarrow pr_1 \\ & U & \end{array}$$

(in other words, f is the projection of a differentiable (*resp.* $\mathcal{C}^\infty$) fibre bundle).

If M is a manifold with boundary ∂M, and if both $f : M \to N$ and $f| : \partial M \to N$ are proper submersions, then the above diagram respects the boundary and the interior $M \backslash \partial M$ of M.

We give also the following result[12] :

Proposition 3.1.1. If $f : M \to N$ is a locally trivial differentiable (*resp.* $\mathcal{C}^\infty$) fibration and if N is contractible, then this fibration is trivial.

For a given $f \in \mathcal{O}(\mathbb{C}^n)$ the Ehresmann fibration theorem cannot be applied directly since in general (when $n \geq 2$) such a map is not proper (just think of $f(z_1, z_2) = z_1 - z_2$) and furthermore f has as a rule a nonempty set $Sing_f \subset \mathbb{C}^n$ of critical points.

We now consider a $z_0 \notin Sing_f$. We introduce the open ball $B = B(z_0, \varepsilon) \subset \mathbb{C}^n$ and the closed ball $\overline{B}$ whose boundary is $\partial\overline{B} = \{z \in \mathbb{C}^n, \|z - z_0\| = \varepsilon\}$. We introduce also the open disc $D = D(t_0, \eta) \subset \mathbb{C}$ where $t_0 = f(z_0)$ and we note

$$X = B \cap f^{-1}(D), \qquad \overline{X} = \overline{B} \cap f^{-1}(D).$$

We assume that $\overline{X}$ contains no critical point of f : this is true at least for $\varepsilon > 0$ small enough since $z_0 \notin Sing_f$.

Also we assume that $f^{-1}(t_0)$ and $\partial\overline{B}$ intersect transversally : apart from its singular points, the locus $f^{-1}(t_0)$ can be seen as a $\mathcal{C}^\infty$ submanifold of $\mathbb{R}^{2n}$ of real dimension $2(n-1)$ while $\partial\overline{B}$ can be considered as a $\mathcal{C}^\infty$ submanifold of $\mathbb{R}^{2n}$ of real dimension $2n-1$. Then $f^{-1}(t_0)$ and $\partial\overline{B}$ cut transversally if

$$\forall\, z \in f^{-1}(t_0) \cap \partial\overline{B}, \qquad T_z(f^{-1}(t_0)) + T_z\partial\overline{B} = \mathbb{R}^{2n}.$$

It can be shown that this is true at least for $\varepsilon > 0$ small enough. Now by the implicit function theorem, $f^{-1}(t)$ and $\partial\overline{B}$ will intersect transversally for any $t \in D$ provided that $\eta > 0$ is chosen small enough.

With these hypotheses, one obtains that $\overline{X} = \overline{B} \cap f^{-1}(D)$ is a $\mathcal{C}^\infty$ manifold with boundary $\partial\overline{X} = \partial\overline{B} \cap f^{-1}(D)$. Applying the Ehresmann fibration theorem to $f|\begin{array}{c}\overline{X}\\ \downarrow\\ D\end{array}$ one obtains:

Theorem 3.2. *We note* $B = B(z_0, \varepsilon) \subset \mathbb{C}^n$ *an open ball and* $D = D(t_0, \eta) \subset \mathbb{C}$ *with* $t_0 = f(z_0)$. *We note*

$$X = B \cap f^{-1}(D), \qquad \overline{X} = \overline{B} \cap f^{-1}(D)$$

and we assume that $\overline{X}$ *contains no critical point of* f. *Then* ε *and* η *can be chosen so that* $f|\begin{array}{c}\overline{X}\\ \downarrow\\ D\end{array}$ *and* $f|\begin{array}{c}X\\ \downarrow\\ D\end{array}$ *are locally* $\mathcal{C}^\infty$ *trivial fibrations.*

As a matter of fact, because D is contractible, we deduce from Proposition 3.1.1 that the fibre bundle $f|\begin{array}{c}X\\ \downarrow\\ D\end{array}$ is trivial : there exists a $\mathcal{C}^\infty$ diffeomorphism Φ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\Phi} & D \times X_{t_0} \\ f|\searrow & & \swarrow pr_1 \\ & D & \end{array}, \qquad X_t = B \cap f^{-1}(t) \tag{38}$$

3.1.3. *Applications*

We assume that the conditions of Theorem 3.2 are fulfilled. We start with a given $(n-1)$-cycle $\gamma(t_0)$ of X_{t_0},

$$\gamma(t_0) = \sum n_i \sigma_i \ \in Z_{n-1}(X_{t_0})$$

and with (38) we define for $t \in D$:

$$\gamma(t) = \sum n_i \Phi_t^{-1}(\sigma_i) = C_{q-1}(\Phi_t^{-1})(\gamma(t_0))$$

where we have written $\Phi_t^{-1}(.) = \Phi^{-1}(t, .)$ and used the notations of A.3. What we get is a (so called) “horizontal deformation” of $\gamma(t_0)$, that is $\gamma(t_0)$ has been deformed continuously (in fact in a $\mathcal{C}^\infty$ manner) into a $(n-1)$-cycle $\gamma(t)$ of X_t.

Remark The $\mathcal{C}^\infty$ diffeomorphim $\Phi_t^{-1} : X_{t_0} \to X_t$ gives rise to an isomorphism of homology groups

$$\Phi_t^{-1}{}_\star : H_q(X_{t_0}) \to H_q(X_t), \qquad \forall\, q$$

and the disjoint union $\bigsqcup_{t\in D} H_q(X_t)$ of all these groups makes a local system on D which is a constant sheaf[c].

Proposition 3.2.1. If $\omega \in \Omega^{n-1}(X)$ is a holomorphic differential $(n-1)$-form in X, and with the above notations and hypotheses, the integral $J(t) = \int_{\gamma(t)} \omega$ defines a $\mathcal{C}^\infty$ function in D.

Proof. By Proposition 3.0.2, the integral $J(t) = \int_{\gamma(t)} \omega$ depends only on the homology class $[\gamma(t)] \in H_{n-1}(X_t)$. In the homology class $[\gamma(t_0)] \in H_{n-1}(X_{t_0})$ we choose a piecewise differentiable $(n-1)$-cycle γ_{t_0} and we note

$$\gamma(t) = \sum n_i \Phi_t^{-1}(\sigma_i) = \sum n_i \sigma_i(t).$$

By construction, each $(q-1)$-simplex $\sigma_i(t)$ is a $\mathcal{C}^\infty$ function of t and since

$$\int_{\sigma_i(t)} \omega = \int_{\Delta^{q-1}} \sigma_i(t)^\star \omega$$

one concludes with the dominated Lebesgue theorem. □

Proposition 3.2.2. With the above notations and hypotheses, the integral $J(t) = \int_{\gamma(t)} \omega$ defines a holomorphic function in D.

Proof. To be holomorphic is a local property. For $t_0 \in D$ we consider $t \in D$ near t_0. We introduce a small oriented circle S^1 in D which surrounds t_0 and t in the natural way (cf. Fig. 10).
We construct a cycle $\delta_t \gamma(t) \in Z_n(X \backslash X_t)$ in the following way: if

$$\gamma(t) = \sum n_i \sigma_i, \qquad S^1 : \Delta^1 \to D$$

[c]Note $\mathcal{F} = \bigsqcup_{t\in D} H_q(X_t)$. If $U \in D$ is an open neighbourhood of t_0, then $\mathcal{F}(U)$ is the set of the sections $\gamma : t \in U \mapsto \gamma(t) \in \mathcal{F}$ defined as $\gamma(t) = \Phi_t^{-1}{}_\star(\gamma(t_0)) \in H_q(X_t)$ where $\gamma(t_0)$ is some given element of $H_q(X_{t_0})$.

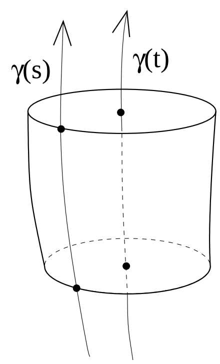


Fig. 13. The cycles $\gamma(t) \in Z_{n-1}(X_t)$ and $\delta_t\gamma(t) \in Z_n(X\backslash X_t)$.

we define (with Φ as in (38))

$$\delta_t\,\sigma_i : \Delta^1 \otimes \Delta^{n-1} \to X\backslash X_t, \qquad \delta_t\,\sigma_i = \Phi^{-1}(S^1, \sigma_i)$$

($\Delta^1 \otimes \Delta^{n-1}$ is here the oriented product of Δ^1 and Δ^{n-1}) and[d]

$$\delta_t\gamma(t) = \sum n_i\,(\delta_t\,\sigma_i).$$

The reader may think of $\delta_t\gamma(t)$ as the union $\bigcup_{s\in S^1} \gamma(s) \subset X\backslash X_t$, see Fig. 13.

In that way, one gets a homomorphism

$$\delta_t : H_{n-1}(X_t) \to H_n(X\backslash X_t)$$

which is just a particular case of the so-called Leray coboundary or tube operator[1,19,25] . With this definition, we now gives a lemma:

Lemma 3.2.1. For t near t_0,

$$J(t) = \frac{1}{2i\pi}\int_{\delta_t\gamma(t)} \frac{df \wedge \omega}{f-t}.$$

Before to give the proof (from Ref. 1), it is worth noting that the integral $\displaystyle\int_{\delta_t\gamma(t)} \frac{df \wedge \omega}{f-t}$ only depends on the homology class $[\delta_t\gamma(t)] \in H_n(X\backslash X_t)$ because $\dfrac{df \wedge \omega}{f-t} \in \Omega^n(X\backslash X_t)$ is closed.

[d]Note that $\delta_t\,\sigma_i$ can be seen as an element of $C_n(X\backslash X_t)$ in the sense of appendix A : just consider for instance the map $\sum_{j=0}^n \lambda_j e_j \in \Delta^n \mapsto \Big(\lambda_0 e_0 + (1-\lambda_0)e_1, \sum_{j=1}^n (\lambda_j + \frac{\lambda_0}{n})e_{j-1}\Big) \in \Delta^1 \times \Delta^{n-1}$ whose inverse map is obvious.

Proof. By Fubini one gets

$$\frac{1}{2i\pi}\int_{\delta_t\gamma(t)}\frac{df\wedge\omega}{f-t}=\frac{1}{2i\pi}\int_{S^1}\left(\int_{\gamma(s)}\omega\right)\frac{ds}{s-t}$$

We write this as

$$\begin{aligned}&\frac{1}{2i\pi}\int_{\delta_t\gamma(t)}\frac{df\wedge\omega}{f-t}\\&\quad=\underbrace{\frac{1}{2i\pi}\int_{S^1}\left(\int_{\gamma(t)}\omega\right)\frac{ds}{s-t}}_{A_1}+\underbrace{\frac{1}{2i\pi}\int_{S^1}\left(\int_{\gamma(s)}\omega-\int_{\gamma(t)}\omega\right)\frac{ds}{s-t}}_{A_2}\end{aligned}$$

By the Cauchy formula (because $\displaystyle\int_{\gamma(t)}\omega$ does not depend on s) one obtains

$$A_1=\frac{1}{2i\pi}\int_{S^1}\frac{ds}{s-t}\int_{\gamma(t)}\omega=\int_{\gamma(t)}\omega.$$

By Proposition 3.2.1 we know that $\displaystyle\int_{\gamma(t)}\omega$ is a $\mathcal{C}^\infty$ function in t, so that by Taylor,

$$\int_{\gamma(s)}\omega-\int_{\gamma(t)}\omega=Cste.(s-t)+O([s-t|^2)$$

where $Cste$ is a constant complex number. Using this result in A_2 then making the radius of the circle S^1 tend to 0 one gets $A_2=0$. ☐

This lemma proves Proposition 3.2.2: in the integral representation

$$J(t)=\frac{1}{2i\pi}\int_{\delta_t\gamma(t)}\frac{df\wedge\omega}{f-t}$$

the cycle $\delta_t\gamma(t)$ does not depend on t for t near t_0 in D. Since the integrand is holomorphic in t, one concludes with the dominated Lebesgue theorem.

3.2. *The polynomial case*

In this subsection we assume that $f:\mathbb{C}^n\to\mathbb{C}$ is a (non constant) polynomial function.

3.2.1. *A fibration theorem and consequences*

In the polynomial case we have the following fibration theorem[6,31] :

Theorem 3.3. *Assume that $f : \mathbb{C}^n \to \mathbb{C}$ is a polynomial function. Then there exists a finite set $Atyp_f = \{t_1, \cdots, t_N\} \subset \mathbb{C}$ of "atypical values" such that the restriction map* $f| \begin{array}{c} \mathbb{C}^n \backslash f^{-1}(Atyp_f) \\ \downarrow \\ \mathbb{C}\backslash Atyp_f \end{array}$ *is a locally $\mathcal{C}^\infty$ trivial fibration.*

Remark To make a link between Theorems 3.3 and 3.1, let us introduce a compactification of f which can be defined as follows. If $d = \deg f$, define

$$G(z, z_0, t) = z_0^d f(\frac{z}{z_0}) - tz_0^d = f^d(z) + z_0 f^{d-1}(z) + \cdots + z_0^d f^0(z) - tz_0^d$$

($z_0^d f(\frac{z}{z_0})$ is the projectivization of f by the new variable z_0; we have noted by f^l the homogeneous part of degree l of f) and introduce the set

$$M = \{((z : z_0), t) \in \mathbb{P}^n_{\mathbb{C}} \times \mathbb{C},\ G(z, z_0, t) = 0\}$$

where $\mathbb{P}^n_{\mathbb{C}}$ is the n-dimensional complex projective space. We consider the embedding $e : z = (z_1, \cdots, z_n) \in \mathbb{C}^n \mapsto (z : 1) = (z_1 : \cdots : z_n : 1) \in \mathbb{P}^n_{\mathbb{C}}$ and the map $E : z \in \mathbb{C}^n \mapsto \Big(e(z), f(z)\Big) \in \mathbb{P}^n_{\mathbb{C}} \times \mathbb{C}$, so that M is the Zariski closure of $E(\mathbb{C}^n)$ in $\mathbb{P}^n_{\mathbb{C}} \times \mathbb{C}$. We thus get a commutative diagram

$$\begin{array}{lcl} \mathbb{C}^n & \xrightarrow{E} & M \ni ((z : z_0), t) \\ f\downarrow & & \quad\downarrow p \\ \mathbb{C} & \xrightarrow{id} & \quad\mathbb{C} \ni t \end{array}$$

A compactification of $f : \mathbb{C}^n \to \mathbb{C}$ is given by the *proper* map $p : M \to \mathbb{C}$ which is a locally trivial fibration except over a set of points, and this translates to the fibration f (see Refs. 31 and 6.. One has to see M as a Whitney stratified space – see B. Teissier, this volume – and instead of Theorem 3.1 one has in fact to use the Thom-Mather first isotopy lemma). Of course $f(Sing_f) \subset Atyp_f$. However, $Atyp_f$ may contains other values which come from a set $Sing_f^\infty$ of singular points at infinity. This is related to the fact that the hypersurface $M \subset \mathbb{P}^n_{\mathbb{C}} \times \mathbb{C}$ has a singular locus, namely $\Sigma \times \mathbb{C}$ where Σ is the following algebraic subset of the hyperplane at infinity $H^\infty = \{x_0 = 0\} \subset \mathbb{P}^n_{\mathbb{C}}$,

$$\Sigma = \{(z : z_0) \in \mathbb{P}^n_{\mathbb{C}},\ \text{grad} f^d = f^{d-1} = z_0 = 0\} \subset M. \tag{39}$$

What have been done in §3.1.2 can be repeated. Theorem 3.3 implies that the union $\displaystyle\bigsqcup_{t \in \mathbb{C}\backslash Atyp_f} H_{n-1}(f^{-1}(t), K)$ ($K = \mathbb{Z}$ or $\mathbb{C}$) makes a local system of groups on $\mathbb{C}\backslash Atyp_f$ (a locally constant sheaf but not constant one in general) and, for a given $\omega \in \Omega^{n-1}(\mathbb{C}^n)$, the mapping

$$L_\omega : [\gamma(t)] \in H_{n-1}(f^{-1}(t)) \mapsto \int_{\gamma(t)} \omega \tag{40}$$

defines a representation of this local system into the sheaf $\mathcal{O}_{\mathbb{C}\backslash Atyp_f}$ of holomorphic functions on $\mathbb{C}\backslash Atyp_f$:

Corollary 3.3.1. For $\omega \in \Omega^{n-1}(\mathbb{C}^n)$ we consider

$$J(t) = \int_{\gamma(t)} \omega, \qquad [\gamma(t)] \in H_{n-1}(f^{-1}(t))$$

as defining a germ of holomorphic functions at $t_0 \in \mathbb{C}\backslash Atyp_f$. Then $J(t)$ extends to a multivalued function on $\mathbb{C}\backslash Atyp_f$.

Using the fibration Theorem 3.3 one can consider the continuous horizontal deformation of a $(n-1)$-cycle when t follows a closed path $\lambda \in \pi_1(\mathbb{C}\backslash Atyp_f, t_0)$. We thus have an action

$$\mathcal{M} : \lambda \in \pi_1(\mathbb{C}\backslash Atyp_f, t_0) \mapsto \rho(\lambda) = Aut\Big(H_{n-1}(f^{-1}(t_0), K)\Big)$$

where the automorphism $\mathcal{M}(\lambda)$ is the **geometric monodromy** associated with the loop λ. We can define also the action

$$\rho : \pi_1(\mathbb{C}\backslash Atyp_f, t_0) \to \mathrm{Aut}_{\mathcal{O}(\mathbb{C}\backslash Atyp_f)-\mathrm{alg.}}(\mathcal{O}_{\widetilde{\mathbb{C}\backslash Atyp_f}, t_0})$$

and we see that analysing the action of the monodromy operator $\rho(\lambda)$ on the germ $J(t) = \displaystyle\int_{\gamma(t)} \omega$ just reduces by the representation (40) in analysing the action of the geometric monodromy operator $\mathcal{M}(\lambda)$ on $[\gamma(t_0)]$:

$$\rho(\lambda)\Big(\int_{\gamma(t_0)} \omega\Big) = \int_{\mathcal{M}(\lambda)[\gamma(t_0)]} \omega. \tag{41}$$

3.2.2. *The finite rank case*

We now **assume** that $H_{n-1}(X_{t_0}) \cong Z^\mu$ for $t_0 \in \mathbb{C}\backslash Atyp_f$, so that $\Big(H_{n-1}(f^{-1}(t))\Big)_{t \in \mathbb{C}\backslash Atyp_f}$ makes a local system of free $\mathbb{Z}$-modules of rank μ on $\mathbb{C}\backslash Atyp_f$.
This is always true when $n = 1$, but also for instance when the set of atypical values $Atyp_f$ arises *only from isolated critical points at finite distance* (in particular $Atyp_f = f(Sing_f)$). In that case and for $n \geq 2$, $\mu = \dim_{\mathbb{C}} \dfrac{C[z]}{(\partial f)}$

is the total Milnor number ((∂f) is the Jacobian ideal) and is the sum of the Milnor numbers at each singular point (cf. Ref. 30), $\mu = \sum_{z_\star \in Sing_f} \mu_{z_\star}$, $\mu_{z_\star} = \dim_{\mathbb{C}} \dfrac{C\{z - z_\star\}}{(\partial f)}$. In a more general case for $n \geq 2$, still assuming that f has only isolated singular points *including those at infinity* (for instance (39) defines a finite set), then for any $t \in \mathbb{C}\backslash Atyp_f$, $H_{n-1}(X_t) \cong Z^\mu \oplus A$ where μ is the total Milnor number while A is some finitely generated abelian group.[7] (We shall return to that point in §4).

Let us then define a basis $\mathcal{B}(t_0) = (\gamma_1(t_0), \cdots, \gamma_\mu(t_0))$ of $(n-1)$-cycles whose classes generate $H_{n-1}(f^{-1}(t_0))$ as a free $\mathbb{Z}$-module of rank μ. The continuous horizontal deformation of $\mathcal{B}(t_0)$ along a loop $\lambda \in \pi_1(\mathbb{C}\backslash Atyp_f, t_0)$ provides another basis $\mathcal{M}(\lambda)\Big(\mathcal{B}(t_0)\Big)$ of $H_{n-1}(f^{-1}(t_0))$: the geometric monodromy operator $\mathcal{M}(\lambda)$ is represented in the basis $\mathcal{B}(t_0)$ by an invertible matrix $M(\lambda) \in GL_\mu(\mathbb{Z})$.

With $Atyp_f = \{t_1, \cdots, t_N\}$, the fundamental group $\pi_1(\mathbb{C}\backslash Atyp_f, t_0)$ of $\mathbb{C}\backslash Atyp_f$ with respect to the base point t_0 is the free group generated by the family of loops $(\lambda_i)_{1\leq i\leq N}$ drawn on Fig. 14 : the loop λ_i follows a path l_i which goes from t_0 to some $\widetilde{t_i}$ near t_i, then λ_i turns around t_i in the positive oriented direction and finally goes back to t_0 following l_i in the inverse way.

We return to $J(t) = \displaystyle\int_{\gamma(t)} \omega$ as in Corollary 3.3.1. By analytic continuations along the path l_i, we can assume that $t_0 = \widetilde{t_i}$ and think of $J(t)$ as an element of $\widetilde{\mathcal{O}_{D^\star_{t_i}, t_0}}$ where D_{t_i} is a small disc centred on t_i. From the fact that $H_{n-1}(f^{-1}(t_0))$ is a free $\mathbb{Z}$-module of rank μ, we know that the geometric monodromy operator $\mathcal{M}(\lambda_i)$ is the zero of a polynomial (for instance its

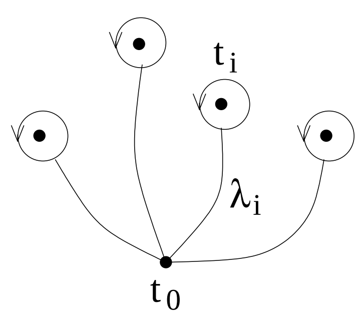


Fig. 14. The generators of the fundamental group $\pi_1(\mathbb{C}\backslash Atyp_f, t_0)$.

characteristic polynomial, by Cayley-Hamilton). By (41) this translates into the fact that $J(t)$ is of finite determination at t_i (The characteristic polynomial $P \in Z[X]$ provides a relationship between the various determinations J, $\rho(\lambda_i)(J)$, ..., $\rho(\lambda_i)^\mu(J)$). Then (cf. L. Narváez, this volume):

Proposition 3.3.1. We assume that $f : \mathbb{C}^n \to \mathbb{C}$ is a polynomial function with $Atyp_f = \{t_1, \cdots, t_N\}$ for its set of atypical values. For $t_0 \in \mathbb{C}\backslash Atyp_f$ we assume that $H_{n-1}(X_{t_0}) \cong Z^\mu$ and we fix $\omega \in \Omega^{n-1}(\mathbb{C}^n)$. Then any analytic continuations l_iJ near any t_i of the germ $J(t) = \int_{\gamma(t)} \omega$, $[\gamma(t)] \in H_{n-1}(f^{-1}(t))$ of analytic functions at t_0 is of finite determination at t_i and one can write locally l_iJ as a finite sum

$$l_iJ(t) = \sum_{\alpha \in \mathbb{C}, l \in \mathbb{N}} \phi_{\alpha,l}(t - t_i)^\alpha \mathrm{Log}^{\,l}(t - t_i)$$

where the $\phi_{\alpha,l}$ are uniform near t_i.

Next we give a (consequence of a) result due to Nilsson[23,24] :

Theorem 3.4. *We assume that ω is a polynomial differential $(n-1)$-form. Then, with the notations and hypotheses of Proposition 3.3.1, for each $i = 1, \cdots, N$, and for any sectorial neighbourhood $\Sigma(t_i; r, a, b)$ of t_i,*

$$\Sigma(t_i; r, a, b) = \{t \in \mathbb{C},\ 0 < |t - t_i| < r,\ a < \arg t < b\}$$

with $r > 0$ small enough, $b - a < 2\pi$, there exist $M \in \mathbb{N}$ and $C > 0$ such that

$$\forall\, t \in \Sigma(t_i; r, a, b),\ \ |l_iJ(t)| \le C|t - t_i|^{-M}.$$

Proposition 3.3.1 and Theorem 3.4 imply that each $l_iJ(t)$ belongs to the Nilsson class (cf. L. Narváez, this volume). Consequently:

Corollary 3.4.1. With the hypotheses of Proposition 3.3.1 and Theorem 3.4, the multivalued function $J(t)$ is a solution of a linear differential equation with at most regular singular points at $t_1, \cdots, t_N$.

3.2.3. *Examples*

First example We consider $f : (p, q) \in \mathbb{C}^2 \mapsto p^2 + V(q)$, $V(q) \in \mathbb{C}[q]$. In classical mechanics, p is the momentum and $V(q)$ stands for the potential function. For a non-critical E (E= the energy) one has $H_1(f^{-1}(E)) \cong \mathbb{Z}^{m-1}$ where m is the degree of V.

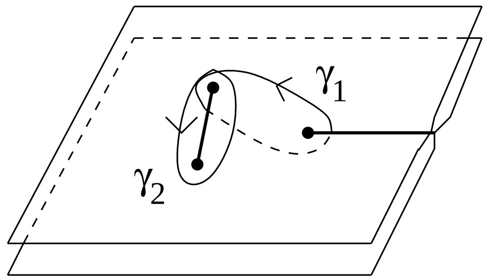


Fig. 15. The level manifold $f^{-1}(E)$ viewed as the Riemann surface of $p = (E - V(q))^{1/2}$ for $V(q) = q^3$ and $E = 1$. The homology classes of the 2 cycles γ_1 and γ_2 drawn generate $H_1(f^{-1}(E))$.

We now assume that $V(q) = q^3$ so that $E = 0$ is the sole critical value. We are interested in the following "period integral"

$$J(E) = \int_{\gamma(E)} \omega$$

where ω is a polynomial differential 1-form (when $\omega = pdq$ the period integral is the "action" in classical mechanics). One can thus apply Proposition 3.3.1 and Theorem 3.4.

We analyse the geometric monodromy; We consider for $E_0 = 1$ the basis $\mathcal{B}(E_0) = ([\gamma_1(E_0)], [\gamma_2(E_0)])$ of $H_1(f^{-1}(E))$ as drawn on Fig. 15. If λ_0 is the natural loop which generates $\pi_1(\mathbb{C}^\star, 1)$, then the associated geometric monodromy operator $\mathcal{M}(\lambda_0)$ satisfies

$$\mathcal{M}(\lambda_0) : ([\gamma_1(E_0)], [\gamma_2(E_0)]) \mapsto ([\gamma_2(E_0)], [\gamma_2(E_0) - \gamma_1(E_0)]).$$

(Just see how the *zeros* of $q^3 = E$ are exchanged). The geometric monodromy operator $\mathcal{M}(\lambda_0)$ is thus represented in the basis $\mathcal{B}(E_0)$ by an invertible matrix $M(\lambda_0) = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \in GL_2(\mathbb{Z})$, whose characteristic polynomial is $P(X) = X^2 - X + 1$. Thus, if $J(E)$ is defined as a germ of holomorphic functions at E_0, one has

$$\rho(\lambda_0)^2(J) - \rho(\lambda_0)(J) + J = 0.$$

In particular $\rho(\lambda_0)^6(J) = J$ so that $J(E)$ reads:

$$J(E) = \sum_n a_n E^{n/6}$$

where $n \in \mathbb{N}$ apart from a finite set of negative values. One can be more precise: since the eigenvalues of $\mathcal{M}(\lambda_0)$ are $X_1 = e^{2i\pi/6}$, $\quad X_2 = e^{-2i\pi/6}$

one deduces that there exists $l_0 \in \mathbb{Z}$ such that

$$J(E) = \sum_{l \geq l_0} \alpha_l E^{\frac{5}{6}+l} + \sum_{l \geq l_0} \beta_l E^{\frac{7}{6}+l}$$

(cf. L. Narváez and M. Granger, this volume, see also Ref. 1). In fact $l_0 \geq 0$ as we shall see in a moment (cf. Theorem 3.6), and the result extends when ω is a holomorphic 1-form.

Second example We now consider the so-called Broughton's polynomial,[6] $f : (z_1, z_2) \in \mathbb{C}^2 \mapsto z_2(1 + z_1 z_2)$. This polynomial has no critical point, however its set of atypical values is $Atyp_f = \{0\}$: the level set $f^{-1}(0)$ is the disjoint union of the two connected components locus, $z_2 = 0$ which is isomorphic to $\mathbb{C}$ and $\{z_1 = -1/z_2,\, z_2 \in \mathbb{C}^\star\}$ which is isomorphic to $\mathbb{C}^\star$, while for $t \neq 0$ the fibre $f^{-1}(t)$ has only one connected component

$$\text{for } t \neq 0, \qquad f^{-1}(t) = \{z_1 = (t - z_2)/z_2^2,\, z_2 \in \mathbb{C}^\star\} \tag{42}$$

which is isomorphic to $\mathbb{C}^\star$.
Using (39) we can remark that $(z_1 : z_2 : z_0) = (1 : 0 : 0)$ is the sole candidate for being a singular point a infinity (see Ref. 30).
From (42) we see that $H_1(f^{-1}(t)) \cong \mathbb{Z}$ for $t \neq 0$. Indeed, $H_1(f^{-1}(t))$ can be generated for instance by the oriented cycle

$$\gamma(t) = \{z_1 = (t - z_2)/z_2^2,\, |z_2| = 1\} \tag{43}$$

As a consequence, the geometric monodromy operator $\mathcal{M}(\lambda_0)$ associated with the natural loop λ_0 which generates $\pi_1(\mathbb{C}^\star, 1)$ is the identity. Therefore, if ω is a polynomial 1-form, then $J(t) = \displaystyle\int_{\gamma(t)} \omega$ with $[\gamma(t)] \in H_1(f^{-1}(t))$ is a holomorphic univalued function on $\mathbb{C}^\star$ and by Proposition 3.3.1 and Theorem 3.4, there exists $n_0 \in \mathbb{Z}$ such that $J(t) = \displaystyle\sum_{n \geq n_0} a_n t^n$. In fact

$$J(t) = \sum_{n \geq 0} a_n t^n.$$

This result can be obtained by simple calculations by first observing that for $t \neq 0$ the mapping

$$\Psi_t : (z_1, z_2) \mapsto (t^{-1} z_1, t z_2)$$

defines a diffeomorphism from the level complex curve $f^{-1}(1)$ to the level complex curve $f^{-1}(t)$. This means that, starting with a 1-cycle $\gamma(t_0)$ on a generic fibre $f^{-1}(t_0)$, its horizontal deformation $\gamma(t)$ will extend toward infinity in the z_1 direction when $t \to 0$. Let us see how this geometric information translates for $J(t)$. One has

$$J(t) = \int_{\gamma(t)} \omega = \int_{\gamma(1)} \Psi_t^\star \omega.$$

In particular, using also (43),

$$\begin{aligned}
&\text{for } \omega = z_1^m z_2^n dz_1 \text{ then } \int_{\gamma(t)} \omega = t^{n-m-1} \int_{\gamma(1)} \omega = t^{n-m-1} \oint \frac{(1-z_2)^m (z_2-2)}{z_2^{2m-n+3}}\, dz_2 \\
&\text{for } \omega = z_1^m z_2^n dz_2 \text{ then } \int_{\gamma(t)} \omega = t^{n-m+1} \int_{\gamma(1)} \omega = t^{n-m+1} \oint \frac{(1-z_2)^m}{z_2^{2m-n}}\, dz_2,
\end{aligned} \tag{44}$$

thus the conclusion ($\oint \frac{(1-z_2)^m(z_2-2)}{z_2^{2m-n+3}}\, dz_2 = 0$ if $n - m - 1 < 0$ and $\oint \frac{(1-z_2)^m}{z_2^{2m-n}}\, dz_2 = 0$ if $n - m + 1 < 0$).

3.3. *Localisation near an isolated singularity*

We assume in this subsection that $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ represents a germ of holomorphic functions at 0, $f(0) = 0$. We assume furthermore that 0 is an isolated critical point for f.

The following theorem is due to Milnor,[22] see also Ref. 12 :

Theorem 3.5 (Milnor). *We note $B = B(0, \varepsilon) \subset \mathbb{C}^n$ the open ball and $D = D(0, \eta) \subset \mathbb{C}$ the open disc. We set*

$$X = B \cap f^{-1}(D), \qquad \overline{X} = \overline{B} \cap f^{-1}(D), \qquad X_t = B \cap f^{-1}(t).$$

Then for $\varepsilon > 0$ small enough and for and $0 < \eta = \eta(\varepsilon) \ll \varepsilon$ the restrictions $\begin{array}{l} \overline{X}\backslash f^{-1}(0) \\ f| \downarrow \\ D^\star \end{array}$ *and* $\begin{array}{l} X\backslash f^{-1}(0) \\ f| \downarrow \\ D^\star \end{array}$ *are locally $\mathcal{C}^\infty$ trivial fibrations. Also :*

(1) The Milnor fibration does not depend of (ε, η), in the sense that two Milnor fibrations given by two allowed pairs (ε_1, η_1) and (ε_2, η_2) are

equivalent : $X_{(\varepsilon_1,\eta_1)}$ *and* $X_{(\varepsilon_2,\eta_2)}$ *are diffeomorphic and the following diagram commutes:*

$$\begin{array}{ccc} X_{(\varepsilon_1,\eta_1)}\backslash f^{-1}(0) & \overset{\approx}{\longrightarrow} & X_{(\varepsilon_2,\eta_2)}\backslash f^{-1}(0) \\ f|\downarrow & & \downarrow f| \\ D^\star_{\eta_1} & \overset{\approx}{\longrightarrow} & D^\star_{\eta_2} \end{array}$$

(2) X *is contractible.*

(3) For $t \in D^\star$ *and* $H_{n-1}(X_t) \cong Z^\mu$ *for* $n \geq 2$, $H_0(X_t) \cong Z^{\mu+1}$ *for* $n = 1$, *where* $\mu = \dim_{\mathbb{C}} \dfrac{C\{z\}}{(\partial f)}$ *is the Milnor number at* 0.

In the conditions of the theorem, X is called a **Milnor ball** while the homology group $H_{n-1}(X_t)$ is called the **vanishing** homology group of the singularity $z = 0$.

Thanks to this theorem 3.5, what have been done in the previous sections can be repeated for

$$J(t) = \int_{\gamma(t)} \omega, \qquad [\gamma(t)] \in H_{n-1}(X_t)$$

where $\omega \in \Omega^{n-1}(X)$. More precisely one has the following properties, see Malgrange,[20] see also Refs. 1, 2 and 3.

Theorem 3.6. *If* $\omega \in \Omega^{n-1}(X)$ *and* $J(t) = \displaystyle\int_{\gamma(t)} \omega, \qquad [\gamma(t)] \in H_{n-1}(X_t)$, *then* $J(t)$ *viewed as a germ of holomophic function at* $t_0 \in D^\star$ *extends as a multivalued holomorphic function* $J \in \widetilde{\mathcal{O}_{D^\star,t_0}}$ *and* J *is of finite determination at* 0.
Also J *belongs to the Nilsson class at* 0. *More precisely, for any sectorial neighbourhood* $\Sigma(r,a,b)$ *of* 0,

$$\Sigma(r,a,b) = \{t \in D,\ 0 < |t| < r,\ a < \arg t < b\} \subset D$$

with $r > 0$ *small enough,* $b - a < 2\pi$, *there exists* $C > 0$ *such that*

$$\forall\, t \in \Sigma(r,a,b),\ \ |J(t)| \leq C.$$

Moreover

$$\lim_{t\to 0,\, t\in\Sigma(r,a,b)} J(t) = 0$$

and

$$J(t) = \sum_{\alpha\in\mathbb{Q}^{\star+},\, l\in[[0,n-1]]} \phi_{\alpha,l} t^\alpha \mathit{Log}^{\,l}(t), \qquad \phi_{\alpha,l} \in \mathbb{C}\{t\}.$$

4. On the Asymptotics of Laplace-Type Integrals

We would like to analyse the asymptotics when $|k| \to +\infty$ of Laplace-type integrals

$$I(k) = \int_\Gamma e^{-kf}\sigma$$

where $f \in \mathcal{O}(\mathbb{C}^n)$ and $\sigma \in \Omega^n(\mathbb{C}^n)$. What we have in mind is to generalise the steepest-descent method discussed in §2.2 on the Airy example.

4.1. *Allowed chains of integration*

In a similar way to what we have done in §2.2, we first have to define a space of allowed endless contours of integration running between valleys at infinity where $\Re(kf) \to +\infty$.

We assume that

$$\theta = \arg(k) \in \frac{\mathbb{R}}{2\pi\mathbb{Z}}$$

is fixed and we introduce the following half-planes: for any $r > 0$ we set

$$S_R^+ = S_R^+(\theta) = \{t \in \mathbb{C}, \Re(te^{i\theta}) \geq R\}, \qquad S_R^- = S_R^-(\theta) = \{t \in \mathbb{C}, \Re(te^{i\theta}) \leq R\} \tag{45}$$

For $f \in \mathcal{O}(\mathbb{C}^n)$ we introduce the family $\Psi = \Psi(\theta)$ of closed subsets $A \in \mathbb{C}^n$ defined as follows :

$$A \in \Psi \Leftrightarrow \forall\, R > 0,\ A \cap f^{-1}(S_R^-) \text{ is compact.} \tag{46}$$

This family Ψ obviously satisfies the properties (A.1) of §A.5. This means that Ψ is a family of supports in $\mathbb{C}^n$ in the sense of homology theory which allows to define the chain-complex $(C_\bullet^\Psi(\mathbb{C}^n), \partial_\bullet)$ of $\mathbb{Z}$-modules and its associated homology groups

$$H_q^\Psi(\mathbb{C}^n) = \frac{Z_q^\Psi(\mathbb{C}^n)}{B_q^\Psi(\mathbb{C}^n)}.$$

Since for any $c = \sum_i n_i\sigma_i \in C_n^\Psi(\mathbb{C}^n)$ one has $[c] \in \Psi$ one deduces from (46) that

$$\Re(kf(z)) \to +\infty \ \text{ when } \ \|z\| \to +\infty,\ z \in [c].$$

In particular $|e^{-kf(z)}| = e^{-\Re(kf(z))}$ is exponentially decreasing when $\|z\| \to +\infty$, $z \in [c]$. Nevertheless, to make the integral

$$\int_c e^{-kf}\sigma = \sum_i n_i \int_{\sigma_i} e^{-kf}\sigma = \sum_i n_i \int_{\Delta^n} e^{-kf(\sigma_i)}\sigma_i^\star(\sigma)$$

absolutly convergent for some $c = \sum_i n_i\sigma_i \in C_n^{\Psi}(\mathbb{C}^n)$, one needs also to control the growth of the differential forms $\sigma_i^{\star}(\sigma)$. This why we restrict ourself to the polynomial case in the following theorem, see Pham[26,27] :

Theorem 4.1. *Assume that $f \in \mathbb{C}[z]$ is a polynomial function on $\mathbb{C}^n$ and that σ is a polynomial differential n-form on $\mathbb{C}^n$, $\sigma = g dz_1 \wedge \cdots \wedge dz_n$, $g \in \mathbb{C}[z]$. Then*

(1) The integral $\int_{\mathbf{\Gamma}} e^{-kf}\sigma$ along a n-cycle $\mathbf{\Gamma} \in Z_n^{\Psi}(\mathbb{C}^n)$ is well defined for $|k| > 0$, $\arg(k) = \theta$ and depends only on the homology class $[\mathbf{\Gamma}] \in H_n^{\Psi}(\mathbb{C}^n)$.

(2) One has the following isomorphism

$$H_q^{\Psi}(\mathbb{C}^n) \cong \varprojlim_{R \geq R_0} H_q(\mathbb{C}^n, f^{-1}(S_R^+)), \qquad \forall q$$

where the inverse limit is considered for $R \geq R_0$ large enough

We say more about the item 2. of the Theorem 4.1. When f is a polynomial function we know from Theorem 3.3 that there exists a finite set $Atyp_f \subset \mathbb{C}$ such that the restriction map $f| \downarrow \begin{array}{c}\mathbb{C}^n \backslash f^{-1}(Atyp_f) \\ \\ \mathbb{C}\backslash Atyp_f\end{array}$ is a locally $\mathcal{C}^\infty$ trivial fibration. In particular, if $R_0 > \max_{t_i \in Atyp_f} |t_i|$ then for any $R' \geq R > R_0$, the pairs $(\mathbb{C}^n, f^{-1}(S_{R'}^+))$ and $(\mathbb{C}^n, f^{-1}(S_R^+))$ are homotopy equivalent so that

$$\forall\, R' \geq R > R_0, \ \forall q, \qquad H_q(\mathbb{C}^n, f^{-1}(S_R^+)) \cong H_q(\mathbb{C}^n, f^{-1}(S_{R'}^+)).$$

We detail that point. Because $S_{R_0}^+$ is contractible, there exists a $\mathcal{C}^\infty$-diffeomorphism Φ such that the following diagram commutes (for a chosen $t_0 \in S_{R_0}^+$):

$$\begin{array}{ccc} f^{-1}(S_{R_0}^+) & \xrightarrow{\Phi} & S_{R_0}^+ \times f^{-1}(t_0) \\ f| \searrow & & \swarrow pr_1 \\ & S_{R_0}^+ & \end{array} \tag{47}$$

Now for $R' \geq R > R_0$ it is easy to defined a homeomorphism h such that

$$h : (\mathbb{C}, S_R^+) \to (\mathbb{C}, S_{R'}^+), \qquad h|_{\mathbb{C}\backslash S_{R_0}^+} = id.$$

(for instance if $te^{i\theta} = a + ib$, define

$$h : (a,b) \in \mathbb{R}^2 \mapsto (h_1(a), b), \qquad h_1(a) = \begin{cases} a \text{ if } a \leq R_0 \\ R_0 + \dfrac{R' - R_0}{R - R_0}(a - R_0) \text{ if } a \geq R_0 \end{cases}$$

This being done we lift this homeomorphism by the fibration f, setting:

$$\begin{cases} H : (\mathbb{C}^n, f^{-1}(S_R^+)) \to (\mathbb{C}^n, f^{-1}(S_{R'}^+)) \\ H = id \ \text{ on restriction to } \ \mathbb{C}^n \backslash f^{-1}(S_{R_0}^+) \\ H = \Phi^{-1} \circ (h \times id) \circ \Phi \ \text{ elsewhere} \end{cases}$$

$$\begin{array}{ccccccc} (f^{-1}(S_{R_0}^+), f^{-1}(S_R^+)) & \xrightarrow{\Phi} & (S_{R_0}^+, S_R^+) \times f^{-1}(t_0) & \xrightarrow{h \times id} & (S_{R_0}^+, S_{R'}^+) \times f^{-1}(t_0) & \xrightarrow{\Phi^{-1}} & (f^{-1}(S_{R_0}^+), f^{-1}(S_{R'}^+)) \\ & f| \searrow \ \swarrow pr_1 & & & & pr_1| \searrow \ \swarrow f| & \\ & (S_{R_0}^+, S_R^+) & & \xrightarrow{h} & & (S_{R_0}^+, S_{R'}^+) & \end{array}$$

By its very construction H is a homeomorphism.

Instead of working with the integrals $\displaystyle\int_{\mathbf{\Gamma}} e^{-kf}\sigma$ with $[\mathbf{\Gamma}] \in H_n^{\Psi}(\mathbb{C}^n)$, Theorem 4.1 allows us to consider rather the integrals

$$I(k) = \int_{\Gamma} e^{-kf}\sigma, \qquad [\Gamma] \in H_n(\mathbb{C}^n, f^{-1}(S_R^+))) \tag{48}$$

for some $R \geq R_0$. This means working with a class of functions defined modulo some exponentially decreasing functions when $|k| \to +\infty$, $\arg k = \theta$. Indeed, assume that Γ, $\Gamma' \in Z_n(\mathbb{C}^n, f^{-1}(S_R^+)))$ define the same class in $H_n(\mathbb{C}^n, f^{-1}(S_R^+))$,

$$\Gamma - \Gamma' = c + \partial b, \qquad c \in C_n(f^{-1}(S_R^+)), \quad b \in C_{n+1}(\mathbb{C}^n).$$

We remark that $\displaystyle\int_{\partial b} e^{-kf}\sigma = \int_b d(e^{-kf}\sigma) = 0$ because $e^{-kf}\sigma \in \Omega^n(\mathbb{C}^n)$ is closed. Moreover, since $c \in C_n(f^{-1}(S_R^+))$ one has, for some fixed $r > 0$:

$$\exists C = C(r) > 0, \qquad \left| \int_c e^{-kf}\sigma \right| \leq Ce^{-R|k|}, \qquad |k| \geq r, \ \arg(k) = \theta.$$

Also, Theorem 4.1 implies that a n-cycle $\mathbf{\Gamma} \in Z_n^{\Psi}(\mathbb{C}^n)$ can be represented by a relative n-cycle $\Gamma \in Z_n(\mathbb{C}^n, f^{-1}(S_R^+)))$ and, thanks to the convergence :

$$\exists C > 0, \qquad \left| \int_{\mathbf{\Gamma}} e^{-kf}\sigma - \int_{\Gamma} e^{-kf}\sigma \right| \leq Ce^{-R|k|}, \qquad |k| \geq r, \ \arg(k) = \theta.$$

Consequently :

Corollary 4.1.1. We assume that $f \in \mathbb{C}[z]$ and that $\sigma = g dz_1 \wedge \cdots \wedge dz_n$, $g \in \mathbb{C}[z]$. We fix some $R > R_0$ and $r > 0$. Then for $|k| \geq r$, $\arg(k) = \theta$ the space of functions $\mathbf{I}(k) = \displaystyle\int_{\mathbf{\Gamma}} e^{-kf}\sigma$, $[\mathbf{\Gamma}] \in H_n^{\Psi}(\mathbb{C}^n)$ can be identified with the space of functions $I(k) = \displaystyle\int_{\Gamma} e^{-kf}\sigma$, $[\Gamma] \in H_n(\mathbb{C}^n, f^{-1}(S_R^+))$ modulo the exponentially decreasing functions of type R in $|k|$.

Remark : By the long exact homology sequence of the pair $(\mathbb{C}^n, f^{-1}(S_R^+))$ (cf. Proposition A.0.3), one easily obtains that (for $n \geq 2$)

$$H_n(\mathbb{C}^n, f^{-1}(S_R^+)) \cong H_{n-1}(f^{-1}(S_R^+))$$

because $\mathbb{C}^n$ is contractible. Then using (47) one deduces that

$$H_n(\mathbb{C}^n, f^{-1}(S_R^+)) \cong H_{n-1}(f^{-1}(t)), \qquad t \notin Atyp_f. \tag{49}$$

4.2. *The steepest-descent method*

We still assume that $f \in \mathbb{C}[z]$ so that the restriction map

$$\begin{array}{l} \mathbb{C}^n \setminus f^{-1}(Atyp_f) \\ f| \downarrow \\ \mathbb{C} \setminus Atyp_f \end{array}$$

is a locally $\mathcal{C}^\infty$ trivial fibration, where $Atyp_f = \{t_1, \cdots, t_N\} \subset \mathbb{C}$ is the finite set of atypical values. Our aim here is to analyse the homology group $H_n(\mathbb{C}^n, f^{-1}(S_R^+)))$, having in mind to extend the steepest-descent method discussed in §2.2 for the Airy example.
Following ideas developed in Ref. 27, in the t-plane we draw the family $(L_i)_{1 \leq i \leq N}$ of closed half-lines $L_i = t_i + e^{-i\theta}\mathbb{R}^+$ for all $t_i \in Atyp_f$. We assume also that θ has been chosen generically so that no Stokes phenomenon is currently occuring, i.e. all these half-lines are two-by-two disjoint. To every $t_i \in Atyp_f$ we associate a closed neighbourhood T_i of L_i, retractable by deformation onto L_i. It will be assumed that all these T_i are disjoint from one another, as shown in Fig. 16.

