Problemas de Tesis y avances.

Arturo Giles Flores

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1 Quasiordinary Hypersurfaces

We will first consider reduced quasiordinary hypersurfaces of a special type,

$$g := y^m - z_1^{m_1} z_2^{m_2} \cdots z_n^{m_n}$$

in $\mathbb{C}^{n+1}$ with coordinates $(y, z_1, \ldots, z_n)$. We can assume that $m_1 \geq \cdots \geq m_n$, and $m \geq 2$.

Proposition 1.1. *Let $(X, 0) \subset (\mathbb{C}^{n+1}, 0))$ be the germ of hyperfurface singularity defined by $g = 0$. If X has no exceptional tangents, then it is Whitney equisingular with its tangent cone.*

To start working in these setting, we must first cite a theorem by [Ban], which says in particular that for a hypersurface of this type if $m_i \geq m$ for all i, then it has no exceptional tangents

Note that in this case, the tangent cone $C_{X,0}$ is the coordinate hyperplane defined by $y^m = 0$.

Remark 1.2. *The specialization space of X to its tangent cone $C_{X,0}$ is also a quasiordinary hypersurface of the same type and has no exceptional tangents.*

This just comes from the fact that the specialization space $f : \mathfrak{X} \to \mathbb{C}$ is defined in $C^{n+1} \times \mathbb{C}$ by the equation:

$$G(y, z_1, \ldots, z_n, t) := t^{-m} g(ty, tz_1, \ldots, tz_n) = y^m - z_1^{m_1} z_2^{m_2} \cdots z_n^{m_n} t^{(\sum m_i) - m}$$

Note, then, that the tangent cone $C_{\mathfrak{X},0}$ is again $y^m = 0$ and is also equal to $C_{X,0} \times \mathbb{C}$.

Lemma 1.3. *Let $\mathfrak{X}^0$ denote the open set of smooth points of $\mathfrak{X}$, and let Y denote the analytic subspace $0 \times \mathbb{C} \subset \mathfrak{X}$. Then, the pair of strata $(\mathfrak{X}^0, Y)$ satisfies Whitney's condition a) at the origin.*

Proof. Let $C(\mathfrak{X}) \subset \mathbb{C}^{n+2} \times \check{\mathbb{P}}^{n+1}$ denote the conormal space of $\mathfrak{X}$, and let us consider the following diagram:

$$\begin{array}{ccc} C(\mathfrak{X}) & & C(Y) \\ \downarrow{\scriptstyle \kappa} & & \downarrow{\scriptstyle h} \\ \mathfrak{X} & \hookleftarrow & Y \end{array}$$

Then, Whitney's condition a) at the origin is equivalent to the set theoretic inclusion

$$|\kappa^{-1}(0)| \subset |h^{-1}(0)|$$

Let $((y, z_1, \ldots, z_n, t), [a : b_1 : b_2 : \ldots : b_n : c])$ be the coordinates of $\mathbb{C}^{n+2} \times \check{\mathbb{P}}^{n+1}$. Now, since Y is the t axis, the conormal space $C(Y)$ is defined by the equations $y = z_1 = \cdots = z_n = c = 0$, and for $h^{-1}(0)$ we just add the equation $t = 0$. But, since by remark 1.2 $\mathfrak{X}$ has no exceptional tangents, we know that $|\kappa^{-1}(0)|$ is only the dual of the projectivized tangent cone, which in this case is the single point set $\{(0), [1 : 0 : \ldots : 0]\}$, which is contained in $h^{-1}(0)$. □

To complete the demonstration of Proposition 1.1, we still have to prove that the aforementioned pair of strata satisfy Whitney's condition b) at the origin. For this, we are going to use the following result

Proposition 1.4. *[L-T, 1.3.8, pg 550] The following conditions are equivalent:*

1. *The pair of strata $(\mathfrak{X}^0, Y)$ satisfy Whitney's condition b) at the origin.*
2. *In every point p of $\kappa^{-1}(0)$, the ideal I defining the intersection $C(\mathfrak{X}) \bigcap C(Y)$ is integral over the ideal J defining the analytic space $\kappa^{-1}(Y)$.*

With this in mind, we can now proceed to prove the following lemma:

Lemma 1.5. *Let $\mathfrak{X}^0$ denote the open set of smooth points of $\mathfrak{X}$, and let Y denote the analytic subspace $0 \times \mathbb{C} \subset \mathfrak{X}$. Then, the pair of strata $(\mathfrak{X}^0, Y)$ satisfies Whitney's condition b) at the origin.*

Proof.
In the proof of lemma 1.3 we have seen that if we set

$$((y, z_1, \ldots, z_n, t), [a : b_1 : b_2 : \ldots : b_n : c])$$

as the coordinates of $\mathbb{C}^{n+2} \times \check{\mathbb{P}}^{n+1}$, then $|\kappa^{-1}(0)|$ is the single point $H = (0), [1 : 0 : \ldots : 0]$. According to proposition 1.4 what we have to check is a local condition, so a natural alternative is to take the local coordinates, corresponding to the only chart of $\mathbb{C}^{n+2} \times \check{\mathbb{P}}^{n+1}$ where we can see H, that is

$$(y, z_1, \ldots, z_n, t, b_1/a, \ldots, b_n/a, c/a)$$

With this local coordinates, and using the notation of proposition 1.4, we get that:

- H is now the origin.
- The ideal $J = \langle y, z_1, \ldots, z_n \rangle \, O_{C(\mathfrak{X}),H}$.
- The ideal $I = \langle y, z_1, \ldots, z_n, c/a \rangle$.

