On B. Segre and the Theory of Polar Varieties

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It is an honour for me to lecture at the University of Bologna, especially since two great algebraic geometers, Beppo Levi and Beniamino Segre, whose work on singularities has had lasting influence, were Professors here.

Both Beppo Levi (see [14]) and Beniamino Segre made use of polar varieties of projective varieties in their studies of resolution of singularities of surfaces. Segre also introduced the Segre classes, which are similar in spirit to the polar classes, replacing tangent spaces by secant lines, and provide numerical invariants of embeddings. He used them to give an alternate construction of the characteristic classes of non singular projective varieties that had been defined by Todd using polar classes.

1. Local polar varieties

Let me begin by recalling the more modern definition of *local* polar varieties, according to [23], [11], [25],[6]. Consider a diagram of complex analytic spaces and maps, all assumed to be "sufficiently small" representatives of germs

$$\begin{array}{ccc} X & \hookrightarrow & S \times \mathbf{C}^N \\ \downarrow f & \swarrow & \\ S & & \end{array}$$

assume that there is a dense open analytic subset $X^\circ \subset X$ such that the restriction of f to X° is flat with nonsingular fibers purely of dimension d. Let us now fix an integer k, $o \leq k \leq d$ and take a linear projection $p\colon \mathbf{C}^N \to \mathbf{C}^{d-k+1}$. The closure $P_k(f;p)$ in X of the critical locus of the restriction $Id_S \times p|X^\circ$ is by definition the polar variety of f associated to the given installation in $S \times \mathbf{C}^N$ and the projection p. One usually considers the polar varieties associated to " sufficiently general" projections.

One finds it helpful to build the relative conormal space $C_f(X)$ of X in $S \times \mathbf{C}^N$, which is the closure in $S \times \mathbf{C}^N \times \check{\mathbf{P}}^{N-1}$ of the set of couples (x, H) where $x \in X^\circ$ and H is a direction of hyperplane in $\mathbf{C}^N$ tangent at x to the fiber $f^{-1}(f(x))$. One then has the following diagram:

$$\begin{array}{lcl} C_f(X) & \hookrightarrow & S \times \mathbf{C}^N \times \check{\mathbf{P}}^{N-1} \\ \downarrow \kappa_f & & \downarrow \\ X & \hookrightarrow & S \times \mathbf{C}^N \\ \downarrow f & \swarrow & \\ S & & \end{array}$$

If one denotes by $\lambda_f \colon C_f(X) \to \check{\mathbf{P}}^{N-1}$ the natural projection, and by $L^{d-k} \subset \check{\mathbf{P}}^{N-1}$ the set of hyperplanes containing ker p, then one has for a sufficiently general projection p the set theoretic equality

$$\kappa_f(\lambda_f^{-1}(L^{d-k})) = P_k(f; p) .$$

However it is certain that one should also study the total family, parametrized by the projections, of the polar varieties of a given type.

In the special case where S is a point and X is the cone in $\mathbf{C}^N$ on a reduced projective variety in $\mathbf{P}^{N-1}$, one recovers the classical theory of polar loci.

Originally in [11] and [23], the definition made use of the relative Nash modification of X (see [26], [25], p.417) instead of the relative conormal space; one can then in a completely analogous way also define more generally polar varieties corresponding to a general Schubert cycle associated to a set of incidence dimensions $\mathbf{a} = (a_1, \dots, a_d)$ and a given flag of linear subspaces. Since they depend upon the flag, what we get is actually a system of subvarieties of X parametrized by a flag manifold. In order to have a well-defined object it was deemed in the classical projective and «absolute» case to be necessary to introduce an equivalence relation on the set of algebraic varieties such that at least for «almost all» flags the corresponding polar varieties are all equivalent. Motivated by the search for numerical invariants, Segre, Severi and Todd considered rational equivalence classes, which, however, forget all but the simplest geometric characters (such as dimension and degree) of an algebraic variety.

The theory of equisingularity allows one to try and preserve much more and still have well-defined objects, since for instance, for «almost all» flags, the corresponding polar varieties are all equisingular. In particular, at least the topology of a general polar variety is well-defined.

The purpose of this lecture is to try to draw attention by examples to the importance of the geometric viewpoint in the theory of polar varieties, and to ask a few questions connected with this viewpoint. Here, geometric is meant as opposed to the cohomological, or cyclist, or numerical, viewpoint.

The geometry of the local polar varieties associated to a germ of a complex space or of a complex morphism contains a wealth of information about this germ, and since polar varieties are of lower dimension they can be used for inductions. They can also be used to define numerical invariants considerably more subtle than those arising from the cyclist viewpoint. Since we take a local viewpoint, it is necessary to check that the objects which we define are independant of the choices of local embeddings and coordinates. All the geometric situations considered will be local unless otherwise specified.

