Local polar varieties in the geometric study of singularities

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This paper is dedicated to those administrators of research who realize how much damage is done by the evaluation of mathematical research solely by the rankings of the journals in which it is published, or more generally by bibliometric indices. We hope that their lucidity will become widespread in all countries.

Abstract

This text presents several aspects of the theory of equisingularity of complex analytic spaces from the standpoint of Whitney conditions. The goal is to describe from the geometrical, topological, and algebraic viewpoints a canonical locally finite partition of a reduced complex analytic space X into nonsingular strata with the property that the local geometry of X is constant on each stratum. Local polar varieties appear in the title because they play a central role in the unification of viewpoints. The geometrical viewpoint leads to the study of spaces of limit directions at a given point of $X \subset \mathbf{C}^n$ of hyperplanes of $\mathbf{C}^n$ tangent to X at nonsingular points, which in turn leads to the realization that the Whitney conditions, which are used to define the stratification, are in fact of a lagrangian nature. The local polar varieties are used to analyze the structure of the set of limit directions of tangent hyperplanes. This structure helps in particular to understand how a singularity differs from its tangent cone, assumed to be reduced. The multiplicities of local polar varieties are related to local topological invariants, local vanishing Euler-Poincaré characteristics, by a formula which turns out to contain as a special case a Plücker-type formula for the degree of the dual of a projective variety.

Résumé

Ce texte présente plusieurs aspects de la théorie de l'équisingularité des espaces analytiques complexes telle qu'elle est définie par les conditions de Whitney. Le but est de décrire des points de vue géométrique, topologique et algébrique une partition canonique localement finie d'un espace analytique complexe réduit X en strates non singulières telles que la géométrie locale de X soit constante le long de chaque strate. Les variétés polaires locales apparaissent dans le titre parce qu'elles jouent un rôle central dans l'unification des points de vue. Le point de vue géométrique conduit à l'étude des directions limites en un point donné de $X \subset \mathbf{C}^n$ des hyperplans de $\mathbf{C}^n$ tangents à X en des points non singuliers. Ceci amène à réaliser que les conditions de Whitney, qui servent à définir la stratification, sont en fait de nature lagrangienne. Les variétés polaires locales sont utilisées pour analyser la structure de l'ensemble des positions limites d'hyperplans tangents. Cette structure aide à comprendre comment une singularité diffère de son cône tangent, supposé réduit. Les multiplicités des variétés polaires locales sont reliées à des invariants topologiques locaux, des caractéristiques d'Euler-Poincaré évanescentes, par une formule qui se révèle avoir comme cas particulier une formule du type Plücker pour le calcul du degré de la variété duale d'une variété projective.

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1 Introduction

The origin of these notes is a course imparted by the second author in the "2$^{\text{ndo}}$ Congreso Latinoamericano de Matemáticos" celebrated in Cancun, Mexico on July 20 -26, 2004. The first redaction was subsequently elaborated by the authors.

The theme of the course was the local study of analytic subsets of $\mathbf{C}^n$, which is the local study of reduced complex analytic spaces. That is, we will consider subsets defined in a neighbourhood of a point $0 \in \mathbf{C}^n$ by equations:

$$f_1(z_1, \ldots, z_n) = \cdots = f_k(z_1, \ldots, z_n) = 0$$

$$f_i \in \mathbf{C}\{z_1, \ldots, z_n\},\ f_i(0) = 0,\ i = 1, \ldots, k.$$

Meaning that the subset $X \subset \mathbf{C}^n$ is thus defined in a neighbourhood U of 0, where all the series f_i converge. Throughout this text, the word "local" means that we work with "sufficiently small" representatives of a germ (X, x).

The purpose of the course was to show how to *stratify* X. In other words, partition X into finitely many **nonsingular** complex analytic manifolds $\{X_\alpha\}_{\alpha \in A}$, which will be called **strata**, such that:

i) The closure $\overline{X_\alpha}$ is a closed complex analytic subspace of X, for all $\alpha \in A$.

ii) $\overline{X_\alpha} \setminus X_\alpha$ is a union of strata X_β for all $\alpha \in A$.

iii) Given any $x \in X_\alpha$, the "geometry" of all the closures $\overline{X_\beta}$ containing X_α is locally constant along X_α in a neighbourhood of x.