So, all we have to prove is that the function c/a is integral over J, in $O_{C(\mathfrak{X}),H}$, which according to [Te1, Chap 1, 1.3] is equivalent to proving the existence of a neighborhood U of H in $C(\mathfrak{X})$ and a positive real constant C such that:

$$\left|\frac{c}{a}\right| \leq C \cdot \sup(|y|, |z_i|)$$

Let U be the neighborhood defined by the intersection of $C(\mathfrak{X})$ with the open unitary ball in this coordinates. The points $(p, H) \in U$ can be split in 2, depending on if $p \in \mathfrak{X}$ is a singular point of $\mathfrak{X}$ or not. Then from the equation F, we get that:

$$\nabla G = \left(my^{m-1}, -m_1 z_1^{m_1-1} z_2^{m_2} \cdots z_n^{m_n} t^{\alpha}, \ldots, -m_n z_1^{m_1} \cdots z_n^{m_n-1} t^{\alpha}, \alpha z_1^{m_1} \cdots z_n^{m_n} t^{\alpha-1}\right)$$

where $\alpha = (\sum m_i) - m$. So, the analytic set $|\mathrm{Sing}\mathfrak{X}|$ in $\mathbb{C}^{n+2}$ is defined by the equations

$$\{y = 0, z_1 \cdots z_n t = 0\} \tag{1}$$

and by definition of the conormal space, if $p \in \mathfrak{X}$ is a smooth point, then $y \neq 0$, $z_i \neq 0$, $t \neq 0$, and its fiber in the conormal space is the point $(p, [\nabla F(p)])$. Now, for a smooth point $p \in \mathfrak{X} \bigcap U$, we have that:

$$\begin{aligned} \left|\frac{c}{a}\right| &= \left|\frac{\alpha z_1^{m_1} \cdots z_n^{m_n} t^{\alpha-1}}{my^{m-1}}\right| \cdot \left|\frac{t}{t}\right| \\ &= \left|\frac{\alpha z_1^{m_1} \cdots z_n^{m_n} t^{\alpha}}{my^{m-1} t}\right| \\ &= \left|\frac{\alpha y^m}{my^{m-1} t}\right| = \left|\frac{\alpha y}{mt}\right| \end{aligned}$$

But for points in $\mathfrak{X}$, we have that

$$|y| = |z_1|^{m_1/m} \cdots |z_n|^{m_n/m} |t|^{\alpha/m}$$

where $m_i/m \geq 1$ and $\alpha/m \geq 2$. So we can now get the inequality we were looking for:

$$\begin{aligned} \left|\frac{c}{a}\right| &= \left|\frac{\alpha y}{mt}\right| \\ &= \left|\frac{\alpha}{m}\right| |z_1|^{m_1/m} \cdots |z_n|^{m_n/m} |t|^{\alpha/m-1} \\ &\leq \left|\frac{\alpha}{m}\right| \cdot \sup\{|z_i|\} \text{ since } \ |z_i| < 1, \ |t| < 1 \end{aligned}$$

We are still not finished, for we must still consider the case of the singular points of $\mathfrak{X} \cap U$. However it is somewhat simpler, first by semicontinuity of fiber dimension, for all singular points $p \in \mathfrak{X}$ sufficiently close to the origin, we have that the fiber $\kappa^{-1}(p)$ in the conormal space is zero dimensional, and so they don't have exceptional cones either, meaning $\kappa^{-1}(p)$ is the projective dual of the tangent cone $C_{\mathfrak{X},p}$.

Next by (1), a singular point p has coordinates $p = (0, \lambda_1, \dots, \lambda_n, \gamma)$, with $\lambda_1 \cdots \lambda_n \gamma = 0$. Now, to find out an equation of the tangent cone $C_{\mathfrak{X},p}$, we translate p to the origin, which transforms G into:

$$y^m - (z_1 + \lambda_1)^{m_1} (z_2 + \lambda_2)^{m_2} \cdots (z_n + \lambda_n)^{m_n} (t + \gamma)^{\alpha}$$

If $n \geq 3$, $\alpha > m$ and so we can read from G that the only plausible equations for the tangent cone are of the form:

$$y^m = 0$$

or

$$y^m - \lambda_1^{m_1} \cdots \hat{\lambda_i}^{m_i} \cdots \lambda_n^{m_n} \gamma^{\alpha} z_i^m = 0$$

which, anyway only gives us points of the form $(p, [1 : 0 : \dots : \beta : 0 : \dots : 0])$ in $C(\mathfrak{X})$, that is points where in our chosen local coordinates $c/a = 0$. For the case $n = 2$ we can also get a tangent cone with equation: $y^m - \lambda_1^{m_1} \lambda_2^{m_2} t^m = 0$ In this case, since $|\lambda_i| < 1$ what we will get is points of the form $(p, [1 : 0 : 0 : \beta])$ with $|\beta| \leq \sup \{|\lambda_1|, |\lambda_2|\}$, so when we pass to local coordinates we find again

$$\left| \frac{c}{a} \right| \leq \sup \{|z_1|, |z_2|\}$$

which finishes the proof.

□

Remark 1.6. *The combination of lemmas 1.3 and 1.5 is the proof of the statement 1.1.*

References

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