THEOREM - *Given a complex analytic morphism* $f \colon (X, x) \to (S, s)$ *satisfying the conditions above, the Whitney-equisingularity type of a general polar variety* $P_k(f; D_{d-k+1})$ *depends only upon the analytic type of the map-germ* f *at* x.

The proof is essentially contained in ([25], Chap.4), and let me first give it in the case of a complex analytic map-germ $f:(\mathbf{C}^2,0)\to(\mathbf{C},0)$.

In this case, after a choice of coordinates x,y, given a linear projection $\ell:\mathbf{C}^2\to\mathbf{C}$, the corresponding polar curve is the zero set of the 2-form $df\wedge d\ell$; il $\ell=x+by$, then an equation for our polar is $\frac{\partial f}{\partial y}-b\frac{\partial f}{\partial x}=0$. By the general results on equisingularity we may assume, if $\ell_0=x$ is a sufficiently general projection, that the family of polar curves is an equisingular family of plane curves as b varies near 0. So for $\ell=\ell_0$ we get the equisingularity type of the general polar curve *in the coordinates* x,y. Now to prove that this equisingularity type is independant of what we chose to call linear coordinates, it is sufficient to consider a one-parameter family of projections of the form $\ell_0+vg(v;x,y)$ with $g(v;x,y)\in(x,y)^2\mathbf{C}\{v,x,y\}$, so that we get a family of curves with equation $df\wedge(dx+vdg)=0$. Instead of considering the corresponding determinant, we consider the associated linear system

$$\frac{\partial f}{\partial x}X+\frac{\partial f}{\partial y}Y=0$$

$$(1+v\frac{\partial g}{\partial x})X+v\frac{\partial g}{\partial y}Y=0$$

which defines locally near the point $(0,0,(0:1))\in\mathbf{C}\times\mathbf{C}^2\times\check{\mathbf{P}}^1$ an analytic subspace $Z\subset\mathbf{C}\times\mathbf{C}^2\times\check{\mathbf{P}}^1$ such that its image in $\mathbf{C}\times\mathbf{C}^2$ is the family of our polar curves. Note that the first equation in independant of v, and defines therefore the product by $\mathbf{C}$ of a family of curves Z_1 in $\mathbf{C}^2\times\check{\mathbf{P}}^1$. This family of curves is actually isomorphic with the family of polar curves as $b=-\frac{X}{Y}$ varies, so it is equisingular, and therefore so is its product with $\mathbf{C}$; the product $\mathbf{C}\times\check{\mathbf{P}}^1$ is the singular locus of Z_1, along which Z_1 is Whitney equisingular. Now we remark that the second equation, upon taking $v=0$, describes a non singular hypersurface $Z_2(0)$ which is *transversal* in $\mathbf{C}^2\times\check{\mathbf{P}}^1$ to the surface $\mathbf{C}\times\check{\mathbf{P}}^1$. Now it is a basic property of Whitney equisingularity that all sections by nonsingular spaces of a given dimension transversal to a stratum form themselves an equisingular family, and as v varies, we have here precisely a family of sections of $Z_1(0)$ by non singular hypersurfaces $Z_2(v)$, transversal for $v=0$ and therefore also for v sufficiently small. Finally we see that Z is a family of curves equisingular along $\mathbf{C}\times 0$. It remains to see that for each value of v, the curve $Z(v)$ is isomorphic with the zero set of $df\wedge(dx+vdg)$, but this is obvious. The same proof works if we multiply the first equation by f, and shows that in fact the *equisingularity class* of the *union* of f and the polar curve depends only on the *analytic class of the germ* f. Exactly the same proof works for the general polar variety of dimension d of a complex analytic mapping $f:(\mathbf{C}^{d+1},0)\to(\mathbf{C},0)$.

The general case is somewhat more delicate, but the idea is the same. It suffices to study the following situation: a one-parameter analytic family of maps $p^*(v):\mathbf{C}^N\to\mathbf{C}^{d-k+1}$ such that $p(0)$ is a general linear projection and the corresponding map $P^*(v)=Id_{\mathbf{C}\times S}\times p^*:\mathbf{C}\times S\times\mathbf{C}^N\to\mathbf{C}\times\mathbf{C}^{d-k+1}$. Consider the graph embedding of P^*

$$\begin{array}{ccc}
\mathbf{C}\times S\times\mathbf{C}^N & \hookrightarrow & \mathbf{C}\times S\times\times\mathbf{C}^{d-k+1}\times\mathbf{C}^N \\
\downarrow P^* & \swarrow & \\
\mathbf{C}\times S\times\mathbf{C}^{d-k+1} & &
\end{array}$$

and the product diagram

$$\begin{array}{ccccc}
\mathbf{C} & \times & C_f(X) & \hookrightarrow & \mathbf{C}\times S\times\mathbf{C}^N\times\check{\mathbf{P}}^{N-1} \\
 & \downarrow & Id_{\mathbf{C}}\times\kappa_f & & \downarrow \\
\mathbf{C} & \times & X & \hookrightarrow & \mathbf{C}\times S\times\mathbf{C}^N \\
 & \downarrow & Id_{\mathbf{C}}\times f & \swarrow & \\
\mathbf{C} & \times & S & &
\end{array}$$