We construct a deformation-retraction of $dr : (\mathbb{C}, S_R^+) \to (S_R^+ \bigsqcup_i T_i, S_R^+)$: this means a continous map of pairs such that $dr|_{S_R^+ \bigsqcup_i T_i} = id$ while $i \circ dr$ is homotopic to the identity with $i : (S_R^+ \bigsqcup_i T_i, S_R^+) \hookrightarrow (\mathbb{C}, S_R^+)$ the canonical injection. Here we add furthermore the condition that $dr(\mathbb{C} \setminus \bigsqcup_i T_i) = S_R^+ \setminus \bigsqcup_i T_i$.
Since $\mathbb{C} \setminus \bigsqcup_i T_i$ is contractible, one easily lift dr by the trivialisation

$$\begin{array}{c} f^{-1}(\mathbb{C} \setminus \sqcup_i T_i) \xrightarrow{\Phi} \mathbb{C} \setminus \sqcup_i T_i \times f^{-1}(t_0) \\ f| \searrow \quad \swarrow pr_1 \\ \mathbb{C} \setminus \sqcup_i T_i \end{array}$$

(for a chosen $t_0 \in S_R^+$), thus ob-

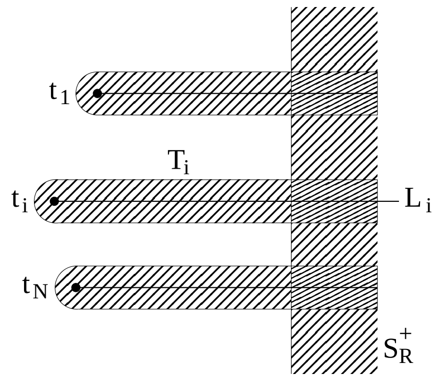


Fig. 16. The family of half-lines L_i and their closed neighbourhoods T_i for $\theta = 0$.

taining a deformation-retraction $Dr : (\mathbb{C}^n, f^{-1}(S_R^+)) \to (\bigcup_i f^{-1}(S_R^+ \cup T_i), f^{-1}(S_R^+))$, by setting:

$$\begin{cases} H = id \quad \text{on restriction to} \quad \bigsqcup\limits_i f^{-1}(T_i) \\ H = \Phi^{-1} \circ (dr \times id) \circ \Phi \quad \text{elsewhere} \end{cases}$$

$$\begin{array}{ccccccc} f^{-1}(\mathbb{C}\backslash \bigsqcup_i T_i) \overset{\Phi}{\to} \mathbb{C}\backslash \bigsqcup_i T_i \times f^{-1}(t_0) & \overset{dr\times id}{\to} & S_R^+\backslash \bigsqcup_i T_i \times f^{-1}(t_0) \overset{\Phi^{-1}}{\to} f^{-1}(S_R^+\backslash \bigsqcup_i T_i) \\ f| \searrow \ \swarrow pr_1 & & pr_1| \searrow \ \swarrow f| \\ \mathbb{C}\backslash \bigsqcup_i T_i & \overset{dr}{\to} & S_R^+\backslash \bigsqcup_i T_i \end{array}$$

This gives the isomorphism (by Proposition A.0.4):

$$H_n(\mathbb{C}^n, f^{-1}(S_R^+)) = H_n\Big(\bigcup_i f^{-1}(S_R^+ \cup T_i), f^{-1}(S_R^+)\Big)$$

Next, defining $A_R = \text{int}S_R^+\backslash \bigcup_i(\text{int}S_R^+ \cap T_i)$ where $\text{int}S_R^+$ is the interior of S_R^+, we note that $(S_R^+ \bigcup_i T_i\backslash A_R, S_R^+\backslash A_R)$ is a deformation retract of $(S_R^+ \bigcup_i T_i, S_R^+)$, Fig. 17.
Now by excision (Corollary A.1.1) the pair $(S_R^+ \bigsqcup_i T_i\backslash A_R, S_R^+\backslash A_R)$ is homotopic to $(\bigsqcup_i T_i, \bigsqcup_i(S_R^+ \cap T_i))$. Lifting this information through the fibration f as previously done, one gets the isomorphism

$$H_n(\mathbb{C}^n, f^{-1}(S_R^+)) = H_n\Big(\bigsqcup_i f^{-1}(T_i), \bigsqcup_i f^{-1}(T_i \cap S_R^+)\Big)$$

Applying the relative Mayer-Vietoris exact homology sequence (Theorem A.2) one deduces that

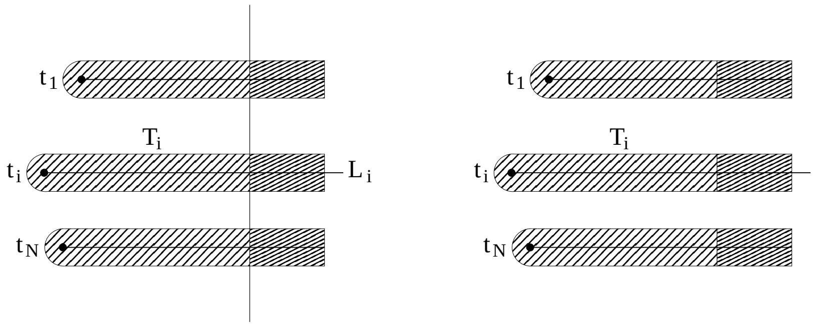


Fig. 17. On the left the pair $(S_R^+ \bigcup_i T_i \backslash A_R, S_R^+ \backslash A_R)$, on the right the pair $(\bigsqcup_i T_i, \bigsqcup_i (S_R^+ \cap T_i))$.

$$H_n(\mathbb{C}^n, f^{-1}(S_R^+)) = \bigoplus_{t_i \in Atyp_f} H_n\Big(f^{-1}(T_i), f^{-1}(T_i \cap S_R^+)\Big) \tag{50}$$

What we have obtained can be formulated as follows (compare to §2.2.3):

Proposition 4.1.1. Any relative n-cycle $\Gamma \in Z_n(\mathbb{C}^n, f^{-1}(S_R^+))$ can be represented
in its homology class $[\Gamma] \in H_n(\mathbb{C}^n, f^{-1}(S_R^+)))$ by a sum $\Gamma = \sum_i \Gamma_i$ with $\Gamma_i \in Z_n(f^{-1}(T_i), f^{-1}(T_i \cap S_R^+))$.

4.3. *Localisation*

For each $t_i \in Atyp_f$, let D_i be an open disc centred at t_i with a small radius η_i and $D_i^\tau = D_i \cap \{t, \Re((t-t_i)e^{i\theta}) \geq \tau\}$, $0 < \tau < \eta_i$. Then by a deformation-retraction whose definition is left to the reader we get

$$H_n(\mathbb{C}^n, f^{-1}(S_R^+)) = \bigoplus_{t_i \in Atyp_f} H_n\Big(f^{-1}(D_i), f^{-1}(D_i^\tau)\Big) \tag{51}$$

At this stage, we have localised our problem at "the target". What we would like to do is to localised the analysis at the source, like what we have done in §2.2.4. For that we have to make some further assumptions on f:
Hypothesis : We assume that the (non-constant) polynomial function $f : \mathbb{C}^n \to \mathbb{C}$ $(n \geq 2)$ has only isolated critical points and no singular point at infinity. In particular $Sing_f$ is a finite subset of $\mathbb{C}^n$ and we have the following *topological triviality property at infinity* :[30] for any $t_0 \in \mathbb{C}$ and for any $R = R(t_0) > 0$ large enough, there exists $\eta(R) > 0$ such that for any $0 < \eta \leq \eta(R)$ the fibres $f^{-1}(t)$, $t \in D_\eta$ cut $\partial\overline{B_R}$ transversally where

B_R is the open ball of radius R centred at 0, and the restriction maps $\begin{array}{l}(\mathbb{C}^n\backslash B_R)\cap f^{-1}(D_\eta)\\ f|\downarrow \\ D_\eta\end{array}$ and $\begin{array}{l}\partial\overline{B_R}\cap f^{-1}(D_\eta)\\ f|\downarrow \\ D_\eta\end{array}$ are $\mathcal{C}^\infty$ trivial fibrations.

For each $t_i \in Atyp_f$ we note $\{z_{ij}\}$ the subset of $Sing_f$ above t_i ($f(z_{ij}) = t_i$). We introduce also

$$X_{ij} = B_{ij} \cap f^{-1}(D_i), \qquad X_{ij}^\tau = X_{ij} \cap f^{-1}(D_i^\tau) \tag{52}$$

where $B_{ij} \subset \mathbb{C}^n$ is an open ball centred at z_{ij} with radius ε small enough so that the B_{ij} are disjoints. Assuming that η_i has been chosen small enough, one can assume that the X_{ij} are Milnor balls (cf. Theorem 3.5).
For $R > R' > R(t_i)$ we note

$$Y_i = f^{-1}(D_i) \cap B_R, \qquad Y_i^\tau = f^{-1}(D_i^\tau) \cap B_R$$

and

$$Z_i = f^{-1}(D_i) \cap (\mathbb{C}^n\backslash B_{R'}), \qquad Z_i^\tau = f^{-1}(D_i^\tau) \cap (\mathbb{C}^n\backslash B_{R'}).$$

The couple $\{(Y_i, Y_i^\tau), (Z_i, Z_i^\tau)\}$ of pairs is an excisive couple of pairs (apply Theorem A.1) so that the relative Mayer-Vietoris exact homology sequence (Theorem A.2) can be applied:

$$\begin{aligned}\cdots \to H_n(Y_i \cap Z_i, Y_i^\tau \cap Z_i^\tau) &\xrightarrow{i_\star} H_n(Y_i, Y_i^\tau) \oplus H_n(Z_i, Z_i^\tau)\\ &\xrightarrow{j_\star} H_q(f^{-1}(D_i), f^{-1}(D_i^\tau))\\ &\xrightarrow{\partial} H_{n-1}(Y_i \cap Z_i, Y_i^\tau \cap Z_i^\tau) \to \cdots\end{aligned}$$

The topological triviality property at infinity implies that $H_n(Z_i, Z_i^\tau) = H_n(Y_i \cap Z_i, Y_i^\tau \cap Z_i^\tau) = H_{n-1}(Y_i \cap Z_i, Y_i^\tau \cap Z_i^\tau) = 0$ so that

$$H_n\Big(f^{-1}(D_i), f^{-1}(D_i^\tau)\Big) = H_n(Y_i, Y_i^\tau)$$

and we thus concentrate on $H_n(Y_i, Y_i^\tau)$. We follow the reasoning of Pham,[27] see also Broughton[6,7] : by the exact homology sequence of a triple (Corollary A.2.1) one obtains ($\forall\, q$):

$$\begin{array}{c}\cdots \to H_n(\bigsqcup_j X_{ij} \cup Y_i^\tau,\, Y_i^\tau) \xrightarrow{i_\star} H_n(Y_i,\, Y_i^\tau) \longrightarrow \\ H_n(Y_i,\, \bigsqcup_j X_{ij} \cup Y_i^\tau) \xrightarrow{\partial} H_{n-1}(\bigsqcup_j X_{ij} \cup Y_i^\tau,\, Y_i^\tau) \to \cdots\end{array} \tag{53}$$

and by excision (by Corollary A.1.1) then by the relative Mayer-Vietoris exact homology sequence one has

$$H_q(\bigsqcup_j X_{ij} \cup Y_i^\tau,\, Y_i^\tau) = H_q(\bigsqcup_j X_{ij},\, \bigsqcup_j X_{ij}^\tau) = \bigoplus_j H_q(X_{ij},\, X_{ij}^\tau) \tag{54}$$

Therefore (53) means that

$$\text{if} \qquad H_q(Y_i,\, \mathbb{X}_i \cup Y_i^\tau) = 0 \qquad \text{with} \qquad \mathbb{X}_i = \bigsqcup_j X_{ij} \tag{55}$$

then

$$H_q(Y_i,\, Y_i^\tau) = \bigoplus_j H_q(X_{ij},\, X_{ij}^\tau).$$

In the pair $(Y_i,\, \mathbb{X}_i \cup Y_i^\tau)$ we excise $\mathbb{X}_i \backslash \partial\mathbb{X}_i$ where $\partial\mathbb{X}_i$ is the boundary of $\mathbb{X}_i$ in Y_i:

$$H_q(Y_i,\, \mathbb{X}_i \cup Y_i^\tau) = H_q(Y_i \backslash \mathbb{X}_i,\, \partial\mathbb{X}_i \cup (Y_i^\tau \backslash \mathbb{X}_i^\tau)) \qquad \text{with} \qquad \mathbb{X}_i^\tau = \bigsqcup_j X_{ij}^\tau. \tag{56}$$

By the exact homology sequence of a triple one has

$$\begin{aligned} \cdots \;\to\; & H_q(\partial\mathbb{X}_i \cup (Y_i^\tau \backslash \mathbb{X}_i^\tau), \partial\mathbb{X}_i) \xrightarrow{i_\star} H_q(Y_i \backslash \mathbb{X}_i, \partial\mathbb{X}_i) \\ \to\; & H_q(Y_i \backslash \mathbb{X}_i, \partial\mathbb{X}_i \cup (Y_i^\tau \backslash \mathbb{X}_i^\tau)) \\ \xrightarrow{\partial}\; & H_{q-1}(\partial\mathbb{X}_i \cup (Y_i^\tau \backslash \mathbb{X}_i^\tau), \partial\mathbb{X}_i) \to \cdots \end{aligned} \tag{57}$$

so that by (56) condition (55) reduces in the isomorphism

$$H_q(Y_i \backslash \mathbb{X}_i, \partial\mathbb{X}_i) = H_q(\partial\mathbb{X}_i \cup (Y_i^\tau \backslash \mathbb{X}_i^\tau), \partial\mathbb{X}_i). \tag{58}$$

But by excision and denoting by $\partial\mathbb{X}_i^\tau$ the boundary of $\mathbb{X}_i^\tau$ in Y_i^τ, (58) reads

$$H_q(Y_i \backslash \mathbb{X}_i, \partial\mathbb{X}_i) = H_q(Y_i^\tau \backslash \mathbb{X}_i^\tau, \partial\mathbb{X}_i^\tau) \tag{59}$$

and this last equality is true by a deformation-retraction argument, because $\begin{array}{c} Y_i \\ f| \downarrow \\ D_i \end{array}$ is a locally trivial fibration (by the Ehresmann fibration theorem 3.1, since Y_i is a manifold), thus a trivial fibration because D_i is contractible.

What we have obtained is in particular the following proposition.

Proposition 4.1.2. Assume that the (non-constant) polynomial function $f : \mathbb{C}^n \to \mathbb{C}$ $(n \geq 2)$ has only isolated critical points and no singular point at infinity (in the sense of the above hypothesis). Then

$$H_n(\mathbb{C}^n, f^{-1}(S_R^+)) = \bigoplus_{t_i \in Atyp_f} \bigoplus_{z_{ij} \in Sing_f} H_n(X_{ij}, X_{ij}^\tau) \tag{60}$$

where X_{ij} is the Milnor ball associated with the critical point $z_{ij} \in Sing_f$, $f(z_{ij}) = t_i$.

Remark : To see what happens in a more general case, see Tibăr.[30]

4.4. *The asymptotics*

To simplify we assume here that $\arg(k) = \theta = 0$. From what precedes and under some convenient hypotheses, we have reduced the analysis of our integral $\mathbf{I}(k) = \displaystyle\int_{\mathbf{\Gamma}} e^{-kf}\sigma$, $[\mathbf{\Gamma}] \in H_n^{\Psi}(\mathbb{C}^n)$ into the analysis of the class of functions

$$I(k) = \int_{\Gamma} e^{-kf}\sigma, \qquad [\Gamma] \in H_n(X, X^{\tau}). \tag{61}$$

Up to a translation at the source and at the target, we can now assume that $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ represents a germ of holomorphic functions at 0 which is an isolated critical point and

$$X = B \cap f^{-1}(D), \qquad X^{\tau} = B \cap f^{-1}(D^{\tau})$$

with

$$B = B(0,\varepsilon) \subset \mathbb{C}^n, \qquad D = D(0,\eta) \subset \mathbb{C}\,,$$
$$D^{\tau} = D \cap \{t,\, \Re(t) \geq \tau\}, 0 < \tau < \eta$$

such that X is a Milnor ball (see Theorem 3.5). For later purpose it will be convenient to note

$$D^{+} = D \cap \{t,\, \Re(t) > 0\}, \qquad X^{+} = B \cap f^{-1}(D^{+}).$$

Since we have localised the problem, one can also assume that $\sigma \in \Omega^n(X)$.

We mention that if $\Gamma, \Gamma' \in Z_n(X, X^{\tau})$ define the same class in $H_n(X, X^{\tau})$, then for some fixed $r > 0$:

$$\exists C = C(r) > 0, \qquad \left| \int_{\Gamma} e^{-kf}\sigma - \int_{\Gamma'} e^{-kf}\sigma \right| \leq C e^{-\tau|k|}\,,$$
$$|k| \geq r,\ \arg(k) = 0$$

(see the reasoning in §4.1) so that $\int_{\Gamma} e^{-kf}\sigma$ and $\int_{\Gamma'} e^{-kf}\sigma$ have the same Poincaré asymptotics when $k \to +\infty$.

4.4.1. *Reduction to an incomplete Laplace transform*

We know from Theorem 3.5 that X is contractible. This imply that $H_n(X) = H_{n-1}(X) = 0$ (we assume that $n \geq 2$) and from the exact homology sequence of the pair (X, X^{τ}) (Proposition A.0.3) one has

$$0 = H_n(X) \xrightarrow{j_\star} H_n(X, X^{\tau}) \xrightarrow{\partial_\star} H_{n-1}(X^{\tau}) \xrightarrow{i_\star} H_{n-1}(X) = 0.$$

Also, since D^+ is contractible, the restriction map $\begin{array}{c} X^+ \\ f| \downarrow \\ D^+ \end{array}$ is a trivial fibration. This induces an isomorphism (because $D^\tau \subset D^+$ is also contractible)

$$H_{n-1}(X^\tau) = H_{n-1}(D^\tau \times X_{t_0}) = H_{n-1}(X_{t_0})$$

for any chosen $t_0 \in D^\tau$. Putting things together, we have obtained an isomorphism

$$H_n(X, X^\tau) \xrightarrow{\partial_{t_0}} H_{n-1}(X_{t_0}), \qquad t_0 \in D^\tau.$$

Consequence: in the class of a given element $[\Gamma] \in H_n(X, X^\tau)$ one can choose a chain $\Gamma_{t_0} \in C_n(X)$ whose boundary $\partial\Gamma_{t_0} = \gamma(t_0)$ belongs to $Z_{n-1}(X_{t_0})$, with $t_0 \in D^\tau$.

In what follows we fix a $t_0 > 0$ in D^τ. By the trivial fibration $\begin{array}{c} X^+ \\ f| \downarrow \\ D^+ \end{array}$ the cycle $\gamma(t_0)$ can be horizontally deformed into $\gamma(t) \in Z_{n-1}(X_t)$, $t \in D^+$. Then :

Theorem 4.2. *With the above notations and hypotheses,*

$$I(k, t_0) = \int_{\Gamma_{t_0}} e^{-kf}\sigma = \int_0^{t_0} e^{-kt} J(t)\, dt$$

with

$$J(t) = \int_{\gamma(t)} \frac{\sigma}{df}|_{f=t} = \sum_{\alpha \in \mathbb{Q}^{\star +},\, l \in [[0, n-1]]} \psi_{\alpha,l} t^{\alpha-1} Log^{\,l}(t), \qquad \psi_{\alpha,l} \in \mathbb{C}\{t\}. \tag{62}$$

Proof. We follow Malgrange.[20] The holomorphic differential form $e^{-kf}\sigma \in \Omega^n(X)$ (for a fixed k) is closed (being of maximal order) and X being a Stein contractible manifold (Theorem 3.5 and Proposition B.1.1), one deduces from Corollary B.3.1 the existence of $\varphi \in \Omega^{n-1}(X)$ such that

$$e^{-kf}\sigma = d\varphi.$$

Thus by the Stokes theorem

$$I(k, t_0) = \int_{\Gamma_{t_0}} e^{-kf}\sigma = \int_{\Gamma_{t_0}} d\varphi = \int_{\gamma(t_0)} \varphi. \tag{63}$$

We note

$$\text{for } t \in D^+, \qquad I(k, t) = \int_{\gamma(t)} \varphi. \tag{64}$$

Note that $I(k,t)$ extends as a multivalued holomorphic function $I(k,t) \in \widetilde{\mathcal{O}_{D^\star,t_0}}$ in t for which Theorem 3.6 can be applied:

$$I(k,t) = \sum_{\alpha\in\mathbb{Q}^+,\, l\in[[0,n-1]]} \phi_{\alpha,l} t^\alpha \mathrm{Log}^{\,l}(t), \qquad \phi_{\alpha,l} \in \mathbb{C}\{t\}. \tag{65}$$

By Lemma 3.2.1 we know that

$$\text{for } t \in D^+, \qquad I(k,t) = \frac{1}{2i\pi} \int_{\delta_t \gamma(t)} \frac{df \wedge \varphi}{f-t}$$

where $\delta_t : H_{n-1}(X_t) \to H_n(X \backslash X_t)$ is the Leray coboundary operator. Therefore using the remark that $d\left(\dfrac{\varphi}{f-t}\right) = \dfrac{d\varphi}{f-t} - \dfrac{df\wedge\varphi}{(f-t)^2}$,

$$\begin{aligned}\text{for } t \in D^+, \qquad \frac{\partial I}{\partial t}(k,t) &= \frac{1}{2i\pi} \int_{\delta_t\gamma(t)} \frac{df\wedge\varphi}{(f-t)^2} \\ &= \frac{1}{2i\pi}\int_{\delta_t\gamma(t)} \frac{d\varphi}{f-t} = \frac{1}{2i\pi}\int_{\delta_t\gamma(t)} e^{-kf}\frac{\sigma}{f-t}.\end{aligned}$$

Introducing $X^\star = X\backslash\{0\}$ we note that the restriction $f|_{X^\star}$ is a submersion. Also, $\sigma \in \Omega^n(X)$ being of maximal order, it can be written locally as $\sigma = df \wedge \beta$. But $X^\star$ is a Stein manifold (by Proposition B.1.2) so that Proposition B.3.1 implies that

$$\exists\, \frac{\sigma}{df} \in \Omega_X^{n-1}(X^\star), \qquad \sigma = df \wedge \frac{\sigma}{df}.$$

Therefore, by Fubini and the Cauchy formula,

$$\text{for } t \in D^+, \qquad \frac{\partial I}{\partial t}(k,t) = e^{-kt}\int_{\gamma(t)} \frac{\sigma}{df}|_{f=t}.$$

From (65) one finally gets that

$$I(k,t_0) = \int_0^{t_0} e^{-kt}\left(\int_{\gamma(t)} \frac{\sigma}{df}|_{f=t}\right) dt$$

with

$$\int_{\gamma(t)} \frac{\sigma}{df}|_{f=t} = \sum_{\alpha\in\mathbb{Q}^{\star+},\, l\in[[0,n-1]]} \psi_{\alpha,l} t^{\alpha-1}\mathrm{Log}^{\,l}(t), \qquad \psi_{\alpha,l}\in\mathbb{C}\{t\}. \qquad \square$$

4.4.2. *The asymptotics*

From Theorem 4.2 we know that

$$I(k,t_0)=\int_{\Gamma_{t_0}} e^{-kf}\sigma=\int_0^{t_0} e^{-kt}J(t)$$

with

$$J(t)=\sum_{\alpha\in\mathbb{Q}^{\star+},\, l\in[[0,n-1]]}\psi_{\alpha,l}t^{\alpha-1}\mathrm{Log}^{\,l}(t),\qquad \psi_{\alpha,l}\in\mathbb{C}\{t\}.$$

Since this is a finite sum, to analyse the asymptotics it is enough to consider the case when

$$J(t)=\sum_{n=0}^{\infty}b_n t^{\alpha-1+n}\mathrm{Log}^{\,l}(t)$$

for some $\alpha\in\mathbb{Q}^{\star+}$ and $l\in[[0,n-1]]$ and we assume that the series expansion converges for $|t|<\tau$, $\tau>t_0$. We also assume that Log (t) is the principal determination of the logarithm function (real for $t>0$).
We adapt what we have done in §2.2.8. For k in the sectorial neighbourdhood of infinity $\Sigma_{r,\delta}$ as defined in (7) one has, for any $N\in\mathbb{N}$:

$$I(k,t_0)=\int_0^{t_0} e^{-kt}\left(\sum_{n=0}^{\infty}b_n t^{\alpha-1+n}\mathrm{Log}^{\,l}(t)\right)dt$$

$$\begin{aligned}I(k,t_0)=&\int_0^{+\infty} e^{-kt}\sum_{n=0}^{N}b_n t^{\alpha-1+n}\mathrm{Log}^{\,l}(t)\,dt\\&-\int_{t_0}^{+\infty} e^{-kt}\sum_{n=0}^{N}b_n t^{\alpha-1+n}\mathrm{Log}^{\,l}(t)\,dt\\&+\int_0^{t_0} e^{-kt}\sum_{n=N+1}^{\infty}b_n t^{\alpha-1+n}\mathrm{Log}^{\,l}(t)\,dt\end{aligned}$$

By (5) this means that

$$I(k,t_0)-\sum_{n=0}^{N}b_n\left(\frac{d}{d\alpha}\right)^l\left(\frac{\Gamma(\alpha+n)}{k^{\alpha+n}}\right)=$$
$$\sum_{n=N+1}^{\infty}b_n\int_0^{t_0}e^{-kt}t^{\alpha-1+n}\mathrm{Log}^{\,l}(t)\,dt-\sum_{n=0}^{N}b_n\int_{t_0}^{+\infty}e^{-kt}t^{\alpha-1+n}\mathrm{Log}^{\,l}(t)\,dt.$$

Therefore, making the change of variable $t = t_0 s$,

$$
\begin{aligned}
&I(k,t_0) - \sum_{n=0}^{N} b_n \left(\frac{d}{d\alpha}\right)^l \left(\frac{\Gamma(\alpha+n)}{k^{\alpha+n}}\right) \\
&\quad = \sum_{p=0}^{l} \binom{l}{p} \mathrm{Log}\,^p(t_0) \Bigg(\sum_{n=N+1}^{\infty} b_n {t_0}^{\alpha+n} \int_0^1 e^{-kt_0 s} s^{\alpha-1+n} \mathrm{Log}\,^{l-p}(s)\, ds \\
&\qquad - \sum_{n=0}^{N} b_n {t_0}^{\alpha+n} \int_1^{+\infty} e^{-kt_0 s} s^{\alpha-1+n} \mathrm{Log}\,^{l-p}(s)\, ds. \Bigg)
\end{aligned}
$$

We now introduce $0 < \epsilon \leq \frac{1}{2}$. In the $\displaystyle\int_0^1$ integral we have $s^{\alpha-1-\epsilon+n} \leq s^{\alpha-\epsilon+N}$ while in the $\displaystyle\int_1^{+\infty}$ integral we have $s^{\alpha-1+\epsilon+n} \leq s^{\alpha-\epsilon+N}$. Then, there exists $A_\epsilon > 0$ such that for $k \in \Sigma_{r,\delta}$,

$$
\begin{aligned}
&\left| I(k,t_0) - \sum_{n=0}^{N} b_n \left(\frac{d}{d\alpha}\right)^l \left(\frac{\Gamma(\alpha+n)}{k^{\alpha+n}}\right) \right| \\
&\qquad \leq A_\epsilon (1 + |\mathrm{Log}\ (t_0)|)^l \left(\sum_{n=0}^{\infty} |b_n| t_0^{n+\alpha} \right) \frac{\Gamma(N+\alpha-\epsilon+1)}{\Big(t_0 \sin(\delta)|k|\Big)^{N+\alpha-\epsilon+1}}. \qquad (66)
\end{aligned}
$$

In other words, for a given $\Sigma_{r,\delta}$,

$$
\forall\, \epsilon \in]0, \frac{1}{2}],\ \exists C > 0,\ \forall N \in \mathbb{N},\ \forall\, k \in \Sigma_{r,\delta},
$$

$$
\left| I(k,t_0) - \sum_{n=0}^{N} b_n \left(\frac{d}{d\alpha}\right)^l \left(\frac{\Gamma(\alpha+n)}{k^{\alpha+n}}\right) \right| \leq C^{N+\alpha-\epsilon+1} \frac{\Gamma(N+\alpha-\epsilon+1)}{|k|^{N+\alpha-\epsilon+1}}. \qquad (67)
$$

This provides the asymptotics we were looking for.

4.5. *An example*

We illustrate the previous considerations with the following simple example. We take

$$
f(z_1, z_2; \beta) = \beta^2 z_1 + z_2 + z_1 z_2^2, \qquad \beta \in \mathbb{C}^\star
$$

which is a deformation of the Broughton's polynomial (see §3.2.3). This polynomial has two atypical values which are the critical values $t = \pm i\beta$. Each generic fibre $f^{-1}(t)$, $t \in \mathbb{C}\backslash\{\pm i\beta\}$ has only one connected component

$$
\text{for } t \neq \pm i\beta, \qquad f^{-1}(t) = \{z_1 = (t - z_2)/(\beta^2 + z_2^2),\ z_2 \in \mathbb{C}\backslash\{\pm i\beta\}\} \qquad (68)
$$

so that $H_0(f^{-1}(t)) \cong \mathbb{Z}$, $H_1(f^{-1}(t)) \cong \mathbb{Z}^2$ and $H_q(f^{-1}(t)) \cong 0$ otherwise.

Note that the singular fibre $f^{-1}(i\beta) = \{z_1 \in \mathbb{C},\, z_2 = i\beta\} \cup \{z_1 = -1/(i\beta + z_2),\, z_2 \in \mathbb{C}\backslash\{-i\beta\}\}$ *resp.* $f^{-1}(-i\beta) = \{z_1 \in \mathbb{C},\, z_2 = -i\beta\} \cup \{z_1 = -1/(-i\beta + z_2),\, z_2 \in \mathbb{C}\backslash\{i\beta\}\}$ can be written as the one point union (wedge) of two connected spaces that intersect at the nondegenerate critical point $(z_1, z_2) = (\frac{i}{2\beta}, i\beta)$ *resp.* $(z_1, z_2) = (-\frac{i}{2\beta}, -i\beta)$. In particular $H_0(f^{-1}(\pm i\beta)) \cong \mathbb{Z}$, $H_1(f^{-1}(\pm i\beta)) \cong \mathbb{Z}$ and $H_q(f^{-1}(\pm i\beta)) \cong 0$ otherwise.

We would like to analyse the asymptotics when $k \to +\infty$ of the following integral

$$\begin{aligned} I_\Gamma(k;\beta) &= \int_\Gamma e^{-kf(z_1,z_2;\beta)} g(z_1, z_2)\, dz_1 \wedge dz_2 \\ &= \int_{\widetilde{\Gamma}} e^{-\beta k f(z_1,z_2;1)} g(\frac{z_1}{\beta}, \beta z_2)\, dz_1 \wedge dz_2 \end{aligned}$$

where $[\Gamma] \in H_2^{\Psi(\beta)}(\mathbb{C}^2)$ and $g \in \mathbb{C}[z_1, z_2]$. Since $H_1(f^{-1}(t)) \cong \mathbb{Z}^2$ for $t \in \mathbb{C}\backslash\{\pm i\beta\}$, we deduce from Theorem 4.1 and (49) that

$$H_2^{\Psi(\beta)}(\mathbb{C}^2) \cong \mathbb{Z}^2. \tag{69}$$

From (68) it is easy to see that $H_2^{\Psi(\beta)}(\mathbb{C}^2)$ is generated by the following two cycles $\Gamma_\pm$ (the so-called Lefschetz thimbles),

$$\begin{aligned} &\Gamma = m_+\Gamma_+ + m_-\Gamma_-, \qquad m_\pm \in \mathbb{Z} \\ &\Gamma_+ = \{z_1 = (i\beta + r - z_2)/(\beta^2 + z_2^2),\, r \in [0, +\infty[,\, |z_2 - i\beta| = \beta\} \\ &\Gamma_- = \{z_1 = (-i\beta + r - z_2)/(\beta^2 + z_2^2),\, r \in [0, +\infty[,\, |z_2 + i\beta| = \beta\} \end{aligned} \tag{70}$$

Note that the data $\left(H_2^{\Psi(\beta)}(\mathbb{C}^2)\right)_{\beta \in \mathbb{C}^\star}$ makes a local system on $\mathbb{C}^\star$ (in fact a constant sheaf of $\mathbb{Z}$-modules).
One can recover (69) using Proposition 4.1.2:

$$H_2^{\Psi}(\mathbb{C}^2) \cong H_2(X_+, X_+^\tau) \oplus H_2(X_-, X_-^\tau)$$

where X_+ (*resp.* X_-) is the Milnor ball associated with the nondegenerate critical point $(z_1, z_2) = (\frac{i}{2\beta}, i\beta)$ (*resp.* $(z_1, z_2) = (-\frac{i}{2\beta}, -i\beta)$). From our previous theoretical analysis, the asymptotics of $I_\Gamma(k;\beta)$ just reduces in a local analysis near each of the two critical points. Simple calculations gives for instance:

for $g(z_1, z_2) = a + bz_1 + cz_2$,

$$I_\Gamma(k;\beta) = m_+ e^{-i\beta k}\pi\left(\frac{a + \frac{i}{2}\frac{b}{\beta} + ic\beta}{\beta k} + \frac{\frac{b}{\beta}}{2(\beta k)^2} + O(k^{-3})\right) + m_- e^{+i\beta k}\pi\left(\frac{-a + \frac{i}{2}\frac{b}{\beta} + ic\beta}{\beta k} - \frac{\frac{b}{\beta}}{2(\beta k)^2} + O(k^{-3})\right) \tag{71}$$

where $m_\pm$ are integers which depend only $[\Gamma]$. It can be shown that the remainder terms $O(k^{-3})$ are in fact zero functions. We do this by direct calculation, using (70) which provides

$$I_{\Gamma_+}(k;\beta) = e^{-i\beta k}\int_0^{+\infty} e^{-kr}\left(\oint\left(a + b\frac{i\beta + r - z_2}{\beta^2 + z_2^2} + cz_2\right)\frac{dz_2}{\beta^2 + z_2^2}\right)dr$$

that is

$$I_{\Gamma_+}(k;\beta) = e^{-i\beta k}\int_0^{+\infty} e^{-kr}\left(a\frac{\pi}{\beta} + b(i\frac{\pi}{2\beta^2} + \frac{\pi}{2\beta^3}r) + ci\pi\right)dr.$$

so that

$$I_{\Gamma_+}(k;\beta) = e^{-i\beta k}\pi\left(\frac{a + \frac{i}{2}\frac{b}{\beta} + ic\beta}{\beta k} + \frac{\frac{b}{\beta}}{2(\beta k)^2}\right).$$

Similarly,

$$I_{\Gamma_-}(k;\beta) = e^{+i\beta k}\pi\left(\frac{-a + \frac{i}{2}\frac{b}{\beta} + ic\beta}{\beta k} - \frac{\frac{b}{\beta}}{2(\beta k)^2}\right).$$

4.6. *To go further*

By their very nature, the complex singular integrals considered in this paper are crossroads for different scientific communities, from pure mathematicians to physicists and chemists. In such various situations one often needs to enlarge the methods presented in this paper, for instance for Laplace-type integrals with boundaries. Also, it allows a theoretical and numerical control of the so-called Stokes phenomenon with a hyperasymptic and resurgent viewpoint. To go futher in that direction, the reader may consult Refs. 10 and 11 and references therein.

Acknowledgments

This paper is based on a course given at the ICTP "School on Algebraic approach to differential equations", Alexandria - Egypt, November 12-24, 2007. The author would like to warmly thank the organizers of this so pleasant School in a so charming place, especially Professors Lê Dung Trang and Mohamed Darwish, as well as the students and colleagues for their many interesting questions and helpful discussions.

Appendix A. Short Introduction to Singular Homology

For this appendix we mainly refer to Refs. 12, 16 and 28. In what follows X is a topological space.

Appendix A.1. *Simplex and chain*

• The **standard (oriented) q-simplex** in the affine space $\mathbb{R}^{q+1}$ is by definition

$$\Delta^q = \{\sum_{j=0}^{q} \lambda_j e_j,\, \sum_{j=0}^{q} \lambda_j = 1,\, 0 \le \lambda_j \le 1\} \subset \mathbb{R}^{q+1}$$

that is the convex hull of $(e_0, \cdots, e_q)$, where $(e_0, \cdots, e_q)$ is the standard basis of $\mathbb{R}^{q+1}$. (The orientation is chosen so that if a is a generic point of Δ^q, then a basis $(v_1, \cdots, v_q) \in T_a\Delta^q$ is positively oriented if the basis $(v_0 = \sum_{j=0}^{q} \frac{1}{q+1} e_j, v_1, \cdots, v_q)$ determines the standard orientation of $\mathbb{R}^{q+1}$).

• A **singular q-simplex of X** is a continuous map

$$\sigma : \Delta^q \mapsto X$$

• A **singular q-chain of X** is a formal linear combination

$$c = \sum_{i=1}^{m} n_i \sigma_i, \qquad n_i \in \mathbb{Z} \qquad (\text{resp. } n_i \in \mathbb{K} = \mathbb{R} \text{ or } \mathbb{C})$$

where the σ_i are singular q-simplices.

• We note by $C_q(X)$ = **the set of all singular q-chains**. We remark that $C_q(X)$ is a $\mathbb{Z}$-module (*resp.* a $\mathbb{K}$-vector space).
Example: $C_0(X) = \{\sum_{finite} n_i[x_i]\}$ where the x_i are points of X.

• The **support** of a singular q-simplex σ is $|\sigma| = \sigma(\Delta^q)$. The **support** of a singular q-chain $c = \sum_{i=1}^{m} n_i\sigma_i$ is $|\sigma| = \bigcup |\sigma_i|$.

Note that the support of a singular q-chain is always a compact subset of X. Also, since any standard q-simplex is path connected, the support of any q-simplex is contained in a path connected component of X.

Appendix A.2. *Boundary operator*

• For $q > 0$ we define for $i = 0, \cdots, q$,

$$F_i^q : \Delta^{q-1} \to \Delta^q, \qquad F_i^q : \sum_{j=0}^{q-1} \lambda_j e_j \mapsto \sum_{0}^{i-1} \lambda_j e_j + \sum_{i}^{q-1} \lambda_j e_{j+1}$$

which maps Δ^{q-1} onto the face $[e_0, \cdots, \widehat{e_i}, \cdots, e_q]$ of Δ^q.
Example: for $q = 2$,

$$\begin{array}{l} F_0^2 : \lambda_0 e_0 + \lambda_1 e_1 \mapsto \lambda_0 e_1 + \lambda_1 e_2 \\ F_1^2 : \lambda_0 e_0 + \lambda_1 e_1 \mapsto \lambda_0 e_0 + \lambda_1 e_2 \\ F_2^2 : \lambda_0 e_0 + \lambda_1 e_1 \mapsto \lambda_0 e_1 + \lambda_1 e_1 \end{array}$$

• If $\sigma : \Delta^q \mapsto X$ is a singular q-simplexe, the $i-th$-**face** $\sigma^{(i)}$ **of** σ is the singular $(q-1)$-simplex

$$\sigma^{(i)} = F_i^{q\star} \circ \sigma = \sigma \circ F_i^q : \Delta^{q-1} \to X.$$

This allows to define the **boundary of** σ :

$$\partial_q \sigma = \sum_{j=0}^{q} (-1)^j \sigma^{(j)}.$$

• If $c = \sum_{i=1}^{m} n_i \sigma_i$ is a singular q-chain, then one defines the **boundary of** c by

$$\partial_q c = \sum_{i=1}^{m} n_i \partial_q \sigma_i.$$

In this way one have a homomorphism (of $\mathbb{Z}$-modules, *resp.* of $\mathbb{K}$-vector spaces)

$$\partial_q : C_q(X) \to C_{q-1}(X)$$

which is the **boundary operator**.
• We set:

$$\begin{cases} C_q(X) = 0 & \text{for} \quad q < 0 \\ \partial_q = 0 & \text{for} \quad q \leq 0 \end{cases}$$

Then it can be shown that:

Proposition A.0.1.

$$\forall\, q, \qquad \partial_q \circ \partial_{q+1} = 0.$$

Definition A.0.1. We note for every q,

$$Z_q(X) = \{q\text{-cycles}\} = \partial_q^{-1}(0) = \ker \partial_q,$$

$$B_q(X) = \{q\text{-boundaries}\} = \partial_{q+1}\Big(C_{q+1}(X)\Big) = im\ \partial_{q+1}.$$

Two q-chains c_1 and c_2 are said to be **homologous** $c_1 \sim c_2$ if $c_1 - c_2$ is a q-boundary.

Proposition A.0.1 say that the sequence

$$\cdots \to C_{q+1}(X) \xrightarrow{\partial_{q+1}} C_q(X) \xrightarrow{\partial_q} C_{q-1}(X) \to \cdots$$

is a *chain complex* $(C_\bullet(X), \partial_\bullet)$ of $\mathbb{Z}$-modules (*resp.* of $\mathbb{K}$-vector spaces). This allows to define:

Definition A.0.2. For every q,

$$H_q(X) = \frac{Z_q(X)}{B_q(X)} \qquad (\text{resp. } H_q(X;K) = \frac{Z_q(X)}{B_q(X)})$$

is the q-th (singular) homology group of X ($\mathbb{Z}$-module, *resp.* K-vector space).

The homology class of a q-cycle c is usually denoted by $[c] \in H_q(X)$.

The graded (by the dimensions q) group $H_\bullet(X) = \bigoplus_q H_q(X)$ is the total (singular) homology group of X.

Example $H_0(X) = \dfrac{Z_0(X)}{B_0(X)} \cong \mathbb{Z}^m$ where m is the number of path-connected components of X. Indeed, one has $Z_0(X) = C_0(X) = \{\sum_{finite} n_i[x_i]\}$ where the x_i are points of X and $B_0(X) = \{\sum_{finite} n_i([x_i^2] - [x_i^1])\}$ where for each i, x_i^1 and x_i^2 are connected by a continuous path in X.

More generally, if X is a Hausdorff topological space and if $\{X_i\}$ are its path-connected components, then

$$H_q(X) = \bigoplus_i H_q(X_i), \qquad \forall\, q.$$

Appendix A.3. *Homomorphism induced by a continuous map*

We consider two topological spaces X and Y and a continuous map

$$f : X \to Y.$$

To a given singular q-simplex $\sigma : \Delta^q \mapsto X$ of X there is a natural way of associating a singular q-simplex of Y,

$$f \circ \sigma : \Delta^q \mapsto Y.$$

We thus have a homomorphism (of $\mathbb{Z}$-modules, *resp.* K-vector spaces)

$$C_q(f) : C_q(X) \to C_q(Y), \qquad C_q(f)\Big(\sum_{finite} n_i \sigma_i\Big) = \sum_{finite} n_i\, f \circ \sigma_i$$

and it can be shown that

$$\partial_q \circ C_q(f) = C_{q-1}(f) \circ \partial_q.$$

We thus have a chain map $C(f)$ which allows to define the homomorphism of homology groups:

$$f_\star : H_q(X) \to H_q(Y).$$

Of course, if Z is another topological space and if $g : Y \to Z$ is a continuous map, then $(g \circ f)_\star = g_\star \circ f_\star$. Also:

Proposition A.0.2. If f, $g : X \to Y$ are homotopic continuous maps, that is there exists a continuous map

$$H : X \times [0,1] \to Y, \qquad H(x,0) = f(x), \qquad H(x,1) = g(x),$$

then $f_\star = g_\star$.

Example Let $A \subset X$ and $i : A \hookrightarrow X$ the canonical injection. Assume that one has a continuous map $f : X \to A$ which is a *deformation-retraction*, that is:

(1) f is the identity on $A : f \circ i = id_A$ (f is a *retraction*).
(2) $i \circ f$ is homotopic to the identity : there exists a continuous map $H : X \times [0,1] \to X$ such that $H(.,0) = id_X$ and $H(.,1) = i \circ f$.

(For instance $f : x \in \mathbb{R}^{n+1}\backslash\{0\} \mapsto x/\|x\| \in S^n$, $H : (x,t) \in \mathbb{R}^{n+1}\backslash\{0\} \times [0,1] \mapsto (1-t)x + tx/\|x\| \in \mathbb{R}^{n+1}\backslash\{0\}$).
From condition 1. one obtains that $f_\star \circ i_\star : H_q(A) \to H_q(A)$ is the identity map. Using Proposition A.0.2, condition 2. implies that $i_\star \circ f_\star : H_q(X) \to H_q(X)$ is also the identity map.

Consequence : if $A \subset X$ is a deformation-retract of X, then the homologies $H_\bullet(A)$ and $H_\bullet(X)$ are isomorphic. (For instance $H_\bullet(S^n) = H_\bullet(\mathbb{R}^{n+1}\backslash\{0\})$).
In particular **if** X **is contractible**, that is can be retract by deformation to a point, then the homology of X is isomorphic to the homology of a point : $\forall\, q \neq 0$, $H_q(X) = 0$, $H_0(X) = \mathbb{Z}$.

Appendix A.4. *Homology of a pair*

Let $A \subset X$ be a subspace of the topological space X and we note $i : A \hookrightarrow X$ the canonical injection. This mapping induces a natural homomorphism $i_\star : C_q(A) \to C_q(X)$ so that the space $C_q(A)$ can be seen as a sub $\mathbb{Z}$-module (*resp.* a sub K-vector space) of $C_q(X)$.

• The quotient space $C_q(X, A) = \dfrac{C_q(X)}{C_q(A)}$ is the space of **relative q-chains of X with respect to A.**

• A q-chain $c \in C_q(X)$ is said to be a **relative q-cycle of X with respect to A** if $\partial_q c \in C_{q-1}(A)$.
We note

$$Z_q(X, A) = \{\text{relative } q\text{-cycles}\}.$$

• A q-chain $c \in C_q(X)$ is said to be a **relative q-boundary of X with respect to A** if there exists a q-chain $c' \in C_q(A)$ such that c and c' are homologous in X : $c - c' \in B_q(X)$.
One notes

$$B_q(X, A) = \{\text{relative } q\text{-boundaries}\}.$$

(See §2.2.2, Fig. 5).

We remark that $B_q(X, A)$ is a sub $\mathbb{Z}$-module, (*resp.* sub K-vector space) of $Z_q(X, A)$. (Indeed, if $c \in B_q(X, A)$, then there exists $c' \in C_q(A)$, there exists $d \in C_{q+1}(X)$ such that $c = c' + \partial_{q+1} d$. Therefore, $\partial_q c = \partial_q c' \in C_{q-1}(A)$).

Definition A.0.3. For every q, the quotient group

$$H_q(X, A) = \frac{Z_q(X, A)}{B_q(X, A)} \qquad (\text{resp. } H_q(X, A; K) = \frac{Z_q(X, A)}{B_q(X, A)})$$

is the q-th relative (singular) homology group ($\mathbb{Z}$-module, *resp.* K-vector space) of X with respect to A, or also the q-th (singular) homology group of the pair (X, A).

The canonical injection $i : A \hookrightarrow X$ induces for any q the natural homomorphism

$$i_\star : H_q(A) \to H_q(X).$$

Since $Z_q(X) \subset Z_q(X, A)$ and $B_q(X) \subset B_q(X, A)$ (note that we have equality when $A = \varnothing$) one has also in a natural way the homomorphism

$$j_\star : [c] \in H_q(X) \mapsto [c] \in H_q(X, A).$$

The following boundary homomorphism

$$\partial_\star : [c] \in H_q(X, A) \mapsto [\partial_q c] \in H_{q-1}(A)$$

is also well-defined : take $c \in Z_q(X, A)$, that is $c \in C_q(X)$ and $\partial_q c \in C_{q-1}(A)$. The homology class $[c] \in H_q(X, A)$ reads $[c] = \{c + c' + d,\ c' \in C_q(A),\ d \in B_q(X)\}$. Therefore

$$\partial_q(c + c' + d) = \underset{\in Z_{q-1}(A)}{\partial_q c} + \underset{\in B_{q-1}(A)}{\partial_q c'}$$

With these definitions, one has:

Proposition A.0.3. The following homology sequence of the pair (X, A) is exact:

$$\cdots \to H_q(A) \xrightarrow{i_\star} H_q(X) \xrightarrow{j_\star} H_q(X, A) \xrightarrow{\partial_\star} H_{q-1}(A) \to \cdots$$

What have been said in §A.3 can be extended for pairs. Let X and Y be two topological spaces and $A \subset X$, $B \subset Y$. From a continuous map of pairs,

$$f : x \in (X, A) \mapsto y \in (Y, B), \qquad f(A) \subset B,$$

one can associated a homomorphism of homology groups

$$f_\star : H_q(X, A) \to H_q(Y, B).$$

Example In §2.2.2 we have introduced the pairs $(\mathbb{C}, \Sigma_R(A_{\theta,\delta}))$. For $R' \geq R$, $(\mathbb{C}, \Sigma_{R'}(A_{\theta,\delta}))$ is a sub-pair of $(\mathbb{C}, \Sigma_R(A_{\theta,\delta}))$ and one has an homeomorphism

$$h : z \in (\mathbb{C}, \Sigma_R(A_{\theta,\delta})) \to \frac{R'}{R} z \in (\mathbb{C}, \Sigma_{R'}(A_{\theta,\delta}))$$

from which is associated an isomorphism

$$h_\star : H_q(\mathbb{C}, \Sigma_R(A_{\theta,\delta}) \to H_q(\mathbb{C}, \Sigma_{R'}(A_{\theta,\delta})).$$

One has the analogue of Proposition A.0.2:

Proposition A.0.4. If f, $g : (X, A) \to (Y, B)$ are homotopic continuous maps of pairs, that is if there exists a continuous map

$$H : (X, A) \times [0, 1] \to (Y, B), \qquad H(x, 0) = f(x), \qquad H(x, 1) = g(x),$$

then $f_\star = g_\star$.

