Limits of tangent spaces

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June 19, 2008

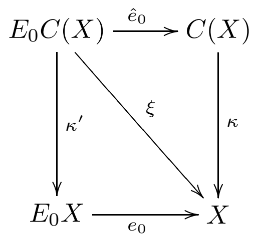
Abstract

The purpose of our poster will be to expose a result of B. Teissier and Lê D. T. which presents the set of limits of tangent hyperplanes to an analytic space at a point as an algebraic space composed of the projectively dual variety of the tangent cone and the projectively dual varieties of certain subcones of the tangent cone.

Keywords: Limits of tangent spaces, conormal space.

To set the working grounds, let us fix a reduced, pure dimensional germ of analytic subspace $(X, 0) \subset (\mathbb{C}^n, 0)$ of dimension d, and by X we will denote a sufficiently small representative of the germ.

Our main object and tool in this setting is the so called normal/conormal diagram:



where $e_0 : E_0X \to X$ is the blowup of the origin in X, $\kappa : C(X) \to X$ denotes the conormal space with its natural map, $\hat{e}_0 : E_0C(X) \to C(X)$ is the blowup of the analytic subpsace $\kappa^{-1}(0)$ and κ' completes the commutative diagram and comes from the universal property of the blowup.

The conormal space, is an analytic subspace of $X \times \check{\mathbb{P}}^{n-1}$, which depends on the embedding of X, with a natural map $\kappa : C(X) \to X$ that has as fibers $\kappa^{-1}(x)$ the set of limits of tangent hyperplanes to X at x. The term tangent hyperplane denotes, the direction of an ambient hyperplane which contains a limit of tangent spaces.

If X is a hypersurface, the conormal map coincides with the Nash modification. In general the geometric structure of the inclusion $\kappa^{-1}(x) \subset \mathbb{P}^{n-1}$ determines the set of limit positions of tangent spaces, i.e., the fiber $\nu^{-1}(x)$ of the Nash modification. The points of $\nu^{-1}(x)$ are in a 1-1 correspondance with the linear projective subspaces $\mathbb{P}^{N-1-d}$ of $\mathbb{P}^{n-1}$ contained in $\kappa^{-1}(x)$.

The result we want to present is the following:

Theorem 0.1. *If D denotes the reduced divisor $|\xi^{-1}(0)|$ and $\cup_\alpha D_\alpha$ its irreducible decomposition then:*

a) Each $D_\alpha \subset \mathbb{P}^{n-1} \times \check{\mathbb{P}}^{n-1}$ is in fact contained in the incidence variety $I \subset \mathbb{P}^{n-1} \times \check{\mathbb{P}}^{n-1}$.

b) Each D_α is lagrangian in I and therefore establishes a projective duality of its images:

$$\begin{array}{ccc} D_\alpha & \longrightarrow & W_\alpha \subset \check{\mathbb{P}}^{n-1} \\ \downarrow & & \\ V_\alpha \subset \mathbb{P}^{n-1} & & \end{array}$$

From the commutativity of the diagram we obtain that $\kappa^{-1}(0) = \bigcup W_\alpha$, and $e_0^{-1}(0) = \bigcup V_\alpha$. These expressions are not necessarily the irreducible decompositions of $\kappa^{-1}(0)$ and $e_0^{-1}(0)$ respectively, since we can't assure that all the V_α (W_α) are of the right dimension. However, it is true that they contain the respective irreducible decompositions. The cones over the V_α which are not of maximal dimension are called exceptional cones. Part b) of the theorem tells us that $\kappa^{-1}(x)$ is an algebraic space composed of the projectively dual variety of the tangent cone and the projectively dual varieties of the exceptional cones.

References

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