We choose as in [25], Chap.4, a Whitney stratification C_α of $C_f(X)$ compatible with the inverse image of the singular locus of X and with the inverse images by κ_f of the strata of a Whitney stratification of $P_k(f,p)$. Now we consider the intersection $\tilde{\mathcal{P}} = \mathbf{C} \times C_f(X) \cap C_{P^\bullet}(\mathbf{C} \times S \times \mathbf{C}^N)$ in $\mathbf{C} \times S \times \mathbf{C}^N \times \check{\mathbf{P}}^{N-1}$ and its natural projection $\mathcal{P} \to \mathbf{C}$.

Since we assume that the original projection $p(0) = p$ is a general linear projection, it is easy to verify that for each sufficiently small v, the image of $\tilde{\mathcal{P}}(v)$ is the polar variety of f with respect to the analytic projection $p(v)$, and the image of $\tilde{\mathcal{P}}$ in $\mathbf{C} \times S \times \mathbf{C}^N$ is the total space $\mathcal{P}$ of the family of polar varieties of f as v varies. Now it is no longer true, as in the curve case,that the fibers $\tilde{\mathcal{P}}(v)$ are isomorphic to the polar varieties, but one knows enough about the map $\tilde{\mathcal{P}}$ in $\mathbf{C} \times S \times \mathbf{C}^N \to \mathcal{P}$ to show the following:

If we choose carefully as above an equisingular stratification of $C_f(X)$, and denote by S_0 the largest stratum contained in $\kappa_f^{-1}(0)$, the product $\mathbf{C} \times S_0$ is a stratum of an equisingular stratification of $\mathbf{C} \times C_f(X)$. The intersection with $C_{P^\bullet}(\mathbf{C} \times S \times \mathbf{C}^N)$ is transversal for $v = 0$, so $\mathbf{C} \times S_0 \cap (\mathbf{C}^N \times L^{d-k})$ is a stratum of an equisingular stratification of $\tilde{\mathcal{P}}$.

Using the characterization of Whitney stratifications by properties of the Auréole (see [12], corollaire 2.2.4.1) instead of the characterization of equimultiplicity as in ([25], p.428) one then deduces from this that $\mathbf{C} \times 0$ is a stratum of a Whitney stratification of $\mathcal{P}$, which proves the theorem.

This proof is given only partially because it could be simplified if one had the answer to the following

Question 1. Let $C(X) \subset \mathbf{C}^N \times \check{\mathbf{P}}^{N-1}$ be the conormal space of $X \subset \mathbf{C}^N$. Let $(X_\alpha)_{\alpha \in A}$ be the canonical Whitney stratification of X. Can one describe a *purely Lagrangean* method to construct the Lagrangean variety $\cup_{\alpha \in A} C(X_\alpha)$. That is, without going down to stratify X, but using for example higher microlocalizations.

This question is in a way a geometric version of the following question, which Thom calls his *Dream of Youth*:

Question 2. (Thom) Given an ideal $I \subset \mathbf{C}\{z_1, \dots, z_n\}$ defining a reduced equidimensional germ $X \subset \mathbf{C}^N$, let

$$X = F_0 \supset F_1 \supset \cdots \supset F_r \supset \emptyset$$

be the canonical Whitney filtration of X (near 0), characterized (see [25], Chap.6) by the fact that the connected components of the differences $F_i \setminus F_{i+1}$ are the strata of the minimal Whitney stratification, and let $I = I_0 \subset I_1 \subset \cdots \subset I_r \subset \mathbf{C}\{z_1, \dots, z_n\}$ be the corresponding sequence of ideals.

The problem is: find a way using Jacobian extensions of ideals (i.e, by adding suitable Jacobian minors of generators), to generate the sequence (I_j) from the ideal I.

I do not know how to answer this exact question (which I have taken the liberty to complexify and slightly adapt from the original) but using the results of [25] I can give (see [27]) a similar result but I have to use not only Jacobian extensions, but also the residuation $(J : K) = \{g \in \mathbf{C}\{z_1, \dots, z_n\} / gK \subset J\}$ of ideals, choices of special system of generators, and «generic» choices of coordinates.