Example Let (Y, B) be a subpair of (X, A) and $i : (Y, B) \hookrightarrow (X, A)$ the canonical injection. Assume that the continuous map $f : (X, A) \to (Y, B)$ is a *deformation-retraction*, that is:

(1) f is the identity on $Y : f \circ i = id_Y$ (f is a *retraction*).
(2) $i \circ f$ is homotopic to the identity.

Then the homologies $H_\bullet(X, A)$ and $H_\bullet(Y, B)$ are isomorphic.

Definition A.0.4. An inclusion map $(Y, B) \hookrightarrow (X, A)$ between topological pairs is called an *excision map* if $Y \backslash B = X \backslash A$.
The couple $\{X_1, X_2\}$ made by two subsets of a given topological space is an *excisive couple* of subsets if the inclusion chain map $C_\bullet(X_1) + C_\bullet(X_2) \subset C_\bullet(X_1 \cup X_2)$ induces an isomorphism of homology.
The couple $\{(X_1, A_1), (X_2, A_2)\}$ of pairs in a given topological space is an *excisive couple of pairs* if both $\{X_1, X_2\}$ and $\{A_1, A_2\}$ are excisive couple of subsets.

Theorem A.1. • *If* $X_1 \cup X_2 = \mathrm{int}_{X_1 \cup X_2} X_1 \cup \mathrm{int}_{X_1 \cup X_2} X_2$, *then* $\{X_1, X_2\}$ *is an excisive couple.*
• $\{X_1, X_2\}$ *is an excisive couple if and only if the excision map* $(X_1, X_1 \cap X_2) \hookrightarrow (X_1 \cup X_2, X_2)$ *induces an isomorphism of homology.*

Corollary A.1.1. We assume that $U \subset A \subset X$ are subsets such that the closure $\overline{U}$ of U is included in the interior $intA$ of A. Then U can be excised, that is

$$H_q(X, A) = H_q(X \backslash U, A \backslash U), \qquad \forall\, q.$$

Proof. The couple $\{X \backslash U, A\}$ is an excisive couple so that the excision map $(X \backslash U, A \backslash U) \hookrightarrow (X, A)$ induces an isomorphism of homology. □

Theorem A.2 (Mayer-Vietoris). • *Assume that* $\{X_1, X_2\}$ *is an excisive couple. Then the following Mayer-Vietoris homology sequence is exact:*

$$\begin{aligned}\cdots \to H_q(X_1 \cap X_2) &\overset{(i_{X_1\star}, -i_{X_2\star})}{\longrightarrow} H_q(X_1) \oplus H_q(X_2) \overset{j_{X_1\star}+j_{X_2\star}}{\longrightarrow} H_q(X_1 \cup X_2)\\ &\overset{\partial}{\longrightarrow} \quad H_{q-1}(X_1 \cap X_2) \to \cdots\end{aligned}$$

where $X_1 \cap X_2 \overset{i_{X_1}}{\hookrightarrow} X_1 \overset{j_{X_1}}{\hookrightarrow} X_1 \cup X_2$ *and* $X_1 \cap X_2 \overset{i_{X_2}}{\hookrightarrow} X_2 \overset{j_{X_2}}{\hookrightarrow} X_1 \cup X_2$ *are the canonical injections.*

• *Assume that* $\{(X_1, A_1), (X_2, A_2)\}$ *is an excisive couple of pairs. Then the following relative Mayer-Vietoris homology sequence is exact:*

$$\begin{aligned}\cdots \to H_q(X_1 \cap X_2, A_1 \cap A_2) &\overset{i_\star}{\to} H_q(X_1, A_1) \oplus H_q(X_2, A_2)\\ &\overset{j_\star}{\to} H_q(X_1 \cup X_2, A_1 \cup A_2)\\ &\overset{\partial}{\to} H_{q-1}(X_1 \cap X_2, A_1 \cap A_2) \to \cdots\end{aligned}$$

Note that for a given triple (X, A, B), $B \subset A \subset X$, the couple of pairs $\{(X, B), (A, A)\}$ and $\{(X, B), (A\cup, A\cup B)\}$ are always an excisive couple of pairs. One thus deduces from the relative Mayer-Vietoris homology exact sequence that:

Corollary A.2.1. • Assume that $B \subset A \subset X$. Then the following homology sequence of the triple (X, A, B) is exact:

$$\cdots \to H_q(A, B) \xrightarrow{i_\star} H_q(X, B) \xrightarrow{j_\star} H_q(X, A) \xrightarrow{\partial} H_{q-1}(A, B) \to \cdots$$

• Assume that A, B are two subsets of X such that $\{A, B\}$ is an excisive couple. Then the following homology sequence of the triad (X, A, B) is exact:

$$\cdots \to H_q(A, A\cap B) \xrightarrow{i_\star} H_q(X, B) \overset{j_\star}{\to} H_q(X, A\cup B) \xrightarrow{\partial} H_{q-1}(A, A\cap B) \to \cdots$$

Appendix A.5. *Homology with support in a family*

It is sometimes needed to extend the definition of homology to homology defined by a family of supports (see[25]).

Let X be a Hausdorff topological space assumed to be locally compact (i.e., every point has a local base of compact neighbourhoods) and paracompact (i.e., every open cover admits an open locally finite refinement).

We introduce a family Φ of closed subsets of X which satisfies the following properties:

$$\begin{cases} (i)\ A,\, B \in \Phi \Rightarrow A \cup B \in \Phi \\ (ii)\ B \text{ closed} \subset A \in \Phi \Rightarrow B \in \Phi \\ (iii)\ \text{Any } A \in \Phi \text{ has a neighbourhood which belongs to } \Phi \end{cases} \tag{A.1}$$

To define what is a **a q-chain c with support in Φ** one modifies the definition of a q-chain as follows :

- c is a formal linear combination $c = \sum_i n_i\sigma_i$ where the sum may be infinite but is locally finite: any point $x \in X$ has a neighbourhood $U_x \subset X$ which meets only a finite number of supports $[\sigma_i]$ (the σ_i are usual singular q-simplices). This implies that $[c] = \bigcup_i [\sigma_i]$ is a closed subset of X.
- moreover $[c] \in \Phi$.

The set of q-chains with support in Ψ makes the **group $C_q^\Phi(X)$ of q-chains with support in Φ**. The boundary operator ∂_q can be defined in an obvious way on $C_q^\Phi(X)$ (∂_q defines a map from $C_q^\Phi(X)$ into $C_{q-1}^\Phi(X)$ by (ii) of (A.1)). To the chain-complex $(C_\bullet^\Phi(X), \partial_\bullet)$ of $\mathbb{Z}$-modules one can then associate the homology groups

$$H_q^\Phi(X) = \frac{Z_q^\Phi(X)}{B_q^\Phi(X)}$$

and $H_q^\Phi(X)$ is called the **q-th homology group with support in Φ**.

Examples • The family $\Phi = c$ of all compact subsets of X satisfies (A.1) and the associated homology coincides with the singular homology : $H_q^c(X) = H_q(X)$.

• The family $\Phi = F$ of all closed subsets of X satisfies (A.1) : $H_\bullet^F(X)$ is the homology with closed support (or Borel-Moore homology).

Appendix A.6. *Homology and fibre bundle*

We just mention here the following fundamental tool in algebraic topology:

Theorem A.3 (Homotopy lifting Theorem). *Assume that M and N are topological spaces such that N is a paracompact Hausdorff space. Assume that $f \begin{array}{c} M \\ \downarrow \\ N \end{array}$ is the projection of a fibre bundle. Then f has the homotopy lifting property : for any given topological space X and for*
– any homotopy $H : X \times [0,1] \to N$,
– any continuous map $\widetilde{h_0} : X \to M$ lifting $h_0 = H|_{X\times\{0\}}$,

$\begin{array}{ccc} & & M \\ & \widetilde{h_0} \nearrow & \downarrow f \\ X \times \{0\} & \xrightarrow{h_0} & N \end{array}$, *there exists a homotopy* $\widetilde{H} : X \times [0,1] \to M$ *lifting* H, $\begin{array}{ccc} & & M \\ & \widetilde{H} \nearrow & \downarrow f \\ X \times I & \xrightarrow{H} & N \end{array}$, *with* $\widetilde{h_0} = \widetilde{H}|_{X\times\{0\}}$.

Appendix A.7. *Integration*

We assume here that X is a $\mathcal{C}^\infty$-manifold.

• If ω is a differential q-form on X and if $c = \sum_{i=1}^{m} n_i \sigma_i$ is a q-chain, then by definition

$$\int_c \omega = \sum_{i=1}^{m} n_i \int_{\sigma_i} \omega = \sum_{i=1}^{m} n_i \int_{\Delta^q} \sigma_i^\star \omega$$

($\sigma_i^\star \omega = \sigma_i^\star \circ \omega$) with the following remarks:

- we remind the reader that the Δ^q are oriented;
- to define the $\sigma_i^\star \omega$, we assume that each σ_i is differentiable instead of continuous, that is each σ_i is the restriction to Δ^q of a $\mathcal{C}^\infty$ function $\widetilde{\sigma_i} : U \mapsto X$, $\Delta^q \subset U \subset \mathbb{R}^{q+1}$. In that case the q-chain $c = \sum_{i=1}^{m} n_i \sigma_i$ is said to be *piecewise differentiable.*

Note that any continuous function $\sigma : \Delta^q \mapsto X$ can be approximated by a differentiable map. (Any continous function with compact support can be uniformaly approximated by $\mathcal{C}_c^\infty$ functions, by regularisation).
This has the following consequence :

Proposition A.3.1. Each homology class $[c] \in H_q(X)$ of a q-cycle can be represented by a piecewise differentiable q-cycle c, and each piecewise differentiable null homologous q-cycle is the boundary of a piecewise differentiable $(q+1)$-chain.

In this course, when an integral of a differential form along a chain is considered, one always assume that this chain is piecewise differentiable.

Theorem A.4 (Stokes theorem). *If φ is a differential $(q-1)$-form on X and if c is a q-chain, then*

$$\int_c d\varphi = \int_{\partial c} \varphi.$$

Corollary A.4.1. If φ is a closed differential q-form on X $(d\varphi = 0)$, then the integral $\int_c \varphi$ along a q-cycle c only depends on the homology class $[c] \in H_q(X)$.

Appendix B. Stein Manifolds and Some Consequences

We refer to Refs. 15 and 14 for this appendix.

Appendix B.1. *Definition and main properties*

In what follows it will be assumed that one works with paracompact complex manifolds. We start with the following definition of Stein manifolds:

Definition B.0.1. A complex manifold X is called :
• *holomorphically spreadable* if for any point $x_0 \in X$ there are holomorphic functions $f_1, \cdots, f_N$ on X such that x_0 is isolated in the set

$$N(f_1, \cdots, f_N) = \{x \in X,\, f_1(x) = \cdots = f_N(x) = 0\}.$$

• *holomorphically convex* if for any compact set $K \subset X$ the holomorphically convex hull $\widehat{K} = \{x \in X,\, |f(x)| \leq \sup_X |f| \text{ for every f } \in \mathcal{O}(X)\}$ is also compact.
• *a Stein manifold* if X is connected and is holomorphically spreadable and holomorphically convex.

Typical Stein manifolds are domains of holomorphy:

Theorem B.1. *We assume that $D \subset \mathbb{C}^n$ is a domain, that is D is a connected open set of $\mathbb{C}^n$. Then:*

$$\begin{gathered} D \textit{ is a Stein manifold} \\ \Updownarrow \\ D \textit{ is holomorphically convex} \\ \Updownarrow \\ D \textit{ is a domain of holomorphy, that is there exists a function } h \in \mathcal{O}(D) \\ \textit{that is completely singular at every point of } \partial D. \end{gathered}$$

We provide some example of domains of holomorphy.

Proposition B.1.1. • Every domain in $\mathbb{C}$ is a domain of holomorphy.
• Every affine convex open subset of $\mathbb{C}^n$ is a domain of holomorphy (for instance $\mathbb{C}^n$ and any open ball of $\mathbb{C}^n$ are domains of holomorphy).
• Every cartesian product of domains of holomorphy is a domain of holomorphy.
• If $G_1 \subset \mathbb{C}^n$ and $G_2 \subset \mathbb{C}^p$ are domains of holomorphy and if $f : G_1 \to \mathbb{C}^p$ is a holomorphic map, then $f^{-1}(G_2) \cap G_1$ is a domain of holomorphy.

Example : A Milnor ball is a domain of holomorphy (and thus a Stein manifold).

Next we give some properties for Stein manifolds.

Proposition B.1.2. • Every closed submanifold of a Stein manifold is Stein.
• If X is a Stein manifold and if $f \in \mathcal{O}_X(X)$, then $X \backslash N(f)$ is Stein.
• If X is a complex manifold and if U_1, $U_2 \subset X$ are two open Stein manifold, then $U_1 \cap U_2$ is Stein.
• Every cartesian product of Stein manifolds is a Stein manifold.
• If $f : X \to Y$ is a holomorphic submersion between complex manifolds. If X is Stein and if $Z \subset Y$ is a Stein submanifold, then $f^{-1}(Z)$ is a Stein manifold.

To go further, one needs the so-called theorem B of Cartan-Serre. We first give some definitions.
We consider a complex manifold X and a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules. Let $s_1, \cdots, s_p \in \mathcal{F}(U)$ and consider the homomorphism

$$\sigma_U : (g_{1x}, \cdots, g_{px}) \in (\mathcal{O}_X|_U)^p \mapsto \sigma_U(g_{1x}, \cdots, g_{px}) = \sum_{i=1}^{p} g_{ix} s_{ix} \in \mathcal{F}|_U \tag{B.1}$$

(we note $\mathcal{F}|_U$ the restriction sheaf to the open set $U \subset X$). One says that $\mathcal{F}|_U$ is *generated* by the sections $s_1, \cdots, s_p$ if σ_U is surjective. (Every stalk $\mathcal{F}_x$, $x \in U$ is generated as a $\mathcal{O}_x$-module by the germs $s_{1x}, \cdots, s_{px}$).
The sheaf $\mathcal{F}$ is said to be *finite at* $x \in X$ if there exists a neighbourhood U of x and a finite number of sections $s_1, \cdots, s_p \in \mathcal{F}(U)$ which generate $\mathcal{F}|_U$.
The **sheaf $\mathcal{F}$ is said to be finite on** X if it is finite at every point $x \in X$.
If σ_U is an $\mathcal{O}_X|_U$-homomorphism as defined by (B.1), then the following

sheaf of $O_X|_U$-modules in $(\mathcal{O}_X|_U)^p$

$$\mathcal{R}el(s_1, \cdots, s_p) = \ker \sigma_U = \bigcup_{x \in U} \{(g_{1x}, \cdots, g_{px}) \in (\mathcal{O}_x)^p, \sum_{i=1}^{p} g_{ix} s_{ix} = 0\}$$

is called the *sheaf of relations* of $s_1, \cdots, s_p$.
The sheaf $\mathcal{F}$ is a *finite relation sheaf at* $x \in X$ if for every open neighbourhood U of x and for arbitrary sections $s_1, \cdots, s_p \in \mathcal{F}(U)$ the sheaf of relations $\mathcal{R}el(s_1, \cdots, s_p)$ is finite at x.
The **sheaf $\mathcal{F}$ is said to be a finite relation sheaf** if it is a finite relation sheaf at every $x \in X$. The **$\mathcal{O}_X$-sheaf $\mathcal{F}$ over X is said to be coherent** if it is finite and a finite relation sheaf.
Example : if X is a complex manifold of dimension n, then the $\mathcal{O}_X$-sheaves Ω_X^p are coherent (since Ω_X^p is locally free of rank $\binom{n}{p}$).

Theorem B.2 (Theorem B). *We consider a complex manifold X. Then X is Stein if and only if for every coherent $\mathcal{O}_X$-sheaf $\mathcal{F}$ over X one has $H^q(X, \mathcal{F}) = 0$, $\forall q > 0$.*

Here we do not define cohomology with values in a sheaf. It will be enough for our purpose to mention that

$$H^0(X, \mathcal{F}) = \mathcal{F}(X)$$

where $\mathcal{F}(X)$ is the set of all global sections of the sheaf $\mathcal{F}$.

Appendix B.2. *Some applications*

Appendix B.2.1. *First applicaton*

Let X be a complex manifold of dimension n. We note Ω_X^p the $\mathcal{O}_X$-sheaf of germs of holomorphic p-forms on X. Denoting by $\underline{\mathbb{C}}$ the constant sheaf, one has the following sequence of sheaf:

$$0 \to \underline{\mathbb{C}} \xrightarrow{i} \mathcal{O}_X = \Omega_X^0 \xrightarrow{d} \Omega_X^1 \xrightarrow{d} \Omega_X^2 \xrightarrow{d} \cdots \xrightarrow{d} \Omega_X^n \xrightarrow{d} 0 \qquad \text{(B.2)}$$

By the holomorphic Poincaré lemma we know that if $U \subset X$ is a star-shaped open set then any holomorphic closed p-form is exact; consequently (B.2) is an exact sequence (in which case (B.2) is a so-called *resolution* of the constant sheaf $\underline{\mathbb{C}}$). Furthermore if we assume that X is Stein, then the $\mathcal{O}_X$-sheaves Ω_X^p are coherent sheaves and by Theorem B.2 we get

$$\forall\, q > 0,\, \forall p \geq 0,\;\; H^q(X, \Omega^p_X) = 0.$$

In such case the resolution (B.2) is said to be *acyclic* and one has the following de Rham theorem:

Theorem B.3 (de Rham theorem). *We assume that X is a complex manifold of dimension n. If (B.2) is an acyclic resolution of the constant sheaf $\underline{\mathbb{C}}$ then there exists natural $\mathbb{C}$-isomorphisms*

$$\begin{aligned} H^0(X, \mathbb{C}) &\cong \ker\Big(d : \mathcal{O}(X) \to \Omega^1(X)\Big) \\ H^p(X, \mathbb{C}) &\cong \frac{\ker\Big(d : \Omega^p(X) \to \Omega^{p+1}(X)\Big)}{\mathrm{Im}\,\Big(d : \Omega^{p-1}(X) \to \Omega^p(X)\Big)}, \qquad p \geq 1. \end{aligned}$$

This is true in particular when X is a Stein manifold.

In this theorem $H^\bullet(X, \mathbb{C})$ stands for the singular cohomology deduced from the singular homology by duality (one defines the set of q-cochains as $C^q(X) = \mathrm{Hom}(C_q(X), \mathbb{C})$ and the coboundary operator δ as the dual map of the boundary operator ∂, $< \sigma, \delta C > = < \partial\sigma, C >$). By the so-called universal coefficient theorem for cohomology (cf.[28]) one has the exact sequence

$$0 \to \mathrm{Ext}(H_{q-1}(X), \mathbb{C}) \to H^q(X, \mathbb{C}) \to \mathrm{Hom}(H_q(X), \mathbb{C}) \to 0 \qquad \text{(B.3)}$$

where $\mathrm{Ext}(H_{q-1}(X), \mathbb{C}) = 0$ if $H_{q-1}(X)$ is a free $\mathbb{Z}$-module. In particular if X is contractible then $\forall\, q \neq 0$, $H_q(X) = 0$, $H_0(X) = \mathbb{Z}$ so that $\forall\, q \neq 0$, $H^q(X, \mathbb{C}) = 0$, $H^0(X, \mathbb{C}) = \mathbb{C}$. Therefore by Theorem B.3:

Corollary B.3.1. If X is a contractible Stein manifold then every closed holomorphic form is exact.

Appendix B.2.2. *Second application*

Assume that X is a complex manifold of dimension n and $f : X \to T \subset \mathbb{C}$ a (non constant) holomorphic map, where T is an open connected set.

We define the following *sheaf of holomorphic relative forms*[e]:

$$\Omega^0_{X/T} = \mathcal{O}_X, \qquad \Omega^q_{X/T} = \frac{\Omega^q_X}{df \wedge \Omega^{q-1}_X}, \ \ q \geq 1. \tag{B.4}$$

Note that we have a (de Rham relative) chain complex

$$\Omega^0_{X/T} \xrightarrow{d_{X/T}} \Omega^1_{X/T} \xrightarrow{d_{X/T}} \cdots \xrightarrow{d_{X/T}} \Omega^{n-1}_{X/T} \xrightarrow{d_{X/T}} \Omega^n_{X/T}$$

where the differential $d_{X/T}$ is defined in a natural way ($d_{X/T}[\omega] = d_{X/T}[\omega + df \wedge \Psi] = [d(\omega + df \wedge \Psi)] = [d\omega - df \wedge d\Psi] = [d\omega]$).

From the very definition (B.4) one has the following exact sequence of sheaves,

$$0 \to df \wedge \Omega^{q-1}_X \xrightarrow{i} \Omega^q_X \to \Omega^q_{X/T} \to 0. \tag{B.5}$$

We know that the Ω^q_X are coherent sheaves of $\mathcal{O}_X$-modules. Since $df \wedge \Omega^{q-1}_X$ is the image of Ω^{q-1}_X by the $\mathcal{O}_X$-linear map $df \wedge . : \Omega^{q-1}_X \to \Omega^q_X$ one deduces that $df \wedge \Omega^{q-1}_X$ are also coherent sheaves. In the exact sequence (B.5) since two sheaves are coherent, then the third sheaf $\Omega^q_{X/T}$ is also coherent as a consequence of the so-called "Three lemma".[14]

We now add the assumption that $X' \subset X$ is a Stein (sub)manifold. From the short exact sequence of sheaves (B.5) we derive the following long sequence of cohomology:

$$\begin{aligned} \cdots \to H^0(X', df \wedge \Omega^{q-1}_X) &\to H^0(X', \Omega^q_X) \\ \to H^0(X', \Omega^q_{X/T}) &\to H^1(X', df \wedge \Omega^{q-1}_X) \to \cdots \end{aligned} \tag{B.6}$$

and by Theorem B.2 we know that $H^1(X', df \wedge \Omega^{q-1}_X) = 0$. This implies that the homomorphim $\Omega^q_X(X') \to \Omega^q_{X/T}(X')$ is a surjective map ($\forall\, q$).

We have also the following result which is used in this course:

Proposition B.3.1. We assume that X is a complex manifold of dimension n and that $f : X \to T \subset \mathbb{C}$ a (non constant) holomorphic map where

[e]If $\mathcal{F}$ is a sheaf over X and $\mathcal{G}$ a subsheaf the collection of quotient spaces $\left(\dfrac{\mathcal{F}(U)}{\mathcal{G}(U)}\right)_{U \text{ open } \subset X}$ makes as a rule only a presheaf and $\dfrac{\mathcal{F}}{\mathcal{G}}$ is the sheaf generated by this presheaf. The collection of canonical homomorphisms $\mathcal{F}(U) \to \dfrac{\mathcal{F}(U)}{\mathcal{G}(U)}$ allows to define a sheaf homomorphism $\phi : \mathcal{F} \to \dfrac{\mathcal{F}}{\mathcal{G}}$ which is surjective, that is for all x, $\phi_x : \mathcal{F}_x \to \dfrac{\mathcal{F}}{\mathcal{G}}_x$ is surjective. However the homomorphisms $\phi(U) : \mathcal{F}(U) \to \dfrac{\mathcal{F}}{\mathcal{G}}(U)$ are not surjective in general.

T is an open connected set. We assume furthermore that $X' \subset X$ is a Stein (sub)manifold and that the restriction $f|_{X'}$ is a submersion.
Then if $\omega \in \Omega_X^{n-1}(X')$ is $d_{X/T}$-closed, then there exists $\dfrac{d\omega}{df} \in \Omega_X^{n-1}(X')$ such that $d\omega = df \wedge \dfrac{d\omega}{df}$.

Proof. The following sequence of sheaves

$$0 \to \Omega_X^0 \xrightarrow{df\wedge\cdot} \Omega_X^1 \xrightarrow{df\wedge\cdot} \cdots \xrightarrow{df\wedge\cdot} \Omega_X^{n-1} \xrightarrow{df\wedge\cdot} \Omega_X^n \to 0$$

is exact on X': this is a consequence of Lemma 2.0.1 since $f|_{X'}$ is a submersion. This implies that the following short sequence of sheaves is also exact on X' :

$$0 \to \Omega_{X/T}^{n-2} \xrightarrow{df\wedge\cdot} \Omega_X^{n-1} \xrightarrow{df\wedge\cdot} df \wedge \Omega_X^{n-1} \to 0$$

We thus derive the following long sequence of cohomology,

$$\cdots \to H^0(X', \Omega_X^{n-1}) \to H^0(X', df \wedge \Omega_X^{n-1}) \to H^1(X', \Omega_{X/T}^{n-2}) \to \cdots, \tag{B.7}$$

and since X' is Stein while $\Omega_{X/T}^{n-2}$ is a coherent sheaf we deduce that the

$$(df \wedge .)_\star : H^0(X', \Omega_X^{n-1}) \to H^0(X', df \wedge \Omega_X^{n-1})$$

is a surjective homomorphism.
Now, if $\omega \in \Omega_X^{n-1}(X')$ is $d_{X/T}$-closed, then $d\omega$ can be seen as an element of $H^0(X', df \wedge \Omega_X^{n-1})$ and the above result provides the existence of $\dfrac{d\omega}{df} \in \Omega_X^{n-1}(X')$ such that $d\omega = df \wedge \dfrac{d\omega}{df}$. □

References

1. V. Arnold, A. Varchenko, S. Gussein-Zadé, *Singularities of differentiable maps. Vol. II. Monodromy and asymptotics of integrals.* Monographs in Mathematics, 83. Birkhäuser Boston, Inc., Boston, MA (1988).
2. V. Arnold, V. Goryunov, O. Lyashko, V. Vassiliev, *Singularities I.* Encyclopaedia of Mathematical Sciences, Vol. 6, Springer-Verlag, Berlin and New York (1993).
3. V. Arnold, V. Goryunov, O. Lyashko, V. Vassiliev, *Singularities II.* Encyclopaedia of Mathematical Sciences, Vol. 39, Springer-Verlag, Berlin and New York (1993).
4. M.V. Berry, *The Riemann-Siegel expansion for the zeta function: high orders and remainders.* Proc. Roy. Soc. London Ser. A **450** Proc. Roy. Soc. Lond. A **434**, no. 1939, 439–462 (1995).

5. M.V. Berry, C.J. Howls, *Hyperasymptotics for integrals with saddles.* Proc. Roy. Soc. Lond. A **434**, 657-675 (1991).
6. S.A. Broughton, *On the topology of polynomial hypersurfaces.* Proc. Symp. of pure Math. **40**, A.M.S. Pub., 167-178 (1983).
7. S.A. Broughton, *Milnor numbers and the topology of polynomial hypersurfaces.* Invent. Math. **92**, No.2, 217-241 (1988).
8. E.T. Copson, *Asymptotic expansions.* Cambridge University Press (1965).
9. E. Delabaere, C.J. Howls, *Hyperasymptotics for multidimensional Laplace integrals with boundaries.* Toward the exact WKB analysis of differential equations, linear or non-linear (Kyoto, 1998) **119**, 145–163, Kyoto Univ. Press, Kyoto (2000).
10. E. Delabaere, C.J. Howls, *Global asymptotics for multiple integrals with boundaries.* Duke Math. J. **112** (2002), no. 2, 199–264.
11. E. Delabaere, C.J. Howls, *Addendum to the hyperasymptotics for multidimensional Laplace integrals.* In Analyzable functions and applications, 177–190, Contemp. Math., **373**, Amer. Math. Soc., Providence, RI, (2005).
12. W. Ebeling, *Functions of several complex variables and their singularities.* Translated from the 2001 German original by Philip G. Spain. Graduate Studies in Mathematics, 83. American Mathematical Society, Providence, RI, (2007).
13. M.V. Fedoryuk, *Metod perevala (The saddle-point method).* zdat. "Nauka", Moscow (1977).
14. H. Grauert, R. Remmert, *Theory of Stein spaces.* Translated from the German by Alan Huckleberry. Reprint of the 1979 translation. Classics in Mathematics. Springer-Verlag, Berlin, (2004).
15. K. Fritzsche, H. Grauert, *From holomorphic functions to complex manifolds.* Graduate Texts in Mathematics **213**, Springer-Verlag, New York, (2002).
16. M. J. Greenberg, *Lectures on algebraic topology.* W. A. Benjamin, Inc., New York-Amsterdam (1967).
17. C.J. Howls *Hyperasymptotics for multidimensional integrals, exact remainder terms and the global connection problem.* Proc. Roy. Soc. Lond. A **453**, 2271-2294 (1997).
18. D. Kaminski, R.B. Paris, *Asymptotics and Mellin-Barnes integrals.* Encyclopedia of Mathematics and its Applications, 85. Cambridge University Press, Cambridge, (2001).
19. J. Leray, *Le calcul différentiel et intégral sur une variété analytique complexe (Problème de Cauchy III).* Bull. Soc. Math. France, **87**, 81-180 (1959).
20. B. Malgrange, *Intégrales asymptotiques et monodromie.* Ann. Sci. cole Norm. Sup. (4) **7** (1974), 405–430 (1975).
21. B. Malgrange, *Sommation des séries divergentes.* Expositiones Mathematicae **13**, 163-222 (1995).
22. J. Milnor, *Singular points of complex hypersurfaces.* Ann. Math. Stud. 61. Princeton Univ. Press: Princeton (1968).
23. N. Nilsson, *Some growth and ramification properties of certain integrals on algebraic manifolds.* Ark. Mat. **5**, 463–476 (1965).

24. N. Nilsson, *Monodromy and asymptotic properties of certain multiple integrals.* Ark. Mat. **18**, no. 2, 181–198. (1980).
25. F. Pham, *Intégrales singulières.* Savoirs actuels, EDP Sciences/CNRS Editions (2005).
26. F. Pham, *Vanishing homologies and the n variable saddlepoint method.* Proc. Symposia in Pure Math. **40**, part 2 319–333 (1983).
27. F. Pham, *La descente des cols par les onglets de Lefschetz, avec vues sur Gauss-Manin.* in Systèmes différentiels et singularités, Astérisque **130** (1985).
28. E.H. Spanier, *Algebraic topology.* Corrected reprint of the 1966 original. Berlin: Springer-Verlag (1995).
29. E. M. Stein, *Singular integrals and differentiability properties of functions.* Princeton Mathematical Series, No. 30 Princeton University Press, Princeton, N.J. (1970).
30. M. Tibăr, *Polynomials and vanishing cycles.* Cambridge Tracts in Mathematics **170**, Cambridge University Press (2007).
31. J-L. Verdier, *Stratifications de Whitney et théorème de Bertini-Sard.* Inventiones math. **36**, 295-312 (1976).

HYPERGEOMETRIC FUNCTIONS AND HYPERPLANE ARRANGEMENTS

MICHEL JAMBU

Laboratoire J.A. Dieudonné, UMR 6621, Université de Nice, France
E-mail: jambu@unice.fr

This is a short presentation of some relations between the arrangements of hyperplanes and the theory of hypergeometric functions. All of this paper is coming from the illuminating monograph written by P. Orlik and H. Terao[9] where all the detailed proofs could be found. Moreover, we focus our interest on the part of this theory which consists of the computation of the cohomology groups, part of the hypergeometric pairing. We mainly consider the non-resonant weights and we just give some insight on the resonant case.

1. Classical Hypergeometric Functions

1.1. *Classical hypergeometric series (Gauss)*

The series which has become known as the ordinary **hypergeometric series** or the Gauss series is denoted

$$F[(a,b);c;x] = 1 + \frac{ab}{c}\frac{x}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{x^2}{2!} + \cdots$$

When we introduce Appel's notation $(a,n) = a(a+1)(a+2)\cdots(a+n-1)$ we can write

$$F[(a,b);c;x] = \sum_{n\geq 0} \frac{(a,n)(b,n)}{(c,n)}\frac{x^n}{n!}$$

Notice that:

- If $a = 0$ or $b = 0$ or is negative, then F is a polynomial.
- If $c = 0$ or is negative, then F is not defined.
- The series is convergent for $|x| < 1$.

The sum of the series inside its circle of convergence is called the **hypergeometric function**, and the same name is used for its analytic continuation outside the circle of convergence.

Barnes constructed more general hypergeometric series with p numerators parameters $(a) = (a_1, \ldots, a_p)$ and q denominators parameters $(c) = (c_1, \ldots, c_q)$

$$ {}_pF_q[(a);(c);x] = \sum_{n\geq 0} \frac{(a_1,n)\cdots(a_p,n)}{(c_1,n)\cdots(c_q,n)} \frac{x^n}{n!} $$

Therefore, the original Gauss series is ${}_2F_1$. It is also possible to express many functions as hypergeometric series. Let us give some few examples:

$$ e^x = \sum_{n\geq 0} \frac{x^n}{n!} = {}_0F_0[x] $$

$$ (1-x)^{-a} = \sum_{n\geq 0} (a,n)\frac{x^n}{n!} = {}_1F_0[a;x] $$

$$ \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = {}_0F_1[1/2; -x^2/4] $$

$$ \sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = x.{}_0F_1[3/2; -x^2/4] $$

$$ \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots = x.{}_2F_1[(1,1);2;-x] $$

$$ \mathrm{Li}_2(x) = \sum_{n\geq 0} \frac{x^n}{n^2} = x.{}_3F_2[(1,1,1);(2,2);x] $$

1.2. *Hypergeometric differential equation (Gauss)*

The differential equation:

$$ x(x-1)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + [c-(1+a+b)]\frac{\mathrm{d}y}{\mathrm{d}x} - aby = 0 $$

is satisfied by $F[(a,b);c;x]$.

1.3. *Hypergeometric integral (Euler)*

Let be the Gamma function $\Gamma(x) = \displaystyle\int_0^\infty e^{-u}u^{x-1}du$ and set

$$ I(x) = \int_0^1 u^{a-1}(1-u)^{c-a-1}(1-xu)^{-b}du $$

Then $I(x)$ converges if $\mathrm{Re}(a) > 0$ and $\mathrm{Re}(c-a) > 0$.
Moreover, $(1-xu)^{-b} = \sum_{n\geq 0}(b,n)u^n \frac{x^n}{n!}$

$$\begin{aligned} \text{so} \quad I(x) &= \sum_{n\geq 0}(b,n)x^n \int_0^1 u^{a+n-1}(1-u)^{c-a-1}du \\ &= \sum_{n\geq 0}\frac{(b,n)}{n!}x^n \frac{\Gamma(a+n)\Gamma(c-a)}{\Gamma(c+n)} \\ &= \frac{\Gamma(c-a)\Gamma(a)}{\Gamma(c)}F[(a,b);c;x] \end{aligned}$$

Thus we obtain the integral representation

$$\frac{\Gamma(c-a)\Gamma(a)}{\Gamma(c)}F[(a,b);c;x] = \int_0^1 u^{a-1}(1-u)^{c-a-1}(1-xu)^{-b}du$$

provided that $|x| < 1$, $\mathrm{Re}(a) > 0$, and $\mathrm{Re}(c-a) > 0$.

2. Modern Approach

This point of view is mainly due to Aomoto and Kita[1] on one hand, and Gelfand[7] and Varchenko[11] on the other hand.
Let $M_x = \mathbb{C} \setminus \{0, 1, x^{-1}\}$, $x \neq 0, 1$, and $\underline{\lambda} = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{C}^3$. Let

$$\Phi(u; \underline{\lambda}; x) = (1-u)^{\lambda_1} u^{\lambda_2} (1-xu)^{\lambda_3}$$

which defines a *multivalued* holomorphic function on M_x.
In order to write down suitable integrals using homology and cohomology, we must introduce *twisted* version where twisting comes from the change of Φ as we prolong it by analytic continuation while moving around $0, 1, x^{-1}$, so we have to introduce a rank one local system $\mathcal{L}$ on M_x given by the representation

$$\rho : \pi_1(M_x) \longrightarrow \mathrm{Aut}(\mathbb{C}) = Gl(1;\mathbb{C}) \simeq \mathbb{C}^*$$

$$\gamma_j \quad \longmapsto \exp(-2\pi\sqrt{-1}\lambda_j);\ j = 1, 2, 3$$

for any meridian loop γ_j about the hyperplane $H_j, j = 1, 2, 3$ where $H_1 = \{1\}, H_2 = \{0\}, H_3 = \{x^{-1}\}$.
More generally, let us define the **multidimensional** hypergeometric functions as follows.

Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be an affine arrangement of hyperplanes in $V \simeq \mathbb{C}^l$ and $M = V \setminus \bigcup_{i=1}^{n} H_i$. For each hyperplane H_i, choose a degree one polynomial α_i such that $H_i = \mathrm{Ker}\alpha_i$ and let $\underline{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ be a collection of weights. Define

$$\Phi(u; \underline{\lambda}) = \prod_{i=1}^{n} \alpha_i^{\lambda_i} := \Phi_\lambda$$

Φ_λ is a multivalued holomorphic function on, M
A *generalized hypergeometric integral* is of the form

$$\int_\sigma \Phi_\lambda \eta$$

where σ is a suitable domain of integration and η is a holomorphic form on M. As for the classical hypergeometric integral, we have to introduce a rank one local system $\mathcal{L}_\lambda$ on M defined with the monodromy $\exp(-2\pi\sqrt{-1}\lambda_i)$ around the hyperplanes H_i, $i = 1, \ldots, n$.
The need to calculate the local system cohomoloy $H^\bullet(M, \mathcal{L}_\lambda)$ arises in several problems: the Aomoto-Gelfand theory of multivariable hypergeometric integrals;[1,7] representation theory of Lie algebras and quantum groups and solutions of the Knizhnik-Zamolodchikov differential equations in conformal field theory;[11] determining the cohomology groups of the Milnor fiber of the non-isolated hypersurface singularity at the origin obtained by coning the arrangement.

3. Local Systems

Definition 3.1. A locally constant sheaf on M of complex vector spaces of dimension r is called **complex local system** of rank r on M.

Example 3.1. Let $r = 1$, $M = \mathbb{C}^*$, and U an open set in $\mathbb{C}^*$. Define $\mathcal{L}(U) = \mathbb{C}\{\text{branch of } \sqrt{u} \text{ on } U\}$. Then $\mathcal{L}(\mathbb{C}^*) = \{0\}, \mathcal{L}(U) \simeq \mathbb{C}$ if U is simply connected and $\mathcal{L}_x \simeq \mathbb{C}$. Then we get the monodromy representation

$$\rho : \pi_1(\mathbb{C}^*; 1) \longrightarrow Gl(1; \mathbb{C}) \simeq \mathbb{C}^*$$

$$\gamma \longmapsto -1$$

where γ is a generator of $\pi_1(\mathbb{C}^*; 1)$.

Example 3.2. Let $r = 1$, $M = \mathbb{C}^*$, and U an open set in $\mathbb{C}^*$. Define $\mathcal{L}(U) = \mathbb{C}\{\text{branch of } u^{-\lambda} \text{ on } U\}$ where $\lambda = -\dfrac{1}{2\pi\sqrt{-1}}\log\mu$, $\mu = e^{-2\pi\sqrt{-1}\lambda}$. Then $\mathcal{L}(\mathbb{C}^*) = \{0\}$, $\mathcal{L}(U) \simeq \mathbb{C}$ if U is simply connected and $\mathcal{L}_x \simeq \mathbb{C}$. Then we get the monodromy representation

$$\rho : \pi_1(\mathbb{C}^*; 1) \longrightarrow Gl(1; \mathbb{C}) \simeq \mathbb{C}^*$$

$$\gamma \longmapsto \mu$$

where γ is a generator of $\pi_1(\mathbb{C}^*; 1)$.

Remark 3.1. There is a bijection between the isomorphism classes of rank r local systems on M and the isomorphism classes of representations $\pi_1(M) \to Gl(r; \mathbb{C})$.

Our central object is the local system $\mathcal{L}_\lambda$ on M defined as follows.
Let $\mathcal{L}_\lambda(U) = \{f : U \to \mathbb{C}, \text{holomorphic such that } df + \omega_\lambda \wedge f = 0\}$ where $\omega_\lambda = \dfrac{d\Phi_\lambda}{\Phi_\lambda} = d(\log(\Phi_\lambda))$.

Proposition 3.1. *$\mathcal{L}_\lambda$ defines a local system of rank 1 on M with the monodromy representation $\rho : \pi_1(M) \longrightarrow Gl(1; \mathbb{C}) \simeq \mathbb{C}^*$ where $\rho(\gamma_i) \mapsto e^{-2\pi\sqrt{-1}\lambda_i}$ and γ_i are the generators of $\pi_1(M)$.*

Proof. Denote $\nabla_\lambda = d + \omega_\lambda \wedge$.

$$\nabla_\lambda(\Phi_\lambda^{-1}) = -\Phi_\lambda^{-2} d\Phi_\lambda + \frac{d\Phi_\lambda}{\Phi_\lambda}.\Phi_\lambda^{-1} = 0$$

If $\nabla_\lambda(f) = 0$, then $df = -f\omega_\lambda = -f\dfrac{d\Phi_\lambda}{\Phi_\lambda}$. Thus $d(f\Phi_\lambda) = df\Phi_\lambda + fd\Phi_\lambda = 0$ so $f \in \mathbb{C}\Phi_\lambda^{-1}$. Cover M with contractible open sets to see that $\mathcal{L}_\lambda$ is locally constant. □

Let us point out that for any $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$, the local system associated to $\lambda + m$ coincides with that associated to λ.

4. Hypergeometric Pairing

We have to interpret hypergeometric integrals as the result of the hypergeometric pairing.

The rank one local system $\mathcal{L}_\lambda$ on M defines the cohomology groups $H^p(M, \mathcal{L}_\lambda)$. On the other hand, the dual local system $\mathcal{L}_\lambda^\vee$ defines the homology groups $H_p(M, \mathcal{L}_\lambda^\vee)$. In fact, it is shown that we have a perfect pairing

$$H^p(M, \mathcal{L}_\lambda) \times H_p(M, \mathcal{L}_\lambda^\vee) \longrightarrow \mathbb{C}$$

$$(\eta, \sigma) \longmapsto \int_\sigma \Phi_\lambda \eta$$

In this paper, we focus on the determination of the cohomology groups $H^p(M, \mathcal{L}_\lambda)$.

5. Cohomology Groups $H^p(M, \mathcal{L}_\lambda)$

Let $\mathcal{O}$ denote the sheaf of germs of holomorphic functions on M and let $\Omega^\bullet$ be the de Rham complex of germs of holomorphic differentials on M, where $\Omega^0 = \mathcal{O}$. Then we see that $\nabla_\lambda : \Omega^0 \to \Omega^1$ is a flat connection whose kernel is $\mathcal{L}_\lambda$. Extend to a derivation of degree one. The sequence

$$0 \to \mathcal{L}_\lambda \longrightarrow \Omega^0 \xrightarrow{\nabla_\lambda} \Omega^1 \xrightarrow{\nabla_\lambda} \cdots \xrightarrow{\nabla_\lambda} \Omega^l \to 0$$

is exact.

Theorem 5.1. *(Holomorphic de Rham theorem)*

$$H^p(M, \mathcal{L}_\lambda) \simeq H^p(\Gamma(M; \Omega^\bullet), \nabla_\lambda)$$

where Γ *denotes global sections on* M*.*

Let $\Omega^p(*\mathcal{A})$ be the group of p-rational forms on M with poles on $\bigcup_{i=1}^n H_i$. Then $\Omega^p(*\mathcal{A}) \subset \Gamma(M; \Omega^\bullet)$. But $\omega_\lambda \in \Omega^p(*\mathcal{A})$ then $(\Omega^\bullet(*\mathcal{A}), \nabla)$ is a complex.

Theorem 5.2. *(Algebraic de Rham theorem (Deligne, Grothendieck))*

$$H^p(\Gamma(M; \Omega^\cdot), \nabla_\lambda) \simeq H^p(\Omega^p(*\mathcal{A}), \nabla_\lambda)$$

The problem is still difficult. We have to reduce the case of arbitrary poles on $N = \bigcup_{i=1}^n H_i$ to the case of order one (i.e. logarithmic). Following Deligne's results, we must compactify M with a normal crossing divisor. We embed $V \subset \mathbb{C}P^l$ by adding the infinite hyperplane, H_∞. Define the projective closure of $\mathcal{A}$, as $\mathcal{A}_\infty$. The divisor $N(\mathcal{A}_\infty)$ may have non-normal crossings. Before going further, let us review some basic notions on hyperplane arrangements.

6. Hyperplane Arrangements

6.1. *Generalities*

We refer the reader to Ref. 8 as a general reference on arrangements.
Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an arrangement of hyperplanes over $\mathbb{C}$, where H_i are affine hyperplanes of $\mathbb{C}^l$. $\mathcal{A}$ is also called l-arrangement. $\mathcal{A}$ is said to be **central** if $\bigcap_{H\in\mathcal{A}} H \neq \emptyset$. Define the complement $M(\mathcal{A}) = \mathbb{C}^l \setminus \bigcup_{i=1}^{n} H_i$ and $L(\mathcal{A})$ the poset of nonempty intersections of hyperplanes of $\mathcal{A}$ with reverse order

$$X \leq Y \text{ if } Y \subseteq X$$

Notice that the rank denoted rk satisfies $\mathrm{rk}(X) = \mathrm{codim}(X)$. An element $X \in L(\mathcal{A})$ is called an **edge** of $\mathcal{A}$. The rank of $\mathcal{A}$ is the maximal number of linearly independent hyperplanes in $\mathcal{A}$. In the following, we will denote $r = \mathrm{rk}(\mathcal{A})$.
Let $N(\mathcal{A}) = \bigcup_{H\in\mathcal{A}} H$ be the divisor of $\mathcal{A}$. For each hyperplane H_i, choose a degree one polynomial α_i such that $H_i = \mathrm{Ker}\alpha_i$. The product $Q(\mathcal{A}) = \prod_{i=1}^{n} \alpha_i$ is a defining polynomial for $\mathcal{A}$.
The affine arrangement $\mathcal{A}$ gives rise to a central arrangement $\mathbf{c}\mathcal{A}$ in $\mathbb{C}^{l+1}$, called the **cone** over $\mathcal{A}$. Let $\tilde{Q}$ be the homogenized $Q(\mathcal{A})$ with respect to the new variable u_0. Then $Q(\mathbf{c}\mathcal{A}) = u_0\tilde{Q}$.
Conversely, given a central arrangement $\mathcal{A}$ in $V = \mathbb{C}^{l+1}$ and $H \in V$, we define an affine arrangement $\mathbf{d}_H\mathcal{A}$, called the **decone** of $\mathcal{A}$ with respect to H.
Embed $V = \mathbb{C}^l$ in the complex projective space $\mathbb{C}P^l$ and call the complement of V the infinite hyperplane denoted $\overline{H}_\infty$. Let $\overline{H}$ be the projective closure of H and write $N(\mathcal{A}_\infty) = (\cup_{H\in\mathcal{A}})\overline{H}) \cup \{\overline{H}_\infty\}$. Then $Q(\mathcal{A}_\infty) = Q(\mathbf{c}\mathcal{A})$.
We construct the projective quotient $\mathbf{P}\mathcal{A}$ and choose coordinates so that $\mathbf{P}H = \mathrm{ker}u_0$ is the hyperplane at infinity. By removing it, we obtain an affine arrangement $\mathbf{d}_H\mathcal{A}$.
Let $\mu : L(\mathcal{A}) \to \mathbb{Z}$ be the **Möbius function** of $L(\mathcal{A})$ defined by $\mu(V) = 1$, and for $X > Y$ by the recursion $\sum_{Y\leq X} \mu(Y) = 0$.

Definition 6.1. The **characteristic polynomial** of $\mathcal{A}$ is defined as

$$\chi(\mathcal{A}, t) = \sum_{X\in L(\mathcal{A})} \mu(X) t^{\dim X}.$$

Given an edge $X \in L(\mathcal{A})$, we define the subarrangement $\mathcal{A}_X$ of $\mathcal{A}$ by $\mathcal{A}_X = \{H \in \mathcal{A} \mid X \subseteq H\}$ and the arrangement $\mathcal{A}^X$ called **restriction** of $\mathcal{A}^X$ to X by $\mathcal{A}^X = \{X \cap H \mid H \in \mathcal{A} \text{ and } X \cap H \neq \emptyset\}$.
The *deletion-restriction triple* $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ is a nonempty arrangement $\mathcal{A}$ and $H \in \mathcal{A}$ together with $\mathcal{A}' = \mathcal{A} \setminus \{H\}$ and $\mathcal{A}'' = \mathcal{A}^H$.

