(3.2) Lemma Let A be a local noetherian ring. Let $U \subseteq \mathrm{Spec}A$ be an open such that for each $\mathfrak{p} \in U$ the ring $A_\mathfrak{p}$ satisfies the condition (S_n) of Serre. Assume that the set of prime ideals $\mathfrak{p}$ such that $A_\mathfrak{P}$ satisfies the condition (S_{n+1}) is open. Then the set $P = \{\in U : \mathrm{hcod}(A_\mathfrak{p}) = n \text{ and } \dim A_\mathfrak{p} > n\}$ is finite.

Proof Let $A_{n+1} = \{\mathfrak{p} \in U : A_\mathfrak{p}$ does not satisfy the condition $(S_{n+1})\}$. Because of the hypothesis, the set A_{n+1} is closed in U. If $\mathfrak{p} \in A_{n+1}$ then it is clear that $\dim(A_\mathfrak{p}) > n$ because $A_\mathfrak{p}$ satisfies the condition (S_n). Let $\mathfrak{p}_1, ..., \mathfrak{p}_r$ be minimal prime ideals of A_{n+1} and $\mathfrak{a} = \mathfrak{p}_1 \cap ... \cap \mathfrak{p}_r$. There exists elements $x_1, ..., x_n \in \mathfrak{a}$ which form a prime sequence of length n in $A_\mathfrak{p}$. Let P' be the set of associated prime ideals of $x_1 A + \cdots + A_{n+1}$. P' is finite. Furthermore $P \subset P'$ because every $\mathfrak{p} \in P$ is contained in A_{n+1} and thus it contains the elements $x_1, ..., x_n$ which form a prime sequence in $A_\mathfrak{p}$. Because for $\mathfrak{p} \in P$ we have that $\mathrm{hcod}(A_\mathfrak{p}) = n$, then P must be associated to $x_1 A + \cdots + x_n A$. The statement (3.2) is proved.

(3.3) Theorem Let A be a local excellent ring, $\mathfrak{I} \subseteq \mathfrak{M}_A$ be an ideal and $U \subset D(\mathfrak{I})$ be an open subset. Let $A_\mathfrak{p}$ verify the conditions (R_k) and (S_r) of Serre for $\mathfrak{p} \in U$. Then there exists $x \in A$ such that $(A/xA)_\mathfrak{p}$ satifies the conditions (R_k) and (S_r) of Serre for every prime ideal $\mathfrak{P} \in U \cap V(x)$.

Proof Because A is excellent, the set $\mathrm{Sing}\, U = \{\mathfrak{p} \in U : A_\mathfrak{p}$ is not regular$\}$ is closed. Let P_1 be the finite set of the minimal prime ideals of $\mathrm{Sing}U$. Thanks to the Lemma (3.2) the set $P_2 = \{\mathfrak{p} \in U : \mathrm{hcod}(A_\mathfrak{p}) = r \text{ and } \dim A_\mathfrak{p} > r\}$ is finite. We find because the Theorem (3.1) an element $x \in I$ such that $\mathrm{Reg}(A/xA) \cap U \supseteq \mathrm{Reg}(A) \cap V(x) \cap U$ and the element x avoids the prime ideals of p_1 and P_2 and the minimal prime ideals of U. This element satisfies the statement of the Theorem as one can easily convice himself.