Here is a question which is in a way intermediate between these two:

Question 3. Given a projective variety $V \subset \mathbf{P}^N$, consider the canonical Whitney stratification (W_α) of its dual variety $\check{V} \subset \check{\mathbf{P}}^N$. How can one build from equations of V the totality of the duals $\check{W}_\alpha \subset \mathbf{P}^N$?

As an exercise, given a projective plane curve, try to build from its equation those of all the special lines associated with it : inflection tangents, double tangents, etc.., only by using Jacobian extensions, etc.., without dualizing.

2. The contact of the relative polar varieties

In fact this study was begun over a hundred years ago, in the case of plane curves, by H.J. Stephen Smith in [20], in connection with the study of Plücker formulas. Then the italian geometers, for whom the contact was best expressed in terms of common infinitely near points, tried to find the infinitely near points common to a curve and its general polar. Since the infinitely near points of the polar depend upon the analytic type of the singularity and not only on its tree of infinitely near points (that is, its equisingularity class), this viewpoint led essentially to frustration, at least until the very recent results of Casas ([1]) concerning the polar curve of a plane curve singularity which is «generic» among those belonging to a given equisingularity class.

However, there are some specific results at least in the case of hypersurfaces with isolated singularities, to which we will come back.

For the purpose of resolution of singularities, however, this fundamental impossibility to determine the equisingularity type of polar varieties (absolute or relative) from just the equisingularity type of the given germ of variety or map may not hamper the use of polar varieties as providers of inductive steps. Here is the prime example, after Segre [19] :

2.1. Contact of polar varieties and resolution of singularities

Given a projective hypersurface $X \in \mathbf{P}^N$ with equation $F(X_1, \ldots, X_{N+1}) = 0$, let $\xi = (\xi_1, \ldots, \xi_{N+1})$ be a point of $\mathbf{P}^N$, then the polar hypersurface of X with respect to the point ξ is the hypersurface with equation $\sum_1^{N+1} \xi_i \frac{\partial F}{\partial X_i} = 0$; it is the projective hypersurface corresponding to therelative polar variety $P_1(F; p)$ associated to the map $F: \mathbf{C}^{N+1} \to \mathbf{C}$ and the projection $p: \mathbf{C}^{N+1} \to \mathbf{C}^N$ corresponding to ξ .

In [19], Segre proposed to correct in the case of surfaces an attempt of Derwidue in [2] (see Zariski's Math. Review [31]) to resolve singularities of projective varieties, and convince the reader that one could after corrections extract a proof at least in the case of surfaces in characteristic zero (which at that time was already known by the work of Zariski). The main steps of the argument are as follows:

1) a computation of the strict transform of the polar curve under blowing up of a non singular center along which the given hypersurface is equimultiple, *at a point where the multiplicity has not decreased* and the constatation that at such a point the strict transform of a general polar is a polar of multiplicity $m - 1$ of the strict transform.

2) In the case of surfaces, one can by blowing up points reduce to the case where the equimultiplicity locus is nonsingular, and by 1), one sees that it is sufficient to show that one can make the multiplicity of a general polar decrease by a finite number of blowing ups along non singular equimultiplicity loci. One thus reduces to the case where the given hypersurface has singularities of multiplicity two only.

3) A direct computational treatment of this last case.

Let $f(y_1, \ldots, y_t, z_1, \ldots, z_n) = 0$ be a local equation for a hypersurface in $\mathbf{C}^N$, where

$N = n + t$. Let assume that $f(y_1, \ldots, y_t, 0, \ldots, 0) = 0$ and consider locally around 0 the blowing up $\pi: Z \to \mathbf{C}^N$ of the subspace of $\mathbf{C}^N$ defined by the ideal $(z_1, \ldots, z_n)$. In a typical chart of Z we have coordinates $y'_1, \ldots, y'_t, z'_1, \ldots, z'_n$ and the map π described by

$$y_j \circ \pi = y'_j \qquad \text{for } 1 \leq j \leq t$$
$$z_1 \circ \pi = z'_1$$
$$z_i \circ \pi = z'_i z'_1 \qquad \text{for } i \neq 1 .$$

The strict transform of f by π is decribed in the same chart by

$$f'(y'_1, \ldots, y'_t, z'_1, \ldots, z'_n) = z_1'^{-m} f(y'_1, \ldots, y'_t, z'_1, z'_2 z'_1, \ldots, z'_n z'_1) .$$