6.2. *Orlik-Solomon algebra*

The Orlik-Solomon algebra is combinatorially defined. Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an affine arrangement and let $\{e_1, \dots, e_n\}$ be a set. Let $E = \bigwedge_{\mathbb{C}}(e_1, \dots, e_n)$ be the free exterior algebra over $\mathbb{C}$. If $S = (i_1, \dots, i_p)$ is an ordered p-tuple, denote the product $e_{i_1} \wedge \cdots \wedge e_{i_p}$ by e_S. Then the **Orlik-Solomon algebra** is defined as

$$A^\bullet(\mathcal{A}) := E/\mathcal{J}$$

where $\mathcal{J}$ ideal generated by all e_S with $\bigcap_{i \in S} H_i = \emptyset$ and the relations of the form:

$$\sum_{j=1}^{s} (-1)^{i-1} e_{i_1} \dots \widehat{e}_{i_j} \dots e_{i_s}$$

for all $1 \leq i_1 < \dots < i_s \leq n$ such that $\mathrm{rk}(H_{i_1} \cap \cdots \cap H_{i_s}) < s$, i.e. the hyperplanes $\{H_{i_1}, \cdots, H_{i_s}\}$ are dependent and where $\widehat{}$ indicates an omitted factor.
Notice that for a central arrangement, $\bigcap_{i \in S} H_i \neq \emptyset$ for any S.
We will denote a_H (resp. a_S) the image of e_H (resp. e_S) under the natural projection and denote $b_p(\mathcal{A}) = \dim A^p(\mathcal{A})$ the p-th Betti number of $A^\bullet$.
Let $P(\mathcal{A}, t) = \sum_{p \geq 0} b_p(\mathcal{A}) t^p$ be the **Poincaré polynomial** of $A^\bullet$. It is shown that $P(\mathcal{A}, t) = \sum_{X \in L(\mathcal{A})} \mu(X)(-t)^{\mathrm{codim}(X)}$.
Moreover, $P(c\mathcal{A}, t) = (1+t)P(\mathcal{A}, t)$.
$P(\mathcal{A}, t)$ is closely related to the characteristic polynomial $\chi(\mathcal{A}, t)$. As an important property of this algebra is the following theorem due to Orlik and Solomon.

Theorem 6.1. *The cohomology algebra and the Orlik-Solomon algebra are isomorphic as graded algebras*

$$A^\bullet(\mathcal{A}) \cong H^\bullet(M(\mathcal{A}); \mathbb{C}).$$

6.3. *Brieskorn algebra*

Define the **Brieskorn algebra** denoted $\mathrm{B}^\bullet(\mathcal{A})$ as the $\mathbb{C}$-algebra generated by 1 and the forms $\omega_H = \dfrac{d\alpha_H}{\alpha_H} = d\log(\alpha_H)$, $H \in \mathcal{A}$.

The inclusion $\mathrm{B}^\bullet(\mathcal{A}) \subset \Omega_M^\bullet$ induces isomorphisms of graded algebras

$$\mathrm{B}^\bullet(\mathcal{A}) \simeq H^\bullet(M(\mathcal{A}), \mathbb{C})$$

6.4. *Dense edges*

Let $\mathcal{A}$ be a central arrangement in V. We call $\mathcal{A}$ **decomposable** if $\mathcal{A}$ is the disjoint union of two nonempty subarrangements $\mathcal{A}_1$ and $\mathcal{A}_2$.

Definition 6.2. $X \in L(\mathcal{A})$ is called **dense edge** in $\mathcal{A}$ if and only if the central arrangement $\mathcal{A}_X$ is not decomposable. Let $\mathrm{D}_j(\mathcal{A})$ denote the set of dense edges of dimension j and let $\mathrm{D}(\mathcal{A}) = \bigcup\limits_{j\geq 0} \mathrm{D}_j$.

The divisors $N(\mathcal{A})$ and $N(\mathcal{A}_\infty)$ do not have normal crossings along a dense edge.

Example 6.1. An l-arrangement $\mathcal{A}$ is called a **general position** arrangement if for every subset $\{H_1, \ldots, H_q\} \subseteq \mathcal{A}$ with $q \leq l$, then $r(H_1 \cap \ldots \cap H_q) = q$ and when $q > l$, $H_1 \cap \ldots \cap H_q = \emptyset$. For such a arrangement $\mathcal{A}$, the set of dense edges $D(\mathcal{A}) = \emptyset$.

Lemma 6.1. *Let $\mathcal{A}$ be an nonempty central arrangement with $H \in \mathcal{A}$. If $\mathcal{A}'$ and $\mathcal{A}''$ are decomposable, then $\mathcal{A}$ is decomposable.*

Example 6.2. Let $\mathcal{A}$ be the Selberg arrangement defined by

$$Q(\mathcal{A}) = u_1(u_1 - 1)u_2(u_2 - 1)(u_1 - u_2).$$

Label the hyperplanes in the order given by the factors in Q and write j in place of H_j. Let

$$Q(\mathcal{A}_\infty) = u_0 u_1(u_1 - u_0)u_2(u_2 - u_0)(u_1 - u_2) = u_0 Q(\mathcal{A}).$$

Then $\mathrm{D}_1(\mathcal{A}) = \{1, 2, 3, 4, 5\}$, $\mathrm{D}_2(\mathcal{A}) = \{135, 245\}$. The additional dense edges in its projective closure $\mathcal{A}_\infty$ are $\{\infty, 12\infty, 34\infty\}$.

6.5. *The β invariant*

In higher dimensions, it is difficult to determine the dense edges. The following provides a numerical criterion to decide which edges are dense. Recall that the characteristic polynomial of the arrangement $\mathcal{A}$ is defined as $\chi(\mathcal{A}, t) = \sum_{X \in L(\mathcal{A})} \mu(X) t^{\dim(X)}$.

Remark 6.1. Let $\mathcal{A}$ be a central arrangement, then $\chi(\mathcal{A}, 1) = 0$.

Definition 6.3. Let $\mathcal{A}$ be an arrangement of rank r. Define its **beta invariant** by

$$\beta(\mathcal{A}) = (-1)^r \chi(\mathcal{A}, 1)$$

Remark 6.2. $\beta(\mathcal{A}) = |e(M)|$, where $e(M)$ is the Euler characteristic of the complement. $\beta(\mathcal{A})$ is a combinatorial invariant.

Remark 6.3. If $\mathcal{A}$ is a complexified real arrangement (i.e. the polynomials defining the hyperplanes have real coefficients), then $\beta(\mathcal{A})$ is the number of bounded chambers of the complement of the real arrangement.

Theorem 6.2. *Let $\mathcal{A}$ be an arrangement and let $X \in L(\mathcal{A})$. The following conditions are equivalent:*

1. *X is dense,*
2. *$\mathcal{A}_X$ is not decomposable,*
3. *$\beta(\mathbf{d}\mathcal{A}_X) \neq 0$,*
4. *$\beta(\mathbf{d}\mathcal{A}_X) > 0$.*

The proof mainly uses the properties of the triple $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ where $\mathcal{A}' = \mathcal{A} \setminus \{H\}$ for a suitable H.

7. Resonance

Let $\lambda_\infty = -\sum_{H \in \mathcal{A}} \lambda_H$ be the *weights* of H_∞. For $X \in L(\mathcal{A}_\infty)$, define $\lambda_X \in \mathbb{C}$ by

$$\lambda_X = \sum_{X \subset H} \lambda_H, \quad H \in \mathcal{A}.$$

Let $\omega_\lambda = \sum_{H \in \mathcal{A}} \lambda_H \omega_H = d(\log \Phi_\lambda)$. Since $\omega_\lambda \wedge \omega_\lambda = 0$, wedge product with ω_λ provides a finite dimensional subcomplex $(\mathrm{B}^\bullet, \omega_\lambda \wedge)$ of $(\Omega^\bullet(*\mathcal{A}), \nabla_\lambda)$:

$$0 \longrightarrow \mathrm{B}^0 \xrightarrow{\omega_\lambda \wedge} \mathrm{B}^1 \xrightarrow{\omega_\lambda \wedge} \dots \xrightarrow{\omega_\lambda \wedge} \mathrm{B}^l \longrightarrow 0$$

Theorem 7.1 (Refs. 4,10). *Assume that $\lambda_X \notin \mathbf{Z}_{>0}$, for every dense edge $X \in L(\mathcal{A}_\infty)$. Then, for every p*

$$H^p(M, \mathcal{L}_\lambda) \simeq H^p(\mathrm{B}^\bullet(\mathcal{A}), \omega_\lambda \wedge).$$

Notice that the map $a_H \mapsto \omega_H$ induces an isomorphism of graded algebras $A^\bullet(\mathcal{A}) \to B^\bullet(\mathcal{A})$.
Let $a_\lambda = \sum_{H \in \mathcal{A}} \lambda_H a_H$. Since $a_\lambda \wedge a_\lambda = 0$, exterior product with a_λ provides a complex $(A^\bullet(\mathcal{A}), a_\lambda \wedge)$

$$0 \longrightarrow A^0(\mathcal{A}) \xrightarrow{a_\lambda \wedge} A^1(\mathcal{A}) \xrightarrow{a_\lambda \wedge} \dots \xrightarrow{a_\lambda \wedge} A^l(\mathcal{A}) \longrightarrow 0$$

Corollary 7.1 (Ref. 9). *Assume that $\lambda_X \notin \mathbf{Z}_{>0}$, for every dense edge $X \in L(\mathcal{A}_\infty)$ Then, for every p*

$$H^p(M, \mathcal{L}_\lambda) \simeq H^p(\mathrm{B}^\bullet(\mathcal{A}), \omega_\lambda \wedge) \simeq H^p(A^\bullet(\mathcal{A}), a_\lambda \wedge).$$

Example 7.1. Let $\mathcal{A}$ be a general position arrangement. Then for all $p < l$, $H^p(A^\bullet(\mathcal{A}), a_\lambda \wedge) = 0$.

This completes the transformation of the analytic problem into a problem in combinatorics.
We mention the following easy result which explains the assumption on the weights. This result explains why it is assumed $\lambda_\infty = -\sum_{H \in \mathcal{A}} \lambda_H$.

Proposition 7.1 (Refs. 5,12). *If $\sum_{H \in \mathcal{A}_\infty} \lambda_H \neq 0$, then $H^\bullet(A^\bullet, a_\lambda \wedge) = 0$.*

Proof. Let ∂ be the map defining by $\partial(a_i a_S) = a_S - a_i \partial a_S$. Let denote d_λ the left multiplication by a_λ. It follows that $d_\lambda \partial + \partial d_\lambda = (\sum_{H \in \mathcal{A}_\infty} \lambda_H) id$, so that $(\sum_{H \in \mathcal{A}_\infty} \lambda_H)^{-1} \partial$ is a chain contraction of $(A^\bullet, a_\lambda \wedge)$. Thus $H^\bullet(A^\bullet, a_\lambda \wedge) = 0$. □

The case $\underline{\lambda} = 0$ implies the well-known result that $A^\bullet(\mathcal{A})$ is isomorphic to the ordinary cohomology $H^\bullet(M; \mathbb{C})$.
The following result is due to S. Yuzvinsky.

Theorem 7.2 (Ref. 12). *Assume that $\lambda_X \notin \mathbf{Z}_{\geq 0}$, for every dense edge $X \in L(\mathcal{A}_\infty)$. Then*

$$H^q(A^\bullet(\mathcal{A}), a_\lambda \wedge) = 0 \text{ for } q \neq r$$

and

$$\dim H^r((A^\bullet(\mathcal{A}), a_\lambda \wedge) = \dim H^r(M, \mathcal{L}_\lambda) = \mid e(M) \mid$$

Up to now, we considered some systems of weights which defined a subset of $\mathbb{C}^n$:

$$\mathbf{W}(\mathcal{A}) = \{\underline{\lambda} \in \mathbb{C}^n \mid \lambda_X \notin \mathbb{Z}_{\geq 0} \quad \text{for every dense edge } X \text{ of } \mathcal{A}_\infty\}$$

For $\underline{\lambda} \in \mathbf{W}(\mathcal{A})$, the local system cohomology groups are independent of the weights. There exists a maximal dense open subset $\mathbf{U}(\mathcal{A}) \supseteq \mathbf{W}(\mathcal{A})$ where the local system cohomology groups are independent of the weights. We call weights $\underline{\lambda} \in \mathbf{U}(\mathcal{A})$ **non-resonant**. Notice that there are in general non-resonant weights which do not lie in the set $\mathbf{W}(\mathcal{A})$.
Much is known in such a case which is called of non-resonant weights. But substantially less is known about resonant weights. This interest has been a motivating factor in many works and we will refer to the paper of D. Cohen and P. Orlik.[3] They study the local system cohomology using stratified Morse theory following D. Cohen.[2] They construct a **universal complex** $(K^\bullet_\Lambda(\mathcal{A}), \Delta^\bullet(x))$, where $x = (x_1, \ldots, x_n)$ are non-zero complex variables, $\Lambda = \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ denote the ring of complex Laurent polynomials, $K^q_\Lambda(\mathcal{A}) = \Lambda \otimes_{\mathbb{C}} K^q_\Lambda(\mathcal{A}) \simeq \Lambda^{b_q(\mathcal{A})}$ where $b_q(\mathcal{A}) = \dim A^q(\mathcal{A}) = \dim H^q(M;\mathbb{C})$ is the q-th Betti number of M with trivial local coefficients $\mathbb{C}$ and $\Delta^\bullet(x)$ are Λ-linear with the property that the specialization $x_j \mapsto t_j = \exp(-2\pi i \lambda_j)$ calculates $H^\bullet(M, \mathcal{L}_\lambda)$.
There is also a similar universal complex, called the **Aomoto complex** $(A^\bullet_R(\mathcal{A}), a_y\wedge)$ where $y = (y_1, \ldots, y_n)$ are variables, $R = \mathbb{C}[y_1, \ldots, y_n]$ is the polynomial ring, where $A^q_R(\mathcal{A}) = R \otimes_{\mathbb{C}} A^q(\mathcal{A})$ and boundary maps are given by $p(y) \otimes \eta \mapsto y_i p(y) \otimes a_{H_i} \wedge \eta$.
For $\lambda \in \mathbb{C}^n$, the specialization $y \mapsto \lambda$ of the Aomoto complex yields the Orlik-Solomon algebra $(A^\bullet(\mathcal{A}), a_\lambda \wedge)$.

Theorem 7.3 (Ref. 3). *For any arrangement $\mathcal{A}$, the Aomoto complex $(A^\bullet_R(\mathcal{A}), a_y\wedge)$ is chain equivalent to the linearization of the universal complex $(K^\bullet_\Lambda(\mathcal{A}), \Delta^\bullet(x))$.*

Recall that, for $m \in \mathbb{Z}^n$, the local system associated to $\lambda + m$ coincides with that associated to λ. Finally, some lower and upper bounds could be determined for arbitrary weights as following:

$$\sup_{\mathrm{m} \in \mathbb{Z}^{\mathrm{n}}} \dim H^p(A^\bullet, a_{\lambda+m}\wedge) \leq \dim H^p(M, \mathcal{L}_\lambda) \leq \dim H^p(M;\mathbb{C})$$

Thus a system of weights $\underline{\lambda} \in \mathbb{C}^n$ is *non-resonant* if the Betti numbers of M with coefficients in the local system $\mathcal{L}_\lambda$ are minimal.

8. The NBC Complex

Let $\mathcal{A}$ be an affine arrangement of hyperplanes over $\mathbb{C}$, $\mathcal{A} = \{H_1, \ldots, H_n\}$, define the linear order on $\mathcal{A}$ by $H_i \prec H_j$ if $i < j$. An inclusion-minimal dependent set is called a **circuit**. A **broken circuit** is a set S for which there exist $H \prec \min(S)$ such that $\{H\} \cup S$ is a circuit. The collection of the empty set and the nonempty subsets of $\mathcal{A}$ which have nonempty intersection and contain no broken circuit is called NBC($\mathcal{A}$). Since NBC($\mathcal{A}$) is closed under taking subsets, it forms a pure $(r-1)$-dimensional simplicial complex.

Example 8.1. Let $\mathcal{A}$ be the Selberg arrangement as defined above. The 1-simplices of NBC($\mathcal{A}$) are $\{\{1;3\},\{2,3\},\{1,4\},\{2,4\},\{1,5\},\{2,5\}\}$.

Theorem 8.1 (Ref. 9). *Let $\mathcal{A}$ be an l-arrangement of hyperplanes of rank $r \geq 1$. Then NBC($\mathcal{A}$) has the homotopy type of a wedge of spheres, $\bigvee_{\beta(\mathcal{A}} S^{r-1}$.*

Idea of the proof: If v is a vertex of NBC($\mathcal{A}$) then its **star**, denoted st(v), consists of all the simplexes whose closure contains v. The closure $\overline{\mathrm{st}(v)}$ is a cone with cone point v. Let $(\mathcal{A},\mathcal{A}',\mathcal{A}'')$ be a triple with respect to the last hyperplane H_n. Then prove that $\mathrm{NBC}(\mathcal{A}'') \simeq \overline{\mathrm{st}(H_n)} \cap \mathrm{NBC}(\mathcal{A}')$. Consider the Mayer-Vietoris sequence for the excisive couple $\{\overline{\mathrm{st}(H_n)}, \mathrm{NBC}(\mathcal{A}')\}$ to get the long exact sequence and use the fact that $\overline{\mathrm{st}(H_n)}$ is contractible. □

Theorem 8.2 (Ref. 9). *Let $\mathcal{A}$ be an l-arrangement of hyperplanes of rank $r \geq 1$. Then*

$$\begin{aligned} H^p(\mathrm{NBC}(\mathcal{A})) &= 0 && \text{if } p \neq r-1 \\ &= \text{free of rank } \beta(\mathcal{A}) && \text{if } p = r-1 \end{aligned}$$

Idea of the proof: Use induction on r and consider the long exact sequence of the previous theorem. □

A maximal independent set is called a **frame**. An $(r-1)$-dimensional simplex of NBC($\mathcal{A}$) is called an **nbc frame**. Following Ziegler,[13] let define a subset $\beta\mathbf{nbc}(\mathcal{A})$ of $\mathbf{nbc}(\mathcal{A})$ of cardinality $\beta(\mathcal{A})$.
Let define a $\beta\mathbf{nbc}(\mathcal{A})$ frame B as a frame which is a **nbc** frame such that for every $H \in B$, there exists $H' \prec H$ in $\mathcal{A}$ with $(B \setminus \{H\} \cup \{H'\}$ is a frame. Let $\beta\mathbf{nbc}(\mathcal{A})$ be the set of all $\mathbf{nbc}(\mathcal{A})$ frames. When $\mathcal{A}$ is empty, we agree that $\beta\mathbf{nbc}(\mathcal{A})= \emptyset$.
Let $(\mathcal{A},\mathcal{A}',\mathcal{A}'')$ be a triple with respect to the last hyperplane H_n. Then there is a disjoint union $\beta\mathbf{nbc}(\mathcal{A}) = \beta\mathbf{nbc}(\mathcal{A}') \cup \overline{\beta\mathbf{nbc}(\mathcal{A}'')}$ where $\overline{\beta\mathbf{nbc}(\mathcal{A}'')} = \{\{\nu(B''), H_n\} \mid b'' \in \beta\mathbf{nbc}(\mathcal{A}'')\}$ and $\nu(X) = \min(\mathcal{A}_X)$.
For an nbc frame $B \in \mathbf{nbc}(\mathcal{A})$, $B^* \in C^{r-1}(\mathrm{NBC}(\mathcal{A}))$, denotes the $(r-1)$-cochain dual to B.

Theorem 8.3 (Ref. 6). *The set $\{[B^*] \mid B \in \beta\mathbf{nbc}(\mathcal{A})\}$ is a basis for $H^{r-1}(\mathrm{NBC}(\mathcal{A}))$.*

Example 8.2. Let $\mathcal{A}$ be the Selberg arrangement as defined above. Then $\beta\mathbf{nbc}(\mathcal{A}) = \{\{2,4\},\{2,5\}\}$. So the cohomology classes $[\{2,4\}]^*$ and $[\{2,5\}]^*$ form a basis for $H^1(\mathrm{NBC}(\mathcal{A}))$.

Let us back to the combinatorial complex $(A^\bullet(\mathcal{A}), a_\lambda\wedge)$.

Theorem 8.4 (Ref. 9). *Let $\mathcal{A}$ be an affine arrangement with projective closure $\mathcal{A}_\infty$. Assume that $\lambda_X \neq 0$ for every dense edge $X \in D(\mathcal{A}_\infty)$, then*

$$H^p(A^\bullet(\mathcal{A}), a_\lambda\wedge) \simeq H^{p-1}(\mathrm{NBC}(\mathcal{A}), \mathbb{C})$$

Let $B = \{H_{i_1}, \cdots, H_{i_r}\}$ be a $\beta\mathbf{nbc}$ frame and $X_1 > \cdots > X_r$ where $X_p = \bigcap_{k=p}^{r} H_{i_k}$ for $1 \leq p \leq r$. Define $\zeta(B) = \bigwedge_{p=1}^{r} \omega_\lambda(X_p)$ where $\omega_\lambda(X) = \sum_{H\in\mathcal{A}_X} \lambda_H \omega_H \in \mathrm{B}^1(\mathcal{A})$.

Theorem 8.5 (Ref. 9). *Let $\mathcal{A}$ be an affine arrangement of rank r with projective closure $\mathcal{A}_\infty$. Assume that $\lambda_X \notin \mathbf{Z}_{\geq 0}$ for every $X \in \mathrm{D}(\mathcal{A}_\infty)$. Then the set*

$$\{\zeta(B) \in H^r(M, \mathcal{L}_\lambda) \mid B \in \beta\mathbf{nbc}(\mathcal{A})\}$$

is a basis for the only nonzero local system cohomology group, $H^r(M, \mathcal{L}_\lambda)$.

Thus, there is an explicit isomorphism between the only non trivial cohomology group $H^r(M, \mathcal{L}_\lambda)$ and $H^{r-1}(\mathrm{NBC}(\mathcal{A}), \mathbb{C})$ under the non resonance conditions.

Example 8.3. Let $\mathcal{A}$ be the Selberg arrangement once more. Assume the weights of the dense edges satisfy suitable conditions. Then

$$\zeta(\{2,4\}) = (\lambda_2\omega_2 + \lambda_4\omega_4 + \lambda_5\omega_5)\lambda_4\omega_4 = \lambda_2\lambda_4\omega_{24} - \lambda_4\lambda_5\omega_{45}$$

$$\zeta(\{2,5\}) = (\lambda_2\omega_2 + \lambda_4\omega_4 + \lambda_5\omega_5)\lambda_5\omega_5 = \lambda_2\lambda_5\omega_{25} + \lambda_4\lambda_5\omega_{45}$$

provide a basis for $H^2(M, \mathcal{L}_\lambda)$.

References

1. K. Aomoto and M. Kita, *Hypergeometric Functions* (in Japanese), (Springer-Verlag, 1994).
2. D. Cohen, *Adv. Math.* **97**, 231 (1993).
3. D. Cohen and P. Orlik, *Mathematical Research Letters* **7**, 299 (2000).
4. H. Esnault, V. Schechtman and E. Viehweg, *Invent. Math.* **109**, 557 (1992); Erratum, ibid **112**, 447 (1993).
5. M. Falk, *Annals of Combinatorics* **1**, 135 (1997).

6. M. Falk and H. Terao, *Trans. AMS* **349**, 189 (1997).
7. I. M. Gelfand, *Soviet Math. Dokl.* **33**, 573 (1986).
8. P. Orlik and H. Terao, *Arrangements of Hyperplanes*, *Grundlehren Math. Wiss.*, Vol. **300** (Springer-Verlag, Berlin, 1992).
9. P. Orlik and H. Terao, *Arrangements and Hypergeometric Integrals*, MSJ Memoirs, *Mathematical Society of Japan* **9** (2001).
10. V. Schechtman, H. Terao and A. Varchenko, *J. Pure Appl. Algebra* **100**, 93 (1995).
11. A. N. Varchenko, *Multidimensional Hypergeometric Functions and Representation Theory of Lie groups*, *Adv. Studies in Maths. Physics* **21** (World Scientific Publishers, 1995).
12. S. Yuzvinsky, *Comm. Algebra* **23**, 5339 (1995).
13. G. Ziegler, *J. Alg. Combinatorics* **1**, 283 (1992).

BERNSTEIN-SATO POLYNOMIALS AND FUNCTIONAL EQUATIONS

MICHEL GRANGER
Université d'Angers, LAREMA, UMR 6093 du CNRS
2 Bd Lavoisier 49045 Angers France

February 11, 2010

These notes are an expanded version of the lectures given in the frame of the I.C.T.P. School held at Alexandria in Egypt from 12 to 24 November 2007.

Our purpose in this course was to give a survey of the various aspects, algebraic, analytic and formal, of the functional equations which are satisfied by the powers f^s of a function f and involve a polynomial in one variable $b_f(s)$ called the Bernstein-Sato polynomial of f. Since this course is intended to be useful for newcomers to the subject we give enough significant details and examples in the most basic sections, which are sections 1, 2, and also 4. The latter is devoted to the calculation of the Bernstein-Sato polynomial for the basic example of quasi-homogeneous polynomials with isolated singularities. This case undoubtedly served as a guide in the first developments of the theory.

We particularly focused our attention on the problem of the meromorphic continuation of distribution f_+^s in the real case, which in turn motivated the problem of the existence of these polynomials, without forgetting related questions like the Mellin transform and the division of distributions. See the content of section 3. The question of the analytic continuation property was brought up as early as 1954 at the congress of Amsterdam by I.M. Gelfand. The meromorphic continuation was proved 15 years later independently by Atiyah and I.N. Bernstein-S.I. Gel'fand who used the resolution of singularities. The existence of the functional equations proved by I.N. Bernstein in the polynomial case allowed him to give a simpler proof which does not use the resolution of singularities. His proof establishes at the same time relationship between the poles of the continuation and the zeros of the Bernstein Sato polynomial. The already known rationality of the poles gave a strong reason for conjecturing the famous result about the rationality of the zeros of the b-function which was proved by Kashiwara and Malgrange.

We also want mention another source of interest for studying functional equations due to Mikio Sato. It concerns the case of the semi-invariants of prehomogeneous actions of an algebraic group, and especially of a reductive group. In the latter case the functional equation is of a very particular type and the name b-function frequently employed as a shortcut the for Bernstein-Sato Polynomial, comes from this theory. We give the central step of the proof of the existence theorem in the reductive case in section 2.4.

Let us summarize the contents of the different sections. In section 1, we give the basic definitions and elementary facts about b-functions with emphasis on first hand examples. In section 2 we recall the proof of Bernstein for the existence theorem in the polynomial case. Although also treated by F. Castro in this volume we give it for the sake of completeness and also to make it clear that the case of multivariable Bernstein-Sato polynomials can be solved with the same proof in the algebraic case. In section 3 we give a detailed proof of the analytic continuation property using the functional equation and we make a comparison with the proof which uses the Mellin transform and asymptotic expansions in the way Atiyah and Bernstein-Gel'fand first did. In section 4, we give a proof of the calculation of the Bernstein polynomial in the quasi-homogeneous case. In so doing we give a large view of a preprint less accessible to the public than a published version which treats directly the more complicate case of a singularity non-degenerate with respect to its Newton polygon. In section 5 we give the main steps of the proof of the existence theorem for the local analytic case. This proof is originally due to Kashiwara and uses a fairly large amount of material from analytic $\mathcal{D}$-module theory. In order to make this section 5 more readable we gathered a summary of the necessary material in section 7 refering to the literature for the details. Finally in section 6 we give an account without proof of a very fundamental property of Bernstein-Sato polynomials, the fact that their roots are negative rational number. This result using different methods, is basically due to B. Malgrange and M. Kashiwara.

I am aware of the fact that these notes do not cover all the aspects of the subject or recent developments like the theory of the V filtration in the continuity of Kashiwara and Malgrange results, the microlocal aspects, the computational aspects in the algebraic case, the relative b-functions and their link with deformations, the prehomogeneous space aspects. I refer the reader to the bibliography and the references it contains for further reading.

I wish to end this introduction first by thanking Professor Lê Duńg Tráng who conceived and organized the school, as well as the localorganizers and especially Professor Mohamed Darwish for their hospitality and the way they took care of all the practical details of our stay in Alexandria.

Contents

1 Introduction to Functional Equations **228**

1.1 *Definitions* 228
1.2 *A review of a number of elementary facts about b-functions* 230
1.3 *A first list of examples* 231
1.4 *Remarks on variants of the definition* 232

2 On the existence of a functional equation in the polynomial case **233**

2.1 *Holonomic modules* 233
2.2 *Bernstein equation* 235
2.3 *Generalisations* 236
2.4 *Semi-invariants of prehomogeneous spaces* 238

3 Bernstein-Sato polynomials and analytic continuation of f^s **241**

3.1 *Roots of b and analytic continuation of $Y(f)f^s$* 241
3.2 *Asymptotic expansion and Mellin transforms* 243
3.3 *Application to analytic continuation* 245
3.4 *Application to the division of distribution* 247

4 Quasi-homogeneous and semi-quasi-homogeneous isolated singularities **248**

4.1 *Quasi-homogeneous polynomials* 248
4.2 *Semi-quasi-homogeneous germs* 251
4.3 Calculation of the Bernstein-Sato polynomial of a quasi-homogeneous polynomial 253

5 Proof of the existence of the b-function for a an analytic germ **256**

5.1 *The proof in the analytic case* 256
5.2 *An application: Holonomicity of the module $\mathcal{O}[\frac{1}{f}]$* 259
5.3 Links between various Bernstein-Sato polynomials 260

6 Rationality of the zeroes over any algebraically closed field of characteristic zero **261**

6.1 *The rationality of the zeros* 261
6.2 *Derived results* 262

7 Introduction to analytic differential modules. Abridged version **263**
7.1 *The ring of differential operators* 263
7.1.1 *Definition of $\mathcal{D}(U)$ on an open subset of $\mathbb{C}^n$* 263
7.1.2 Behaviour under change of coordinates 264
7.1.3 Sheaf of differential operators on an analytic manifold 266
7.1.4 *Principal symbols, and graded associated sheaves* . . 266
7.1.5 *Coherence* 268
7.2 *Coherent $\mathcal{D}$-modules or differential systems* 270
7.2.1 *What is a differential system?* 270
7.2.2 *Regular connections* 271
7.3 *Good filtrations and coherence conditions* 273
7.4 *Characteristic variety of a differential system* 276
7.4.1 *Case of a monogeneous module $\mathcal{M} = \frac{\mathcal{D}}{\mathcal{I}}$* 276
7.4.2 *General case* 277
7.4.3 *A finiteness property* 278

8 Complements **282**
8.1 *Appendix A: Mellin and Laplace transforms* 282
8.1.1 *Integrals depending on a parameter* 282
8.1.2 *Analyticity of Mellin transforms* 282
8.2 Appendix B: Regular sequences, and application to the annihilator $Ann_{\mathcal{D}} f^s$, in the isolated case 285

1 Introduction to Functional Equations

1.1 *Definitions*

Let us consider the ring $\mathcal{O} = \mathbb{C}\{x_1, \cdots, x_n\}$ of germs of functions defined by a convergent power series at the origin of $\mathbb{C}^n$:

$$f(x_1, \cdots, x_n) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha \underline{x}^\alpha \text{ with } \underline{x}^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

Let $\mathcal{D}$ be the ring of differential operators with coefficients in $\mathcal{O}$ i.e. the set of sums P of monomials in the variables x_i and in the partial derivatives denoted ∂_i or $\frac{\partial}{\partial_{x_i}}$

$$P = P(\underline{x}, \underline{\partial}_x) = \sum_{\beta \in \mathbb{N}^n} f_\beta(\underline{x}) \underline{\partial}_x^\beta \text{ with } \underline{\partial_x}^\beta = \partial_1^{\beta_1} \cdots \partial_n^{\beta_n}.$$

Given a germ of an analytic function $f \in \mathcal{O} = \mathbb{C}\{x_1, \cdots, x_n\}$ there exists a non zero polynomial $e(s)$ and an analytic differential operator $P(s) \in \mathcal{D}[s]$ polynomial in the indeterminate s such that

$$\boxed{P(s)f^{s+1} = e(s)f^s} \tag{1}$$

The set of polynomials $e(s)$ for which an equation of the type (1) exists is clearly an *ideal* B_f of $K(s)$, which is principal since $K[s]$ is a principal domain.

The *Bernstein-Sato polynomial* of f is by definition the monic generator of this ideal denoted by $b_f(s)$ or simply $b(s)$:

$$B_f = K[s] \cdot b_f(s)$$

Here are some variants and extensions of this definition

1) Algebraic case. When $f \in K[X_1, \cdots, X_n]$ we consider the Weyl algebra $A_n(K)$ i.e. the set of operators $P(\underline{x}, \underline{\partial}_x) = \sum_{\beta \in \mathbb{N}} f_\beta(\underline{x}) \underline{\partial}_x^\beta$ with polynomial coefficients $f_\beta \in K[X_1, \cdots, X_n]$. The global or algebraic Bernstein-Sato polynomial is the monic generator of the ideal of polynomials $e(s)$ included in a functional equation as in (1) but with a polynomial operator $P(s) \in A_n(K)[s]$.

2) We may also consider the formal analogue where f and the coefficients of $P(s)$ are in $K[[X_1, \cdots, X_n]]$.

3) A generalisation: let $f_1, \cdots, f_p$ be p elements in $\mathbb{C}\{x_1, \cdots, x_n\}$. Then there exists a non zero polynomial $b(s_1, \cdots, s_p) \in \mathbb{C}[s_1, \cdots, s_n]$, and a functional equation:

$$\boxed{b(s_1, \cdots, s_p) f_1^{s_1} \cdots f_p^{s_p} = P(s_1, \cdots, s_p) f_1^{s_1+1} \cdots f_p^{s_p+1}} \tag{2}$$

with

$$b \in \mathbb{C}[s_1, \cdots, s_p], \quad P \in \mathcal{D}[s_1, \cdots, s_p]$$

The set of polynomials $b(s_1, \cdots, s_p)$ as in (2) is an ideal $\mathcal{B}_{(f_1, \cdots, f_p)}$ of $K[s_1, \cdots, s_p]$. There is an algebraic variant of this notion of a Bernstein-Sato ideal $\mathcal{B}_{(f_1, \cdots, f_p)}$.

History of the existence theorem

This polynomial was simultaneously introduced by Mikio Sato in a different context in view of giving functional equations for relative invariants of prehomogeneous spaces and of studying zeta functions associated with them, see [33], [34]. The name b-function comes from this theory and the so-called a,b,c functions of M. Sato, see [35] for definitions. The existence of a nontrivial equation as stated in (1), was first proved by I.N. Bernstein in the polynomial case, see [6]. The polynomials $b_f(s)$ are called Bernstein-Sato polynomials in order to take this double origin into account. The analytic local case is due to Kashiwara in [21]. An algebraic proof by Mebkhout and Narvaez can be found in [32]. The formal case is given by Björk in his book [9]

Nontrivial polynomials as in (2) were first introduced by C. Sabbah, see [36]. In the algebraic case the proof is a direct generalisation of the proof of Bernstein-Sato. For the analytic case see [36], completed by [3] where is shown the necessity of using a division theorem proved in [1]. There is by [36], [3] a functional equation (2) in which the polynomial $b(s_1, \cdots, s_p)$ is a product of a finite number of affine forms. It is as far as I know an unsolved problem to state the existence of a system of generators of $\mathcal{B}_{f_1,\cdots f_p}$ made of polynomials of this type.

1.2 *A review of a number of elementary facts about b-functions*

- The functional equation (1) is an identity in $\mathcal{O}[s, \frac{1}{f}] \cdot f^s$ which is the rank one free module over the ring $\mathcal{O}[s, \frac{1}{f}]$, with s as an indeterminate. The generator is denoted f^s viewed as a symbol, in order to signpost the fact that we give this module the $\mathcal{D}[s]$-module structure in which the action of the derivatives on a generic element $g(x,s) \cdot f^s$ is:

$$\frac{\partial}{\partial x_i} \cdot g(x,s) f^s = \Big[\frac{\partial g}{\partial x_i} + g \cdot \frac{s \frac{\partial f}{\partial x_i}}{f}\Big] f^s$$

  We may notice that the multiplication by s on this module is $\mathcal{D}$-linear.

- We consider the submodule $\mathcal{D}[s] \cdot f^s$ generated by f^s. The equation (1) means that the action of s on the quotient:

$$\boxed{\tilde{s} : \frac{\mathcal{D}[s] \cdot f^s}{\mathcal{D}[s] \cdot f^{s+1}} \longrightarrow \frac{\mathcal{D}[s] \cdot f^s}{\mathcal{D}[s] \cdot f^{s+1}}}$$

  which is the $\mathcal{D}$-linear map

$$[P(s) f^s] \to [sP(s) f^s].$$

  admits a minimal polynomial hence that the module $\frac{\mathcal{D}[s] \cdot f^s}{\mathcal{D}[s] \cdot f^{s+1}}$ is finite over $\mathcal{D}$.

  It is a remarkable fact due to Kashiwara that $\mathcal{D}[s] \cdot f^s$ itself is finite over $\mathcal{D}$. We sometimes denote $f^m \cdot f^s = f^{s+m}$, which corresponds to the intuitive meaning and the change of s into $s+m$ induces a $\mathcal{D}$-linear automorphism of $\mathcal{O}[s, \frac{1}{f}] \cdot f^s$, which restrict to $\mathcal{D}[s] \cdot f^s \to \mathcal{D}[s] \cdot f^{s+m}$ given by $P(s) f^s \to P(s+m) f^{s+m}$.

- Recall that there is a unique maximal ideal in $\mathcal{O}$:

$$\mathfrak{M} = \{ f \in \mathcal{O} \mid f(0) = 0 \}$$

- If $f \notin \mathfrak{M}$ is a unit in $\mathcal{O}$ and $b_f = 1$. Similarly, in the algebraic case $b_f = 1$ if f is a constant.

- If f is in the maximal ideal of $\mathcal{O}$ in the local analytic case, or if f is not a constant in the polynomial case, we obtain by setting $s = -1$ in the functional equation: $P(-1) \cdot 1 = b(-1) \frac{1}{f}$, and this implies $b(-1) = 0$. We write usually in this case $b(s) = (s+1)\tilde{b}(s)$.

Setting $s = -1$ in the equation now gives $P(-1) \cdot 1 = 0$ and therefore

$$P(-1) = \sum_{i=1}^{n} A_i \frac{\partial}{\partial_{x_i}}, \quad P(s) = (s+1)Q(s) + \sum_{i=1}^{n} A_i \frac{\partial}{\partial x_i}$$

Carrying this over to the functional equation leads to the following result:

Lemma 1.1 *The polynomial $\tilde{b}(s)$ is the minimal polynomial of the action of s on $(s+1)\frac{\mathcal{D}[s] \cdot f^s}{\mathcal{D}[s] \cdot f^{s+1}}$. This is the same as the unitary minimal polynomial such that there is a functional equation:*

$$\tilde{b}(s) f^s = \big[\sum_{i=1}^{n} Q(s) \cdot f + A_i(s) \frac{\partial}{\partial x_i} \big] \cdot f^s$$

We summarize this fact by writing $\tilde{b}(s) f^s \in \mathcal{D}[s](f + J(f)) \cdot f^s$, with $J(f)$ the jacobian ideal of f generated by $\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}$.

1.3 *A first list of examples*

- When f is smooth we have $b_f(s) = s+1$. This can be seen easily by reducing the calculation to the case $f = x_1$. The converse is true: the equality $b_f(s) = s+1$ may only happen in the smooth case. The result can be found in the preprint [10] by Briançon and Maisonobe.

- When $f = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ we obtain by a straightforward calculation:

$$\frac{1}{\prod_{i=1}^n \alpha_i^{\alpha_i}} \prod_{i=1}^n \left(\frac{\partial}{\partial_{x_1}}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial_{x_n}}\right)^{\alpha_n} \cdot f^{s+1}$$
$$= \prod_{i=1}^n \left(\prod_{k=0}^{\alpha_i - 1} \left(\mathrm{s} + 1 - \frac{k}{\alpha_i} \right) \right) \cdot f^s$$

  It is an easy exercise to prove that this equation does yield the minimal polynomial for a monomial.

- Let $f = x_1^2 + \cdots + x_n^2$ and let Δ be the Laplacian operator then there is a functional equation leading to the polynomial $b(s) = (s+1)(s+\frac{n}{2})$. To be precise:

$$\Delta f^{s+1} = (s+1)(4s+2n) \cdot f^s$$

  The minimality of the polynomial is not so obvious but can be deduced from calculations in the section 4 below. This is an example of a semi invariant for the action on $\mathbb{C}^n$ of the complex orthogonal group which is reductive.

- Finding by blind calculatory means a functional equation in more general cases is virtually impossible. The case of $f = x^2 + y^3$ is already challenging. In section 4 we shall treat the case of all quasi-homogeneous singularities which includes all the Pham-Brieskorn polynomials

$$x_1^{a_1} + \cdots + x_n^{a_n}$$

1.4 *Remarks on variants of the definition*

Let $f \in K[x_1, \cdots, x_n]$ be a polynomial with coefficients in K. For a given $a \in K^n$ we may consider various Bernstein-Sato polynomials:

1) The usual Bernstein-Sato polynomial $b_f = b_{alg}$, such that:

$$b_f(s) f^s \in A_n(K)[s] \cdot f^{s+1}$$

2) The local algebraic Bernstein-Sato polynomial b_{loc}, such that:

$$b_{loc,a}(s) f^s \in A_n(\mathbb{C})_{\mathfrak{m}_a}[s] \cdot f^{s+1}$$

with a functional equation having its coefficients in the algebraic local ring at a.

3) If $K = \mathbb{C}$ we may consider the local analytic Bernstein-Sato polynomial b_{an} at a, characterised by

$$b_{an,a}(s) f^s \in \mathcal{D}_{\mathbb{C}^n,a}[s] \cdot f^{s+1}$$

Obviously we have the divisibility relations $b_{an,a}|b_{loc,a}$, and $b_{loc,a}|b_{alg}$, but there is in in fact a much more precise relation

Proposition 1.2

$$b_{an,a} = b_{loc,a}\,, \quad b_f = \operatorname*{lcm}_{a\in K^n}(b_{loc,a}) \text{ when } K \text{ is algebraically closed}$$

The proof of these results may be found in a more general setting in [11], including the case of multivaviables Bernstein-Sato polynomials $b(s_1, \cdots, s_p)$. We shall give the proof of proposition 1.2 in section 5.3.

Remark 1.3 *If we consider the ring of differential operators with coefficients in a ring of analytic functions $\mathcal{O}(U)$ on an open set in $\mathbb{C}^n$ this proposition 1.2 is no longer valid. It is not even true that for $f \in \mathcal{O}(U)$ there is always a functional equation globally written on U.*

2 On the existence of a functional equation in the polynomial case

In this section we recall the proof by Bernstein-Sato of the existence of the functional equation in the polynomial case. I put off giving a sketch of the proof in the analytic case which requires more sophisticated tools, to the last part of these notes. This section also covered in the course of F. Castro, is inserted here for the sake of completeness and because it is a natural and basic question. I also give a first approach in the algebraic frame to fundamental notions (dimension, multiplicity, holonomicity) which will be reformulated with more sophisticated tools in the analytic case.

2.1 *Holonomic modules*

We consider the Bernstein filtration of $A_n(K)$:

$$\Gamma_k(A_n(K)) = \left\{ \sum_{(\alpha,\beta\in\mathbb{N}^n),|\alpha|+|\beta|\leq k} a_{\alpha,\beta}x^\alpha\partial^\beta \right\}$$

Let us first recall general results about holonomic modules on the Weyl algebra $A_n(K)$:

Theorem 2.1 *1) Let M be a finitely generated module on the Weyl algebra $A_n(K)$, endowed with a good filtration $F_\bullet(M)$. Then there is a polynomial $\chi(\Gamma, M)$, such that for large k:*

$$\dim F_k(M) = \chi(\Gamma, M)(k)$$

2) The leading term of this polynomial is independent of the choice of a filtration. If we denote it $e(M)\frac{k^d}{d!}$*, then the coefficient* $e(M)$ *is an integer called the multiplicity of* M*, and the degree* d *is called the dimension of* M*.*

Proposition 2.2 (basic properties) *Let* M *be a finitely generated module on the Weyl algebra* $A_n(K)$ *and* N *a sub-module then we have:*

$$d(M) \leq \max(d(N), d(M/N))$$

$$\textit{if } d(N) = d(M/N) \textit{ then } e(M) = e(M) + e(M/N)$$

Theorem 2.3 (Bernstein inequality) *For any finitely generated module we have the inequalities:*

$$n \leq d(M) \leq 2n$$

Proposition-definition 2.4 *A module* M *such that* $d(M) = n$ *is called holonomic if* $d(M) = n$*. A holonomic module has finite length.*

Let us end this section by a characterizing finitely generated modules which will be useful in the next section.

Lemma 2.5 *Let* M *be an* $A_n(K)$*-module with a filtration* F *compatible with the Bernstein filtration on* $A_n(K)$ *and such that for some constants* $c_1 > 0$ *and* $c_2 > 0$ *and any* $j \in \mathbb{N}$

$$\dim F^k(M) \leq c_1 \frac{k^n}{n!} + c_2(k+1)^{n-1}$$

then M *is finitely generated and holonomic with multiplicity* $e(M) \leq c_1$*.*

Proof **Assume first that M is finitely generated:** By the hypothesis on M we may choose a good filtration Ω. The fact that Ω is good implies the existence of an integer q such that for all $k \in \mathbb{N}$:

$$\Omega_k(M) \subset F_{k+q}(M)$$

In particular $\chi(\Omega, M)(k) \leq c_1 \frac{(k+q)^n}{n!} + c_2(k+q+1)^{n-1}$, and the degree of χ is at most n, so that $d(M) = n$ and M has the minimal dimension. Looking at the leading term we also see that $e(M) \leq c_1$.

Reduction to the finite case: Let $N \subset M$ be a finitely generated submodule. By applying part 1 to N with the filtration $F_\bullet \cap N$ we see that N is holonomic of multiplicity $\leq c_1$. This implies because of 2.4 that any strictly ascending sequence:

$$0 \neq N_1 \subset N_2 \subset \cdots \subset N_r$$

of submodules of M has length $r \leq c_1$, so that M must be finitely generated. And we are reduced to part 1. $\square$

2.2 *Bernstein equation*

Theorem 2.6 *Let $f \in K[x_1, \cdots, x_n]$ be a non zero polynomial. There is a polynomial $b \in K[s]$ in one indeterminate s, and a differential operator $P(s) \in A_n(K)[s]$ such that*

$$P(s)f^{s+1} = b(s)f^s$$

Proof We work in $K[x_1, \cdots, x_n, \frac{1}{f}]$, given with its natural structure of a module over the algebra $A_n(K)$, and also with the modules $K[s][x_1, \cdots, x_n, \frac{1}{f}]f^s$ and $K(s)[x_1, \cdots, x_n, \frac{1}{f}]f^s$ seen as modules over the algebras $A_n(K)[s]$ and $A_n(K)(s)$. We remark that $A_n(K)[s]$ is not a Weyl algebra over a field but that $A_n(K)(s)$ is the Weyl algebra for the field $K(s)$ so that we can apply dimension theory to this field as well.

Lemma 2.7 *The module $M = K[x_1, \cdots, x_n, \frac{1}{f}]$ is a holonomic module over $A_n(K)$.*

Let N be the total degree of the polynomial f. We define a filtration on M:

$$F_k(M) = \{\frac{g(x)}{f(x)^k} \mid \deg g \leq k(N+1)\}$$

The space $F_k(M)$ is isomorphic to the space of all polynomials of degree $\leq k(N+1)$ in n variables. Thus:

$$\dim F_k(M) = \binom{k(N+1)+n}{n}$$

and looking at this expression as a polynomial in k with a constant c_2 depending on N and n we obtain:

$$\dim F_K(M) \leq (N+1)^n \frac{k^n}{n!} + c_2(k+1)^{n-1}$$

It remains to be noticed that $F_k(M)$ is a filtration of M as an $A_n(K)$-module and that $M = \bigcup_{k\in\mathbb{Z}} F_k(M)$. The conclusion of the lemma follows after 2.5, moreover with the inequality:

$$e(M) \leq (\deg f + 1)^n$$

Lemma 2.8 *The module $M_f = K[x_1, \cdots, x_n, \frac{1}{f}](s)f^s$ is a holonomic module over $A_n(K(s))$.*

The proof is similar except for the fact that the filtrations and dimensions concern vector spaces over the ring of fractions $K(s)$. We set

$$F_k(M_f) = \{\frac{g(x,s)}{f(x)^k} f^s \mid \deg_x g \leq k(N+1)\}$$

In order to check that we obtain a filtration, assume that $\deg_x g \leq k(N+1)$. Then we have

$$\frac{\partial}{\partial x_i} \cdot \frac{g(x,s)}{f(x)^k} f^s = [\frac{\frac{\partial g}{\partial x_i}}{f^k} + g\frac{\frac{\partial f}{\partial x_i}}{f^{k+1}}] f^s$$

and the verification to be made is just that $\deg(f \cdot \frac{\partial g}{\partial x_i} + g \cdot \frac{\partial f}{\partial_{x_i}}) \leq (k+1)(N+1)$

The proof of the existence of a functional equation can now be ended in the following way: The module M_f contains the descending sequence of submodules

$$M_f \supset A_n(K(s)) \cdot f \cdot f^s \supset \cdots \supset A_n(K(s)) \cdot f^m \cdot f^s \supset \cdots$$

Since the module M_f is holonomic hence of finite length this sequence is stationary. There is an integer m such that $A_n(K(s)) \cdot f^m \cdot f^s = A_n(K(s)) \cdot f^{m+1} \cdot f^s$ or equivalently:

$$f^m \cdot f^s \in A_n(K(s)) \cdot f^{m+1} \cdot f^s$$

This means the existence of a differential operator $P_1(s)$ with coefficients in $K(s)$ such that

$$f^m \cdot f^s = P_1(s) \cdot f^{m+1} \cdot f^s \tag{3}$$

We may also notice that the map $A_n(K(s)) \cdot f^s \longrightarrow A_n(K(s)) \cdot f^m \cdot f^s$, given by

$$P(s) f^s \to P(s+m) \cdot f^m \cdot f^s$$

is an isomorphism of $A_n(K)$-modules, its inverse being $Q(s) \cdot f^{s+m} \longrightarrow Q(s-m) f^s$. We convert the equation (3) into an equation $f^s = P_2(s) \cdot f f^s$, with $P_2(s) = P_1(s-m)$, and setting $P_2(s) = \frac{P(s)}{b(s)}$ with P now having polynomial coefficients, we obtain the desired functional equation. $\square$

2.3 *Generalisations*

Theorem 2.9 *Let $f_1, \cdots, f_p \in K[x_1, \cdots, x_n]$ be p non zero polynomials. There is a polynomial $b \in K[\underline{s}]$ depending on the p variables $\underline{s} = (s_1, \cdots, s_p)$, and a differential operator $P(\underline{s}) \in A_n(K)[s_1, \cdots, s_p]$ such that*

$$P(\underline{s}) f_1^{s_1+1} \cdots f_p^{s_p+1} = b(\underline{s}) f_1^{s_1} \cdots f_p^{s_p}$$

These functional equations were considered by Claude Sabbah in [36] in the analytic context. The proof for the polynomial case is a simple adaptation of the proof in the previous section, while the proof in the analytic case is much more intricate. For the proof in the analytic case, see the forthcoming section 5 in the case $p = 1$ and for $p \geq 2$ see the remarks at the end of section 1.1.

