Problemas de Tesis y avances.

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1 Singular Locus of Codimension 1 and Whitney Equisingularity.

Let $(X, 0) \subset (\mathbb{C}^n, 0)$ be a reduced and equidimensional $d-$dimensional germ of singularity with singular locus Σ of codimension 1.

Proposition 1.1. *If the germ $(\Sigma, 0)$ is smooth, and the pair $(X, \Sigma)_0$ satisfy Whitney's condition a) at the origin, then the tangent cone $C_{X,0}$ consists of a finite number of $d-$planes.*

Proof. Set $(y_1, \dots, y_{d-1}, z_d, \dots, z_n)$ as coordinates of $\mathbb{C}^n$. We can assume that $\mathbb{C}^n = \Sigma \times \mathbb{C}^{n-d+1}$, so the conormal space of Σ in $\mathbb{C}^n \times \check{\mathbb{P}}^{n-1}$ can be written as $\Sigma \times \check{\mathbb{P}}^{n-d}$, where $\check{\mathbb{P}}^{n-d}$ is the projective space of all hyperplanes of $\mathbb{C}^n$ containing Σ. Now, if we take a look at the diagram:

$$\begin{array}{ccc} C(X) & & C(\Sigma) \\ \downarrow{\scriptstyle \kappa} & & \downarrow{\scriptstyle h} \\ X & \hookleftarrow & \Sigma \end{array}$$

Then, Whitney's condition a) at the origin is equivalent to the set theoretic inclusion

$$|\kappa^{-1}(0)| \subset |h^{-1}(0)| = \{0\} \times \check{\mathbb{P}}^{n-d}$$

But, since for a nonsingular point $x \in X$, $\kappa^{-1}(x)$ is of dimension $n-d-1$, then by semicontinuity of fiber dimension we obtain $n - d \geq \dim \kappa^{-1}(0) \geq n - d - 1$.

Remark 1.2. *Recall that if $X \subset \mathbb{P}^n$ and the codimension of X is $r+1$, then the dual variety $\check{X}$ is ruled by projective linear subspaces of dimension r.*

In our case the projectivized tangent cone $\mathbb{P}C_{X,0} \subset \mathbb{P}^{n-1}$ is of codimension $n-d$, so its dual is ruled by projective linear subspaces of dimension $n-d-1$. Recall that $\check{C}_{X,0}$ is contained in $\kappa^{-1}(0)$. With this mind we can analyze the two cases:

1) If dim $\kappa^{-1}(0) = n - d - 1$, then $\kappa^{-1}(0)$ is the union of a finite number of projective linear subspaces of dimension $n-d-1$, each of which correspond to the dual of an irreducible component of the tangent cone, that is to a d-plane of $\mathbb{C}^n$.

2) If dim $\kappa^{-1}(0) = n - d$, then $\kappa^{-1}(0) = \{0\} \times \check{\mathbb{P}}^{n-d}$. This tells us that $(X, 0)$ has Σ as an exceptional tangent. On the other hand if Z is an irreducible component of the tangent cone then necessarily $\check{Z} \subset \check{\mathbb{P}}^{n-d}$ and $\check{Z} \neq \check{\mathbb{P}}^{n-d}$ because $Z \supset \Sigma$ So, we get that dim $\check{Z} = n - d - 1$, and again by the remark we get that it must be a projective linear subspaces of dimension $n - d - 1$ which corresponds to a d-plane of $\mathbb{C}^n$.

□

Let us suppose for a moment, that the pair $(X, \Sigma)_0$ also satisfies Whitney's condition b). Then, we get that the partition $X = X^o \sqcup \Sigma$ is a Whitney stratification of X, and by a theorem of Hironaka [Hi2, Theorem 6.1], we get that X is normally pseudoflat along Σ at 0. Let us recall, the concept of normal pseudoflatness with the following definition-proposition.

Definition 1.3. *[He-Or, 1.4.9, pg 574] Let X be a reduced complex space, $Y \hookrightarrow X$ a closed complex subspace, and $y \in Y$ such that the germ (X, y) is equidimensional. Let $\nu : C_{X,Y} \to Y$ be the normal cone of X along Y, then X is normally pseudoflat along Y at y if and only if one of the following equivalent conditions hold:*

1. *ν is universally open near y.*
2. *The dimension of the fibre $\nu^{-1}(z)$ does not depend on z neay y.*
3. *The dimension of the fibre, dim $\nu^{-1}(z) =$ dim $(X, y)-$ dim(Y, y).*

Once we have stated this, we are now in position to prove the following result.

Proposition 1.4. *In the same context, if the pair $(X, \Sigma)_0$ satisfy Whitney's conditions a) and b) at the origin, then the germ $(X, 0)$ has no exceptional tangents.*

Proof. Let $\nu : C_{X,\Sigma} \to \Sigma$ be the non-projectivized normal cone of X along Σ. As we stated just after the proof of proposition 1.1 the Whitney conditions of $(X, \Sigma)_0$ imply that X is normally pseudoflat along Y at 0, which by definition 1.3 tells us that the fiber $\nu^{-1}(0)$ is of dimension 1. However, remember that the fibers are conical, so the fiber $\nu^{-1}(0)$ has no choice but to be a finite number of lines.

On the other hand, if T is a limit of tangent spaces to X at 0, then by Whitney's condition a) we have that $\Sigma \subset T$. Retaking the notation of proposition 1.1, let $\{a_n\} = \{(\underline{y}, \underline{z})_n\}$ be a sequence of smooth points of X, tending to the origin such that the sequence of tangent spaces $T_{a_n}X^0$ tends to T. Define the sequence the sequence $\{b_n\} = \{(\underline{y}, 0)_n\} \subset \Sigma$ tending to 0. Now, by Whitney's condition b), we have that:

$$\lim \overline{a_n b_n} = \lim[0 : z]_n = l \subset T$$

where $\overline{a_n b_n}$ denotes the direction of the line going through a_n and b_n, and [:] are homogeneous coordinates of $\mathbb{P}^{n-1}$. Note, that each line $[0 : z]_n$ is perpendicular to Σ, which implies that its limit l is also perpendicular, and since T is of dimension d and Σ of dimension $d-1$ this tells us that $T = \Sigma \oplus l$.

Finally, by construction, any such l belongs to the fiber $\nu^{-1}(0)$, which is a finite set, so there is only a finite set of limits of tangent spaces to X at 0, and they must necessarilly coincide with the irreducible components of the tangent cone $C_{X,0}$ which we already know by proposition 1.1 consists of a finite number of d planes. In particular, all this implies that there are no exceptional tangents. □

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