Let us now compute the composition with π of the partial derivatives

$$\frac{\partial f}{\partial z_1} \circ \pi = m z_1'^{m-1} f' - z_1'^{m} \frac{\partial f'}{\partial z'_1} - \sum_{i \geq 2} z'_i z_1'^{m} \frac{\partial f'}{\partial z'_i}$$
$$\frac{\partial f}{\partial z_i} \circ \pi = z_1'^{m-1} \frac{\partial f'}{\partial z'_i} \ (2 \leq i \leq n)$$
$$\frac{\partial f}{\partial y_s} \circ \pi = z_1'^{m} \frac{\partial f'}{\partial y'_s} \ (1 \leq s \leq t)$$
$$f \circ \pi = z_1'^{m} f' .$$

So that if we take an element of the ideal $(f, \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}, \frac{\partial f}{\partial y_1}, \ldots, \frac{\partial f}{\partial y_t})$, say $g = af + \sum_1^n b_i \frac{\partial f}{\partial z_i} + \sum_1^t c_s \frac{\partial f}{\partial y_s}$, we find the identity

$$z_1'^{-(m-1)} g \circ \pi = ((a \circ \pi) z'_1 + m b_1 \circ \pi) f' - (b_1 \circ \pi) z'_1 \frac{\partial f'}{\partial z'_1}$$
$$+ \sum_{i=2}^{n} (b_i \circ \pi - (b_1 \circ \pi) z'_1 z'_i) \frac{\partial f'}{\partial z'_i} + \sum_{s=1}^{t} (c_s \circ \pi) z'_1 \frac{\partial f'}{\partial y'_s} .$$

So if g is of order $m - 1$ at 0, its strict transform is in the ideal $(f', \frac{\partial f'}{\partial z'_1}, \ldots, \frac{\partial f'}{\partial z'_n}, \frac{\partial f'}{\partial y'_1}, \ldots, \frac{\partial f'}{\partial y'_t})$.

There are several observations to make:

1) The behaviour of g is actually governed by that of $\sum_1^n b_i \frac{\partial f}{\partial z_i} + \sum_1^t c_s \frac{\partial f}{\partial y_s}$, and one may as well call polar variety the hypersurface defined by $g = 0$ at least in the case where a and the b's and c's are constants.

2) If g is of the form

$$g = \sum_2^n b_i \frac{\partial f}{\partial z_i} + \sum_1^t c_s \frac{\partial f}{\partial y_s} \text{ with } b_i,\ c_s \in \mathbf{C}$$

then

$$z_1'^{-(m-1)} g' = \sum_2^n b_i \frac{\partial f'}{\partial z_i} + \sum_1^t c_s z'_1 \frac{\partial f'}{\partial y'_s}$$

so we have a very similar expression for the strict transform.
3) If f is equimultiple along Y and the multiplicity of f' at the point x' with coordinates $z'_1 = \ldots = z'_n = y'_1 = \ldots = y'_t = 0$ is again $m-1$, then the exceptional divisor $z'_1 = 0$ is *transversal* to $f' = 0$ in Z at x'.
4) If f is equimultiple along Y, then so is an element

$$g = af + \sum_1^n b_i \frac{\partial f}{\partial z_i} + \sum_1^t c_s \frac{\partial f}{\partial y_s}$$

which is of multiplicity $m-1$ at the point 0.

These observations result from straightforward computations, and they are part of the stock-in-trade of all resolvers of singularities. In particular they have been vastly generalized by Hironaka.

Another important fact discovered by Segre (in the projective situation; it has been rediscovered in the local situation several times since) is that in the blowing-up $\pi: X' \to X$ of X along a non singular center Y *such that X is equimultiple along Y*, the multiplicity of X' at any of its points x' is at most equal to the multiplicity of X at $\pi(x')$.

Segre uses this to show that (in modern language), the problem to prove that a permissible succession of blowing ups along isolated m-uple points and nonsingular m-uple curves is necessarily finite can be reduced to the case of a hypersurface of multiplicity 2. The solution of this problem implies resolution, and the proof in the case where $m=2$ is fairly easy. Segre uses a case-by-case approach.

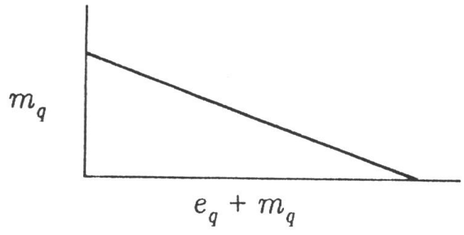
The idea of the reduction is that in view of the preceding results, as long as the multiplicity does not drop, the multiplicity of a general polar, which is $m-1$, does not drop either. Of course in higher dimensions, the choice of centers of blowing-ups becomes a major problem. In any case, Segre's work suggests that the use of the polar hypersurface may offer some alternative to the «maximal contact» of Hironaka. In some sense, the general polar hypersurface of f has «maximal contact» at 0 with $f = 0$ among those of multiplicity $m-1$. In the case of locally irreducible curves one can make this really precise using a result of Merle and the beautiful theory of maximal contact of singular curves of Monique Lejeune-Jalabert (see [13] and [16]). Notice also that if you iterate the polar curve operation for an irreducible plane curve until you get a non singular curve, this curve does have maximal contact at 0. It is a good exercise at this point to prove resolution of singularities of plane curves using Segre's method.