Here is a proof in the algebraic case: Let us consider the following module over the Weyl algebra $A_n(K)(s_1, \cdots, s_p)$ of the field of fraction $K(s_1, \cdots, s_p)$:

$$M_{f_1,\cdots,f_p} = A_n(K(s_1, \cdots, s_p))[x_1, \cdots, x_n, \frac{1}{f_1 \cdots f_p}] \cdot f_1^{s_1} \cdots f_p^{s_p}$$

The action of the derivative is given is a way similar to the one dimensional case at the beginning of the proof of theorem 2.6. We use the abbreviations $\underline{s} = (s_1, \cdots, s_p)$, $f^{\underline{s}} = f_1^{s_1} \cdots f_p^{s_p}$, $f^{\underline{s}+\mathbf{1}} = f_1^{s_1+1} \cdots f_p^{s_p+1}$ and $\frac{1}{f} = \frac{1}{f_1 \cdots f_p}$.

$$\frac{\partial}{\partial x_i} \cdot g(x, \underline{s}) f^{\underline{s}} = [\frac{\partial g}{\partial x_i} + g \sum_{k=1}^{p} s_k \frac{\frac{\partial f_k}{\partial x_i}}{f_k}] \cdot f^{\underline{s}}$$

Let M be a holonomic $A_n(K)$-module and let $u \in M$. We are also interested in equations of the following type:

$$P(\underline{s}) u f^{\underline{s}+1} = b(\underline{s}) u f^{\underline{s}}$$

This is an equation in the module $M \otimes_{K[\underline{x}]} K[\underline{s}][x_1, \cdots, x_n, \frac{1}{f}] f^{\underline{s}}$ which is a module over $A_n(K)[\underline{s}]$, with the natural action of the partial derivatives given by:

$$\frac{\partial}{\partial x_i}(u \otimes g(x, \underline{s}) f^{\underline{s}}) = (\frac{\partial}{\partial x_i} \cdot u) \otimes g(x, s) f^{\underline{s}} + u \otimes \frac{\partial}{\partial x_i} \cdot g(x, \underline{s}) f^{\underline{s}}$$

Theorem 2.10 *Let $f \in K[x_1, \cdots, x_n]$ be a non zero polynomial, and let u be an element in a holonomic $A_n(K)$-module. Then there is a polynomial $b \in K[\underline{s}]$, and a differential operator $P(\underline{s}) \in A_n(K)[\underline{s}]$ such that*

$$P(s) \cdot (u \otimes f^{\underline{s}+\mathbf{1}}) = b(s) u \otimes f^{\underline{s}}$$

The proof is quite similar to the previous ones. We consider a filtration $F_k(M)$, and the good filtration of $K(\underline{s})[x_1, \cdots, x_n, \frac{1}{f}] f^{\underline{s}}$ already considered in the previous section. We shall work in the $A_n(K(\underline{s}))$-module:

$$\begin{aligned} K(\underline{s}) \otimes_{K[\underline{s}]} (M \otimes_{K[\underline{x}]} K[\underline{s}][x_1, \cdots, x_n, \frac{1}{f}] f^{\underline{s}}) \\ = M(\underline{s}) \otimes_{K(\underline{s})[\underline{x}]} K(s)[x_1, \cdots, x_n, \frac{1}{f}] f^{\underline{s}} \end{aligned}$$

where $M(\underline{s}) = M \otimes_K K(\underline{s})$ has the obviously good filtration $F_k(M) \otimes_K K(\underline{s})$. Then we see easily that the filtration:

$$F_k(M(\underline{s}) \otimes K(\underline{s})[\underline{x}, \frac{1}{f}]f^{\underline{s}}) = \sum_{p+q=k} F_p(M(\underline{s})) \otimes F_q(K(\underline{s})[\underline{x}, \frac{1}{f}]f^{\underline{s}})$$

is good and that its Hilbert polynomial χ is related to the Hilbert polynomials χ_1 and χ_2 of M and $K(\underline{s})[\underline{x}, \frac{1}{f}]f^{\underline{s}}$ by the inequality:

$$\chi(k) \leq \sum_{p+q=k} \chi_1(p)\chi_2(q)$$

from which the result follows: the module $M(\underline{s}) \otimes K(\underline{s})[\underline{x}, \frac{1}{f}]f^s$ is holonomic and we just have to look now at the decreasing sequence of submodules:

$$A_n(K(\underline{s})) \cdot \left[u \otimes K(\underline{s})[\underline{x}, \frac{1}{f}]f^m f^{\underline{s}}\right]$$

to obtain the conclusion in the same way as above.

Final remarks.- In the analytic case the simple algebraic proof developped above fails already in the standard case with $p = 1$ and $u = 1 \in \mathcal{O}$. The reason as explained in [32] is that there is not a simple way to involve the field $K(s)$, because the formation of rings of analytic operators with series coefficients does not commute with base change $K(s) \otimes_K$ as in the algebraic case. In loc.cit. this failure is in a sense repaired in the case $p = 1$, and a purely algebraic proof of the existence of a Bernstein-Sato polynomial is given, but with much more sophisticated tools than in the proof above. I am not aware of such a proof in the analytic multifunctions case.

2.4 *Semi-invariants of prehomogeneous spaces*

We refer for the detailed definitions and the main properties of prehomogeneous spaces to the book of T. Kimura [23].

Recall that a *prehomogeneous space* is just an algebraic action $G \xrightarrow{\rho} GL(V)$ of an algebraic group G on a K-vector space V which admits a Zariski open orbit U. A *semi invariant* is a rational function $f \in K(V)$ such that there existe a one dimensional representation or character $\chi : G \to K^\star = GL(1)$ such that

$$\forall x \in V, \forall g \in G, \quad f(\rho(g)x) = f(x)$$

On a prehomogeneous space the character χ determines the semi invariant f, in particular a semi invariant is associated to the character 1 if and only if it is constant. The equations of the one codimensional components S_i of the complement $V \setminus U$ of the open orbit, are irreducible homogeneous polynomials and are semi invariants. In fact all the irreducible semi invariants are of this type. See [23, Theorem 2.9.] for details.

Let us now focus on the case of a reductive complex algebraic groups. One shows that such a group is the Zariski closure of a compact subgroup H that we may assume included in the unitary group $U(n)$, after an appropriate change to coordinates (x) known as unitary. In that situation one shows that the dual action

$$G \xrightarrow{\rho^\star = {}^t\rho^{-1}} GL(V)$$

is a prehomogeneus vector space. On also shows by a straightforward calculation in the dual coordinates (y) that if $h \in H$, then $\rho^\star(h) = \overline{\rho(h)}$ the complex conjugate and that if $f(x)$ is a semi invariant polynomial associated with a character χ, $f^\star(y) = \overline{f(\overline{y})}$ is s semi invariant associated with χ^{-1}.

Proposition 2.11 *In the above situation with homogeneous polynomial semi invariants $f, f^\star$ of degree d, there exist a non zero polynomial $b(s)$ of degree d such that*

$$f^\star(D_x)f(x)^{s+1} = b(s)f(x)^s \quad D_x = \mathit{transpose}(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})$$

Proof The chain rule $\frac{\partial u}{\partial y_i}(g\cdot x) = \sum_j \left(g^{-1}\right)_{j,i} \frac{\partial}{\partial x_j}(u(g\cdot x))$ yields the usual formula for base change on vectors in accordance with the base change $y = g.x$ on coordinates:

$$\begin{pmatrix} \frac{\partial}{\partial y_1} \\ \vdots \\ \frac{\partial}{\partial y_n} \end{pmatrix} = {}^t g^{-1} \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix} \tag{4}$$

Let us condense formula (4) into

$$D_y = {}^t g^{-1}\, D_x = \rho^\star(g) D_x \quad \text{or componentwise :} \ \frac{\partial}{\partial y_i} = ({}^t g^{-1}\, D_x)_i$$

Kimura writes $D_{\rho(g)x}$ for the more explicit $\rho^\star(g)D_x$ which is just a column of differential operators which are linear combinations of the $\frac{\partial}{\partial x_j}$ and $f^\star(D_{\rho(g)x})$, is just $f^\star(\rho^\star(g)D_x)$. In a similar way we associate to any polynomial differential operator its image by g^{-1}

$$P(D_y) = \sum_{I=(i_1,\cdots,i_n)\in\mathbb{N}} C_I (\frac{\partial}{\partial y_1})^{i_1} \cdots (\frac{\partial}{\partial y_n})^{i_n}$$

$$P(D_{\rho(g)x}) = \sum_{I=(i_1,\cdots,i_n)\in\mathbb{N}} c_I(\rho^\star(g)D_x)_1^{i_1}\cdots(\rho^\star(g)D_x)_n^{i_n}$$

obtained by applying the substitution (4) into $f^\star(D_y)$.

Now the chainrule formula reads:

$$\frac{\partial u}{\partial y_i}\circ\rho(g) = \sum_j \left(g^{-1}\right)_{j,i}\frac{\partial}{\partial x_j}(u\circ\rho(g)) = \left(\rho^\star(g)D_x\right)_i(u\circ\rho(g))$$

This reformulation allows us by a straighforward induction on the degree of P to write:

$$P(D_y)(u)\circ\rho(g) = P(D_{\rho(g)x})(u\circ\rho(g))$$

therefore in particular:

$$f^\star(D_y)(u)\circ\rho(g) = f^\star(\rho^\star(g)D_x)(u\circ\rho(g)) = \frac{1}{\chi(g)}f^\star(D_x)(u\circ\rho(g)) \qquad (5)$$

If $u(x)$ is a semi invariant satisfying $u(\rho(g)x) = \chi_u(g)u(x)$ we get from the formula (5) and using the k-linearity of $f^\star(D_x)$:

$$f^\star(\frac{\partial}{\partial y})(u)(\rho(g)(x)) = \frac{1}{\chi(g)}f^\star(D_x)(\chi_u(g)u(x)) = \frac{\chi_u(g)}{\chi(g)}f^\star(\partial_x)(u)(x)$$

In particular applied to $u = f(y)^{s+1}$, we obtain

$$f^\star(\frac{\partial}{\partial y})(f(y)^{s+1})\circ\rho(g) = \frac{\chi(g)^{s+1}}{\chi(g)}f^\star(\partial_x)(f(x)^{s+1})$$

so that

$$\Big(\frac{1}{f(y)^s}f^\star(\frac{\partial}{\partial y})(f(y)^{s+1})\Big)_{|y=\rho(g)x} = \frac{1}{f(x)^s}f^\star(\partial_x)(f(x)^{s+1})$$

This proves that $x \to \frac{f^\star(\partial_x)(f(x)^{s+1})}{f(x)^s}$ is an absolute invariant hence a constant depending on $b(s)$. The fact that $b(s)$ is a polynomial is clear and we refer to [23] for the statement on its degree. $\square$

We refer the interested reader to [35] and its bibliography for more informations on these b-functions as well as for the method of calculation of b_f in the case of an irreducible singular locus S by microlocal calculus and holonoly diagramm. See also recent work about reducible examples involving linear free divisor, in [16], [17].

3 Bernstein-Sato polynomials and analytic continuation of f^s

3.1 *Roots of* b *and analytic continuation of* $Y(f)f^s$

In this section, we assume that the fonction f is real on an open subset U of $\mathbb{R}^n$:

$$f : \mathbb{R}^n \to \mathbb{R}$$

and admits a global Bernstein-Sato polynomial denoted by b. We can always assume this is true on a small enough neighbourhood of each point. We shall see after the next proposition how to deal with a function for which we drop this restriction.

For $\Re s > 0$ let us define the locally integrable function $Y(f)f^s$ or f_+^s

$$Y(f)f^s = \begin{cases} \exp(s \log f) \text{ if } f(x) > 0 \\ 0 \text{ if } f(x) \leq 0 \end{cases}$$

Proposition 3.1 *As a distribution $Y(f)f^s$ admits an analytic continuation with poles on the set:*

$$A - \mathbb{N} := \{s \in \mathbb{C} \,|\, \exists i \in \mathbb{N}, b(s+i) = 0\}$$

where A is the set of zeros of b.

Proof Recall that a distribution is defined by its value for any test function $\varphi \in \mathcal{C}_c^\infty$ with a compact support in the domain of f:

$$\langle Y(f)f^s, \varphi\rangle = \int_{\mathbb{R}^n} \varphi(x) Y(f) f^s dx$$

convergent and holomorphic for $\Re s > 0$.

We may write for all s with $\Re s > 0$:

$$\begin{aligned} b(s) \int_{\mathbb{R}^n} \varphi(x) Y(f) f^s dx &= \int_{f>o} \varphi(x) b(s) f^s dx \\ &= \int_{\mathbb{R}^n} \varphi(x) Y(f) P(s) f^{s+1} dx \\ &= \int_{\mathbb{R}^n} P(s)^\star(\varphi(x)) Y(f) f^{s+1} dx \qquad (6) \end{aligned}$$

The last equality requires a bit more explanation:

The operator $P(s)^\star$ is the adjoint operator of $P(s)$ defined by:

$$P(s) = \sum a_\beta(x,s)\big(\frac{\partial}{\partial x_i}\big)^\beta$$

$$P(s)^\star = \sum (-1)^{|\beta|}\big(\frac{\partial}{\partial x_i}\big)^\beta a_\beta(x,s)$$

We first prove the last equality in (6) for $\Re s$ large enough. Using the fact that if $\Re s \geq \mathrm{ord}(P(s)) := d$ the function $Y(f)f^{s+1}$ is of class $\mathcal{C}^d$ at least, we have

$$Y(f)P(s)f^{s+1} = P(s)(Y(f)f^{s+1})$$

since both members are zero along the hypersurface $f^{-1}(0)$. With each monomial in $P(s)$ we perform a succession of integrations by parts in which the integrated term is zero. By adding up the equalities obtained in this way equation (6) follows.

Now (6) is valid for any s with $\Re s > 0$ by analytic continuation.

We have obtained the following equality:

$$\boxed{\langle Y(f)f^s, \varphi\rangle = \tfrac{1}{b(s)}\langle Y(f)f^{s+1}, P(s)^\star(\varphi(x))\rangle}$$

Since the right hand side is well defined and holomorphic in the open set:

$$\{s \,|\, \Re s > -1\} \setminus b^{-1}(0),$$

we have an *analytic continuation of*

$$\langle Y(f)f^s, \varphi\rangle = \int_{\mathbb{R}^n} \varphi(x)Y(f)f^s dx,$$

as a *meromorphic function* defined on the half-plane $\{s \,|\, \Re s > -1\}$ with poles included in the zeros of b.

Iterating this process we obtain similarly:

$$< Y(f)f^s, \varphi > = \frac{1}{b(s+p-1)\cdots b(s)} \int_{\mathbb{R}^n} P(s+p-1)^\star \cdots P(s)^\star(\varphi(x))Y(f)f^{s+p}dx$$

Since the right-hand side of this equation has a meaning as soon as $\Re s > -p$, we obtain a meromorphic continuation of the distribution $Y(f)f^s$ to the half plane $\Re s > -p$, in which a necessary condition for s to be a pole is

$$b(s+i) = 0, \text{ for some } i \in \{0, \cdots, p-1\}$$

Doing this for all p gives the result. □

If we consider an arbitrary analytic function defined on an open subset of $\mathbb{R}^n$ we can no longer assume the existence of a global Bernstein-Sato polynomial, but we can assume it locally i.e. on each element of a sufficiently fine open covering of U. Finally we obtain:

Corollary 3.2 *Let $f : U \to \mathbb{R}$ a real analytic function defined on an open set $U \subset \mathbb{R}^n$. Then the map $s \to Y(f)f^s$ considered as a distribution depending holomorphically on s in the half-plane $\Re s > 0$, admits an meromorphic continuation to the whole plane $\mathbb{C}$. The precise meaning is that its restriction to any compact set $K \subset U$ admits a discrete set of poles (depending on K) contained in a set of the form:*

$$A - \mathbb{N} := \{s \in \mathbb{C} \mid \exists i \in \mathbb{N}, b(s+i) = 0\}$$

for some finite subset $A \subset \mathbb{Q}_{<0}$.

N.B. In the last statement we anticipate the fact that the roots of the Bernstein-Sato polynomial of an analytic germ are negative rational numbers. See section 6.

3.2 *Asymptotic expansion and Mellin transforms*

In this section we wish to show another aspect of the meromorphic continuation of f^s that we have just obtained: its interpretation as the continuation of a Mellin transform.

Assume that f has no singular point outside of $f^{-1}(0)$. Then there is a unique differential $(n-1)$-form ω defined on the open set $f > 0$ such that $dx = df \wedge \omega$.

Using Fubini theorem we have:

$$\int_{f>0} f^s \varphi dx = \int_{f>0} f^s \varphi df \wedge \omega = \int_0^\infty t^s dt \int_{f=t} \varphi \omega_{|f=t},$$

In this way we describe the function $s \to \langle Y(f)f^s, \varphi\rangle$ as the *Mellin transform* $\Gamma_F(s+1)$ [1] of the following function in which we denote

[1] We may also notice that the stationnary phase integrals are Fourier transforms of the same function by:

$$\int e^{i\tau f} \varphi dx = \int_{-\infty}^{+\infty} e^{i\tau t} dt \Big(\int_{f=t} \varphi \frac{dx}{df}\Big)$$

$\frac{dx}{df} := \omega_{|f=t}$:

$$F(t) = \int_{f=t} \varphi \frac{dx}{df}$$

The definition of the Mellin transform is:

$$\Gamma_F(s) = \int_0^{+\infty} t^{s-1} F(t) dt$$

The function $F(t)$ has an asymptotic expansion with terms of the form $a_{\alpha,q} t^{\alpha} (log)^k$. A precise version of this result can be found in Jeanquartier [18]:

Let $f : X \to \mathbb{R}$ be a real analytic function, on a manifold X. The Dirac distribution $\delta_t(f)$ is defined setting $X_t = f^{-1}(t)$ by the following formula:

$$\langle \delta_t(f), \varphi dx \rangle = \int_{X_t} \omega.$$

where ω is a $n-1$ form such that $\varphi dx = df \wedge \omega$ in a neighbourhood of $supp(\varphi) \cap X_t$

Théorème 3.3 *For any relatively compact open subset $U \subset X$ of X, there is a positive integer q and distributions $A_{j,k} \in \mathfrak{D}'(U)$ such that $\delta_t(f)$ has in the scale of functions $t^{\frac{j-q}{q}} (\log t)^k$ an asymptotic expansion:*

$$\delta_t(f) \underset{t \to 0, t>0}{\sim} \sum_{j=1,2,\cdots;\, 0 \le k \le n-1} A_{j,k} t^{\frac{j-q}{q}} (\log t)^k$$

At the level of usual functions $F(t) = \langle \delta_t(f), \varphi dx \rangle$ has an asymptotic expansion:

$$F(t) \underset{t \to 0, t>0}{\sim} \sum_{j=1,2,\cdots;\, 0 \le k \le n-1} a_{j,k} t^{\frac{j-q}{q}} (\log t)^k,$$

We are mainly interested in the case $X = \mathbb{R}^n$, but even in this case the statement in [18] with an arbitrary variety X is necessary. This variety may even be a non-oriented one and the test function becomes a n-form of odd type. This is already hidden in the case of $\mathbb{R}^n$, because the proof in [18], see also [19, Proposition 4.4.] uses the resolution of singularities.

We shall not give a detailed proof of this theorem but just make the calculation for the crucial step which concerns the function $f(x) = x_1^{k_1} \cdots x_\ell^{k_\ell}$ on the hypercube $\{\underline{x} \in \mathbb{R}^n \mid 0 < x_i \le a, i = 1 \dots n\}$. Using a completion argument one shows that it is sufficient to consider test functions with separated variables i.e. of the type $g(x) = g_1(x_1) \cdots g_\ell(x_\ell)$.

We have

$$dx = dx_1 \wedge \cdots \wedge dx_n = \frac{x_1}{k_1}\frac{df}{f} \wedge dx_2 \wedge \cdots \wedge dx_n$$

$$F(t) = \int_{f(x)=t} g_1(x_1)\cdots g_\ell(x_\ell)\frac{x_1}{k_1 f}dx_2 \wedge \cdots \wedge dx_n$$

$$F(t) = \int_{0<x_i\leq a;1\leq i\leq n} g_1\Big(\big(\frac{t}{x_2^{k_2}\cdots x_\ell^{k_\ell}}\big)^{\frac{1}{k_1}}\Big)g_2(x_2)$$
$$\cdots g_\ell(x_\ell)\frac{\big(\frac{t}{x_2^{k_2}\cdots x_\ell^{k_\ell}}\big)^{\frac{1}{k_1}}}{k_1 t}dx_2 \wedge \cdots \wedge dx_n.$$

$$F(t) = \frac{1}{k_1}t^{\frac{1}{k_1}-1}\int_{0<x_i\leq a;1\leq i\leq n} g_1\Big(\big(\frac{t}{x_2^{k_2}\cdots x_\ell^{k_\ell}}\big)^{\frac{1}{k_1}}\Big)g_2(x_2)$$
$$\cdots g_\ell(x_\ell)\big(\frac{1}{x_2^{k_2}\cdots x_\ell^{k_\ell}}\big)^{\frac{1}{k_1}}dx_2 \wedge \cdots \wedge dx_n.$$

We make an induction on ℓ. The case $\ell = 2$ which already gives a precise idea of the general case is written as follows[2]:

$$F(t) = \frac{1}{k_1}t^{\frac{1}{k_1}-1}\int_{\big(\frac{t}{a^{k_1}}\big)^{\frac{1}{k_2}}}^{a} g_1\Big(\big(\frac{t}{x_2^{k_2}}\big)^{\frac{1}{k_1}}\Big)g_2(x_2)\frac{1}{x_2^{\frac{k_2}{k_1}}}dx_2.$$

From this we derive an asymptotic expansion with exponents of the type $-1 + \frac{j_1}{k_1} + \frac{j_2}{k_2}$, with $j_p > 0$ and possibly degree-one logarithmic terms. □

The asymptotic expansion in the complex domain for integrals on the fiber $f^{-1}(t)$ of an holomorphic map, has been intensively studied by Daniel Barlet and also by Henri Maire. We will not consider this question here but refer to [4] for an existence theorem, and also to further papers, in which D. Barlet studied a number of consequence for the poles of f^λ.

3.3 *Application to analytic continuation*

We shall see in the appendix 8.1 on Mellin transforms that we can derive from the asymptotic expansion of the distribution $t \to \delta_t f$ a meromorphic

[2]Notice also the easiest case of one variable in which the exponents $-1 + \frac{j}{k}$ already clearly appear:

$$\int_{x^k=t}\varphi(x)\frac{dx}{dx^k} = \int_{x^k=t}\frac{\varphi(x)}{kx^{k-1}} = t^{-1+\frac{1}{k}}\varphi(t^{\frac{1}{k}})$$

continuation for $\langle Y(f)f^s, \varphi\rangle$. In fact we have:

$$\begin{aligned}\langle Y(f)f^s, \varphi\rangle &= I(s) + J(s) \text{ with } I(s)\\ &= \int_0^\gamma t^s f(t)dt\,;\; J(s) = \int_\gamma^{+\infty} t^s f(t)dt\end{aligned}$$

and $J(s)$ is an entire function on $\mathbb{C}$.

But conversely although there is a theory of an inverse Mellin transform in the case of $f(t)$ having compact support in $]0, +\infty[$, we cannot hope in the situation of a locally integrable I, to recover an asymptotic expansion at 0 from its Mellin transform.

The following example is noticed in [18]:

Example Let $f(t) = \sin(\frac{1}{t})$. The integral:

$$I(s) = \int_0^1 t^s f(t)dt$$

which is a priori convergent for $Res > -1$ admits an analytic continuation as an entire function on the whole complex plane: indeed a double integration by part yields for $\Re s > 2$ a formula:

$$I(s-2) = s(s+1)I(s) + as + b \quad (a = \sin 1; b = \cos 1)$$

which allows us to define $I(s)$ by analytic continuation successively on each strip $\Re > -2 \cdots \Re > -2p$, as in the case of the usual function Γ, when we use the identity $\Gamma(z+1) = z\Gamma(z)$. But in the present case case no pole at all appears.

Atiyah proved in [2], as well as Bernstein and Gel'fand in [5], using this type of method that the distribution $Y(f)f^s$ has an analytic meromorphic-continuation in $\mathbb{C}$, with poles of order at most n.

The details are explained in the appendix A, see section 8.1. The first and easiest case (without multiple poles) of the method for finding meromorphic continuations out of aymptotic developments is explained in detail in theorem 8.3. The case of higher order poles which come from terms in the asymptotic expansion involving powers of $\log t$ is also treated more succinctly by an argument of derivation.

Conclusion

We described in the two preceding sections two ways for obtaining meromorphic continuations of the distribution $Y(f)f^s$.

- By proving the existence of an asymptotic expansion and using its Mellin transform. We obtain that the poles are rational of order at most the dimension of the ambient space.

- By the existence of Bernstein-Sato polynomial. Here the precision is: the poles are in $A' \setminus \mathbb{N}$, A' the set of *rational* roots of the b function.

As we announced in corollary 3.2 and will prove in the forthcoming section 6 the roots of the b-function are in $\mathbb{Q}_{<0}$. This gives another proof of the rationality of the poles. Similarly the upper bound on the order of the poles is a consequence of the fact that the multiplcity of the roots of b are at most n. This result comes from the geometric interpretation of these roots as exponents r of the eigenvalues of the monodromy $e^{2\pi ir}$. Ultimaltely the result on the order of poles comes from the monodromy theorem which states that these eigenvalues are roots of unity.

3.4 *Application to the division of distribution*

Proposition 3.4 *Let $f : X \to \mathbb{C}$ be a real analytic function on a real analytic manifold X, with f not identically zero on each connected component of X. Then there is a distribution T such that $Tf = 1$.*

Proof Assume first that $f \geq 0$. Then the distribution $Y(f)f^0$ is just the constant 1.

Applying the theorem of meromorphic continuation we get a Laurent expansion of f^s around $s = -1$ as a distribution of the type

$$\boxed{f^s = \sum_{-m}^{\infty} A_k(s+1)^k}$$

with A_k a distribution.

The expansion of $f \cdot f^s = f^{s+1}$ has no pole at $s = -1$, its value at $s = -1$ being $f^0 = 1$, This expansion is also

$$f^{s+1} = \sum_{-m}^{\infty} fA_k(s+1)^k$$

This implies the following relations:

$$fA_k = 0 \text{ for } k < 0 \quad \text{ and } fA_0 = 1$$

and the latter provides an inverse to f as a distribution, as intented. In the general case we take an inverse T to $|f|^2$ and the relation giving the result is $1 = T|f|^2 = T\overline{f} \cdot f$ $\square$

4 Quasi-homogeneous and semi-quasi-homogeneous isolated singularities

This section is inpired by the preprint [7] in which we give an algorithm for the calculation of the Bernstein-Sato polynomial non only for quasi-homogeneous polynomials, but also in the more general situation of a semi-quasi-homogeneous function with an isolated singularity. In [8] we generalised this calculation to any function which is nondegenerate with respect to its Newton polygon. We will give in detail below the proof in the quasi-homogeneous case with hints for the extension to the semi-quasi-homogeneous germs.

In all these examples we see by a direct calculation that the coefficients of the monic Bernstein-Sato polynomial are in $\mathbb{Q}$.

As a preliminary let us recall the example of a monomial in which we see directly that the roots of $b_f(s)$ are rational.

When $f = x_1^{a_1} \cdots x_n^{a_n} := \underline{x}^a$, b_f divides the following product:

$$\Pi_a(s) = \prod_{i=1}^{n} \big(\prod_{k=0}^{a_i-1} (s + 1 - \frac{k}{a_i}) \big)$$

In fact we have the equality $b_f = \Pi_a(s)$. Consider a functional equation

$$\sum_{\alpha,\beta \in \mathbb{N},} a_{\alpha,\beta,j} s^j \underline{x}^\alpha \partial^\beta x^{as+a} = e(s) x^{as}$$

By an identification of the terms having the same degree we can restrict the differential operator to a linear combination of monomials of the type $s^j \underline{x}^{\beta - a} \partial^\beta$.

This implies $\beta_i \geq a_i$ for all i such that $a_{\alpha,\beta,j} \neq 0$. Since any such β yields $\partial^\beta x^{as+a} = e_\beta(s) x^{as}$ with $e_\beta(s)$ a multiple of $\Pi_a(s)$, the polynomial $\Pi_a(s)$ divides $e(s)$ as expected.

4.1 *Quasi-homogeneous polynomials*

We consider a system of weights $\underline{w} = (w_1, \cdots, w_n) \in (\mathbb{Q}_+^\star)^n$. and for $I \in \mathbb{N}^n$ we denote $\langle \underline{w}, I \rangle = w_1 I_1 + \cdots w_n I_n$

Definition 4.1 *The polynomial $f \in \mathbb{C}[x_1, \cdots, x_n]$ is quasi-homogeneous of weight (we say also w-degree) ρ if its expansion has the form:*

$$f = \sum_{\langle \underline{w}, I \rangle = \rho} f_I x^I$$

We denote $\mathbb{C}[X]_\rho$ the finite dimensional space of quasi-homogeneous polynomials of degree ρ

Definition 4.2 *The function germ f defines an isolated singularity if $\{0\}$ is an isolated point in the set defined by the equations*

$$\frac{\partial f}{\partial x_1} = \cdots = \frac{\partial f}{\partial x_n} = 0$$

The ideal $J(f) = (\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n})$ generated by the partial derivatives is called the Jacobian ideal.

Examples:

$f(x,y) = x^a + y^b$, $f(x,y) = x^a + xy^b$, are quasi-homogeneous isolated singularities.

$f(x,y) = x^4+y^5+xy^4$, has a non-quasi-homogeneous isolated singularity at $(0,0)$.

Basic example (Pham-Brieskorn polynomials.)

$$f(x_1, \cdots, x_n) = x_1^{a_1} + \cdots + x_n^{a_n}$$

This polynomial is quasi-homogeneous of weight 1 for the weights $w_i = \frac{1}{a_i}$. The jacobian ideal $J(f)$ is generated by the monomials

$$\frac{1}{a_i} f'_{x_i} = x_i^{a_i - 1}$$

and the quotient is a finite dimensional generated by the set of monomials:

$$M = \{x_1^{k_1} \cdots x_n^{k_n}\ , \quad \text{with } 0 \le k_i \le a_i - 2\}$$

The main result proved in subsection 4.3 is:

Proposition 4.3 *Let f be a quasi homogeneous polynomial with an isolated singularity at $(0,0)$ and w-weight equal to 1. Let M be a monomial basis of the Jacobian quotient $\frac{\mathbb{C}[x_1,\cdots,x_n]}{J(f)}$. Let $|w| = \sum w_i$, and let Π be the set of weights without repetition of elements of M. Then:*

$$b_f(s) = (s+1) \prod_{\rho \in \Pi} (s + |w| + \rho)$$

We have the following elementary properties:

- The hypothesis of isolated singularity implies that the quotient $\frac{\mathcal{O}}{J(f)}$ is finite-dimensional. Therefore M is finite as implicitly stated in proposition 4.3.

- Let us denote by E_ρ a supplementary subspace to $J(f) \cap \mathbb{C}[x]_\rho$ in $\mathbb{C}[x]_\rho$, generated by monomials. Then

$$E = \bigoplus_0^\sigma E_\rho$$

is a set of representative of $\frac{\mathcal{O}}{J(f)}$.

- This reflects a grading:

$$\frac{\mathcal{O}}{J(f)} = \bigoplus_0^\sigma \frac{\mathbb{C}[x]_\rho}{J(f) \cap \mathbb{C}[x]_\rho}.$$

- The dimension of E_ρ depends only on the weights $w_1, \cdots, w_n$ if we normalize the weight of f as being 1.

- Let σ be the maximum of the weights ρ such that $E_\rho \neq 0$ or equivalently such that $\mathcal{O}_{>\rho} := \{g \in \mathcal{O}\, \rho(g) > \rho\}$ is contained in $J(f)$. It is well known that σ is equal to the weight of the Hessian of f:

$$\sigma = \rho\Big(\det(\frac{\partial^2 f}{\partial x_i \partial x_j})\Big) = n - 2\sum_{i=1}^n w_i$$

In the Pham-Brieskorn case $\sigma = (1 - \frac{2}{a_1}) + \cdots + (1 - \frac{2}{a_n})$

- We also have by a theorem of Milnor-Orlik [31]:

$$\mu = \dim(\mathcal{O}/J(f)) = \prod_{i=1}^n (\frac{1}{w_i} - 1)$$

To finish these preliminaries let us quote a very simple division lemma:

Lemma 4.4 *For any $u \in \mathcal{O}$ quasi-homogeneous of degree ρ, then there are a unique $v \in E_\rho$ and $(\lambda_1, \cdots, \lambda_n) \in \mathcal{O}^n$ such that*

$$u = v + \sum_{i=1}^n \lambda_i \frac{\partial f}{\partial x_i}$$

with the following conditions (using the convention that $\rho(0) = +\infty$):

$$\rho(v) = \rho(u) \text{ or } v = 0$$

$$\rho(\lambda_i) = \rho(u) - 1 + w_i \text{ or } \lambda_i = 0$$

4.2 *Semi-quasi-homogeneous germs*

In this section we mention briefly the notion of a semi-quasi-homogeneous germ for which a generalisation of proposition (4.3) for the calculation of the Bernstein-Sato polynomial can be derived. Given the system of positive weights $w_1, \cdots, w_n$ let us define the weight of a series $f = \sum f_I x^I \in \mathcal{O} := \mathbb{C}\{x_1, \cdots, x_n\}$ as the rational number $\rho = \rho(f) := \min\{\langle w, I\rangle \mid I \neq 0\}$. We define the initial part $\mathrm{in}_w f$ of f as the quasi homogeneous polynomial, sum of its lowest degree terms:

$$\mathrm{in}_w f = \sum_{\langle w,I\rangle \text{ minimal}} f_I x^I$$

Definition 4.5 *The function germ f is called semi-quasi-homogeneous of weight ρ if its initial part $\mathrm{in}_w f$ has an isolated singularity at the origin.*

Let us list again a number of elementary properties similar to those in the quasi-homogeneous case:

- Thanks to the definition of initial parts the sequence $(\frac{\partial \mathrm{in} f}{\partial x_1}, \cdots, \frac{\partial \mathrm{in} f}{\partial x_n})$, is a regular sequence, generating the Jacobian ideal of $\mathrm{in} f$. This implies that the initial ideal $\mathrm{in}_w(J(f))$ i.e. the ideal generated by $\{\mathrm{in}_w g, g \in J(f),\}$, is in fact the Jacobian ideal $J(\mathrm{in} f)$.

- This implies also that $\frac{\mathcal{O}}{J(f)}$ and $\frac{\mathcal{O}}{In(J(f))}$ are of the same finite dimension.

- Recall that according to subsection 4.1 there is a grading $\frac{\mathbb{C}[x]}{\mathrm{in}(J(f))} = \bigoplus \frac{\mathbb{C}[x]_\rho}{\mathrm{in}(J(f)) \cap \mathbb{C}[x]_\rho}$ where E_ρ is a supplementary subspace to $\mathrm{in}_w(J(f)) \cap \mathbb{C}[x]_\rho$ in $\mathbb{C}[x]_\rho$. The vector space:

$$E = \bigoplus_0^\sigma E_\rho$$

  is a set of representative of $\frac{\mathbb{C}[x]}{In(J(f))}$.

- By the hypothesis of semi-quasi-homogeneity, E is also a set of representatives of $\frac{\mathcal{O}}{J(f)}$. In fact we have a decreasing filtration of $\mathcal{O}$ by $\mathcal{O}_\rho$ the set of series of weight $\geq \rho$, and an induced filtration $\mathbb{C}\{x_1, \cdots, x_n\} \cap J(f)$ on $J(f)$. The graded module $\frac{\mathbb{C}[x]}{\mathrm{in}(J(f))}$ is just the graded module associated with this filtration:

$$\mathrm{gr}\frac{\mathcal{O}}{J(f)} = \frac{\mathrm{gr}(\mathcal{O})}{\mathrm{gr}(J(f))} = \frac{\mathbb{C}[x]_\rho}{\mathrm{in}(J(f)}$$

- The dimension of E_ρ depends only on the weights $w_1, \cdots, w_n$ as long as we normalize the degree of $\mathrm{in}_w f$ as being 1. This can be seen by considering the graded Koszul complex associated with the regular sequence $(\mathrm{in}_w \frac{\partial f}{\partial_{x_1}}, \cdots, \mathrm{in}_w \frac{\partial f}{\partial_{x_n}})$. It is well known too that the weight σ which is by definition the maximal weight such that $E_\rho \neq 0$ is equal to the weight of the Hessian of f so that:

$$\sigma = \rho\Big(\det(\frac{\partial^2 f}{\partial x_i \partial x_j})\Big) = n - 2\sum_{i=1}^{n} w_i$$

- We have by Milnor-Orlik:

$$\mu = \dim(\mathcal{O}/J(f)) = \prod_{i=1}^{n}(\frac{1}{w_i} - 1)$$

The proof relies on the following simple division lemma in which we have fixed a graded supplementary space E to $\mathrm{in}J(f)$ as above:

Lemma 4.6 *For any $u \in \mathcal{O}$ there are unique $v \in E$ and $(\lambda_1, \cdots, \lambda_n) \in \mathcal{O}^n$ such that*

$$u = v + \sum_{i=1}^{n} \lambda_i \frac{\partial f}{\partial x_i}$$

with the following conditions (using the convention that $\rho(0) = +\infty$):

$$\rho(v) \geq \rho(u) \text{ and } \rho(\lambda_i) \geq \rho(u) - 1 + \alpha_i$$

Semi-quasi-homogeneity and μ-constant deformation.

Consider a semi-quasi-homogeneous germ with initial part f_0. We set $w_i = \frac{k_i}{d}$ and assume $k_1, \cdots, k_n$ to be relatively prime, and f to have weight 1, i.e. degree d with respect to the weights k_i. We may consider the deformation of f_0:

$$f(x,t) = t^{-d} f(t^{k_1}x_1, \cdots, t^{k_n}x_n)$$

This is a deformation of isolated singularities with a constant Milnor number. All the germs $f_t = f(\bullet, t)$, for $t \neq 0$ are isomorphic, and therefore have the same Bernstein-Sato polynomial $b_{f_t} = b_f$ which is in general different from the polynomial $b_{f_0}(s)$ calculated in proposition 4.3 and subsection 4.3. According to the result of Lê Duñg Tráng and C.P. Ramanujan, see [25], this situation implies for $n \neq 3$ at least that the topological type of the germ f_t is constant. It also implies that the monodromies of f and f_0 are conjugate, see [24]. We shall see in section 6 that this allows a comparison of the roots of the two Bernstein-Sato polynomials b_f, b_{f_0}, because the roots of b_f are deeply linked to the eigenvalues of the monodromy. By a result due to B. Malgrange in the isolated case:

$$\exists k \in \mathbb{Z} \mid b_f(r+k) = 0 \Longleftrightarrow \lambda = e^{2i\pi r}$$
$$\text{is an eigenvalue of the monodromy of } f$$

Therefore the polynomial b_f is a product of factors which are all of the form $(s + |w| + \rho + k)$, with $\rho \in \Pi$ and $k \in \mathbb{Z}$ can be thought as a "shift" with respect to the quasi-homogeneous case. The number of factors may be different but all the Bernstein-Sato polynomials have only simple roots and in fact all the shifts are negative bounded by $-n \leq k \leq 0$. We refer to [7] for a precise statement and an explicit algorithm for the calculation. See also the published version [8] which deals directly with the non-degenerate case (for which the fact that the roots have multiplicity one is lost in general). In both papers the generic value of the Bernstein-Sato polynomial of a deformation is calculated in the case of quasi-homogeneous curves.

4.3 Calculation of the Bernstein-Sato polynomial of a quasi-homogeneous polynomial

In this section we assume that f is a quasi homogeneous polynomial with isolated singularities and of w weight equal to 1. Let $\chi = \sum w_i x_i \frac{\partial}{\partial x_i}$, called the Euler vector field. The quasi-homogeneity condition means that

$$\chi(f) = f$$

Let u be a quasi-homogeneous polynomial of weight ρ. We shall use lemma 4.4 which is a w-homogeneous equality in $\mathbb{C}[x]_\rho$ in order to carrt out first an easy calculation in the $\mathcal{D}$-module $\mathcal{D}[s]f^s$. We obtain a polynomial which is a priori a multiple of the b-function.

Lemma 4.7 *Let $u \in \mathcal{O}$ be any function germ. We set $|w| = \sum_{i=1}^n w_i$. Then, for any $\rho \in \mathbb{Q}$:*

$$(s + |w| + \rho)uf^s = \big[\sum_{i=1}^n w_i \frac{\partial}{\partial x_i} \cdot x_i u + (\rho \cdot u - \chi(u))\big] f^s$$

It is just a matter of an elementary calculation noticing that

$$\sum_{i=1}^n w_i \frac{\partial}{\partial x_i} \cdot x_i = \chi + |w|, \quad \chi(uf^s) = suf^s + \chi(u)f^s$$

$$\sum_{i=1}^{n} w_i \frac{\partial}{\partial x_i} \cdot x_i u f^s = (\chi + |w|) \cdot u f^s = (\chi(u) + (|w| + s)u) f^s$$

and reordering.

Let $\Pi = \rho(E) = \{\rho, E_\rho \neq 0\}$ be the set of weights of elements of E, which is a finite subset of $[0, \sigma] \cap \mathbb{Q}$. When $u \in E$ is a quasi-homogeneous element of weight $\rho = \rho(u)$, we can write $\chi(u) = \rho \cdot u$ and by applying the division lemma to each of the monomials $x_i u$ we get:

$$(s + |w| + \rho) u f^s = \big[\sum_{i=1}^{n} w_i \frac{\partial}{\partial x_i} \cdot x_i u\big] f^s = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \cdot v_i \quad (\bmod \mathcal{D} J(f) \cdot f^s)$$

where $v_i \in E$ is quasi -homogeneous of weight $\rho + w_i$.

By iterating this process applied to all the v_i or equivalently by a descending induction on the weight of u we obtain an equality for a quasi homogeneous u:

$$\prod_{\rho \geq \rho(u), \rho \in \Pi} (s + |\alpha| + \rho) u f^s = \sum A_i \frac{\partial f}{\partial x_i} f^s$$

involving operators $A_i \in \mathcal{D}$. Notice moreover that

$$(s+1) \frac{\partial f}{\partial x_i} f^s = \frac{\partial}{\partial x_i} \cdot f^{s+1}$$

Putting all this together we obtain:

$$(s+1) \prod_{\rho \geq \rho(u), \rho \in \Pi} (s + |w| + \rho) u f^s = \sum A_i \frac{\partial}{\partial x_i} \cdot f^{s+1} \in \mathcal{D}[s] f^{s+1}$$

and in particular setting $u = 1$:

$$(s+1) \prod_{\rho \in \Pi} (s + |w| + \rho) f^s \in \mathcal{D}[s] f^{s+1}$$

which proves the fact that $(s+1) \prod_{\rho \in \Pi} (s + |w| + \rho)$ is a multiple of the Bernstein-Sato polynomial of f.

Theorem 4.8 *Let f be a quasi homogeneous isolated singularity of weight 1 for the system of weight $(w_1, \cdots, w_n)$ and E a graded space representative of the quotient $\mathcal{O}/J(f)$. Then the Bernstein-Sato polynomial of f is equal to:*

$$b_f(s) = (s+1) \prod_{\rho \in \rho(E)} (s + |w| + \rho)$$

Proof Let $b(s) = (s+1)\tilde{b}(s)$ be the Bernstein-Sato polynomial of f. We need to prove that $\tilde{b}(-|w| - \rho) = 0$, for all the weight in Π. Let $u \in E$ a monomial of weight $\rho = \rho(u)$. Using the functional equation and multiplying it by u on one side and using the lemma 4.7 on the other side we obtain two equations:

$$(s + |w| + \rho)uf^s = \big[\sum_{i=1}^{n} w_i \frac{\partial}{\partial x_i} \cdot x_i u\big] f^s$$

$$\tilde{b}(s)uf^s = \big[Q(s)f + \sum_{i=1}^{n} A_i \frac{\partial f}{\partial x_i}\big] f^s$$

If $\tilde{b}(-|w| - \rho) \neq 0$, then the polynomials $s + |w| + \rho$ and $\tilde{b}(s)$ would be relatively prime so that we would have a Bézout identity $c_1(s)(s + |w| + \rho) + c_2(s)\tilde{b}(s) = 1$ and therefore:

$$uf^s = \big[c_1(s)(\sum_{i=1}^{n} \alpha_i \frac{\partial}{\partial x_i} \cdot x_i u) + c_2(s)(Q(s)f + \sum_{i=1}^{n} A_i \frac{\partial f}{\partial x_i})\big] f^s$$

Using the fact that $sf^s = \chi(f^s)$ and that $f \in J(f)$ we finally obtain an identity:

$$uf^s = \sum_{i=1}^{n} B_i \frac{\partial f}{\partial x_i} \cdot f^s + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \cdot C_i f^s$$

with $B_i, C_i \in \mathcal{D}$. This implies the fact that the operator

$$Q = u - \sum_{i=1}^{n} B_i \frac{\partial f}{\partial x_i} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \cdot C_i.$$

is in the annihilator of f^s. But we know that the annihilator of f^s in $\mathcal{D}$ is generated by the operators $\frac{\partial}{\partial x_i}\frac{\partial f}{\partial x_j} - \frac{\partial}{\partial x_j}\frac{\partial f}{\partial x_i}$ which would imply $Q \in \sum_{I \in \mathbb{N}} \partial^I \cdot J(f)$ and hence also $u \in J(f)$ which is a contradiction. $\square$

In the last argument we used the following proposition:

Proposition 4.9 *If f is a germ of an isolated singularity the annihilator of f^s in $\mathcal{D}$ (not in $\mathcal{D}[s]$!) is generated by the following operators:*

$$\frac{\partial}{\partial x_i}\frac{\partial f}{\partial x_j} - \frac{\partial}{\partial x_j}\frac{\partial f}{\partial x_i} = \frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial}{\partial x_i}$$

This result is a consequence of the fact that $(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n})$ is a regular sequence. For the sake of completeness and of accessibility we shall give a detailed proof of this result in appendix B, in section 8.2.

5 Proof of the existence of the b-function for a an analytic germ

The existence theorem announced in this section is much more difficult to prove than its algebraic counterpart. We already explained at the end of section 2.3 the difficulty of the purely algebraic proof given in [32]. We are going now to give the basic step of a proof for analytic germs which uses tools from sheaf theory, coherent $\mathcal{D}$-modules and their characteristic variety. The original source for this proof can be found in the paper of M. Kashiwara [21], see also [22] and these notes are inspired by the version of it that we gave in [15, Chapter VI]. We shall not develop these tools in detail here in order to focus on the plan of the proof itself, but refer to section 7 for an abridged version, and to [15] for more details about the theory of analytic $\mathcal{D}$-modules, especially for the notions of characteristic variety, multiplicities, dimension, and holonomic modules.