2.2. Contact of a polar curve of a germ of holomorphic map $f: (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ with the hypersurface $f^{-1}(0)$

This is the only case where we know something precise. This contact is the subject of [24],[26], [28] and is decribed by the *Jacobian Newton polygon*, which is constructed as follows:

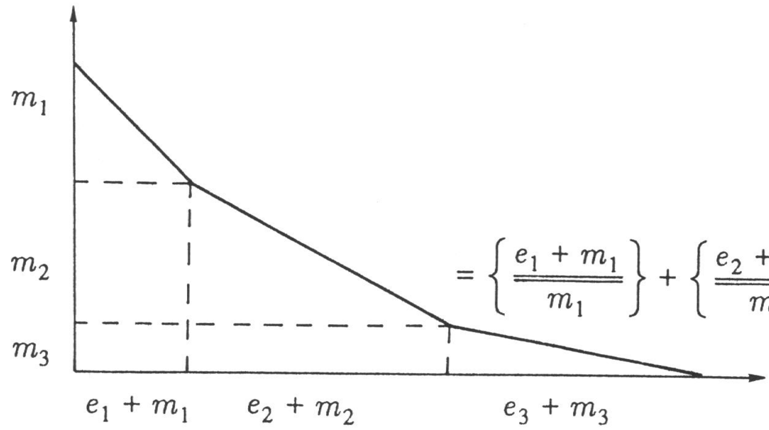
Let $f \in \mathbf{C}\{z_0, \ldots, z_n\}$ describe our map. The relative polar curve of the map with respect to a linear projection $\ell: \mathbf{C}^{n+1} \to \mathbf{C}^n$ is the zero set of the 2-form $df \wedge d\ell$ and if ℓ is sufficiently general, it is a germ of a reduced curve $P_n(f, \ell)$.

I now decompose this germ into its irreducible components $\Gamma_i, 1 \leq i \leq r$. By the definition of polar varieties, none of these components can be included in the hypersurface $f = 0$ (if it were the case, the component would have to be included in the critical locus of f), and each has a multiplicity at 0, which we denote by m_q, and an intersection number at 0 with $f = 0$, necessarily $\geq m_q$, and which we denote by $e_q + m_q$. From these numbers we construct the *contact polygon* at 0 of the hypersurface $f=0$ and the curve $P_n(f,\ell)$ as follows : we construct a Newton polygon by adding the elementary Newton polygons



denoted by $\left\{\dfrac{e_q + m_q}{m_q}\right\}$

We get a Newton polygon, for example



$$= \left\{\frac{e_1 + m_1}{m_1}\right\} + \left\{\frac{e_2 + m_2}{m_2}\right\} + \left\{\frac{e_3 + m_3}{m_3}\right\}$$

Because of the theorem of transversality of relative polar varieties of [24], th.1 (see also [30], [25] Chap.4 and [5] for generalizations) which states that when the base S is a point or a nonsingular curve the tangent cones at 0 of the polar varieties are transversal to the kernels of the corresponding projections, this abstract Newton polygon is actually the Newton polygon of the image in $\mathbf{C}^2$ with coordinates (t_1, t_2) of the polar curve which is the closure of the critical locus outside 0 of the map $(f,\ell): \mathbf{C}^{n+1} \to \mathbf{C}^2$. This Jacobian Newton polygon was introduced in [24] in the case where f has an isolated singularity; it refines in that case the ratio $\frac{\mu^{(n+1)}+\mu^{(n)}}{\mu^{(n)}}$, where $\mu^{(n+1)}$ is the Milnor number of the hypersurface and $\mu^{(i)}$ is the Milnor number of its intersection with a general i-dimensional plane through the origin.

But it has turned out that the slopes of its sides, often called the polar invariants of f , play a basic role in several *a priori* unrelated questions about the singularities of f :

1) the poles of the meromorphic map from $\mathbf{C}^2$ to the space of distributions in $\mathbf{C}^{n+1}$ obtained by analytic continuation from the map defined for $\mathrm{Re}s >> 0, \mathrm{Re}t >> 0$ by

$$(s,t) \mapsto (\phi \mapsto \int_{\mathbf{C}^N} \ell^s f^t \phi)$$

where ℓ is a sufficiently general linear form and ϕ a differentiable function with compact support. Work of Sabbah and Loeser has shown that the poles of this meromorphic map lie on lines in $\mathbf{C}^2$ with slopes given by those of the Jacobian Newton polygon. They actually show much more, see [15], [18].