5.1 *The proof in the analytic case*

Let us first enumerate a number of basic ingredients for the proof of the existence of a functional equation

1. The following theorem will be accepted without proof in these notes, and we refer to [15]:

   Theorem 5.1 *Let $\mathcal{M}$ be a coherent $\mathcal{D}$-module with $c = \text{codim}\mathcal{M}$. Then there exists a filtration by* coherent *$\mathcal{D}$-submodules:*
   $$\mathcal{M} = F_c\mathcal{M} \supset F_{c+1}\mathcal{M} \supset \cdots \supset F_n\mathcal{M}$$
   such that $F_k(\mathcal{M}) = \{m \in \mathcal{M}/\mathrm{codim}\mathcal{D}m \geq k\}$ is the maximal subsheaf of codimension k of $\mathcal{M}$.

   The fact that the maximal $(n-k)$-dimensional coherent subsheaf of $\mathcal{M}$ is, if it exists, given by the condition $\mathrm{codim}\mathcal{D}m \geq k$ on the germ of section m, and that this condition defines a subsheaf of $\mathcal{D}$-module can be seen at once. The point is to prove that the so-defined subsheaf is indeed $\mathcal{D}$-coherent. This extra condition contains in fact an essential difficulty, and its proof requires an analysis of the biduality spectral sequence of $\mathcal{M}$, attached to the derived functors of $Hom\big(Hom(\bullet, \mathcal{D}), \mathcal{D}\big)$ see [15, sections V.5., V.6.]

2. **Proposition 5.2** *Let $\mathcal{M}$ be a germ of an holonomic module and $\varphi : \mathcal{M} \longrightarrow \mathcal{M}$ a $\mathcal{D}$-linear map. Then φ admits a minimal polynomial.*

This means that there exists a polynomial $b \in \mathbb{C}[s]$ such that $b(\varphi) = 0$. The proof is by induction on the number of components (of dimension n) of the characteristic variety of $\mathcal{M}$. We postpone to section 7 an exposition of the necessary material on analytic $\mathcal{D}$-module and a sketch of the proof. See also [15, Proposition 23].

Corollary 5.3 *Let $f \in \mathcal{O}$ be an analytic germ. Then if the $\mathcal{D}$-module $\mathcal{M} := \frac{\mathcal{D}[s]f^s}{\mathcal{D}[s]f^{s+1}}$ is holonomic, the germ f admits a nontrivial Bernstein-Sato polynomial.*

Indeed the map $s : \mathcal{M} \to \mathcal{M}$ given by the multiplication by s is $\mathcal{D}$-linear and the existence of a functional equation associated with $e(s)$ is equivalent to $e(s)(\mathcal{M}) = 0$.

3. Let $\mathcal{M}$ be a $\mathcal{D}$-module, not necessary a coherent one. Then $\mathcal{M}$ is coherent if and only if it admits locally good filtrations.

Theorem 5.4 *For any $f \in \mathcal{O}$ there is a nontrivial functional equation.*

Proof We start with a particular case:
Case 1: f is Euler-homogeneous. We assume here that f is an element of its Jacobian ideal which is equivalent to the fact that there is a vector field $\chi = \sum a_i(x)\frac{\partial}{\partial x_i}$ such that $\chi(f) = f$.

The key step in the proof of the theorem is the following lemma:

Lemma 5.5 *Under the hypothesis of case 1, the module $\mathcal{N} := \mathcal{D}[s] \cdot f^s$ is coherent and of dimension $n+1$. Such a module is said to be subholonomic.*

Proof *of the lemma:* Because of the existence of the vector field χ, we may write:

$$sf^s = s\chi(f)f^{s-1} = \chi(f^s)$$

This shows that $\mathcal{N} = \mathcal{D}f^s$ is of finite type over $\mathcal{D}$. In order to prove that it is coherent we just have to exhibit a good filtration, for which a candidate is $\mathcal{N}_k := \mathcal{D}(k)f^s$, if we can prove that this module is $\mathcal{O}$-coherent. But $\mathcal{N}_k$ is clearly of finite type over $\mathcal{O}$, generated by the finite set of all $\partial^\alpha \cdot f^s$, for $|\alpha| \leq k$, and in order to prove its coherence we just have to notice that it is included as a submodule in a free and finite type hence coherent $\mathcal{O}$-module because we have:

$$\mathcal{N}_k \subset \Big[\sum_{i+j\leq 2k} s^i f^{-j}\mathcal{O}\Big]f^s$$

Now let us consider according to the theorem 5.1 the maximal subholonomic submodule $\tilde{\mathcal{N}} \subset \mathcal{N}$. I claim that if $x \notin f^{-1}(0)$, then the germ of $\widetilde{\mathcal{N}}$ and $\mathcal{N}$ at x are equal. The reason is that the operator:

$$\frac{\partial f}{\partial x_i} \circ \frac{\partial}{\partial x_j} - \frac{\partial f}{\partial x_j} \circ \frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_j} \circ \frac{\partial f}{\partial x_i} - \frac{\partial}{\partial x_i} \circ \frac{\partial f}{\partial x_j}$$

is in the annihilator of f^s so that the germ at x of the characteristic variety of $\mathcal{N}$ is contained in and in fact equal to the set defined by the equations:

$$\frac{\partial f}{\partial x_i}\xi_j - \frac{\partial f}{\partial x_j}\xi_i = 0\,, \quad 1 \leq i < j \leq n.$$

This set is smooth of dimension $n+1$ because $\frac{\partial f}{\partial x_{i_0}}(x) \neq 0$ for some index i_0. Now this proves that $\frac{\mathcal{N}}{\widetilde{\mathcal{N}}}$ is a *coherent* $\mathcal{D}$-module with support in $f^{-1}(0)$. Therefore by the nullstellensatz and since the section $[f^s]$ belongs to an $\mathcal{O}$-coherent module, namely $\mathcal{O} \cdot [f^s]$, we have $f^k \cdot [f^s] = 0$ which means that $f^{k+s} \in \widetilde{\mathcal{N}}$. Furthermore there is an isomorphism:

$$\mathcal{N} \longrightarrow \mathcal{D}f^{s+k} \,:\, Pf^s \to Pf^{s+k}$$

Therefore $\mathcal{N}$ is isomorphic to $\mathcal{D}f^{s+k}$ which is isomorphic to a submodule of $\widetilde{\mathcal{N}}$. This implies that $\mathcal{N}$ is already subholonomic and that $\mathcal{N} = \widetilde{\mathcal{N}}$. This ends the proof of the lemma. $\square$

Under the hypothesis of case 1 we deduce from the lemma the existence of a nontrivial equation with the help of corollary 5.3 by proving the holonomicity of $\mathcal{M}$. This is done as follows. We have an exact sequence:

$$0 \to \mathcal{N}' \to \mathcal{N} \to \mathcal{M} \to 0$$

with $\mathcal{N}' = \mathcal{D}f^{s+1}$. Since $\mathcal{N}'$ and $\mathcal{N}$ are isomorphic, we have at each point $x^\star$ of the characterictic variety of $\mathcal{M}$ the following relations between the local dimensions and multiplicities:

$$d_{x^\star}(\mathcal{N}) = d_{x^\star}(\mathcal{N}') \quad e_{x^\star}(\mathcal{N}) = e_{x^\star}(\mathcal{N}')\,,$$
$$\text{hence } d_{x^\star}(\mathcal{M}) < d_{x^\star}(\mathcal{N}) = d_{x^\star}(\mathcal{N}')$$

which implies $d_{x^\star}(\mathcal{M}) = n$. The module $\mathcal{M}$ is holonomic and therefore a nontrivial functional equation exists.

General case. We set $g(t,x) = (1+t)f(x)$. The germ g is Euler homogeneous using the vector field $\chi = (1+t)\frac{\partial}{\partial t}$ and we have by the case 1 an equation:

$$P(x, \partial_x, t, \partial_t)(s)[(1+t)f]^{s+1} = b(s)(1+t)^s f^s$$

We may order P with respect to the powers ∂_t and we notice that

$$\frac{\partial}{\partial t} g^{s+1} = \frac{1}{1+t}(s+1)g^{s+1}$$

Therefore by descending induction on $\deg_{\frac{\partial}{\partial t}} P(s)$ we may write another equation with an operator independent of $\frac{\partial}{\partial t}$

$$P(x, \partial_x, t)(s)[(1+t)f]^{s+1} = b(s)(1+t)^s f^s$$

and this leads directly to the desired equation by $P(x, \partial_x, 0)(s) f^{s+1} = b(s) f^s$ $\square$

5.2 *An application: Holonomicity of the module* $\mathcal{O}[\frac{1}{f}]$

Theorem 5.6 *The module $\mathcal{O}[\frac{1}{f}]$ is coherent as a module over $\mathcal{D}$.*

Proof Let $b(s)$ be the Bernstein-Sato polynomial of f and let U be an open set containing the origin in which there is a functional equation with converging analytic coefficients. For any $k \in \mathbb{N}$ we have therefore an equation:

$$P(-k-1)\frac{1}{f^k} = b(-k-1)\frac{1}{f^{k+1}}$$

Let k_0 be an integer such that $b(\ell) \neq 0$ if $\ell < k_0$. Then for any $k \geq k_0$ we have:

$$\frac{1}{f^{k+1}} = \frac{1}{b(-k-1)} P(-k-1)\frac{1}{f^k} \in \mathcal{D}\frac{1}{f^k}$$

Finally this proves that $\mathcal{O}[\frac{1}{f}] = \mathcal{D} \cdot \frac{1}{f^{k_0}}$ and $\mathcal{O}[\frac{1}{f}]$ is of finite type over $\mathcal{D}_U$

In order to obtain the coherence we now just need to produce a good filtration of $\mathcal{M} = \mathcal{O}[\frac{1}{f}]$. The filtration by the $\mathcal{O}$-submodules $\mathcal{M}_k = \mathcal{D}_U(k)\frac{1}{f^{k_0}}$ works. The subsheaf $\mathcal{M}_k$ is indeed finitely generated over $\mathcal{O}$ hence coherent being a submodule of the $\mathcal{O}$-module $\mathcal{O}_U \frac{1}{f^{k+k_0}}$ which is itself coherent being free of rank one. $\square$

Theorem 5.7 *The module $\mathcal{O}[\frac{1}{f}]$ is an holonomic $\mathcal{D}$-module.*

Proof Les us consider for a given $\lambda \in \mathbb{C}$ the following $\mathcal{D}$-modules:

$$\mathcal{N}_\lambda := \frac{\mathcal{N}}{(s-\lambda)\mathcal{N}}, \quad \mathcal{D} \cdot f^\lambda$$

the second one being viewed as a submodule of $\mathcal{O}[\frac{1}{f}] \cdot f^\lambda$ with its obvious action. There is a surjective map $\mathcal{N}_\lambda \longrightarrow \mathcal{D} \cdot f^\lambda$ and therefore it is sufficient to prove that $\mathcal{N}_\lambda$ is holonomic because we have seen that for some integer k_0 this surjection becomes:

$$\mathcal{N}_{k_0} \longrightarrow \mathcal{D} \cdot \frac{1}{f^{k_0}} = \mathcal{O}[\frac{1}{f}]$$

Now the annihilator of $u_\lambda := f^s (\text{mod } (s - \lambda))$ contains the following elements:

$$\frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial}{\partial x_i}, \quad f\frac{\partial}{\partial x_i} - \lambda\frac{\partial f}{\partial x_i}$$

from which we derive that $\mathcal{N}_\lambda$ is holonomic outside $f^{-1}(0)$.

Considering the maximal holonomic submodule of $\mathcal{N}_\lambda$ we prove that $\mathcal{N}_\lambda$ is holonomic exactly in the same way as for the proof of the subholonomicity of $\mathcal{D}f^s$ in the lemma 5.5. $\square$

5.3 Links between various Bernstein-Sato polynomials

We are going to prove the results announced in the introduction concerning the link beween various types of global and local Bernstein-Sato polynomials and deduce some properties. We anticipate in this section the result about the rationality of the zeroes of the Bernstein-Sato polynomial which is explained independently in next section 6.

Let $f \in K[x_1, \cdots, x_n]$ and let $\mathcal{P} \subset K[x_1, \cdots, x_n]$ be a prime ideal. We denote by $b_{f,\mathcal{P}}$ the monic generator of the ideal consisting of polynomial $e(s)$ for which there is a functional equation $P(s)f^{s+1} = b(s)f^s$ with operators having their coefficients in the local ring $K[x_1, \cdots, x_n]_\mathcal{P}$. In other words we allow denominators $p(x) \in K[x_1, \cdots, x_n] \setminus \mathcal{P}$.

Clearly if $\mathcal{P}_1 \subset \mathcal{P}_2$, we have the divisibility relations $b_{f,\mathcal{P}_1} | b_{f,\mathcal{P}_2} | b_f$.

Proposition 5.8 *The global Bernstein-Sato polynomial b_f of a non zero polynomial $f \in K[x_1, \cdots, x_n]$ is equal to the* lcm *of all the $b_{f,\mathcal{P}}$ when $\mathcal{P}$ runs over all prime ideals. And in fact it is enough to take only maximal ideals.*

In particular if K is algebraically closed we have:

$$b_f(s) = \text{lcm}\{b_{loc,a} \mid \quad a \in K^n\}$$

where $b_{loc,a} = b_{f,\mathfrak{M}_a}$ is the Bernstein-Sato polynomial attached to the maximal ideak at $a \in K^n$

Proof The divisibility relation

$$b(s) := \mathrm{lcm}\{b_{f,\mathcal{P}}\,|\mathcal{P} \text{ maximal }\}\Big| b_f(s)$$

is obvious and in view of the converse we notice that the set of polynomial $g(x)$, such that

$$g(x)b(s)f^s \in A_n(K)\cdot f^{s+1}$$

is an ideal of I of $K[x_1,\cdots,x_n]$. Since for any maximal ideal $\mathcal{P}$ we have a functional equation $b(s)f^s = \frac{P(s)}{g(x)}f^{s+1}$, with $g \notin \mathcal{P}$ and $P(s) \in A_n(K)[s]$ we obtain $g \in I \setminus \mathcal{P}$ and therefore $I \nsubseteq \mathcal{P}$. This result being true for any maximal ideal we must have $I = K[x_1,\cdots,x_n]$ so that I contains 1 and therefore $b_f\Big|b$ as required. □

There is also a link with the analytic Bernstein-Sato polynomial in the case of $K = \mathbb{C}$.

Proposition 5.9 *Let $f \in \mathbb{C}[x_1,\cdots,x_n]$. The local analytic Bernstein-Sato polynomial $b_{an,a}$ at $a \in \mathbb{C}^n$ is equal to the local algebraic one:*

$$b_{an,a} = b_{f,\mathfrak{M}_a}$$

We shall admit this result for which we refer to [11]. The proof relies on the faithful flatness of $\mathcal{O}_{\mathbb{C}^N,a}$ over $\mathbb{C}[x_1,\cdots,x_n]_{\mathfrak{M}_a}$

6 Rationality of the zeroes over any algebraically closed field of characteristic zero

6.1 *The rationality of the zeros*

There is a deep result due to Malgrange [27] in the isolated case and to Kashiwara [21] in general:

Theorem 6.1 *The roots of the local analytic Berstein polynomial are negative rational numbers.*

This results is linked to the monodromy theorem and had been tested beforehand in various contexts.

- In the quasi-homogeneous case the rationality of the roots can be checked directly as was first noticed by Kashiwara. We gave a proof and an explicit calculation in section 4.

- In [27], B. Malgrange gave a partial proof in the case of isolated singularities. He used the monodromy theorem which states that the eigenvalues of the monodromy are roots of unity and he proved that the spectrum of the monodromy is exactly the set of all $e^{2i\pi r}$ when r runs over all the roots of the b function. The statement is more precise in that b appears as the minimal polynomial of the action of $t\partial_t$ on a saturation of the Brieskorn lattice. The negativity of the roots relies essentially on results in [28].

- The proof of M. Kashiwara in [21] is quite different and is based on the resolution of singularities, the obvious case of a monomial and the theory of direct images of $\mathcal{D}$-modules.

- In [30] B. Malgrange gave a proof for the general case of non isolated singularities in the spirit of [27]. This is the starting point of the theory of V-filtrations due to Malgrange and Kashiwara.

6.2 *Derived results*

Proposition 6.2 *Le K be an arbitrary field of characteristic zero and let $f \in K[x_1, \cdots, x_n]$. Then the roots of the Bernstein-Sato polynomial as well as the roots of any local Bernstein-Sato polynomial $b_{f,a}(s)$ are negative rational numbers.*

Proof

Lemma 6.3 *Let $K \subset L$ be a field extension and let $f \in K[x_1, \cdots, x_n]$. Then the Bernstein-Sato polynomial $b_{K,f}$ of f is equal to the Bernstein-Sato polynomial of the same f seen as having coefficients in L*

It is immediate to see that $b_{L,f} \big| b_{K,f}$. Conversely consider a functional equation

$$b_{L,f} f^s = P(s) f^{s+1}$$

with $P(s) \in A_n(L)$ and let $(e_j)_{i \in J}$ be a basis of L over K. We have decompositions, involving only a finite number of terms in J:

$$b_{L,f} = \sum_{j \in J} b_j(s) e_j \text{ and } P(s) = \sum_{j \in J} P_j(s) e_j,$$

such that far all j, $b_j \in K[s]$ and $P_j \in A_n(K)$. Since f has all its coefficients in K these decompositions are transmitted to the functional equation and we obtain for all j:

$$b_j(s) f^s = P_j(s) f^{s+1}$$

and therefore $b_{K,f} \big| b_j$, so that $b_{K,f} \big| b_{L,f}$ as expected.

Let us now come to the proof of the theorem itself: Let $k \subset K$ be the subfield of K generated by $\mathbb{Q}$ and all the coefficients of f. Because of the lemma the Bernstein-Sato polynomial of f is equal to its Bernstein-Sato polynomial as a polynomial in $k[x_1, \cdots, x_n]$. Now since k is a finitely generated extension of $\mathbb{Q}$ there exists an embedding $k \hookrightarrow \mathbb{C}$, and therefore b_f may be considered as the Bernstein-Sato polynomial of an element of $\mathbb{C}[x_1, \cdots, x_n]$.

Because of the results of the section 5.3, this polynomial is the lcm of local analytic Bernstein-Sato polynomials. The roots of b_f have then the required properties by the results of Kashiwara. □

N.B. This proof fails for the Bernstein equation of a "true formal " i.e. non convergent power series. The existence is proved in the book [9] of J.E. Björk. The rationality of the roots is likely but I do not know of a reference or a proof.

7 Introduction to analytic differential modules. Abridged version

7.1 *The ring of differential operators*

The aim of this section is to define linear differential operators with analytic coefficients on a complex analytic manifold X. For that we shall look at operators on open subsets $U \subset \mathbb{C}^n$, and at their behaviour by change of coordinates.

We shall see that the natural definition for an operator on a real or complex variety $P \in \mathcal{D}(X)$, is as an operator on the sheaf $\mathcal{O}_X$. We refer to the book of R. Godement [14] for general facts about sheaves and to [15] for a more expanded version of this subject.

7.1.1 *Definition of $\mathcal{D}(U)$ on an open subset of $\mathbb{C}^n$*

Let $U \subset \mathbb{C}^n$ be open and let $\mathcal{O}(U)$ be the ring of holomorphic functions on U.

Definition 7.1 *A linear differential operator with analytic coefficients in $R = \mathcal{O}(U)$ resp in.*
$R = \mathbb{C}\{x_1, \cdots, x_n\}$ is a $\mathbb{C}$-linear map $P : R \to R$, described by a finite *sum of the type:*

$$f \to \sum_{\alpha \in \mathbb{N}^n} a_\alpha(x) D^\alpha(f)$$

with $a_\alpha \in R$ and $D^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, $\partial_i = \frac{\partial}{\partial x_i}$. We set $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and define the order of P as the natural integer $ord(P) := \sup\{m \mid \exists \alpha \in \mathbb{N}^n, |\alpha| = m, a_\alpha(x) \neq 0\}$.

We denote $\mathcal{D}(U)$, resp. $\mathcal{D}_0$ the ring of linear operators on U resp. of germs of operators. We also write $\mathcal{D}(m)(U)$ resp. $\mathcal{D}(m)_0$ for the sub-modules of operators of order $\leq m$.

The coefficients a_α in the expansion of an operator P are unique. Indeed if $|\alpha| = m = ord(P)$, we easily get $P(x^\alpha) = \alpha! a_\alpha$, and therefore the highest order coefficient are determined in a unique way by P. We conclude by induction.

As easy consequence we obtain that $\mathcal{D}(m)(U)$ is a free $\mathcal{O}(U)$-module of dimension $\binom{m+n}{n}$

Exercises:

i) Verify that $[D^\alpha, a] = D^\alpha \circ a - a \circ D^\alpha$ has order $\leq |\alpha| - 1$.

ii) Assume that $P = \sum_{|\alpha| \leq p} a_\alpha(x) D^\alpha$ has order exactly p, and similarly that Q has order q. Prove that $[P, Q] = P \circ Q - Q \circ P$ has order $\leq p + q - 1$. Define the symbols of P and Q as the functions on $U \times \mathbb{C}^n$:

$$f(x, \xi) = \sigma(P) = \sum_{|\alpha| = p} a_\alpha(x) \xi^\alpha, \quad g(x, \xi) = \sigma(Q)(x, \xi) = \sum_{|\beta| = q} a_\beta(x) \xi^\beta$$

Let $\sigma_{p+q-1}([P, Q])$ be either 0 or the symbol of $[P, Q]$ if its order is $p + q - 1$. Then we have

$$\sigma_{p+q-1}([P, Q]) = \sum_{i=1}^{n} \frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial x_i} - \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial \xi_i}$$

We shall prove that this formula has an intrinsic meaning on any variety.

Towards the sheaf $\mathcal{D}$

The correspondence $U \to \mathcal{D}(U)$ defines a pre-sheaf of rings on $\mathbb{C}^n$: If $V \subset U$, and $P \in \mathcal{D}(U)$, the restriction $\rho_{U,V} P \in \mathcal{D}(V)$ is just the restricted action of P on $\mathcal{O}(V)$.

The sheaf $\mathcal{D}_{\mathbb{C}^n}$ associated with this presheaf is called the sheaf of linear differential operators. Any section has locally a finite order, and because of the unicity of the expansion and by analytic continuation for the a_α's, the set of section of $\mathcal{D}$ is exactly $\mathcal{D}(U)$ on any connected open set.

7.1.2 Behaviour under change of coordinates

In order to define operators on an analytic variety it is necessary to deal with changes of coordinates. Let

$$U \xrightarrow{\varphi} V$$

be a diffeomorphism between two open subsets of $\mathbb{C}^n$. We denote $\psi = \varphi^{-1}$.

It induces an isomorphism of rings $\mathcal{O}(V) \xrightarrow{\varphi^\star} \mathcal{O}(U)$, by $\varphi^\star(f) = f \circ \varphi$, whose inverse is $\psi^\star$.

Given a linear map $P \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{O}(U), \mathcal{O}(U))$, we define $\varphi_\star(P) := \psi^\star \circ P \circ \varphi^\star : \mathcal{O}(V) \to \mathcal{O}(V)$. This means that that $\varphi_\star(P)$ is inserted in the following commutative diagram:

$$\begin{tikzcd}
\mathcal{O}(U) \arrow[r, "P"] & \mathcal{O}(U) \\
\mathcal{O}(V) \arrow[u, "\varphi^\star"] \arrow[r, "\varphi_\star(P)"] & \mathcal{O}(V) \arrow[u, "\varphi^\star"']
\end{tikzcd}$$

We get an isomorphism of rings whose inverse is $\psi_\star$

$$\mathrm{Hom}_{\mathbb{C}}(\mathcal{O}(U), \mathcal{O}(U)) \xrightarrow{\varphi_\star} \mathrm{Hom}_{\mathbb{C}}(\mathcal{O}(V), \mathcal{O}(V))$$

Proposition 7.2 *The map $P \to \varphi_\star(P)$, can be restricted to an isomorphism of rings from $\mathcal{D}(U)$ to $\mathcal{D}(V)$. On the set of vector fields it coincides with the direct image induced by the tangent map of φ: $\varphi_\star(\xi)(y) := d_x\varphi(\xi(x))$, $\xi \in \Theta(U)$, $y = \varphi(x)$.*

Proof The equality $\varphi_\star(PQ) = \varphi_\star(P)\varphi_\star(Q)$ is a direct consequence of the above diagram and of the definition of PQ as a composition of operators from $\mathcal{O}(U)$ to itself.

In order to prove that $\varphi_\star\mathcal{D}(U) \subset \mathcal{D}(V)$ we just have to verify that $\varphi_\star$ preserves vector fields.

It can be done by the following explicit calculation:

$$\begin{aligned}
\varphi_\star(a(x)\frac{\partial}{\partial x_j})(h)(y) &= a(\varphi^{-1}(y))\frac{\partial}{\partial x_j}(h \circ \varphi)(\varphi^{-1}(y)) \\
&= a(\varphi^{-1}(y))\sum \frac{\partial \varphi_i}{\partial x_j}(\varphi^{-1}(y))\frac{\partial h}{\partial y_i}
\end{aligned}$$

or erasing the variables:

$$\varphi_\star(a\frac{\partial}{\partial x_j}) = (a \circ \psi)\sum(\frac{\partial \varphi_i}{\partial x_j} \circ \psi)\frac{\partial}{\partial y_i}$$

And for an operator $P = \sum_{|\alpha| \leq} a_\alpha D^\alpha$ it gives the formula:

$$\varphi_\star(P) = \sum_{|\alpha| \leq m} a_\alpha \prod_{j=1}^{n} \Big[\sum_{i=1}^{n} \frac{\partial \varphi_i}{\partial x_j}(\psi(y))\frac{\partial}{\partial y_i}\Big]^{\alpha_j}$$

from which incidentally we see that the map $\varphi_\star$ preserves the order of operators. $\square$

7.1.3 Sheaf of differential operators on an analytic manifold

An example. Consider the manifold $X = \mathbb{P}_1$ with the two coordinate sets (U, z), (V, t), and change of coordinates $z \to t = \frac{1}{z}$ on $U \cap V \cong \mathbb{C}^\star$. There is a global vector field $\xi \in \Theta(\mathbb{P}_1)$ defined by:

$$\begin{cases} \xi_{|U} = \frac{\partial}{\partial z} \\ \xi_{|V} = -t^2 \frac{\partial}{\partial t} \end{cases}$$

By the maximum principle $\mathcal{O}(\mathbb{P}_1) = \mathbb{C}$, so that the action of ξ is zero. This shows that a differential operator is not always determined by its action on global functions. However we see easily that the same ξ is determined by its action on all the $\mathcal{O}(U)$'s, for $U \subset \mathbb{P}_1$ an open subset.

Exercise: Describe a similar situation with the elliptic curve $X = \mathbb{C}/\mathbb{Z}^2$ (Hint: there is an atlas whose coordinate changes are only translations).

In general we notice that if U is an open subset of $\mathbb{C}^n$ the operator $P = \sum_{\alpha \in \mathbb{N}^n} x^\alpha \partial^\beta$ acts as well on all the rings $\mathcal{O}(V)$ for $V \subset U$. In fact the definition of P as acting on the ring $\mathcal{O}(U)$ is clearly equivalent to its definition as a morphism of sheaves $\mathcal{O}_U \to \mathcal{O}_U$.

Definition 7.3 *1) A differential operator $P \in \mathcal{D}(X)$ on a complex analytic manifold X is a $\mathbb{C}$-linear homomorpism of sheaves:*

$$\mathcal{O}_X \xrightarrow{P} \mathcal{O}_X$$

whose restriction to each coordinate neighbourhood $U \subset X$ is a linear differential operator in the sense of 7.1.1.

2) We may define $\mathcal{D}(U)$ in the same way for any open set $U \subset X$. The sheaf $\mathcal{D}_X$ is the sheaf of non-commutative rings $U \to \mathcal{D}_X(U) = \mathcal{D}(U)$.

In this definition the restriction is just the restriction $\mathcal{D}(U) \to \mathcal{D}(V)$ of morphism of sheaves $P \to P_{|\mathcal{O}_V}$ and the axioms of sheaves are straightforward. We leave it as a routine exercise.

7.1.4 *Principal symbols, and graded associated sheaves*

Proposition 7.4 *Let $U \in \mathbb{C}^n$ be an open set. Let*

$$\mathcal{D}(m)(U) := \{ \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha \}$$

be the set of operators with order $\leq m$. *Then we obtain a ring filtration i.e.* $\mathcal{D}(m_1)(U)\mathcal{D}(m_2)(U) \subset \mathcal{D}(m_1+m_2)(U)$ *and the associated graded ring*

$$\mathrm{gr}\mathcal{D}(U) = \bigoplus \frac{\mathcal{D}(m)(U)}{\mathcal{D}(m-1)(U)}$$

is commutative isomorphic to $\mathcal{O}(U)[\xi_1, \cdots, \xi_n]$ *with* ξ_i *the class of* $\frac{\partial}{\partial x_i}$.

Proof Denote by $\sigma_m(P)$ the class of $P \in \mathcal{D}(m)(U)$ in the quotient $\frac{\mathcal{D}(m)(U)}{\mathcal{D}(m-1)(U)}$. If $\sigma_m(P) \neq 0$, that is if $P \in \mathcal{D}(m)(U) \setminus \mathcal{D}(m-1)(U)$ we denote it $\sigma(P)$ and call it the principal symbol of P.

It is clear that $\sigma(P) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$, and therefore $\mathcal{O}(U)$ and $\xi_1, \cdots, \xi_n$ generate $\mathrm{gr}\mathcal{D}(U)$ and this ring is commutative because these generators pairwise commute, the only nontrivial case being:

$$\begin{aligned}
[\xi_i, a] &= \sigma_1(\frac{\partial}{\partial x_i})\sigma_0(a) - \sigma_0(a)\sigma_1(\frac{\partial}{\partial x_i}) \\
&= \sigma_1(\frac{\partial}{\partial x_i} \circ a - a \circ \frac{\partial}{\partial x_i}) = \sigma_1(\frac{\partial a}{\partial x_i}) = 0
\end{aligned}$$

The last assertion follows from the unicity of the coefficients a_α of an operator hence also of a symbol of order m. $\square$

Let $T^\star X \to X$ be the conormal bundle on X.

The set of operators $\mathcal{D}_X(m)$ of order $\leq m$, is organized in a sheaf of locally finite module over the sheaf $\mathcal{O}_X$ of holomorphic sections. We obtain the following intrinsic formulation of the proposition 7.4:

Proposition 7.5 *There is a coherent sheaf of rings*

$$\mathrm{gr}\mathcal{D}_X = \bigoplus_{m \in \mathbb{N}} \frac{\mathcal{D}_X(m)}{\mathcal{D}_X(m-1)}$$

which is isomorphic to the sheaf $\pi_\star \mathcal{O}_{[T^\star X]}$ *whose section on* U *are holomophic functions on* $T^\star U$, *polynomial in the fibers.*

The fact that a principal symbol $\sigma(P)$ identifies with a function on $T^\star U \subset T^\star X$, is a matter of control of the behaviour by a coordinate change, and a direct consequence of the proof of proposition 7.2. Heuristically the case of a vector field $v \in \Theta(U)$ is clear because the evaluation on linear forms is a function on the conormal bundle of U. We leave the details and the extension to operators as an exercise.

<u>Consequence</u>: Let us define the Poisson bracket by the formula:

$$\{f, g\} = \sum_{i=1}^{n} \frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial x_i} - \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial \xi_i}$$

Although we used coordinates it has an intrinsic meaning on $T^{\star}X$, this being

$$\{f, g\} = \sigma_{p+q-1}([P, Q]),$$

if $f = \sigma_p P$ and $g = \sigma_q Q$.

7.1.5 *Coherence*

Theorem 7.6 *The sheaves $\mathcal{D}_X$ and $\mathrm{gr}\mathcal{D}_X$ are coherent sheaves of rings.*

We refer to the paper of J.-P. Serre for the definition of coherence of a sheaf of rings and details about this notion. Recall that this notion is of particular interest in that it reduces the notion of a coherent module, to the property of having locally a finite presentation. Left coherent modules are differential systems in the case of $\mathcal{D}$. For $\mathcal{D}_X$ the statement is valid for left cherence and for right coherence as well.

We shall admit the following three results, more detail being available in [15]:

Lemma 7.7 *Let $K \subset \mathbb{C}^n$ be a compact polycylinder, then the ring $\mathcal{O}(K)$ of holomorphic functions in a neighbourghood of K is noetherian.*

Theorem 7.8 *Theorem of Cartan and Oka. For any complex analytic manifold the sheaf $\mathcal{O}_X$ is coherent.*

Theorem 7.9 *Theorem A of H. Cartan. Let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}$-modules on a neighbourhood U of a polycylinder K.*

1) The $\mathcal{O}(K)$-module $\Gamma(K, \mathcal{F})$ is finitely generated

2) The sheaf $\mathcal{F}_{|K}$ is (finitely!) generated by the set $\Gamma(K, \mathcal{F})$ of its global sections.

This last claim means that for any $x \in K$, and any germ of a section of $\mathcal{F}$, $s \in \mathcal{F}_x$ there are $a_1, \cdots, a_q \in \mathcal{F}_x$ and $s_1, \cdots, s_q \in \Gamma(K, \mathcal{F})$ such that

$$s = \sum_{i=1}^{q} a_i s_{i,x}$$

where we denote by $s_{i,x}$ the germ of s_i at x.

In view of giving the sketch of a proof of the coherence theorem we need some facts about sections on a polycylinder K.

1. The homomorphism induced by restrictions:

$$\lim_{U \supset K, U \text{ open}} \mathcal{F}(U) \to \mathcal{F}(K)$$

is an isomorphism. This fact can be found in [14].

2. The module $\mathrm{gr}\mathcal{D}(U)$ of sections of $\mathrm{gr}\mathcal{D}$ over an open chart domain U is just the quotient $\frac{\mathcal{D}_X(m)(U)}{\mathcal{D}_X(m-1)(U)}$ and similarly over a compact polycylinder we have

$$\mathrm{gr}\mathcal{D}(K) \simeq \frac{\Gamma(K, \mathcal{D}_X(m))}{\Gamma(K, \mathcal{D}_X(m-1))}$$

The reason is that on a chart we have a *split* exact sequence of sheaves of free $\mathcal{O}_X$-modules:

$$0 \longrightarrow \mathcal{D}_X(m-1) \longrightarrow \mathcal{D}_X(m) \longrightarrow \frac{\mathcal{D}_X(m)}{\mathcal{D}_X(m-1)} \longrightarrow 0$$

Theorem 7.10 *The ring* $\mathrm{gr}\mathcal{D}_X(K)$ *is noetherian and* $\mathcal{D}_X(K)$ *is left and right noetherian.*

Proof By the description above and proposition 7.4 we have an isomorphism $\mathrm{gr}\mathcal{D}_X(K) \simeq \mathcal{O}(K)[\xi_1, \cdots, \xi_n]$, and the first noetherianity is just a consequence lemma 7.7 and of the usual transfer theorem.

As for the left (or right) noetherianity of $\mathcal{D}_X(K)$ we can deduce it from the noetherianity of $\mathrm{gr}\mathcal{D}_X(K) = \mathrm{gr}(\mathcal{D}_X(K))$ by a proof similar to the proof of the transfer property: take a left ideal $\mathcal{I}$ in $\mathcal{D}_X(K)$ and prove its finiteness from the finiteness of $\mathrm{gr}\mathcal{I}$. Details are left as an exercise, and we refer to our course [15, page 111]. $\square$

Proof of the coherence theorem. We have to prove that the kernel of any morphism of left $\mathcal{D}$-modules:

$$\phi : (\mathcal{D}_U)^q \to (\mathcal{D}_U)^p$$

is coherent. Since the morphism ϕ is just the multiplication on the right by a matrix of operators $(Q_1, \cdots, Q_q) \mapsto (Q_1, \cdots, Q_q)\Big(R_{i,j}\Big)$, and setting $k_0 = \max(\mathrm{ord} R_{i,k})$ we find that $\phi((\mathcal{D}_U(\ell))^q) \subset (\mathcal{D}_U(\ell + k_0))^p$. Therefore $\ker \phi \cap \mathcal{D}_U(\ell)^q$ appears as the kernel of a morphism of locally free finite type modules:

$$\mathcal{D}_U(\ell)^q \to \mathcal{D}_U(\ell + k_0)^p$$

and hence is coherent for each ℓ, by Cartan-Oka theorem. The union over ℓ being $\ker\phi$.

Therefore by theorem A of Cartan, for any polycylinder $K \subset U$, and any $x \in K$ the fiber $\left[\ker\phi \cap \mathcal{D}_U(\ell))^q\right]_x$ is generated by $\Gamma(K, \ker\phi \cap \mathcal{D}_U(\ell))^q \subset \Gamma(K, \ker\phi)$, from which $\ker\phi_x$ is generated by $\Gamma(K, \ker\phi)$.

By the left exactness of $\Gamma(K, \bullet)$, this gives an exact sequence:

$$0 \longrightarrow \Gamma(K, \ker\phi) \longrightarrow \Gamma(K, (\mathcal{D}_U)^q) \simeq \mathcal{D}_X(K)^q$$
$$\longrightarrow \Gamma(K, (\mathcal{D}_U)^q) \simeq \mathcal{D}_X(K)^p$$

and by the noetherianity of $\mathcal{D}_X(K)$ this proves that $\Gamma(K, \ker\phi)$ is generated by a finite number of sections say $s_1, \cdots, s_r$.

Using again the theorem A of Cartan we conclude that the morphism:

$$\mathcal{D}_X(K)^r \to (\ker\phi)_{|K}$$

is surjective which ends the proof. $\square$

7.2 *Coherent $\mathcal{D}$-modules or differential systems*

7.2.1 *What is a differential system?*

Definition 7.11 *A differential system over a complex analytic variety is a sheaf of coherent $\mathcal{D}_X$-modules.*

Since $\mathcal{D}_X$ is a coherent sheaf we have local finite presentations of $\mathcal{M}$

$$\mathcal{D}^q_{|U} \xrightarrow{\phi} \mathcal{D}^p_{|U} \longrightarrow \mathcal{M}_{|U} \longrightarrow 0$$

as a cokernel of maps $\phi(Q_1, \cdots, Q_q) = (Q_1, \cdots, Q_q)\Big(R_{i,j}\Big)$.

Let us consider solutions $\underline{u} = {}^{tr}(u_1, \cdots, u_p)$ in $\mathcal{O}_U$, of the differential system associated with the matrix of ϕ

$$R_{i,1}u_1 + \cdots + R_{i,p}u_p = 0, \text{ for } i = 1, \cdots, q$$

There is a bijection between this set of solutions and the set of left $\mathcal{D}$-linear maps $\mathcal{M}_U \to \mathcal{O}_U$. Indeed for any solution $\underline{u}$ and any $\underline{Q} \in \mathcal{D}^q_{|U}$, we have $\phi(\underline{Q})(\underline{u}) = \Big(\underline{Q}R_{i,j}\Big)\underline{u} = 0$, this meaning that the $\mathcal{D}$-linear homomorphism $\mathcal{D}^p_{|U} \to \mathcal{O}_U$, given by $(P_1, \cdots P_p) \to \sum P_j u_j$, is zero on $Im\phi$, hence factors through $\mathcal{M}_U$. This leads to:

Definition 7.12 *The sheaf of solutions of $\mathcal{M}$ is the sheaf $Sol(\mathcal{M}) = \underline{\mathcal{H}om}(\mathcal{M}, \mathcal{O})$ of germs of homomorphism from $\mathcal{M}$ to $\mathcal{O}$.*

7.2.2 *Regular connections*

Lemma 7.13 *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module which is also coherent as an $\mathcal{O}_X$-module. Then the fiber of $\mathcal{M}$ is a locally free $\mathcal{O}_X$ module of finite rank.*

Proof We may assume that X is a neighbourhood of $0 \in \mathbb{C}^n$. Consider a minimal system of generators $(e_1, \cdots, e_p)$ of the fiber $\mathcal{M}_0$, and let us prove that $(e_1, \cdots, e_p)$ is free over $\mathcal{O}_0$. Otherwise we would have a linear relation:

$$(\star) \qquad \sum_{i=1}^{p} u_i e_i = 0 \text{ with } u_i \in \mathcal{O}_{X,0}$$

Define ν as the minimal valuation amongst the $val(a_i)$, reached say on a_1, that is:

$$\forall i\,,\, u_i \in (x_1, \cdots, x_n)^\nu,\ \text{ and } u_1 \in (x_1, \cdots, x_n)^\nu \setminus (x_1, \cdots, x_n)^{\nu+1}$$

If $\nu > 0$ then we can choose a variable x_k such that the initial part $in_\nu u_1$ depends on it. In that case the valuation of $\frac{\partial u_1}{\partial x_k}$ is exactly $\nu - 1$.

Let us derive $(\star)$:

$$\sum_{i=1}^{p} \frac{\partial u_i}{\partial x_k} e_i + \sum_{i=1}^{p} u_i \frac{\partial}{\partial x_k} e_i$$

Rewriting each $\frac{\partial}{\partial x_k} e_i$ as a linear combination of the e_i we obtain a new linear relation between the generators e_i. In this relation the coefficient of e_1, has the form:

$$u_1' = \frac{\partial u_1}{\partial x_k} + L(u_1, \cdots, u_n)$$

where $L(u_1, \cdots, u_n)$ is a linear combination of the coefficients u_i. This implies that the valuation of this coefficient u_1' is now $\nu - 1$. By repeating this process we might have assumed that $\nu = 0$. But in that case u_1 would be a unit in the local ring $\mathcal{O}_{X,0}$ and e_1 a linear combination of $e_2, \cdots, e_n$ contrary to the minimality.

Now assume that $e_1, \cdots, e_n$ are defined as sections of $\mathcal{M}$ on the open neighbourhood U of 0. Since $\mathcal{M}$ is locally of finite type, we know that they generate the restriction of $\mathcal{M}$ to some $V \subset U$ containing the origin.

Arguing similarly with the sheaf of their relations over $\mathcal{O}$ whose fiber at 0 is zero we can prove that $\mathcal{M}$ is free with basis $\{e_i\}$ on some $W \subset V$. $\square$

Theorem 7.14 *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$ module which is also of finite type and coherent* [3] *over $\mathcal{O}_X$. Then $\mathcal{M}$ is a locally isomorphic to some $(\mathcal{O}_X)^r$ as a $\mathcal{D}_X$ module, with $\mathcal{O}_X$ being equipped with its canonical structure over $\mathcal{D}_X$. We call a $\mathcal{D}$-module which is locally free over $\mathcal{O}_X$ a* regular connection[4].

By the lemma we have a local isomorphism $\mathcal{M} \to (\mathcal{O}_{X|U})^r$ of $\mathcal{O}_{X|U}$-modules, with a basis $(e_1, \cdots, e_r)$.

Due to the fact that $\mathcal{O} \simeq \frac{\mathcal{D}}{\mathcal{D}\cdot(\partial_1,\cdots,\partial_n)}$, the statement of the theorem is equivalent to the existence of an horizontal basis $m_1, \cdots, m_r$ that is a basis such that for all (j,k) we have $\partial_j m_k = 0$.

We refer to our course [15] theorem 4 page 137 to 139.

<u>Exercise</u>: Show that the one variable case in 1) is just Cauchy theorem. Hint: write the action of $\frac{\partial}{\partial x}$ on $\mathcal{M}$ as

$$\nabla \begin{pmatrix} m_1 \\ \vdots \\ m_r \end{pmatrix} = A(x) \begin{pmatrix} m_1 \\ \vdots \\ m_r \end{pmatrix}$$

where $A(x) = a_{i,j}(x)$ is an $n \times n$ matrix with holomorphic coefficients. Show that $\nabla(\sum g_i(x) m_i) = 0$ is equivalent to the linear differential system:

$$\frac{\partial}{\partial x} \begin{pmatrix} g_1 \\ \vdots \\ g_r \end{pmatrix} = -{}^{tr}A(x) \begin{pmatrix} g_1 \\ \vdots \\ g_r \end{pmatrix}$$

The proof in higher dimension is a more technical n-dimensional version of the Cauchy theorem. The following corollary is the first step toward the finiteness property in proposition 5.2 used for the proof of the existence of the Bernstein-Sato polynomial.

Corollary 7.15 *Let $\mathcal{M}$ be a regular connection and let $\varphi : \mathcal{M} \to \mathcal{M}$ be a $\mathcal{D}$-linear homomorphism. Then the germ of φ at any point $p \in X$ admits a minimal polynomial.*

According to theorem 7.14 we may assume that $\mathcal{M}_p = \mathcal{O}^r$ as a $\mathcal{D}_p$-module. Then φ is described by a matrix $(f_{i,j})$ so that the action on the vectors of the canonical basis is

$$\varphi(\epsilon_j) = \sum_{i=1}^{n} f_{i,j} \epsilon_j$$

[3]In fact the coherence hypothesis over $\mathcal{O}_X$ is redundant as we shall see from the existence of good filtrations, see next section theorem 7.18.

[4]We shall see in the next section that a regular connection is the same as a $\mathcal{D}_X$-module whose characteristic variety is the zero section $T^\star_X X$ of the cotangent bundle.

From the fact that for any k, $\partial_k \epsilon_j = 0$, we deduce easily that for all i, j, k, $\partial_k(f_{i,j}) = 0$, so that φ is represented by an $r \times r$ matrix with constant coefficients and as such admits a minimal polynomial $b_\varphi \in \mathbb{C}[s]$, $b_\varphi(\varphi) = 0$

7.3 *Good filtrations and coherence conditions*

Consider the differential system $\mathcal{M} = \frac{\mathcal{D}}{\mathcal{I}}$ associated with a left ideal $\mathcal{I}$ and consider its $\mathcal{O}_X$-submodules induced as quotients of the subsheaves $\mathcal{D}_X(m)$ of operators of order $\leq m$.

Following this model we are led to the definition below:

Definition 7.16 *Let $\mathcal{M}$ be a $\mathcal{D}_X$-module. A filtration on $\mathcal{M}$ is a collection of $\mathcal{O}_X$-submodules $\mathcal{M}_k$ such that*

1. $\mathcal{M} = \bigcup_{k \in \mathbb{N}} \mathcal{M}_k$
2. $\mathcal{M}_k \subset \mathcal{M}_{k+1}$
3. $\forall k, \ell \in \mathbb{N}, \mathcal{D}_X(\ell)\mathcal{M}_k \subset \mathcal{M}_{k+\ell}$

Definition 7.17 *We say that $\{\mathcal{M}_k\}$ is a good filtration on $\mathcal{M}$ if*

1. *Each $\mathcal{M}_k$ is a coherent $\mathcal{O}_X$-module.*
2. $\exists k_0 \in \mathbb{N}\,, \forall \ell \in \mathbb{N}\,, \mathcal{D}_X(\ell)\mathcal{M}_{k_0} = \mathcal{M}_{k_0+\ell}$

Exercise Verify that $\mathrm{gr}\mathcal{M} = \bigoplus_{k \in \mathbb{N}} \frac{\mathcal{M}_k}{\mathcal{M}_{k-1}}$ is a $\mathrm{gr}\mathcal{D}$-module in the obvious way.

Examples

- Let $\mathcal{M}$ be a regular connection. By definition it is an $\mathcal{O}_X$-coherent module and we can take $\mathcal{M}_k = \mathcal{M}$ for all k.
- $\mathcal{O}[\frac{1}{x}]$, sheaf of meromorphic functions on $X = \mathbb{C}$ with finite order pole at 0, filtered by the set $\mathcal{O}[\frac{1}{x}]_{\leq k}$ of functions with poles of order $\leq k$.

Exercises on good filtrations

In both exercises ii) and iii), remember that if $\mathcal{F}$ is coherent as a sheaf of $\mathcal{O}$-module, then a submodule $\mathcal{F}' \subset \mathcal{F}$ is coherent if it admits locally a finite number of generators.

i) Let $\mathcal{M}_k$ be a locally good filtration on $\mathcal{M}$. Then for any compact subset K of X we can find an integer with the property of the definition, globally on K.

ii) Prove that the sub $\mathcal{D}_X$-module $\mathcal{D} \cdot f^s \subset \mathcal{O}[\frac{1}{f}, s]f^s$ has a good filtration.

Hint: Consider the filtration of $\mathcal{D}_X[s](m)$ by the total order with respect to (∂, s) and prove that $\mathcal{D}_X[s](m)f^s$ is $\mathcal{O}_X$-coherent.

iii) In the proof of theorem 5.6 we found k_0 such that $\mathcal{O}[\frac{1}{f}] = \mathcal{D}_X \frac{1}{f^{k_0}}$. Prove in detail that $(\mathcal{D}_X(m)\frac{1}{f^{k_0}})_{m\in\mathbb{N}}$ is good. A key step towards the holonomicity of $\mathcal{O}[\frac{1}{f}]$

Théorème 7.18 *Any coherent left $\mathcal{D}_X$-module admits locally a good filtration*

We just give here an idea of the proof. Given $x \in X$ consider a local presentation:

$$(\mathcal{D}_{|U})^q \xrightarrow{\Phi} (\mathcal{D}_{|U})^p \xrightarrow{\pi} \mathcal{M}_{|U} \longrightarrow 0$$

in a neighbourhood U of x. We set $\mathcal{M}_k = \pi(\mathcal{D}_{|U}(k)^p)$. For this filtration the condition 2. of definition 7.17 is satisfied with $k_0 = 0$ and each $\mathcal{M}_k$ is $\mathcal{O}$-coherent. For the details see [15].

In the same reference the reader will find treated with some details a result converse to the previous result, starting from a coherence criterion associated with graded modules, and then a theorem which asserts the equivalence between the *local existence* of a good filtration and the coherence of a $\mathcal{D}_X$-module. This is the main reason why we did not limit our definition of a good filtration to a priori coherent $\mathcal{D}_X$-modules.

Proposition 7.19 *Let $\mathcal{M}$ be a left $\mathcal{D}_X$-module equipped with a filtration $\{\mathcal{M}_k\}$. Then this filtration is good if and only if* $\mathrm{gr}\mathcal{M}$ *is a coherent* $\mathrm{gr}\mathcal{D}$-*module.*

About this result an easy part of the proof is that definition 7.17 implies the finiteness of the fiber $\mathrm{gr}\mathcal{M}_x$ over $\mathrm{gr}\mathcal{D}_x$.

Théorème 7.20 *A left $\mathcal{D}$-module $\mathcal{M}$ is coherent if and only if it admits good filtrations of a sufficiently small neighbourhood of each point.*

This result allowed us to prove that $\mathcal{O}[\frac{1}{f}]$, and $\mathcal{D}[s]f^s$ are coherent $\mathcal{D}$-modules since we refered to it in section 5 and in theorem 5.6. For the second module we notice that the goodness of an appropriate filtration does not use the existence of the Bernstein-Sato polynomial. All the better so since this coherence is a piece of the proof of the existence of a local Bernstein-Sato polynomial!

A corollary of the proposition 7.19 is the behaviour by exact sequences which leads to an Artin type property necessary in the proof of theorem 7.20:

Proposition 7.21 *Let*

$$0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{N} \xrightarrow{\pi} \mathcal{P} \longrightarrow 0$$

be an exact sequence of left $\mathcal{D}$-modules and let $\{\mathcal{N}_k\}$ be a good filtration on $\mathcal{N}$. Then the induced and quotient filtrations $\{\mathcal{M}_k = \mathcal{N}_k \cap \mathcal{M}\}$ and $\mathcal{P}_k = \pi(\mathcal{N}_k)$ are good.

The induced and quotient filtrations are exactly made for the exactness of the sequence of sheaves

$$0 \longrightarrow \mathrm{gr}\mathcal{M} \longrightarrow \mathrm{gr}\mathcal{N} \xrightarrow{\pi} \mathrm{gr}\mathcal{P} \longrightarrow 0$$

By the proposition 7.19 $\mathrm{gr}\mathcal{N}$ is coherent and it is sufficient to prove the coherence of $\mathrm{gr}\mathcal{P}$, see the details in [15, Proposition 14].

<u>Exercises</u> i) Translate the above proposition into the following Artin type formula:

$$\exists k_1 \in \mathbb{N}, \forall \ell \in \mathbb{N}, \quad \mathcal{N}_{k_1+\ell} \cap \mathcal{M} = \mathcal{D}_X(\ell)(\mathcal{N}_{k_1} \cap \mathcal{M})$$

ii) Prove that any good filtration $\{\mathcal{M}_k\}$ has a local expression of the following type: in a sufficiently small neighbourhood U of any given $x \in X$, there is a presentation of $\mathcal{M}$ as a quotient $\frac{\mathcal{D}^r}{\mathcal{N}}$ of a free module by a coherent submodule $\mathcal{N}$ and a r-uple of integers $\underline{n} = (n_1, \cdots, n_r) \in \mathbb{Z}$ such that we have:

$$\mathcal{M}_k = \pi(\mathcal{D}[\underline{n}](k))\,, \text{ with } \mathcal{D}[\underline{n}](k) = \bigoplus_{i=1}^{r} \mathcal{D}(k - n_i)$$

Hint: choose a local system of homogeneous generators of $\mathrm{gr}\mathcal{M}$ which also generate $\mathcal{M}$.

It is technically important to state a result of local comparison between two good filtrations:

Proposition 7.22 *Let $\mathcal{M}_k = F_k(\mathcal{M})$ and $\mathcal{M}'_k = F'_k(\mathcal{M})$ be two good filtrations on a left $\mathcal{D}_X$-module $\mathcal{M}$. Then, for any $x \in X$, there is a neighbourhood U of X and an integer $k_0 \in \mathbb{N}$ such that*

$$\forall \ell \in \mathbb{N}\,, \quad F_\ell(\mathcal{M}_{|U}) \subset F'_{k_0+\ell}(\mathcal{M}_{|U}) \subset F_{2k_0+\ell}(\mathcal{M}_{|U})$$

We shall omit the easy proof, left as an exercise. See also [15].

7.4 *Characteristic variety of a differential system*

7.4.1 *Case of a monogeneous module* $\mathcal{M} = \frac{\mathcal{D}}{\mathcal{I}}$

Let $\mathcal{I}$ be a coherent left ideal of $\mathcal{D}$. We denote I the graded ideal of $D := \mathrm{gr}\mathcal{D}$ associated with the natural induced filtration on $\mathcal{I}$:

$$I = \mathrm{gr}\mathcal{I} = \bigoplus_{\ell \in \mathbb{N}} \frac{\mathcal{I} \cap \mathcal{D}_X(\ell)}{\mathcal{I} \cap \mathcal{D}_X(\ell - 1)} \simeq \bigoplus \frac{\mathcal{I} \cap \mathcal{D}_X(\ell) + \mathcal{D}_X(\ell - 1)}{\mathcal{D}_X(\ell - 1)}$$

We are in fact in the situation of the proposition 7.21

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{D} \xrightarrow{\pi} \mathcal{M} \longrightarrow 0$$

with the natural filtrations $\mathcal{M}_\ell = \pi(\mathcal{D}_X(\ell)) = \frac{\mathcal{D}_X(\ell)}{\mathcal{I} \cap \mathcal{D}_X(\ell)}$ and $\mathcal{I}_\ell = \mathcal{I} \cap \mathcal{D}_X(\ell)$

We can write $\mathrm{gr}(\mathcal{M}) := \bigoplus \frac{\mathcal{M}_\ell}{\mathcal{I} \cap \mathcal{D}_X(\ell)} \simeq \frac{D}{I}$ according to elementary results on quotients of modules:

Exercise: Verify that the following sequences are exact:

$$0 \longrightarrow \mathcal{I}_\ell \longrightarrow \mathcal{D}(\ell) \longrightarrow \mathcal{M}_\ell \longrightarrow 0$$

$$0 \longrightarrow \mathrm{gr}^\ell \mathcal{I} \longrightarrow \frac{\mathcal{D}(\ell)}{\mathcal{D}(\ell-1)} \xrightarrow{\pi} \frac{\mathcal{M}_\ell}{\mathcal{M}_{\ell-1}} \longrightarrow 0$$

Using proposition 7.21 we can set the following

Definition 7.23 *The sheaf I is a coherent graded sheaf of ideals in $D \subset \mathcal{O}_{T^\star X}$. This ideal defines a conical subset of $T^\star X$ that we denote* $\mathrm{char}(\mathcal{M})$ *and call the characteristic variety associated with $\mathcal{I}$.*

Some Remarks:
- Conical means that $(x, \xi) \in \mathrm{char}(\mathcal{M}) \Rightarrow \forall \lambda \in \mathbb{C}\,,\ (x, \lambda\xi) \in \mathrm{char}(\mathcal{M})$
- Let f_i be a local system of homogeneous generators of I, that is $I_{|U} = (f_1, \cdots, f_r)\mathrm{gr}\mathcal{D}_{|U}$, then we have a set of homogeneous equations for $\mathrm{char}(\mathcal{M})$:

$$f_1(x, \xi) = \cdots = f_r(x, \xi) = 0$$

But if $\mathcal{I} = \mathcal{D} \cdot (P_1, \cdots, P_r)$, the symbols $\sigma(P_1), \cdots, \sigma(P_r)$, may fail to generate I.

We shall prove soon in a more general setting that this variety is intrinsic for $\mathcal{M}$. Here is an example:

Let $\mathcal{M} = \frac{\mathcal{D}}{\mathcal{D} \cdot x}$. Let δ be the class of 1. Since $x\delta = 0$, we have $\delta = -x\delta'$, so that $\mathcal{M} = \mathcal{D} \cdot \delta = \mathcal{D} \cdot \delta'$

This gives two presentations $\mathcal{M} = \frac{\mathcal{D}}{\mathcal{I}1} = \frac{\mathcal{D}}{\mathcal{I}_2}$, with $\mathcal{I}_1 = \mathcal{D} \cdot x$ and $\mathcal{I}_2$ the annihilator of δ'.

Exercise Prove that $\mathcal{I}_2 = \mathcal{D} \cdot x^2 + \mathcal{D} \cdot (x\frac{\partial}{\partial x} + 2)$

Verify that $I_1 = \mathcal{O}_{T^\star X} \cdot x$ and $I_2 = \mathcal{O}_{T^\star X} \cdot (x^2, x\xi)$ define the same subset of $T^\star X$, the conormal set to the origin $T^\star_0 X$.

Let us remark as a transition that $I = \mathrm{gr}\mathcal{I}$ is the annihilator of $\mathrm{gr}\mathcal{M}$. In the next section we generalize this observation.

7.4.2 *General case*

Let us assume first that $\mathcal{M}$ has a global good filtration $\{\mathcal{M}_\ell\}_{\ell\in\mathbb{N}}$.

We set:

$$I = \mathrm{Ann}(\mathrm{gr}\mathcal{M}) = \{\mathrm{f} \in \mathrm{gr}\mathcal{D}\,, \quad \mathrm{f} \cdot \mathrm{gr}\mathcal{M} = 0\}$$

Remark 7.24 *The set I is a homogeneous ideal of D which means that:*

$$I = \bigoplus I_\ell \text{ where } I_\ell = I \cap \mathrm{gr}^\ell \mathcal{D}.$$

This is a general fact about annihilators of graded modules over graded rings, which we leave as an exercise.

Remark 7.25 *If $f = \sigma(P) \in I_\ell$, with P of order d, and $m \in \mathcal{M}_k$, the equation $fm = 0$ is equivalent to*

$$\overline{Pm} = 0\,, \ \ in\ \frac{\mathcal{M}_{k+d}}{\mathcal{M}_{k+d}}$$

that is:

$$P \cdot \mathcal{M}_k \subset \mathcal{M}_{k+d-1}$$

In view of defining the characteristic variety in term of I the following result is important:

Proposition 7.26 *Let $\mathcal{M}$ be a coherent $\mathcal{D}$-module with a good filtration $\{\mathcal{M}_k\}$. Then the ideal $I = \mathrm{Ann}(\mathrm{gr}\mathcal{M})$ is a coherent sheaf of ideals of D.*

To complete the definition we would like to glue the sets of zeros of I on all coordinate sets where we have a good filtration. This requires verifying that this set of zeros as a *reduced analytic set* is independent of the chosen good filtration. This is obtained in the following theorem.

Théorème 7.27 *Let $\mathcal{M}$ be a left coherent $\mathcal{D}_X$-module. Then the radical*

$$\sqrt{I} = \sqrt{\mathrm{Anngr^F}\mathcal{M}}$$

does not depend on the choice of a good filtration F.

The proof is identical to the proof for the algebraic case and we omit it. See the course of F. Castro-Jimenez in this volume.

We shall also admit the following fact (see Gunning and Rossi): if $I \subset D$ is a coherent sheaf of ideals then $\sqrt{I}$ is also a coherent sheaf. We are now ready to give as a conclusion the final definition:

Definition 7.28 *Let $\mathcal{M}$ be a coherent left $\mathcal{D}_X$ module on a complex analytic manifold X. Then there is a coherent sheaf of homogeneous ideals $J \subset \mathrm{gr}\mathcal{D}_X$ such that for any local good filtration of the restriction of $\mathcal{M}$ to an open subset $U \subset X$, we have:*

$$J = \sqrt{\mathrm{Anngr}^{\mathrm{F}}\mathcal{M}}$$

The conical subset of $T^\star X$ defined as the zero locus of J is called the characteristic variety $\mathrm{char}\,\mathcal{M}$ *of the differential system $\mathcal{M}$.*

7.4.3 *A finiteness property*

Various properties of the Characteristic variety $\mathrm{char}(\mathcal{M}) \subset T^\star X$, such as the dimension, the Bernstein-Sato inequality, the involutivity, the multiplicity and the dimension at a point in $T^\star X$ should still be developped and we shall give in this last section a very partial view of these subjects adapted to the use we made of them in section 5.

As we already noticed in section 5 theorem 5.1 and proposition 5.2 are indeed the two pillars on which the existence of the b-function is built.

For the analysis of the local dimension and the multiplicity at any point of the characteristic variety of $\mathcal{M}$, for the study of the homological dimension of $\mathcal{D}$ and the way all these ingredients allow us to build the filtration by the dimension of a coherent module as stated in theorem 5.1, we refer to [15, Chapter V].

In this last section we shall now give a partial account of the list of results which lead quite naturally to a proof of the proposition 5.2 as in [15, IV.2.2.].

Theorem 7.29 *: Bernstein inequality. Let $\mathcal{M} \neq (0)$ be a coherent $\mathcal{D}_X$-module on an n-dimensional manifold X. Then for any $x \in X$ such that the germ at $(x, 0)$ of $\mathrm{char}(\mathcal{M})$ is non zero we have:*

$$\dim_{(x,0)} \mathrm{char}(\mathcal{M}) \geq n = \dim X.$$

It is easy to prove that the support of $\mathcal{M}$ which is the setlocus $\{x \in X \mid \mathcal{M}_x \neq 0\}$ of non zero stalks is equal to $\mathrm{char}(\mathcal{M}) \cap T^\star_X X$. The result follows by descending induction on the dimension of this support using a lemma on the structure of the systems whose support is contained in a hypersurface. See [15, III.3.] for more details on coherence conditions.

Lemma 7.30 *Let $\mathcal{M}$ be a $\mathcal{D}_U$-module, with $U \subset \mathbb{C}^n$ whose support is contained in the hyperplane $H = \{x_n = 0\}$. Then the kernel $\overline{\mathcal{M}}$ of the map $x_n : \mathcal{M} \to \mathcal{M}$ is a coherent $\mathcal{D}_{U\cap H}$-module and we have isomorphisms:*

$$\mathcal{M}_{|U\cap H} \simeq \bigoplus_{k=0}^{\infty} \partial_n^k \overline{\mathcal{M}}\,, \ \ \textit{and } \operatorname{char}(\mathcal{M}) = \operatorname{char}(\overline{\mathcal{M}}) \times \{(0, \xi_n)\}$$

Definition 7.31 *A coherent $\mathcal{D}_X$-module is called holonomic if the characteristic variety* $\operatorname{char}(\mathcal{M})$ *has dimension n.*

Let us notice that Bernstein inequality, in the way we stated it in 7.29 does not guarantee the same lower bound

$$d_{x^\star} \operatorname{char}(\mathcal{M}) \geq n \tag{7}$$

for the local dimension at any point $x^\star \in \operatorname{char}(\mathcal{M})$ which is out of the zero section $T^\star_X X$.

This generalized Bernstein inequality is also true. As a consequence the components of $\operatorname{char}(\mathcal{M})$ have dimension at least n, and in particular the characteristic variety of a holonomic module is of pure dimension n. The inequality (7) follows from a much deeper result: the involutivity of the characteristic variety. Recall that the cotangent bundle is equipped with a canonical 1-form θ and a canonical 2-form $\omega = d\theta$ whose expression in local coordinates are:

$$\theta = \sum \xi_i dx_i\,, \quad \omega = d\theta = \sum_{i=1}^{n} d\xi_i \wedge dx_i.$$

The 2-form ω defines at each point $x^\star$ of $T^\star X$ a nondegenerate bilinear form. Given a smooth subvariety $V \subset T^\star X$ and $p \in V$ we write $(T_pV)^\perp \subset T_p(T^\star X)$ for the orthogonal subspace to T_pV with respect to ω.

Theorem 7.32 *The characteristic variety of any coherent module is involutive.*

Involutivity means that for any p in the smooth part of $\operatorname{char}(\mathcal{M})$, we have:

$$(T_p \operatorname{char}(\mathcal{M}))^\perp \subset T_p \operatorname{char}(\mathcal{M}) \tag{8}$$

A proof of the involutivity theorem can be found in [29]. As explained by Bernard Malgrange in this reference the first proof of the involutivity is due to M. Kashiwara, T. Kawai and M. Sato in [20]. This proof as well as the proof by Malgrange in [29] or the proof inspired by a suggestion of B. Malgrange in [15] uses a localisation of operators in $T^\star X$ known as microlocalisation. For a purely algebraic proof valid for a much larger class of non commutative rings a basic reference is the paper of O. Gabber [13].

Because of the nondegeneracy of ω, Bernstein inequality follows from the inequality of dimensions $2n - \dim T_p\mathrm{char}(\mathcal{M}) \leq \dim T_p\mathrm{char}(\mathcal{M})$ at any smooth point hence everywhere. For a holonomic module we deduce from the inclusion (8) that there is in fact an equality $(T_p\mathrm{char}(\mathcal{M}))^{\perp} = T_p\mathrm{char}(\mathcal{M})$ at each smooth point. A variety with this property is called Lagrangian.

The structure of these varieties in the conical case is easy to deduce from a study of the symplectic form ω and we obtain (see [15, IV.1.] for details):

Theorem 7.33 *The characteristic variety of a holonomic module $\mathcal{M}$ is a conical involutive subset of $T^\star X$. As such it can be written as a union*

$$\mathrm{char}\mathcal{M} = \bigcup_{\alpha\in A} T^\star_{S_\alpha} X$$

where $(S_\alpha)_{\alpha\in A}$ is the locally finite family of closed analytic sets consisting in the projections of the components of $\mathrm{char}(\mathcal{M})$ *by the map $T^\star X \to X$.*

In this theorem $T^\star_{S_\alpha}X$ is the closure of the conormal to the smooth part of S_α. We can also define a refinement of the support of $\mathcal{M}$, $\bigsqcup_{\alpha\in A} S_\alpha = \bigcup_{\beta\in B} Y_\beta$, into a Whitney stratification $(Y_\beta)_{\beta\in B}$. We have then an inclusion:

$$\mathrm{char}(\mathcal{M}) \subset \bigsqcup_{\beta\in B} T^\star_{Y_\beta} X$$

because the second member of this inclusion is closed as a consequence of the Whitney condition. For another aspect of the geometry of the characteristic variety, involving sheaves of solutions, the reader can consult the lectures notes [26] of Lê Duñg Tráng and Bernard Teissier in this volume.

The structure of a $\mathcal{D}$-module whose characteristic variety is the conormal bundle to a smooth subvariety can be described completely from a local view point. We assume for this purpose that $Y = \{\underline{x} \in \mathbb{C}^n \mid x_1 = \cdots = x_p = 0\} \subset X = \mathbb{C}^n$. A typical module having $T^\star_Y X$ as a characteristic variety is:

$$\mathcal{B}_Y(X) = \frac{\mathcal{D}_X}{\mathcal{D}_X(x_1,\cdots,x_p,\frac{\partial}{\partial x_{p+1}},\cdots,\frac{\partial}{\partial x_n})}$$

Proposition 7.34 *Let $\mathcal{M}$ be a $\mathcal{D}$-module whose characteristic variety is the conormal bundle $T^\star_Y X$. Then $\mathcal{M}$ is isomorphic to the direct sum of r copies of the $\mathcal{D}$-module $\mathcal{B}_Y(X)$*

Proof We can apply lemma 7.30 repeatedly. The iterated kernel $\overline{\mathcal{M}} := \{m \in \mathcal{M}_{|Y} \mid x_1 m = \cdots = x_p m = 0\}$ is a coherent $\mathcal{D}_Y$-module, and because of the relation $\mathrm{char}\mathcal{M}_{|Y} = \{(\underline{0^p}, \xi_1, \cdots, \xi_p)\} \times \mathrm{char}\overline{\mathcal{M}}$, we find that:

$$\mathrm{char}\overline{\mathcal{M}}_{|Y} = T^\star_Y Y$$

the conormal section of Y. It is easy to deduce from this fact that $\mathcal{M}_{|Y}$ is a connection see [15, IV.2.], hence is isomorphic to a $\mathcal{O}_Y^r$ as a $\mathcal{D}_Y$-module. The proposition follows by application of an iterated version of lemma 7.30.

$$\mathcal{M}_{|Y} \simeq \bigoplus_{\underline{k} \in \mathbb{N}^p} \partial_1^{k_1} \cdots \partial_p^{k_p} \mathcal{O}_Y^r \simeq \mathcal{B}_Y(X)^r$$

□

The above proposition can be thought as a generalisation of theorem 7.14 which concerns the case $Y = X$. Similarly the generalisation of corollary 7.15 is:

Corollary 7.35 *Let $\mathcal{M}$ be a $\mathcal{D}_X$-module whose characteristic variety is the conormal bundle $T^\star_Y X$ to a smooth submanifold $Y \subset X$ and let $\varphi : \mathcal{M} \to \mathcal{M}$ be a $\mathcal{D}_X$-linear homomorphism. Then the germ of φ at any point $p \in X$ admits a minimal polynomial.*

The proof is identical to the proof of corollary 7.15 once we have proved by a direct calculation that any morphism $\mathcal{B}_Y(X) \to \mathcal{B}_Y(X)$ is a dilatation $m \mapsto \lambda m, \lambda \in \mathbb{C}$.

Let us now recall proposition 5.2, which we are now able to prove as announced at the begining of this last section 7.4.3

Proposition 5.2 *Let $\mathcal{M}$ be a germ of a holonomic module and $\varphi : \mathcal{M} \longrightarrow \mathcal{M}$ a $\mathcal{D}$-linear map. Then φ admits a minimal polynomial.*

We use the notation of theorem 7.33 and consider a component S_{α_0} of the highest possible dimension in the support of $\mathcal{M}$ and a smooth point $p \in S_{\alpha_0} \setminus \bigcup_{\alpha \neq \alpha_0} S_\alpha$. Then we may apply corollary 7.35 to the germ of $\mathcal{M}$ at p. Therefore there exists a polynomial $b_1(s) \in \mathbb{C}[s]$ such that $b_1(\varphi_p) = 0$ and therefore $b_1(\varphi)\mathcal{M}$ does not contain any more p in its support. Since S_{α_0} is irreducible we conclude that in the characteric variety $\mathrm{char}(b_1(\varphi)cM) \subset \mathrm{char}\mathcal{M}$ we may drop one component $T_{S_{\alpha_0}} X$.

Proposition 5.2 follows by induction on the number of local components of $\mathrm{char}\mathcal{M}$.

8 Complements

8.1 *Appendix A: Mellin and Laplace transforms*

8.1.1 *Integrals depending on a parameter*

We consider $\Omega \subset \mathbb{C}$ an open subset of $\mathbb{C}$ and $f : I \times \Omega$ with $I \subset \mathbb{R}$ an interval, a continuous function which is holomorphic with respect to the second variable.

Proposition 8.1 *Assume that for a compact subset $K \subset \Omega$ there is an integrable function $\varphi : I \to \mathbb{R}^+$ such that:*

$$\forall (t, z) \in I \times \Omega\,, \quad |f(t, z)| \leq \varphi(t)$$

then the function

$$g(z) = \int_I f(t, z) dt$$

is holomorphic on Ω its derivatives being

$$g^{(n)}(z) = \int_I \frac{\partial^n f}{\partial z^n}(t, z) dt$$

8.1.2 *Analyticity of Mellin transforms*

Let $f :]0, +\infty[\to \mathbb{C}$ be a continuous function (or simply a locally integrable one). The Mellin transform of f is the function Γ_f defined in some open set $\Omega \subset \mathbb{C}$ (possibly empty!), by the integral:

$$\Gamma_f(s) = \int_0^{+\infty} f(x) x^{s-1} dx$$

If we set $x = e^{2\pi t}$, et $s = -iz$, this integral can also be interpreted as a Laplace transform of $g(t) = f(e^{2\pi t})$, $t \in \mathbb{R}$.

$$\Gamma_f(s) = 2\pi \int_{\infty}^{+\infty} g(t) e^{2\pi t(-iz-1)} e^{2\pi t} dt = 2\pi \int_{\infty}^{+\infty} g(t) e^{-2\pi i t z} dt$$

Thus we have $\Gamma_f(s) = 2\pi \hat{g}(z)$ in which the Laplace transform of g is defined as:

$$\hat{g}(z) = \int_{\infty}^{+\infty} g(t) e^{-2\pi i t z} d$$

Theorem 8.2 *Assume that the integral defining $\Gamma_f(s)$ is absolutely convergent for $\Re(s) = a$ and for $\Re(s) = b > a$. Then this integral is also absolutely convergent for $a \leq \Re(s) \leq b$, and the function Γ_f is holomorphic and bounded in the open strip $B := \{s \in \mathbb{C}\,, \quad a < \Re(s) < b\}$ and continuous in the closed strip $\overline{B}$.*

Since $|x^{s-1}| = x^{\Re s-1}$, the absolute convergence of $\Gamma_f(s)$ depends only on $\Re s$ and for any $x > 0$ and any $s \in \overline{B}$, we have:

$$|x^{s-1}| = x^{\Re s-1} \leq x^{a-1} + x^{b-1}$$

This implies the convergence of the integral and by section 8.1.1 the continuity on $\overline{B}$ (using the Lebesgue convergence theorem) and holomorphy on B. Furthermore Γ_f is bounded since

$$|\Gamma_f(s)| \leq \Gamma_{|f|}(a) + \Gamma_{|f|}(b)$$

The derivative of Γ_f is:

$$\Gamma'_f(s) = \int_0^{+\infty} f(x)(\log x)x^{s-1}dx$$

Examples

1. Let $f: [0, +\infty[\longrightarrow \mathbb{R}$ be rapidly decreasing at $+\infty$, which means that all $x^n D^\alpha(f)$ are bounded, and continuous at 0. Then $\Gamma_f(s)$ is well defined for $\Re s > 0$
2. Let f be with compact support $[a, b] \subset]0, +\infty[$. Then $\Gamma_f(s)$ is an entire function defined in the whole complex plane.

Both examples are direct consequences of section 8.1.1.

We may refine and generalise the first example to a large class of examples for which there is a meromorphic continuation to a half-plane or to the whole complex plane in some cases.

Theorem 8.3 *Let $f:]0, +\infty[\longrightarrow \mathbb{C}$ be locally integrable function rapidly decreasing at infinity, and which admits an asymptotic expansion at zero of the form*

$$f(x) = a_1 x^{s_1} + \cdots + a_n x^{s_n} + O(x^{s_{n+1}})$$

with $\Re s_1 \leq \cdots \leq \Re s_n < \Re s_{n+1}$. Then the integral $\Gamma_f(s)$ is convergent for $\Re s > -\Re s_1$ and admits an analytic continuation as a meromorphic function on the punctured half-plane:

$$\{s \in \mathbb{C} \,|\, \Re s > -\Re s_{n+1}\} \setminus \{-s_1, \cdots, -s_n\}$$

with simple poles having as residues

$$Res_{-s_k}(\Gamma_(f) = a_k$$

Proof For any $s \in \mathbb{C}$ the integral $\int_1^\infty f(x)x^{s-1}ds$ is convergent and defines a holomorphic entire function by section 8.1.1. Therefore in what follows we deal only with the integral $\int_0^1 f(t)dt$.

Furthermore $g(s) = x^{-s_1}f(x)$ is continuous at 0 with $g(0) = a_1$. Then by example 1 above the Mellin transform $\Gamma_g(\tilde{s})$ of g is defined in the half plane $\tilde{s} > 0$, which is equivalent to the fact that Γ_f is defined by a convergent integral on the half plane $\Re(s + s_1) > 0$.

Let us write the asymptotic expansion of f and its integral on $[0, 1]$:

$$f(x) = a_1x^{s_1} + \cdots + a_nx^{s_n} + x^{s_{n+1}}h(x), h(x) \text{ bounded.}$$

$$\int_0^1 f(x)x^{s-1}dx = \int_0^1 a_1x^{s+s_1-1}dx + \cdots + \int_0^1 a_nx^{s+s_n-1}dx + \int_0^1 x^{s+s_{n+1}-1}h(x)dx$$

$$\int_0^1 f(x)x^{s-1}dx = \frac{a_1}{s+s_1} + \cdots + \frac{a_n}{s+s_n} + \int_0^1 x^{s+s_{n+1}-1}h(x)dx$$

This ends the proof of the theorem because, again by example 1 above (or rather a slight generalisation), the function

$$s \to \int_0^1 x^{s+s_{n+1}-1}h(x)dx$$

is holomorphic in the half plane $\Re(s + s_{n+1}) > 0$. $\square$

Corollary 8.4 *If f is $\mathcal{C}^\infty$ on $[0, +\infty[$ and rapidly decreasing at $+\infty$ then Γ_f can be extended to a meromorphic function on $\mathbb{C} \setminus (-\mathbb{N})$ such that*

$$Res_{-n}(\Gamma_(f) = \frac{f^{(n)}(0)}{n!}$$

More generally if we have an asymptotic expansion at any order of the type:

$$f(x) \sim \sum_{i\in\mathbb{N},\ell\leq n_i} a_ix^{s_i}(\log x)^\ell$$

for some sequence with (strictly) increasing $\Re s_i$ tending to $+\infty$ then the function Γ_f can be extended to a meromorphic function on $\mathbb{C} \setminus \{-s_i\}$ with poles at $-s_i$ of order $\leq n_i$. We just have to use exactly the same argument at any order and use the formula for the derivative of the Mellin transform.

8.2 Appendix B: Regular sequences, and application to the annihilator $Ann_{\mathcal{D}} f^s$, in the isolated case

In this appendix we want to show the following theorem, about the annihilator of f^s over $\mathcal{D}$ in the case of an isolated singularity. We denote by $g_1, \cdots, g_n$ a regular sequence in $\mathcal{O} = \mathbb{C}\{x_1, \cdots, x_n\}$, and $\mathcal{A}$ the ideal $(g_1, \cdots, g_n)$.

The following proposition is stated in Yano's paper [38] without proof. Since I do not know a reference for a proof I give one after hand-written notes [12] which Joël Briançon communicated to me a long time ago...

Proposition 8.5 *The following conditions on $g_1, \cdots, g_n$ are equivalent:*

i) For $i = 2, \cdots, n$ g_{i+1} is a non zero divisor in $\frac{\mathcal{O}}{(g_1, \cdots, g_i)}$ (which is the definition of a regular sequence.)

ii) The kernel of the natural map sending ξ_i to g_i:

$$\mathcal{O}[\xi_1, \cdots, \xi_n] \xrightarrow{\varphi} \mathcal{R}_{\mathcal{A}} := \bigoplus_{\nu \in \mathbb{N}} \mathcal{A}^\nu T^\nu$$

is the ideal of $\mathcal{O}[\xi_1, \cdots, \xi_n]$ generated by the set of elements $\{g_i \xi_j - g_j \xi_i \mid 1 \leq i < j \leq n\}$.

These two conditions are also, as stated in Yano's paper, equivalent to a third one: Let $\mathcal{A}$ be the ideal $\sum \mathcal{O} g_i$, then the ring homomorphism:

$$\frac{\mathcal{O}}{\mathcal{A}}[\xi_1, \cdots, \xi_n] \xrightarrow{\psi} \bigoplus_{\nu \in \mathbb{N}} \frac{\mathcal{A}^\nu}{\mathcal{A}^{\nu+1}}$$

determined by $\psi(\xi_i) = \overline{g_i} \in \frac{\mathcal{A}}{\mathcal{A}^2}$ is an isomorphism.

We will not need this latter fact in these notes. The fact that the annhihilator in $\mathcal{D}$ of f^s is generated by the operators $\frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial}{\partial x_j}$ when f is an isolated singularity is then an easy induction on the order of the operator P using the fact that if $P f^s = 0$ the principal symbol $\sigma(P)(x, \xi)$ is in the kernel of the map φ, and the proposition 8.5 applied to $g_i = \frac{\partial f}{\partial x_i}$.

Proof We first recall the relations between the elements of a regular sequence in an arbitrary commutative ring R

Lemma 8.6 *If $g_1, \cdots, g_n$ is a regular sequence in R then the module of relations:*

$$\mathcal{R}el := \{(\lambda_1, \cdots, \lambda_n) \in R^n \mid \sum \lambda_i g_i\}$$

is generated by the elements $g_i\epsilon_j - g_j\epsilon_i$, with $\epsilon_1, \cdots, \epsilon_n$ the canonical basis of R^n.

An equivalent formulation is to say that there exists an antisymetric $n \times n$ matrix A, such that $\lambda = Ag$, with $\lambda = {}^{tr}(\lambda_1, \cdots, \lambda_n)$ and $g = {}^{tr}(g_1, \cdots, g_n)$

The lemma is easily proved by induction on n. First by the regular sequence condition we know that $\overline{\lambda_n} = 0$ in $\frac{\mathcal{O}}{(g_1, \cdots, g_{n-1})}$. Therefore we may write $\lambda_n = \alpha_1 g_1 + \cdots + \alpha_{n-1} g_{n-1}$, and we get a relation between the $n-1$ first g_i:

$$\sum_{i=1}^{n-1} (\lambda_i + \alpha_i g_n) g_i$$

to which we may apply the induction hypothesis. This gives the desired result (details left to the reader). □

Lemma 8.7 *Let $\sum_{|J|=p} \mu_J g^J = 0$ be a relation in $\mathcal{O}$ between the monomial of a fixed degree p in the g_i's. Then there is a matrix $H = a_{J,i}$ with coefficients in $\mathcal{O}$ such that:*

$$\mu_J = \sum_{i=1}^{n} a_{J,i} g_i$$

and for any n-multiindex L of length $p+1$,

$$\sum_{J+\epsilon_i = L} a_{J,i} = 0,$$

Proof We set in an arbitrary way:

$$\mu_J = \sum_{j_k > 0} \mu_{J,k}, \text{ for any } J = (j_1, \cdots, j_n).$$

We get

$$\begin{aligned} 0 \;=\; \sum_{|J|=p} \mu_J g^J &= \sum_{|I|=p-1} \big(\sum_k \mu_{I+\epsilon_k,k} f_k \big) f^I \\ := \sum_{|I|=p-1} \nu_I f^I & \quad \text{with } \nu_I = \sum_k \mu_{I+\epsilon_k,k} f_k \end{aligned}$$

By an induction hypothesis on p this implies:

$$\nu_I = \sum_{k=1}^{n} b_{I,k} g_k$$
$$\sum_{I+\epsilon_k=J} b_{I,k} = 0$$

which can be rewritten in the following way:

$$\sum_{k=1}^{n} (\mu_{I+\epsilon_k,k} - b_{I,k}) f_k = 0$$

and using the case $p = 1$ which is exactly lemma 8.6 leads to:

$$\mu_{I+\epsilon_k,k} - b_{I,k} = \sum h^I_{k,i} f_i$$

a formula which involves several antisymmetric matrix $h^I = (h^I_{k,i})$. Then by an appropriate summation:

$$\sum_{j_k>0} (\mu_{J,k} - b_{J-\epsilon_k,k}) = \mu_J - \sum_{I+\epsilon_k=J} b_{I,k} = \mu_J$$

and finally

$$\mu_J = \sum_{I+\epsilon_k=J} (\mu_{I+\epsilon_k,k} - b_{I,k}) = \sum_{I+\epsilon_k=J} (\sum h^I_{k,i} f_i) = \sum_{i=1}^{n} a_{J,i} f_i$$

with $a_{J,i} = \sum_{I+\epsilon_k=J} h^I_{k,i}$ This proves the lemma 8.7 because of the following last calculation:

$$\sum_{J+\epsilon_i=L} a_{J,i} = \sum_{I+\epsilon_k+\epsilon_i=L} h^I_{k,i} + h^I_{i,k} = 0$$

□

Now we return finally to the proof of the proposition 8.5:

i) $\Rightarrow$ ii)

Let us take a homogeneous element of degree p, $\sum \mu_J \xi_J$ in the kernel of $\mathcal{O}[\xi] \to \bigoplus \mathcal{A}^\nu T^\nu$. By the lemma 8.7 we may write $\mu_J = \sum_i a_{J,i} g_i$ with $\sum_{J+\epsilon_i=L} a_{J,i} = 0$ for any fixed L of length $p+1$.

Let us choose J and i such that $a_{J,i} \neq 0$. We change the relation if there is an index j_k such that $j+k \neq 0$ and $k < i$. Then for such a triple (J,k,i) we have:

$$g_i \xi^J = g_i \xi_k \xi^{J-\epsilon_k} = (g_i \xi_k - g_k \xi_i)\xi^{J-\epsilon_k} + g_k \xi^{J'}$$

with $J' = (j_1, \cdots, j_k - 1, \cdots, j_i + 1, \cdots, j_n)$

Now modulo the ideal of $\mathcal{O}$ generated by the element $g_i\xi_k - g_k\xi_i$ we can replace the term $a_{J,i} g_i \xi_J$ by $a_{J,i} g_k \xi_{J'}$, noticing that because of $\epsilon_i + J = \epsilon_k + J'$ the condition $\sum_{J+\epsilon_i=L} a_{J,i} = 0$ is preserved. Iterating this proces we are reduced to a situation where the only remaining $a_{J,i}$ are for multiindices J of the type:

$$J = (0, \cdots, 0, j_i, \cdots, j_n)$$

Then all the $a_{J,i}$ are zero because of the relation $\sum_{J+\epsilon_i=L} a_{J,i} = 0$ and the fact that in this situation there is for any L a unique J with $\epsilon_i = L$.

ii) $\Rightarrow$ i) We have to prove that g_{i+1} is not a zero divisor in $\frac{\mathcal{O}}{(g_i,\dots,g_i)}$. It is enough to do this for $i+1=n$. If $\overline{u} \cdot \overline{g_n} = 0$ in $\frac{\mathcal{O}}{(g_i,\dots,g_{n-1})}$ this means that $ug_n = \sum_{i=1}^n u_i g_i$ or that $(u \mathrm{mod} \mathcal{A})\xi_{i+1} = \sum_{k=1}^i (u_i \mathrm{mod}\mathcal{A}) g_k \in Ker\psi$. By a direct application of ii) this implies that $u \in cA$.

References

[1] A. Assi, F.J. Castro-Jimenez and M. Granger, The standard fan of an analytic $\mathcal{D}$-module. JPAA $n^\circ 164$ (2001) pp 3-31.

[2] M.F. Atiyah, *Resolution of singularities and division of distribution.* Comm in pure and applied mathematics vol 23, 1970, p. 145-150.

[3] R. Bahloul, *Démonstration constructive de l'existence de polynômes de Bernstein-Sato*, Compositio Math. 41 (2005) 175-191.

[4] D. Barlet, *Développememts asymptotiques des fonctions obtenues par intégration sur les fibres.* Inventiones Mathematicae, **68**, p. 129-174 (1982).

[5] I. N. Bernstein, S.I. Gel'fand *Meromorphy of the function* P^λ. Funct. Anal. Appl. **1**, p. 84-86, 1968.

[6] I. N. Bernstein, *The analytic continuation of generalised functions with respect to a parameter*. Funct. Anal. Appl. 6, 273-285, 1972.

[7] J. Briançon, M. Granger and Ph. Maisonobe, *Sur le polynôme de Bernstein des singularités semi quasi-homogènes*. Prépublication de l'Université de Nice $n°$ 138. Nov. 1986.

[8] J. Briançon, M. Granger, Ph. Maisonobe and M. Miniconi, *Algorithme de calcul du polynôme de Bernstein: cas non dégénéré*. Annales de l'Institut Fourier, tome 39, Fasc. 3.1989. pp. 553-610.

[9] J.E. Björk, *Rings of Differential Operators*. Volume 21 of North-Holland Math. Libr., Amsterdam, Noth-Holland, 1979.

[10] J. Briançon and Ph. Maisonobe, *Caractérisation de l'existence du polynôme de Bernstein relatif.* Progress in Mathematics, vol. 134, 1996. 215-236. Acte du colloque de La Rabida.

[11] J. Briançon and Ph. Maisonobe, *Remarques sur l'idéal de bernstein associé à des polynômes*. Prépublications $n°$560 de l'Université de Nice. Mai 2002.

[12] J. Briançon, *Suites régulières et relations.* Hand written notes.

[13] O. Gabber, *The integrability of the characteristic variety.* American Journal of Math., **103**, (1981), p. 445-468.

[14] R. Godement, *Théorie des faisceaux.* Hermann, Paris, 1964.

[15] M. Granger and Ph. Maisonobe, *A basic course on differential modules.* In: Eléments de la théorie des systèmes différentiels, tome 1: $\mathcal{D}_X$-modules cohérents. Travaux en cours $n°$ 46 Hermann, 1993, tome I, pp. 103-167.

[16] M. Granger and M. Schulze, *On the symmetry of b-functions of linear free divisors.* arXiv:0807.0560. to appear in the publications of RIMS (Kyoto).

[17] I de Gregorio, David Mond and Christian Sevenheck, *Linear free divisors and Frobenius manifolds.* Compositio Mathematica, volume 145, $n°$ 5, p. 1305-1350, 2009

[18] P. Jeanquartier, *Développement asymptotique de la distribution de Dirac attaché à une fonction analytique.* note aux CRAS t.271 (9 dec 1970)

[19] P. Jeanquartier, *Transformation de Mellin et développements.* L'Enseignement Mathématique, Vol.25 (1979)

[20] M. Kashiwara, T. Kawai and M. Sato, *Microfunctions and pseudo-differential equations.* Springer Lectures notes in Math., n°287 (1973), p.264-529.

[21] M. Kashiwara, *B-functions and holonomic systems.* Inventiones Mathematicae **38**, 33-53 (1976).

[22] M. Kashiwara, *On the holonomic systems of linear differential equations II.* Inventiones Mathematicae **49**, 121-135 (1978).

[23] T. Kimura, *Introduction to Prehomogeneous Vector Spaces.* Translations of Mathematical monographs, vol 215, Amer. Math. Soc. (2003)

[24] Lê Duñg Tráng and C.P. Ramanujan, *Topologie des singularités des hypersurfaces complexes.* In Singularités à Cargèse, Astérisque n° 7 et 8. SMF Paris (1973).

[25] Lê Duñg Tráng and C.P. Ramanujan, *The invariance of Milnor's number implies the invariance of the topological type.* Am. Journ. of Math. **98**(1976) $n^\circ 1$, 67-78.

[26] Lê Duñg Tráng and B. Teissier, *Geometry of characteristic varieties.* This volume.

[27] B. Malgrange, *Le polynôme de Bernstein d'une singularité isolée.* Lectures notes in Mathematics **459**, 98-119 Springer (1974).

[28] B. Malgrange, *Intégrales asymptotiques et monodromie.* Annales scientifiques de l'EN.S. 4e série, tome 7, no 3 (1974), p. 405-430.

[29] B. Malgrange, *L'involutivité des caractéristiques des systèmes différentiels et microdifférentiels.* in Séminaire Bourbaki, Springer Lectures notes in Math., n°522 (1978).

[30] B. Malgrange, *Polynômes de Bernstein-Sato et cohomologie évanescente.* in Analysis and topology on singular spaces, II, III (Luminy, 1981) 243–267, Astérisque, 101-102, Soc. Math. France, Paris, 1983.

[31] J. Milnor and P. Orlik, *Isolated singularities defined by weighted homogeneous polynomials. 1969*, Topology **9**, (1970), p.385-393

[32] Z. Mebkhout and L. Narvaez-Maccarro, *La théorie du polynôme de Bernstein-Sato pour les algèbres de Tate et de Dwork-Monsky-Washnitzer.* Annales scientifiques de l'ENS t.**24**, $n°2$ (1991), 227-256.

[33] M. Sato, *Theory of prehomogeneous vector spaces.* noted by T. Shintani in Japanese, Sugaku non ayumi **15**, 85-157 (1970.)

[34] M. Sato and T. Shintani, *On zeta-function associated with prehomogeneous vector spaces.* Annals of Math., **100**, p. 131-170 (1974).

[35] M. Sato, M. Kashiwara, and T. Kimura and T. Oshima, *Micro-Local Analysis of Prehomogeneous Vector Spaces.* Inventiones Mathematicae 62 p 117 - 179, (1980)

[36] C. Sabbah, *Proximité evanescente I. La structure polaire d'un $\mathcal{D}$-module.* Compositio Math. 62 (1987) 283-319.

[37] J.-P. Serre *Faisceaux algébriques cohérents.* Annals of Mathematics, **61** (1955)

[38] T. Yano, *On the theory of b-functions.* Publ. RIMS, Kyoto Univ. 14 (1978) 111-202.

DIFFERENTIAL ALGEBRAIC GROUPS

B. MALGRANGE

UFR de Mathématiques, UMR 5582,
Institut Fourier, Université Grenoble 1 -CNRS,
38402 Saint-Martin d'Hères Cedex, France

Given a family of algebraic groups $G \to S$ depending on a parameter S, one studies systems of partial differential equations with variables in S and values in G, which are compatible with the group structure. Two cases are especially studied: the commutative case, and the simple case.

Keywords: Systems of partial differential equations. Differential groups; differential Lie algebras; connections

1. Introduction

Roughly speaking, the question is the following: given an algebraic variety S over $\mathbb{C}$, and a "family of algebraic groups $\pi : G \to S$ with parameter in S", study differential equations over the sections of π which are compatible with the group structure, in a suitable sense.

The question is implicit in the work of E. Cartan on "infinite Lie groups": the differential groups are, in fact, a special case of the "intransitive case" of his theory.[1] But it has been developed in an independent way by E. Kolchin[2] and his students, namely P. Cassidy and A. Buium. It has been used for a "parametric differential Galois theory" by P. Cassidy and M. Singer[3] (see other references later in the text).

My aim here is to give a rough idea of this theory. This work should not be considered as original, except for the presentation of the definitions and of some results.

2. Systems of partial differential equations

I will use the usual language of the theory of schemes (here, over $\mathbb{C}$), as in[4] or.[5] A "variety" means a reduced scheme of finite type over $\mathbb{C}$; in general I will suppose it is separated and irreducible (if it is not the case I will say it explicitly). If V is a variety, I will write often "$a \in V$" for "$a \in V(\mathbb{C})$".

2.1 Suppose we are given two smooth (= nonsingular) algebraic varieties X and S and a morphism $\pi : X \to S$ surjective and smooth (= everywhere of maximal rank, or submersive), we want first to define "systems of partial differential equations on the sections of π".

For that purpose, one introduces, for all $k \geq 0$, the space $J_k(\pi)$ of *jets of order k of sections of* π. This definition is well known in differential geometry, and is developed in many treatises (however, in differential geometry, it is generally used in an analytic or C^∞ context; but there is no problem to transpose it in an algebraic context, and I will not insist on this point).

Then, one has the following definition

Definition 2.2. A system of p.d.e.'s on the sections of π is a projective system $\{Y_k\}$ of closed subschemes of $(J_k(\pi))$ with the following properties

(i) For $k \geq 1$, denote by p_k the projection $J_k(\pi) \to J_{k-1}(\pi)$. Then, one has $p_k Y_k \subset Y_{k-1}$, and the map $O_{Y_{k-1}} \to p_{k*} O_{Y_k}$ is injective.
(ii) Y_{k+1} is contained in the first prolongation $pr_1 Y_k$ of Y_k.

To explain what mean these properties, by taking local charts, we can make the following hypotheses:

(a) S is a finite étale covering of $\mathbb{C}^n - Z$, Z a hypersurface (i.e. Z is defined by $f = 0$, $f \in \mathbb{C}[s_1 \dots s_n]$.
(b) X is contained as closed subvariety in $S \times \mathbb{C}^p$.

First, if $X = S \times \mathbb{C}^p$, the space $J_k(\pi)$ is the space whose coordinates are (s, x_j^α), with $s = (s_1, \dots, s_n) \in S, x_j^\alpha \in \mathbb{C}$, with $1 \leq j \leq p, \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n, |\alpha| = \alpha_1 + \cdots + \alpha_n \leq k$. Denote by A_k the space of regular functions on $J_k(\pi)$, i.e. $A_k = O(S)[x_j^\alpha]$, $O(S)$ the space of regular functions on S. Then Y_k is defined by an ideal $\mathcal{J}_k \subset A_k$, and the conditions (i) and (ii) mean the following

(i) One has $\mathcal{J}_{k+1} \cap A_k = \mathcal{J}_k$ (note that we have an obvious inclusion $A_k \subset A_{k+1}$)
(ii) One has $\mathcal{J}_{k+1} \supset pr_1 \mathcal{J}_k$, where $pr_1 \mathcal{J}_k$ is defined in the following way.