2) The best possible exponents θ_1 and θ_2 in the Łojasiewicz inequalities (near 0)

$$||\mathrm{grad} f(z)|| > C||z||^{\theta_1} \textit{ and } ||\mathrm{grad} f(z)|| > C'|f(z)|^{\theta_2} .$$

3) The different rates at which in a 1-parameter Morsification $f_v = f + v\ell$, where ℓ is a general linear form, the Morse critical points $c_i(v)$ of f_v tend to 0 as v tends to 0 , i.e the best possible exponents in the inequalities

$$||c_i(v)|| \leq C_i|v|^{r_i} .$$

The point here is that we have the

THEOREM ([24]). *In a Whitney-equisingular family of hypersurfaces* $f(v; z_1, \ldots, z_{n+1})$ *with isolated singularity at* 0 *(for each* v *), the Jacobian Newton polygon is constant.*

For irreducible germs of plane curves Merle has given in [16] the expression for the Jacobian Newton polygon in terms of the Puiseux exponents and it turns out that in this case the Jacobian Newton polygon *determines* the Puiseux exponents and therefore the topology of the curve. In fact much more is true: for each of the Puiseux exponents of an irreducible plane curve X , the general polar curve has a bunch of irreducible components whose Puiseux expansion coincides (up to a suitable ramification of the parameter) with the Puiseux expansion of X *up to* (but excluding) that exponent. So the polar really has very high contact with the curve.

In the reducible case, some components of the polar curve may not have high contact with the curve, but except in very special cases some components do have high contact. There is a general result in this direction:

Fact 1. Let us assume that $f: \mathbf{C}^{n+1} \to \mathbf{C}$ has an isolated singularity at 0 , and that the general polar curve is not tangent to $f = 0$ at 0 (that is, no tangent line to the curve is in the tangent cone of $f = 0$ at 0). Since, by the results of [29] and [24], the multiplicity of the polar curve is $\mu^{(n)}$ and its intersection multiplicity with $f = 0$ is $\mu^{(n+1)} + \mu^{(n)}$, while the multiplicity at 0 of $f = 0$ is $\mu^{(1)} + 1$, transversality is equivalent to the equality $\mu^{(n+1)} = \mu^{(1)}\mu^{(n)}$. By the results of ([29], Chap.2), this implies that the blowing-up of the origin resolves the singularity of $f = 0$, or if one prefers, that $f = 0$ is equisingular with its tangent cone at 0 .

Fact 2. If we now take a hypersurface, say $f(v; x, y, z) = 0$ in $\mathbf{C}^4$, which has a nonsingular one dimensional singular locus along the v axis, we can think of it as a one parameter family of hypersurfaces with isolated singularity at the origin. If the general polar curve of f is empty, if the sections $v = v_0$ of the general polar surface have constant contact polygon with $f(v_0; x, y, z)$ as v_0 varies,, then according to [24], the μ^* sequence is constant along the v axis and according to a theorem of Laufer in [8], our hypersurface has a simultaneous resolution of singularities along the v axis.
In view of Segre's proof, these two facts suggest the

Problem 5. Study the behaviour of the polar varieties of codimension > 1 under equimultiple blowing-up with non singular center.

There is also a similar problem for the absolute polar varieties; here the references for motivation are [9] and [20].

In fact, (see [28], [16] [18]) one can put both the absolute and the relative case in the same frame, and generalize the Jacobian Newton polygon as follows: Consider an analytic map $F: X \to \mathbf{C}^2$; choose coordinates t_1, t_2 on $\mathbf{C}^2$ and set $f_1 = t_1 \circ F, f_2 = t_2 \circ F$. Then the strict critical locus $S(F)$ of F is by definition the closure in X of the set of nonsingular points of X where the fibers of f_1 and f_2 are not both non singular and meeting transversally. The Newton polygon in the coordinates t_1, t_2 is a generalization of the Jacobian Newton polygon and we call it the Jacobian Newton polygon of f_1, f_2 on X .

Question 6. Is it true that in an equisingular deformation of X and F this Jacobian Newton polygon is constant ?

(The definition of equisingularity here is left to the reader).

The recent results of Gaffney in [3] and [4] make it plausible that, at least when X is a complete intersection wih isolated singularity, a proof can be given along the lines of [24].

REFERENCES

[1] E. Casas-Alvero : Infinitely near singularities and singularities of polar curves, *Preprint, Universitat de Barcelona, dep. Alg. i Geom.*1988

[2] L. Derwidué: Le problème de la réduction des singularités de variétés algébriques, *Math. Annalen* **123** (1951), 302-330.