First note that the vector fields $\frac{\partial}{\partial s_1}, \dots, \frac{\partial}{\partial s_n}$ are well defined on S (there are vector fields on $\mathbb{C}^n$, which can be lifted to S). Then, one defines the derivation D_i: $A_k \to A_{k+1}$ by

$$D_i f = \frac{\partial f}{\partial s_i} + \sum \frac{\partial f}{\partial x_j^\alpha} x_j^{\alpha+\varepsilon_i}, \quad \varepsilon_i = (0, \dots, 1, \dots, 0)$$

Then $pr_1 \mathcal{J}_k$ is the ideal of A_{k+1} generated by $\mathcal{J}_k$ and $D_i \mathcal{J}_k$, $1 \leq i \leq n$.

The union $\mathcal{J} = \cup \mathcal{J}_k$ is an ideal of $A = \varinjlim A_k$ $(= \cup A_k)$. Property (i) means that $\mathcal{J}_k = \mathcal{J} \cap A_k$. Property (ii) means that $\mathcal{J}$ is a *differential ideal*, i.e. that $D_i \mathcal{J} \subset \mathcal{J}$, $1 \leq i \leq n$.

Now, the case where X is a closed subvariety of $S \times \mathbb{C}^p$ reduces immediately to a special case of the preceding one: one considers just the differential ideals which contain the equations of X (and, of course, their derivatives).

2.3. *A few remarks on these definitions*

(i) I will not make the (very boring) verification that the property (ii), as defined, is compatible with the change of charts. Without giving details, I mention another definition of the first prolongation. One has an inclusion $J_{k+1} \to J_1(J_k(\pi))$ (the "J_1" is taken on the projection "source"). Then, one has $pr_1 Y_k = J_1(Y_k) \times_{J_1(J_k)} J_{k+1}$ (I omit the "π").

(ii) The reason of the definition of the D_i is the following: if $s \mapsto \{X_j(s)\}$ is a section of π, then, in the local coordinates considered above $\mathcal{J}$ give a system of p.d.e. by substituting $\partial^\alpha X_j$ to x_j^α $(\partial^\alpha = \partial_{s_1}^{\alpha_1} \dots \partial_{s_n}^{\alpha_n})$. Then, by the chain rule, one has $D_i f(s, \partial^\alpha X_j) = \frac{\partial}{\partial s_i}[f(s, \partial^\alpha X_j)]$.

(iii) Of course, this section is *a solution* of the system (in short, a solution of $\mathcal{J}$) if, for all $f \in \mathcal{J}$, one has $f(s, \partial^\alpha X_j) = 0$.
In general, with the definition 2.2, a solution is defined in the following way: a section σ of π is a solution of $Y = \{Y_k\}$ iff, $\forall$ k, $j^k \sigma$ is a section of Y_k.
Note also the following fact: when we speak of solutions, we can take algebraic sections, but also analytic germ of sections, (or even formal sections). In this paper, unless otherwise stated explicitly, *solution will mean "germ of analytic solution"*.

(iv) If we have a differential ideal $\mathcal{J}$, its solutions depend only on its radical $\mathcal{J}^{\mathrm{red}} = \{f; f^k \in \mathcal{J}$ for some $k\}$. Now, by a well-known result by Ritt[6] or,[7] $\mathcal{J}^{\mathrm{red}}$ is also a differential ideal.
Applying this result in the general case 2.2, this means the following: if $Y = \{Y_k\}$ is a system of p.d.e.'s, then $Y^{\mathrm{red}} = \{Y_k^{\mathrm{red}}\}$ is also a system of p.d.e.'s (since, by definition, Y_k^{red} is defined by the radical of the sheaf of ideals defining Y_k). Therefore, from now on, we can limit ourselves to reduced systems of p.d.e.'s

3. Differential Groups

Suppose now that X is a "group over S". More precisely, writing G instead of X, it means the following: first, we assume, as before that G and S are smooth and that $\pi : G \to S$ is smooth and surjective. Furthermore, we assume given 3 maps

(i) $\mu : G \underset{S}{\times} G \to G$, compatible with the projections over S
(ii) $\lambda : G \to G$, compatible with π
(iii) $\varepsilon : S \to G$ section of π

We assume also that these morphisms give respectively, in each fiber $\pi^{-1}(s), s \in S(\mathbb{C})$ the product, the inverse, and the unit of a group structure (I leave to the reader to write the explicit conditions).

We give now a (reduced) system of p.d.e.'s $Y = \{Y_k\}$ on π. We are looking to conditions ensuring the following property:

3.1 *If σ and σ' are sections* (i.e. germs of analytic sections) *of π which are solutions of Y, then $\mu(\sigma, \sigma')$ and $\lambda(\sigma)$ are also solutions of Y.* [I suppose, of course, that σ and σ' are germs at the same point $s \in S(\mathbb{C})$.]

An obvious, but important observation is the following: it is sufficient to impose these conditions for the germs at points $s \in V$, where V is a dense set in $S(\mathbb{C})$, (dense, for the "transcendental", or analytic topology); actually the conditions to be a solution are closed: if they are satisfied on V, they will be satisfied everywhere. A fortiori, it is sufficient to impose those conditions at the points of a Zariski open dense set V.

In particular, the condition (3.1) will be satisfied by the following definition

Definition 3.2. Given $\pi : G \to S$, and $Y = \{Y_k\}$ as before, we say that they define a structure of differential (algebraic) group if there is a Zariski open dense set $U \subset S$ such that the restrictions $Y_k|U$ satisfy the following properties

(i) The $Y_k|U$ are smooth, and the mappings $Y_{k+1}|U \to Y_k|U$ smooth and surjective.
(ii) Over U, Y_{k+1} is contained in the prolongation $pr_1 Y_k|U$, with equality if $k \gg 0$.
(iii) $Y_k|U$ is a subgroup over S of $J_k(\pi)|U$.

Only (iii) needs an explanation: the structure of group over S of G extends in an obvious way to $J_k(\pi)$. Therefore, the notion of subgroup over U of $J_k(\pi)$ is just the obvious one (with however, the slight modification that we do not suppose necessarily Y_k irreducible). It just means that the "$\mu, \lambda, \varepsilon$" of $J_k(\pi)$ restricts to $Y_k|U$.

A priori, the conditions of (3.2) could seem rather restrictive. In fact, they are reasonably general; the reasons imply some arguments of differential algebra which are beyond the scope of this course. A more general case, for Lie pseudogroups instead of groups is explained in[8]). However, I do not know if they are absolutely general, in other words if (3.1) implies (3.2).

4. Differential Lie Algebras

Given S as before, and $\dot{\pi} : L \to S$ a (finite dimensional) vector bundle on S, a structure of Lie algebra of L over S is, of course, a bilinear map $[\cdot,\cdot] : L \underset{S}{\times} L \to L$ satisfying the Jacobi identity. Now, linear differential systems on S could be defined as before, but we would need "linear subspaces of $J_k(\pi)$ with variable fibers". It is simpler to work in terms of D_S-modules, where D_S is the sheaf of scalar linear differential operators on S (in the sequel, I will say "D_S-module" for "sheaf of D_S-modules"). Denote by $\mathrm{Diff}(L, \mathbb{1}_S)$ the sheaf of linear differential operators from sections of L to functions. Then $\mathrm{Diff}(L, \mathbb{1}_S)$ is a left D_S-module, and a linear differential system on L is simply defined by a coherent D_S-submodule M of $\mathrm{Diff}(L, \mathbb{1}_S)$.

The solutions of M are the sections of L annihilated by M. Here, we are interested in their stability by $[\cdot,\cdot]$. The question is simpler than the case of groups, and we will find a necessary and sufficient condition.

For that purpose, we have to recall a generic property of D-modules. To simplify the notations, I write D for D_S and Diff for $\mathrm{Diff}(L, \mathbb{1}_S)$. These spaces are provided with filtrations, respectively D_k and Diff_k, by differential operators of order $\leq k$. Put $M_k = M \cap \mathrm{Diff}_k$, and $N_k = \mathrm{Diff}_k/M_k$. Put also $\bar{\mathcal{D}}_k = \mathcal{D}_k/\mathcal{D}_{k-1}, \bar{\mathcal{D}} = \oplus\bar{\mathcal{D}}_k$, and also $\bar{N}_k = N_k/N_{k-1}, \bar{N} = \oplus\bar{N}_k$. Then the result is the following.

Proposition 4.1. *There is a Zariski-open dense subset U of S such that the $N_k|U$ and the $\bar{N}_k|U$ are locally free over O_S (= the sheaf of regular functions on S).*

The filtrations $\{M_k\}$ and $\{N_k\}$ are good, therefore $\bar{N}$ is a coherent module over $\bar{\mathcal{D}}$ (see the courses on D-modules in this school). Now, according to a result of Grothendieck,[9] (see exposé 4, lemma 6.7 of "generic flatness") there is a Zariski open dense subset U of S such that $\bar{N}|U$ is flat, and even free over $O_S|U$. Then the $\bar{N}_k$ are also flat; since they are coherent, it implies that they are locally free. It follows easily that the N_k are also locally free.

For another argument, see.[10] On M_k, this means that, over U "M_k is a subbundle of Diff_k over S", of constant rank over each point $0 \in S$.

Now, $J_k(L)$ is naturally the dual of Diff_k: in local coordinates, if S is étale over $\mathbb{C}^n - Z$, and $L \simeq S \times \mathbb{C}^p$, then a section of Diff_k is written $\Sigma a_{j,\alpha}\partial^\alpha$, $a_{j,\alpha}$ functions on S, and a section of $J_k(\pi)$, is a collection x_j^α of functions on S; the duality is given by $\Sigma a_{j,\alpha} x_j^\alpha$. In this duality, the orthogonal of M_k (= the dual of N_k) is Y_k, the space of jets of order k of solutions of the differential system defined by M.

The properties of M_k are translated in the following:

(i) The Y_k are vector bundles (in the usual sense, of constant rank), and the projections $Y_{k+1} \to Y_k$ are surjective.
(ii) Y_{k+1} is contained in $pr_1 Y_k$ with equality for $k \gg 0$.
(This last property is the translation of the fact that one has $M_{k+1} \supset D_1 M_k$, with equality for $k \gg 0$. I omit the details.)

Now, M being given, I fix a U with these properties. Note that the bracket $[\cdot,\cdot]$ extends obviously to $J_k(\pi)$.

Definition 4.2. In the preceding situation, we say that M defines a structure of differential Lie algebra if, for all k, Y_k is stable by the bracket of $J_k(\pi)$. This implies obviously that, on S (not only on U), the solutions are stable by $[\cdot,\cdot]$. Conversely, suppose that this is true, then, on U (4.2) is satisfied because of the following property.

4.3. *In the preceding situation, given a point $s \in U$, and a $p \in Y_k(\mathbb{C})$ over s, there exists a germ of analytic solution of M at s whose jet of order k at s is p.*

This is a special case of a theorem of existence of analytic solutions for p.d.e.'s. See the precise statement and the proof in,[11] th.III.4.1.

4.4. *Convention*

Taking into account 3.2 and 4.2, we will now work "generically on S", i.e. we will identify structures which coincide on a (not precised) Zariski open dense subset of S. For instance, if necessary, we can suppose that 3.2 or 4.2 are true, not only on U, but on S itself.

4.5. *Lie groups and Lie algebras*

Given a Lie group Γ, recall that one defines its Lie algebra as the tangent space at identity $T_e\Gamma$. It is supplied by a structure of Lie algebra by the following trick: one identifies it with the space of left invariant vector fields on Γ; then the Lie bracket on vector fields furnishes the required structure (one could also take the right invariant vector fields. The consideration of the inverse $g \mapsto g^{-1}$ shows easily that one would obtain the opposite Lie algebra law on $T_e\Gamma$).

Similarly, if we have a group over $S : G \xrightarrow{\pi} S$ in the sense of §3, one has a structure of Lie algebra over S on the bundle of vertical vectors on G along ε (or equivalently, on the normal bundle along the section ε). I denote it by Lie G.

Suppose that we have an algebraic subgroup $G' \subset G$ over S (I suppose again G' smooth, and the map $\pi|G'$ smooth surjective; but I dot not suppose necessarily G' connected). Then, to G' correspond a Lie subalgebra L' over S, i.e. a subbundle (of "constant rank", as explained after 4.1), stable by the bracket of L.

If we have a Lie subalgebra L' of L in the preceding sense, there does not exist necessarily a corresponding G'. There can exist also several G'; but, in this case, their connected components of identity ($= \varepsilon$) are identical.

Now, suppose that $G \to S$ is provided with a structure of differential group, as in 3.2. Restricting S if necessary to a Zariski open dense set, we can suppose that the properties (i) to (iii) are satisfied on S.

We will provide Lie $G = L$ with a structure of differential Lie algebra. For that purpose, denote π' the projection $L \to S$, and note that there is a canonical isomorphism Lie $J_k(\pi) \simeq J_k(\pi')$ (I leave the simple verification to the reader). Then the collection of Lie Y_k give a collection of subbundles of $J_k(\pi')$; and one verifies that *they define a structure of differential Lie algebra on* L.

I omit the verification (there is essentially one point to verify, i.e. the commutation of "take the first prolongation" and "take the Lie algebra"). Of course, a special case is the trivial one, where we take $Y_k = J_k(\pi)$, and then Lie $Y_k = J_k(\pi')$. In this case, the solutions are all the sections of π, and similarly for the Lie algebra.

I will denote by $\tilde{G}$ this differential group, and I will consider the other $Y = \{Y_k\}$ as subgroups of $\tilde{G}$. Following a terminology of Kolchin, I will say that "Y *is dense in* $\tilde{G}$" if (in restriction to some U Zariski open dense set of S), one has $Y_0 = G(= J_0(\pi))$. The main purpose of these lectures is to give, in several cases, a description of dense differential subgroups of $\tilde{G}$. If there is no possible confusion, I write G instead of $\tilde{G}$.

4.6. N.B. As I said at the beginning the differential algebraic groups have been defined and studied by E. Kolchin and his students. But their definition is technically quite different of the one given here, as, instead of a variety S, they work on a differential field K, which they suppose generally *differentially closed*. Therefore, the translation of their results in the present language is not always easy. In the first approximation, it would be simpler to consider that one has two theories, close to each other, but distinct. Some effort to connect them more closely would be useful.

5. First Examples

For a study of these examples in the point of view of differential fields, see.[12]

5.1. *Additive groups*

I recall first some simple facts (see any course on algebraic groups). Let $\mathbb{A}^1$ be the "affine space over $\mathbb{C}$", i.e. $\mathbb{C}$ itself provided with the ring $\mathbb{C}[x]$ (in other words, $\mathbb{A}^1 = \operatorname{spec}\mathbb{C}[x]$). Now the additive group G_a is simply $\mathbb{A}^1$ with the addition $(x, x') \mapsto x + x'$.

The only closed subgroups of G_a^m are its vector subspaces. This can be seen as follows: first the Lie algebra is $\mathbb{C}^m$ (i.e. the same space) with the trivial bracket; its Lie subalgebras are therefore the vector subspaces. To this Lie algebra corresponds one connected subgroup, i.e. the same vector space. To finish, we have to see that all the closed subgroups of G_a^m are connected; taking the quotient by the connected component of 0, i.e. by a vector subspace we are reduced to prove the following result: G_a^m has no discrete (closed) subgroups; but such a group would be finite and $\mathbb{C}^m$ has no finite subgroups.

Now, let S be a smooth algebraic variety as before, and take $G = S \times G_a^m$, with the obvious projection π over S. Applying similar arguments to the $J_k(\pi)$, one finds the following result: the differential algebraic subgroups of G are the linear partial differential systems on S with value in $\mathbb{C}^m$, in other words the left D_S-submodules of D_S^m. Of course, here, according to our conventions, we identify two such systems which coincide on a Zariski dense open subset of S. In other words, we get only D-modules at the generic point of S.

5.2. Here is a remark which will be useful in the next sections. Let $G \xrightarrow{\pi} S$ be a group over S, with G connected. Then, if $Y = \{Y_k\}$ is a differential group dense in G, Y is connected (i.e. the Y_k are connected). Therefore, it is determined by its Lie algebra. This is seen by recurrence on k, by using the following fact: for all $k \geq 0$, the kernel $\ker(Y_{k+1} \to Y_k)$ is connected; actually this is a subgroup over S of $\ker(J_{k+1}(\pi) \to J_k(\pi))$. But this is a family over S of additive groups, and therefore its subgroups are connected.

5.3. *Multiplicative group*

The multiplicative group G_m is $\mathrm{spec}\mathbb{C}[x, x^{-1}]$; the underlying space is therefore $\mathbb{C}^*$. The law is the multiplication $(x, x') \mapsto xx'$.

I am interested in differential subgroups of $S \times G_m$. First, it is easy to see that the non-dense ones are simply the subgroups $S \times \mu_p$. μ_p the group of p-roots of unity. To determine the dense subgroups, I first look at the Lie algebra $S \times \mathbb{C}$ (with trivial bracket). Then, the corresponding differential Lie algebra is a linear differential system, in other words an ideal (= sheaf of ideals) $\mathcal{J} \subset D_S$. According to 5.2 such a $\mathcal{J}$ gives at most one differential subgroup, we have to determine those which give actually one such subgroup.

To do that, I look first at the analytic situation: using the exact sequence $0 \to 2\pi i\mathbb{Z} \to \mathbb{C} \xrightarrow{\exp} \mathbb{C}^* \to 0$, one finds that $S \times \mathbb{C}$ is a covering of $S \times \mathbb{C}^*$ (I write here S for S^{an}, i.e. S considered as an analytic variety). Lifting the system from $S \times G_m$ to $S \times \mathbb{C}$, we find a connected differential system with Lie algebra $\mathcal{J}$; therefore, on $S \times \mathbb{C}$, our system is $\mathcal{J}$ itself. To descent to $S \times \mathbb{C}^*$, it is necessary that the $2\pi ik$, $k \in \mathbb{Z}$, are solutions. This implies that all the constants $C \in \mathbb{C}$ are solutions.

We will write it more explicitly. We can suppose that S is a finite étale covering of $\mathbb{C}^n - Z$, Z a hypersurface. Denote by $s_1, \ldots, s_n$ the coordinates in $\mathbb{C}^n$. The condition on $\mathcal{J}$ is the following: if $p = \Sigma a_\alpha \partial^\alpha$ is in $\mathcal{J}$, then one has $a_0 = 0$. In other words, one has $\mathcal{J} \subset \mathcal{J}_0$, $\mathcal{J}_0$ the ideal generated by the ∂_i.

Now, it is easy to see that this condition is sufficient. To do that, for $\alpha = (\alpha_1, \ldots, \alpha_n)$, define $D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n}$, where the D_i's are defined in §2. If x is the coordinate in $\mathbb{C}^*$, we write $D^\alpha x = x^\alpha$ (I will write x^i for x^{ε_i}).

Then, one verifies easily that the system obtained on $S \times \mathbb{C}^*$ is generated by the following equations: for each monomial $\alpha \neq 0$, choose an $\alpha_i \neq 0$ and write $\alpha = \beta + \varepsilon_i$. Then, to each $p = \Sigma a_\alpha \partial^\alpha \in \mathcal{J}$, we associate $\Sigma a_\alpha D^\alpha(x^{-1}x^i)$. The differential ideal obtained will not depend on the choice of β since one has $D_j(x^{-1}x^i) = D_i(x^{-1}x^j)$.

6. Family of Elliptic Curves

Let S be as before, a smooth algebraic variety over $\mathbb{C}$, and let Γ be an abelian variety (e.g. an elliptic curve). Writing $\Gamma = \mathbb{C}^n/\Lambda$, Λ a lattice, we can easily describe the differential subgroups of $G = S \times \Gamma$ by the same method used in 5.3. I will not give the details.

Here, we are interested in a more general case, where G is a family of abelian varieties over S, varying with $s \in S$. This subject is very rich and related to many other ones; here we will give only a very short introduction to it.

For simplicity, I will limit myself to a family of elliptic curves, e.g. the Legendre family. It is defined in the affine coordinates (y, z) by the equation $z^2 = y(y-1)(y-s)$, $s \in \mathbb{C} - \{0, 1\} = S$.

First I recall some basic facts on elliptic curves, and especially on this family (see any course on elliptic curves)

(i) Fix $s \in S$; we add to the curve a point at ∞ [by passing to homogeneous coordinates Y, Z, U, the equation is $Z^2U = Y(Y-U)(Y-sU)$ and the point is $(0:1:0)$]. Then, we have a projective non singular curve G_S of genus one; if we choose an origin, it has a structure of commutative group. If we choose the origin at infinity, the law is given by the following rule: $a + b + c = 0$ iff a, b, and c are on the same line in $\mathbb{P}^2(\mathbb{C})$.
Now, varying $s \in S$, we get a group G over S, whose restriction to s is G_s.

(ii) For $s \in S$, G_s, as analytic variety, is the quotient of $\mathbb{C}$ by a lattice (= a discrete subgroup of rank 2) Λ_s.
In particular, only the ratio of the periods is well defined, modulo the action of $PG\ell(2, Z)$. But we have a canonical choice: we can choose t, the coordinate of $\mathbb{C}$ in order to have $dt =$ (inverse image of) $\frac{dy}{z}$, $\frac{dy}{z}$ the differential of first kind on G_s. This is the celebrated theorem of Abel on inversion of elliptic integrals.
Then Λ_s is the lattice of periods $\int_\gamma \frac{dy}{z}, \gamma \in H_1(G_s, \mathbb{Z})$. Locally on S, for the transcendental topology, we have a canonical isomorphism $H_1(G_s, \mathbb{Z}) \simeq H_1(G_{s'}, \mathbb{Z})$, and the periods "vary holomorphically with s".
Finally, the group law of G_s is the (quotient of) the addtion $(t, t') \mapsto t + t'$.

(iii) Note also that, for general s and s', G_s is not isomorphic to $G_{s'}$.
[The condition of isomorphism is $\varphi(s) = \varphi(s'), \varphi(s) = (s^1 - s + 2)^3/ [(s+1)(2s-1)(s-2)]^2$. This can be seen by reducing the equation, by translation in y, to the form $z^2 = y^3 + ay + b$, in which case the condition for isomorphism is $a^3/b^2 = a'^3/b'^2$. I will not use this fact.]

Now, to find a dense structure of differential group on G, we can try to do as in 5.3. We have a linear system on Lie $G = S \times \mathbb{C}$, i.e. a left ideal $\mathcal{J}$ of D_S. Considering now $S \times \mathbb{C}$ as the covering of G, we look at conditions to descend this system to G. To have that, it is necessary, when s varies, that *the periods are solutions of* $\mathcal{J}$. But the equation satisfied by the periods is classical: this is the special case of the Gauss hypergeometric equation of parameters $\left(\frac{1}{2}, \frac{1}{2}, 1\right)$. Explicitly, this equation is (6.1) $s(1-s)t'' + (1-2s)t' - \frac{1}{4}t = 0$.

Denote $\mathcal{J}_0$ the ideal generated by this equation. Then, the condition is that $\mathcal{J} \subset \mathcal{J}_0$. Incidentally $\mathcal{J}_0$ is just the special case of what is called "Gauss-Manin" or "Picard-Fuchs" equation for general families of varieties. See the other courses of this school.

Now comes the difficult point, i.e. to prove that, under these conditions, the differential system on G defined by $\mathcal{J}$ is algebraic. I will speak only on $\mathcal{J}_0$ (in the other cases, there are probably similar results; but I have no proof or reference).

For $\mathcal{J}_0$, the result is proved in[13] and.[14] The first reference uses the formalism of "crystals" by Grothendieck; the second one is more elementary. The subject is closely related to the so-called "maximal extension by an additive group" of an elliptic curve, or, more generally, an abelian variety. It is also related to isomonodromic deformations of rank one bundles provided with an integrable connection. See loc. cit.

Following,[14] I will call "multiplicative Gauss-Manin connection" the differential group on G corresponding to $\mathcal{J}_0$.

The subject is also closely related to another one, seemingly very different: the theory of "algebraically integrable hamiltonian systems". On this subject, see e.g.[15] or.[16] Again, I will give only a simple example, the pendulum.

This is the motion of a particle on a vertical circle, under the action of the weight. With suitable normalizations, the circle is given in the (x, y) plane by the equation $x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$ or $x^2 + y^2 - y = 0$; the force is vertical, along the y-axis. Then, t being the time, the energy integral is $x'^2 + y'^2 + y = s$.

Derivating the first equation, one finds $xx' + \left(y - \frac{1}{2}\right) y' = 0$; eliminating x and x', one gets $y'^2 = 4y(y-1)(y-s)$. Further change of time reduces finally to $y'^2 = y(y-1)(y-s)$.

For s fixed, this is just the motion on the curve G_s, with $\frac{dy}{dt} = z$ or $dt = \frac{dy}{z}$; therefore, it is linearized by the covering $\mathbb{C} \to G_s$ considered before (if one prefers, it is given by the corresponding Weiestrass p function).

The variation of period when s varies satisfies the equation (6.1) (in general, in the theory of integrable systems, the authors omit to consider this point).

The multiplicative Gauss-Manin connection can also be interpreted as the differential Galois group of the hamiltonian system of the pendulum; I hope to explain that in a future publication.

7. Connections

Our next aim is to study the simple (or semi simple) case. For that purpose, some preliminaries are necessary.

7.1. *Connections in the sense of Ehresmann (or "foliated bundles")*

This is simpler to explain first in the analytic case. First I do it locally; let U (resp. V) be an open subset for the usual (or "transcendental") topology of $\mathbb{C}^n$ (resp. $\mathbb{C}^p$), and let π be the projection $U \times V \to U$. Denote the coordinates on U (resp. V) by $s_1, \ldots, s_n$ (resp. $x_1, \ldots, x_p$). Then a π-connection (in the sense of Ehresmann) is simply a system of p.d.e.'s of the type

$$\partial x_j / \partial s_i = a_{ij}(s, x), \quad a_{ij} \text{ holomorphic on } U \times V$$

This system is also expressed in the following way:

(i) By the family of vector fields $\frac{\partial}{\partial s_i} + \sum\limits_j a_{ij} \frac{\partial}{\partial x_j} = \xi_i$.

(ii) By the family of 1-form orthogonal to the $\xi_i : \omega_j = dx_j - \sum\limits_i a_{ij} ds_i$.

The *integrability condition* is just the Frobenius condition for the ξ_i's, or the ω_j's. Due to the special form of ξ_i, it means just that $[\xi_i, \xi_j] = 0 \; \forall \; i, j$. Explicitly, one has

$$\frac{\partial a_{jk}}{\partial s_i} - \frac{\partial a_{ik}}{\partial s_j} + \sum_\ell \left(a_{i\ell} \frac{\partial a_{jk}}{\partial x_\ell} - a_{j\ell} \frac{\partial a_{ik}}{\partial x_\ell} \right) = 0$$

This is equivalent to give a foliation on $U \times V$ with leaves étale over U or, as one says, a "foliated bundle".

Now, this notion can be defined directly without local coordinates: let S and X be smooth analytic $\mathbb{C}$-varieties and $\pi : X \to S$ a surjective submersion. Let $a \in X$, with $\pi(a) = s$; then we have an exact sequence

$$0 \to v_a X \to T_a X \xrightarrow{\pi'} T_s S \to 0$$

(π' the tangent map to π; v mean here "vertical").

Then, a connection can be defined as a splitting of this exact sequence, depending analytically on $a \in X$. One sees at once that this is equivalent, in local coordinates, to the notion defined above. Considering this splitting, either as a lifting of $T_s S$ to $T_a X$, or as a projection of $T_a X$ to $v_a X$, we get the interpretations (i) and (ii) above. In particular, the second interpretation gives a form Ω on X with values in vertical vectors.

We say that this connection is "flat" or "integrable" or "without curvature" if the integrability condition considered above is satisfied. Although I will not need this fact, I mention that the integrability condition can be written $[\Omega, \Omega] = 0$, where $[\cdot, \cdot]$ is the Nijenhuis bracket on vector valued forms; one could also, more generally, define the curvature as $[\Omega, \Omega]$.

To end this section, note that the notion of connection and its integrability can be defined similarly in the algebraic context: here, S and X are smooth algebraic varieties over $\mathbb{C}$, and $\pi : X \to S$ is smooth and surjective. Of course, here, we require that the data defining the connection, e.g. the form Ω, are algebraic. Again, a flat connection defines a foliated bundle (this is just a different name for the same notion).

7.2. *Connections and differential equations*

An algebraic connection (flat or not) on $\pi : X \to S$ is a special kind of first order differential equation: more precisely, they are the closed subvarieties $Y_1 \subset J_1(\pi)$ such that the projection $Y_1 \to X$ is an isomorphism.

The condition of flatness is equivalent to the fact that the first prolongation $Y_2 = pr_1, Y_1$ is surjective over Y_1; it is sufficient to verify this in local coordinates for the underlying analytic connection, which is almost obvious.

Then, if the connection is flat, all the prolongations (defined by recurrence by $Y_{k+1} = pr_1 Y_k$) are isomorphic to each other by the projections $Y_{k+1} \to Y_k$. Again, it is sufficient to prove this in local analytic coordinates; in that case, it follows from the following well-known fact: with the notations of the beginning of 7.1, if the integrability condition is satisfied, by a fibered local analytic change of coordinates, or can reduce the system to the case $a_{ij} = 0$.

Then we can identify the connection with the differential system $\{Y_i\}$, with $Y_0 = X$. The solutions of the connection, called also "horizontal sections" coincide with the solutions of this differential system.

7.3. *From now on, the connections will be supposed flat*

We will now consider the case where, on X, there is some additional structure; we want that the connection be "adapted" to this structure. We will not try to give general definitions, but only examples.

7.3.1. Suppose $X \to S$ is a vector bundle. Then we will say that the connection is adapted if the corresponding differential system is a linear system (= the Y_k are linear subspaces of $J_k(\pi)$.) It is sufficient that Y_1 is a linear subspace of $J_j(\pi)$.

To give explicit conditions, I suppose that S is an étale covering of $\mathbb{C}^n - Z$, Z a hypersurface, and that $X = S \times \mathbb{C}^p$.

Then the equations of the connection can be written

$$\frac{\partial x}{\partial s_i} + A_i(s)x = 0, \;\; x = (x_1, \dots, x_p)^T, \;\; A_i \text{ with values in } \mathrm{End}(\mathbb{C}^p) = \mathcal{G}\ell(p, \mathbb{C}).$$

If we write $\Omega = \Sigma A_i(s) ds_i$, the integrability condition is $d\Omega + \Omega \wedge \Omega = 0$.

7.3.2. Suppose further that X is a Lie algebra over S. We will say that the connection is compatible with the structure of Lie algebra if the corresponding differential system is a Lie subalgebra of $X \to S$. It is necessary and sufficient to require that Y_1 is a Lie subalgebra over S of $J_1(\pi)$.

Explicitly, in the case considered above, the condition is that $A_i(s)$ is, for $s \in S$, a derivation of the Lie algebra, i.e. $A_i(s)[x, y] = [A_i(s)x, y] + [x, A_i(s)y]$.

7.4. We are now interested in the case where $X = G$ is a group over S, in the sense of §3. We will say that a connection on $\pi : G \to S$ is adapted (or compatible) with the group structure if the corresponding differential system is a differential subgroup of G (note that, here, it is necessarily dense). Again here, the following fact is easy to verify: the connection is compatible with the group structure iff the corresponding subvariety $Y_1 \subset J_1(\pi)$ is a subgroup of $J_1(\pi)$ over S.

If one has $G = S \times \Gamma$, Γ an algebraic group over $\mathbb{C}$, the connection can be written $d\gamma - \Omega(\gamma, s)$, Ω regular on G with values in the vertical vector fields. The condition of compatibility is $\Omega(\gamma\gamma', s) = \gamma\Omega(\gamma', s) + \Omega(\gamma, s)\gamma'$. (This expression is clear if the group is linear. Otherwise, one should interpret $\gamma\cdot$ as "left translation by γ" and $\cdot\gamma'$ as "right translation by γ'".)

If we prefer to have a form with values on Lie Γ, write $\gamma^{-1}d\gamma - \gamma^{-1}\Omega(\gamma, s)$, etc.

7.4.1. *Example [cf.[14]]*

Take $\Gamma = G_m \times G_a = \mathbb{C}^* \times \mathbb{C}$ with variables (y, x), and $S = \mathbb{C}$. Suppose the connection is given by $(dx - xds, dy - xyds)$. It is easy to verify its compatibility with the group structure.

If we integrate these equations or, as one says, if we take *the flow* of the connection we get $x = x_0 e^{s-s_0}, y = y_0 e^{x-x_0}$. We get an *analytic automorphism* of Γ, which is not algebraic.

This is related to the following fact: for the group Γ, the functor $\underline{\mathrm{Aut}}\Gamma$ is not representable (in particular, there are infinitesimal automorphisms which are not in the Lie algebra of the group $\mathrm{Aut}\Gamma$, cf.[14] for a systematic discussion of this problem.

7.4.2. In the same spirit, let me mention briefly some other facts, also discussed by Buium in.[14]

Let $G \to S$ be a connected group over S, and let Ω be a flat connection on $G \to S$, compatible with the group structure.

(i) The fibers G_s are all analytically isomorphic. This follows from a result by Hamm, see.[14]

(ii) If G is affine over S, then the G_s are algebraic. This follows from a theorem of Hochschild-Mostow,[17] which says that two connected affine algebraic $\mathbb{C}$-groups analytically isomorphic are also algebraically isomorphic (but it does not mean that all analytic isomorphisms are algebraically isomorphic, cf.[18]).

(iii) If G is not affine, the result is not true. Actually we have seen implicitly a counter-example in §6: the "multiplicative Gauss-Manin connection" considered on G (notations of §6) is not a connection on G. But it is a second order differential operator, which can be considered as a connection on $J_1(\pi) = G^1$. Each fiber is a commutative group, extension of G by an additive group (its "universal" extension, cf. loc. cit.). If G_s is not isomorphic to $G_{s'}$, G_s^1 is not isomorphic to $G_{s'}^1$ (use the Chevalley-Barsotti theorem: the connected maximal affine subgroup is unique, and therefore also the quotient, which is abelian). But they are analytically isomorphic according to Hamm's theorem. Actually, it is not difficult to see that they are all analytically isomorphic to $(\mathbb{C}^*)^2$.

7.5. *Connections on principal bundles*

Let Γ be a connected algebraic group over $\mathbb{C}$, and let $\pi : P \to S$ be a (right) principal bundle, or, if one prefers, an $S \times \Gamma$ torsor. A connection on P is compatible with the structure of principal bundle, by definition, if the corresponding variety $Y_1 \subset J_1(\pi)$ is stable by the obvious action of Γ on $J_1(\pi)$.

If the bundle is trivial, i.e. $P = S \times \Gamma$, with the obvious right action of Γ, the form of the connection Ω verifies $\Omega = d\gamma + \alpha\gamma$, α a form with values on Lie Γ (for the notation $\alpha\gamma$, see the beginning of 7.4). The integrability condition can be written $d\alpha + \alpha \wedge \alpha = 0$, in the case where Γ is linear. In general, one should write $\frac{1}{2}[\alpha, \alpha]$ instead of $\alpha \wedge \alpha$.

Now comes a remark which will be used in §8. In the preceding situation, denote by Aut P the bundle of fibered automorphisms of P commuting with the action of Γ. Then the connection Ω gives a connection on Aut P *compatible* with its group structure.

I shall verify it only in the case where $P = S \times \Gamma$, which is the case I will use. Then, one has *Aut* $P = S \times \Gamma$ since the automorphisms of Γ commuting with right translations are just left tranlations. If $\Omega = d\gamma + \alpha\gamma$, then the connection on Aut P is given by $d\gamma + \gamma\alpha - \alpha\gamma$. It is immediate to verify directly that the product of horizontal sections is a horizontal section.

The corresponding connection on Lie Aut P is given by "the same formula", better written here $d\gamma + [\gamma, \alpha]$. One has the following result.

Theorem 7.6. *If Γ is semi-simple and connected, all connections on $S \times \Gamma$ (resp. $S\times$ Lie Γ) compatible with the group structure (resp. with the Lie algebra structure) are obtained in this way.*

It is sufficient to prove the result for Lie algebras (since connected differential subgroups, in particular those coming from a connection on $S \times \Gamma$ are determined by the Lie algebra). If we write $L =$ Lie Γ, a connection compatible with the structure of Lie algebra on $S \times L$ is given by $d\ell + \alpha$, α a form on S with values in End$L$, satisfying $\alpha[\ell, \ell'] = [\alpha(\ell), \ell'] + [\ell, \alpha(\ell')]$. Now the result follows from the fact that all derivations of L are interior.

7.7. *Connections with respect to a foliation*

We need a more general definition of connections. Let us give a *foliation* of S, i.e. a subsheaf $N \subset \Omega^1_s$, which is "a subbundle" (i.e. Ω^1_s/N is locally free; cf. 4.1) and satisfies the Frobenius condition $dN \subset N \wedge \Omega^1_s$.

Let $F = N^\perp$ be the subbundle of $T(S)$ orthogonal to N. Then, roughly speaking an F-connection on $\pi : X \to S$ is something like a connection, but one derives only along F.

In the analytic case, in suitable local coordinates, F is defined by $ds_1 = \cdot = ds_k = 0$. With the same notations as in 7.1, this means that we will have only derivations with respect to $s_{k+1}, \dots, s_n$, i.e. a system of equations of the type

$$\frac{\partial x_j}{\partial s_i} = a_{ij}(s, x), \quad k+1 \leq i \leq n, \quad 1 \leq j \leq p.$$

In other words, $s_1, \dots, s_k$ act here only as parameters.

In the global analytic or algebraic case $\pi : X \to S$, we have to consider $T_F(X) = \pi'^{-1}(F)$ (π', the differential of π), and to consider splittings of the sequence $0 \to v(X) \to T_F(X) \to F \to 0$.

I will indicate briefly how the considerations of the preceding sections of §7 generalize here. First, the integrability condition is generalized in an obvious way.

Again, an F-connection defines a (smooth) subvariety Y_1 of $J_1(\pi)$. If the integrability condition is satisfied, then all the prolongations Y_k are smooth and surjective on each other. Therefore, a flat (= integrable) F-connection defines a differential system on $\pi : X \to S$.

The compatibility with a structure of group, Lie algebra, or principal bundle is defined as above. Finally the theorem 7.6 is still true with "F-connection" instead of "connection". The verification is not difficult, and I leave it to the reader.

8. Simple Groups

For the general theory of linear algebraic groups, see e.g.[19] or.[20] Here, the result is the following.

Theorem 8.1. *Let S be a smooth algebraic variety over $\mathbb{C}$ as before, and let L be a simple Lie algebra over $\mathbb{C}$. Then the only structures of differential Lie algebra on $S \times L$, dense in $S \times L$, are the connections with respect to foliations of S.*

As for groups, "dense" means, of course, that there are no equations of order 0.

Suppose now that Γ is a connected algebraic group over $\mathbb{C}$, Lie with $\Gamma = L$. Then, combining 8.1 and 7.6 (extended to F-connections), we have a description of dense differential groups on $S \times \Gamma$: all of them are F-connections for a suitable F. In particular, we have a bijection between dense differential Lie algebras on $S \times L$, and dense differential Lie groups of $S \times \Gamma$.

The proof follows an argument by Kiso.[21] See historical remarks 8.3. I will follow the notations of §4 (with here $S \times L$ instead of L). Suppose that we have a dense differential Lie algebra given by a D_S-module $M \subset \mathrm{Diff}(S \times L, \mathbb{1}_s) =$ "Diff". We consider its induced filtration M_k. As in §4, we write $N_k = \mathrm{Diff}_k / M_k$, $\bar{N}_k = N_k / N_{k-1}$.

By generic flatness (prop. 4.1), we can suppose, by restricting S, that the N_k and $\bar{N}_k$ are locally free O_S-modules. On the other hand, by restricting again S, we can suppose that it is finite and étale over $\mathbb{C}^n - Z$, Z a hypersurface. Then $\bar{N} = \oplus \bar{N}_k$ is a $O_S[\xi_1, \dots, \xi_n]$-module, with $\xi_i = gr \frac{\partial}{\partial s_i}$.

Consider now the duals N_k^* over O_S of N_k; it can be identified with $Y_k \subset J_k(\pi)$, the space of k-jets of solutions of M (here, π is the projection $S \times L \to S$). Similarly, the dual of $\bar{N}_k$ is $\bar{Y}_k = \ker(Y_k \to Y_{k-1})$. By the hypothesis of density, we have $\bar{Y}_0 = Y_0 = S \times L$.

Now, we have two structures on $\bar{Y} = \oplus \bar{Y}_k$

(a) A structure of $O_S[\xi_1, \dots, \xi_1]$-module, obtained by duality over O_S of the structure of $\bar{N}$. It is graded by the opposite degree $-k$.
(b) A structure of graded Lie algebra, i.e. $[\bar{Y}_k, \bar{Y}_\ell] \subset \bar{Y}_{k+\ell}$, obtained from the structure of Lie algebras of the Y_k's.

Now, the proof goes as follows.

(i) $\bar{Y}_1 \subset \bar{J}_1(\pi) = L \otimes T^*S$ has the following form (after restricting S): There exists E, vector subbundle of T^*S such that $\bar{Y}_1 = L \otimes E$.
This follows from the next lemma

Lemma 8.1. *Let V be a vector space over $\mathbb{C}$, and consider the representation $ad \otimes id$ of L on $L \otimes V$. Then the invariant subspaces of $L \otimes V$ are the $L \otimes W$, W a vector subspace of V.*

Since L is simple, the representation is completely reducible. Therefore, it suffices to study the irreducible case.
Then, let F be an irreducible invariant subspace of $L \otimes V$; choose a basis $v_1, \dots, v_p$ of V. For $a \in F$, write $a = \ell_1 \otimes v_1 + \dots + \ell_p \otimes v_p$, and denote u_i the map $a \mapsto \ell_i$ from F to L. As F is irreducible, u_i is identically 0, or is an isomorphism. Now, if u_i and u_j are isomorphisms, according to Schur lemma, one has $u_j u_i^{-1} = \lambda \cdot id$, $\lambda \in \mathbb{C}$. The result follows at once.

(ii) One has $\bar{Y}_k = L \otimes S^k E$, as subspace of $\bar{J}_k(\pi) = L \otimes S^k(T^*S)$.
Again the proof can be seen fiber by fiber. I will give it for $k = 2$; the general case is similar. On one side, the module structure implies, $\xi_i \bar{Y}_2 \subset \bar{Y}_1 = L \otimes E$, $1 \le i \le n$. I leave to the reader to verify that this means $\bar{Y}_2 \subset L \otimes S^2 E$. On the other side, the fact that $[L, L] = L$ and $[\bar{Y}_1, \bar{Y}_1] \subset \bar{Y}_2$ imply $\bar{Y}_2 \supset L \otimes S^2 E$.

These results show that our system of equations D is generated by the first order ones; choose a basis of vector fields $X_1, \ldots, X_q$ orthogonal to L; then a basis of the first order equations is $X_i + a_i$, a_i functions on S with values in End$L$ (for X_i fixed, there is a unique a_i, because there are no equations of order 0). To end the proof, we have two facts to prove

(iii) The X_i's (or E) verify the Frobenius condition. Then our system is an F connection, F the foliation defined by E.

(iv) This connection is integrable.

One could prove (iii) by using the "integrability of characteristics" (see the other courses of this school). But it is simpler to prove simultaneously (iii) and (iv): consider $[X_i + a_i, X_j + a_j]$: this is a first order equation which belongs to our system. Therefore, it has to be a linear combination on O_S of the $X_i + a_i$. Both results follow immediately; this ends the proof of 8.1.

8.2. *Generalizations*

Let me indicate them very briefly.

(i) Let Γ be a connected semi-simple group over $\mathbb{C}$. Then, the dense differential group structures on Γ can be obtained as follows.
First, one looks at differential Lie algebra structures on Lie $\Gamma = L_1 \times \cdots \times L_q$, L_i simple. Arguments similar to 8.1 show that these structures are products of F_i-connections on $S \times L_i$ (the F_i can be different from each other). Then, if Γ'_i is the connected component of the group Aut L_i, this differential structure can be lifted to $\Gamma' = \Gamma'_1 \times \cdots \times \Gamma'_p$. Call $Y' = \{Y'_k\}$ this structure. Finally, Γ is a finite covering of Γ' by the adjoint action; then one lifts Y' to $S \times \Gamma$ by $Y_k = Y'_k \underset{\Gamma'}{\times} \Gamma$ [note that one has $J_k(\pi) = J_k(\pi') \underset{\Gamma'}{\times} \Gamma$, with π (resp. π') the projection $S \times \Gamma \to S$ (resp. $S \times \Gamma' \to S'$)]. I leave the details to the reader.

(ii) More generally, let G be a semi-simple group defined on $\mathbb{C}(S)$, the field of rational functions of S. Then, if we replace $\mathbb{C}(S)$ by a suitable finite extension, i.e. if we replace S by S', étale finite covering of $U \subset S$, Zariski dense, we are reduced to (i). However, to finish, we would have to examine descent conditions to go back from S' to S. I will not examine this here.

8.3. *Historical remarks*

Theorems type 8.1 seem to have been considered for the first time by E. Cartan.[1] They come in his theory of "infinite Lie groups", as "intransitive simple groups". Here is what he says:
"Les groupes infinis simples qui ne sont isomorphes à aucun groupe transitif . . . se partagent en deux catégories

(a) Les groupes simples proprement dits: on les obtient en prenant un groupe simple transitif (fini ou infini), et en faisant dépendre de la manière la plus générale possible les éléments arbitraires de p variables invariantes par le groupe.
(b) Les groupes simples improprement dits . . . "

My comment: Cartan considers only the local analytic situation with respect to the parameter (here, denoted S). Now, locally, a flat F-connection is simply a (trivial) connection with respect to some of the variables. Cartan neglects them, seeming to consider that "this does not change the group".

This question has been re-examined by Kiso.[21] He gives also a local analytic statement with respect to the parameters. However, his method gives the more precise result stated here.

Note that Kiso, and Morimoto[22] study also the case of intransitive "infinite groups", whose transitive part is simple, or even primitive. The results, as claims Cartan, are similar. In the algebraic case, they could probably also be expressed in terms of F-connections.

Finally, the question of simple, or, more generally, semi-simple differential algebraic groups has also been studied by Cassidy.[12,23] The results are very similar. But, as I said in 4.6, I found difficult to compare precisely these results with those of Kiso, mainly because they are stated in the very different context of differentially closed differential fields.

References

1. E. Cartan, Ann. Ecole Normale Supérieure **26**, 93 (1909).
2. E. Kolchin, *Differential Algebraic Groups* (Academic Press, New York, 1985).
3. P. Cassidy and M. Singer, Galois theory of parameterized differential equations, Differential equations and quantum groups, IRMA Lecture Notes in Math. and Th. Physics 99, Strasbourg (2007) European Math. Soc.
4. R. Hartshorne, *Algebraic Geometry*, (Graduate Texts in Maths. 52, Springer-Verlag, 1977).
5. D. Mumford, *The Red Book of Varieties and Schemes*, Lecture Notes in Math. 1358 (Springer-Verlag, 1988).
6. J. F. Ritt, Amer. Math. Soc. Coll. Publ. **33** (1950) and Dover (1965).

7. E. Kolchin, *Differential Algebra and Algebraic Groups* (Academic Press, New York, 1973).
8. B. Malgrange, Monographie de l'Ens. Math. **38**, 465 (2001).
9. A. Grothendieck, Séminaire de Géométrie Algébrique, Buressur-Yvette (1960-1961).
10. H. Goldschmidt, Inv. Math. **9**, 165 (1970).
11. B. Malgrange, Systèmes différentiels involutifs, Panoramas et synthèses, No.19 Soc. Math. France (2005).
12. P. Cassidy, Amer. J. Math. **94**, 891 (1972).
13. B. Mazur and W. Messing, Universal extensions and one dimensional crystalline cohomology, Lecture Notes in Math. 330 (Springer-Verlag, 1974).
14. A. Buium, Differential algebraic groups of finite dimension, Lecture Notes in Math, 1506 (Springer-Verlag, 1991).
15. F. Calogero, Classical many-body problems amenable to exact treaments, Lecture Notes in Physics, Monographs (Springer-Verlag, 2001).
16. O. Babelon, D. Bernard and M. Talon, *Introduction to Classical Integrable Systems*, Cambridge Monographs on Math. Physics (Cambridge University Press, 2003).
17. G. Hochschild and D. Mostow, Amer. J. Math. **93**, 111 (1961).
18. G. Hochschild and D. Mostow, J. Algerbra **25-1**, 146 (1973).
19. A. Borel, *Linear Algebraic Groups*, 2nd edition (Graduate Texts in Maths. 126, Springer-Verlag, 1991).
20. T. A. Springer, *Linear Algebraic Groups*, (Progress in Maths. 9, Birkhäuser, 1981).
21. K. Kiso, Jap. J. Math. **5-1**, 101 (1979).
22. T. Morimoto, J. Math. Soc. Japan **29**, 35 (1977).
23. P. Cassidy, J. Algebra **121-1**, 169 (1989).