[3] Terence Gaffney : Integral closure of modules and Whitney equisingularity, *preprint, Northeastern University*

[4] Terence Gaffney : Multiplicities and equisingularity of ICIS germs, *preprint, Northeastern University*

[5] J.P.G. Henry et M. Merle: Limites d'espaces tangents et transversalité de variétés polaires, *Algebraic Geometry, Proceedings, La Rábida, 1981*, Springer L.N.M., No.961, 189-199

[6] J.P.G. Henry, M. Merle et C. Sabbah: Sur la condition de Thom stricte pour un morphisme analytique complexe, *Ann. Sci. E.N.S.*, **17**, 1984, 227-268.

[7] Patrick Du Val: Benjamino Segre, *Bull. London Math. Soc.*, **11** (1979), 215-235

[8] Henry Laufer: Strong simultaneous resolution for surface singularities, *preprint*, Stony Brook, 1984

[9] Dũng Tráng Lê: Exposants polaires et résolution des surfaces, *Communication au 3ème Congrès des mathématiciens vietnamiens, 1985.*

[10] Dũng Tráng Lê, Françoise Michel, Claude Weber, Courbes polaires et topologie des courbes planes. A paraître dans *Annales scientifiques de l'E.N.S.*

[11] Dũng Tráng Lê et B. Teissier: Variétés polaires locales et classes de Chern des variétés singulières, *Annals of Math.*, **114**, 1981, 457-491.

[12] Dũng TrángLê et B. Teissier: Limites d'espaces tangents en géométrie analytique, *Commentarii Math. Helv.*, **63**, 4, 1988, 540-578.

[13] Monique Lejeune-Jalabert , *Thèse* Paris 7,1973.

[14] Beppo Levi : Sur la résolution des points singuliers des surfaces algébriques, *Comptes rendus Ac. Sci. Paris*, **134**, 1902, p.224.

[15] François Loeser : Fonctions dzeta locales d'Igusa à plusieurs variables, intégration dans les fibres, et discriminants, *Ann. Sci. E.N.S.*, **22**, 3,1989, 435-471,

[16] Michel Merle: Invariants polaires des courbes planes, *Inventiones Math.*, **41**, 1977, 103-111.

[17] Françoise Michel, Claude Weber, *Livre à paraître sur la topologie des courbes planes.*

[18] Claude Sabbah : Proximité évanescente 1: Structure polaire des D-modules (en collaboration avec F. Castro). *Compositio Math.*, **62**, 1987, 283-328.

[19] Beniamino Segre: Sullo scioglimento delle singolarità delle varietà algebriche, *Ann. di Mat.* **4**, 33, 5-48.

[20] H.J. Stephen Smith : On the higher singularities of plane curves, *Proc. London Math. Soc.*, **5**, 1873, 153-182.

[21] Mark Spivakovsky: Resolution of singularities by normalized Nash blowing-up, To appear, *Annals of Math.*

[22] Bernard Teissier: *appendice à* [32].

[23] Bernard Teissier: Variétés polaires locales et conditions de Whitney, *C.R.A.S., Paris* **280**, 1980.

[24] Bernard Teissier : Variétés polaires 1: invariants polaires des singularités d'hypersurfaces, *Inv. Math.* **40**, 1977, 267-292

[25] Bernard Teissier : Variétés polaires 2: multiplicités polaires, sections planes, et conditions de Whitney. *Algebraic Geometry, Proceedings, La Rábida, 1981* Springer L.N.M., No.961 (1982), 314-491.

[26] Bernard Teissier: The hunting of invariants in the geometry of discriminants. *real and complex singularities*, Sijthoff and Noordhoof 1977.

[27] Bernard Teissier: Analytic singularities: topology, algebra, algebraization, *Stone Lectures at Northeastern*, May1988, in preparation.

[28] Bernard Teissier : Polyèdre de Newton Jacobien et équisingularité, *Séminaire sur les singularités*, Publications de Paris 7, 1980.

[29] Bernard Teissier: Cycles évanescents, sections planes, et conditions de Whitney, "Singularités à Cargèse, *Asterisque* No.7-8, S.M.F. 1973.

[30] Bernard Teissier: Variétés polaires locales: Quelques résultats, Journées complexes, Nancy 1980. *Publications de l'Institut Elie Cartan*, 1981.

[31] Oscar Zariski: *Math. Reviews*, **13**, No.1, Jan. 1952, p.67.

[32] Oscar Zariski: *Modules de branches planes*, Cours au centre de Mathématiques de l'Ecole Polytechnique, 1973. Réedition Hermann 1986.