If by "geometry" we mean the embedded local topological type at $x \in X_\alpha$ of $\overline{X_\beta} \subset \mathbf{C}^n$ and its sections by affine subspaces of $\mathbf{C}^n$ of general directions passing near x or through x, which we call the *total local topological type*, there is a **minimal** such partition, in the sense that any other partition with the same properties must be a sub-partition of it. Characterized by differential-geometric conditions, called Whitney conditions, bearing on limits of tangent spaces and of secants, it plays an important role in generalizing to singular spaces geometric concepts such as Chern classes and integrals of curvature. The existence of such partitions, or stratifications, without proof of the existence of a minimal one, and indeed the very concept of stratification[1], are originally due to Whitney in [Whi1], [Whi2]. In these papers Whitney also initated the study in complex analytic geometry of limits of tangent spaces at non singular points. In algebraic geometry the first general aproach to such limits is due to Semple in [Se].

In addition to topological and differential-geometric characterizations, the partition can also be described algebraically by means of **Polar Varieties**, and this is one of the main points of these lectures.

Apart from the characterization of Whitney conditions by equimultiplicity of polar varieties, one of the main results appearing in these lectures is therefore the equivalence of Whitney conditions for a stratification $X = \bigcup_\alpha X_\alpha$ of a complex analytic space $X \subset \mathbf{C}^n$ with the local topological triviality of the closures $\overline{X_\beta}$ of strata along each X_α which they contain, as well as the local topological triviality along X_α of the intersections of the $\overline{X_\beta}$ with (germs of) general nonsingular subspaces of $\mathbf{C}^n$ containing X_α.

Other facts concerning Whitney conditions also appear in these notes, for example that the Whitney conditions are in fact of a lagrangian nature, related

[1] Which was subsequently developed, in particular by Thom in [Th] and Mather in [Ma].

to a condition of relative projective duality between the irreducible components of the normal cones of the $\overline{X_\beta}$ along the X_α and of some of their subcones, on the one hand, and the irreducible components of the space of limits of tangents hyperplanes to $\overline{X_\beta}$ at nonsingular points approaching X_α, on the other. This duality, applied to the case where X_α is a point, gives a measure of the geometric difference between a germ of singular space at a point and its tangent cone at that point, assumed to be reduced. Among the important facts concerning polar varieties of a germ (X, x) is that their multiplicity at a point depends only on the *total local topological type* of the germ.

Applying this to the cone over a projective variety V gives a formula for the degree of the dual variety, assumed to be a hypersurface, which depends only on the local topological characters of the incidence between the strata of the minimal Whitney stratification of V and the Euler characteristics of these strata and their general linear sections. In particular we recover with topological arguments the formula for the class of a projective hypersurface with isolated singularities.

The original idea of the course was to be as geometric as possible. Since many proofs in this story are quite algebraic, using in particular the notion of integral dependence on ideals and modules (see [Te3], [Ga1]), they are often replaced by references. Note also that in this text, intersections with linear subspaces of the nonsingular ambient space are taken as reduced intersections.

We shall begin by trying to put into historical context the appearance of polar varieties, as a means to give the reader a little insight and intuition into what we will be doing. A part of what follows is taken from [Te5]; see also [Pi2].

It is possible that the first example of a polar variety is in the treatise on conics of Apollonius. The cone drawn from a point 0 in affine three-space outside of a fixed sphere around that sphere meets the sphere along a circle C. If we consider a plane not containing the point, and the projection π from 0 of the affine three-space onto that plane, the circle C is the set of critical points of the restriction of π to the sphere. Fixing a plane H, by moving the point 0, we can obtain any circle drawn on the sphere, the great circles beeing obtained by sending the point to infinity in the direction perpendicular to the plane of the circle.

Somewhat later, around 1680, John Wallis asked how many tangents can be drawn to a nonsingular curve of degree d in the plane from a point in that plane and conjectured that this number should always be $\leq d^2$. In modern terms, he was proposing to compare the visual complexity of a curve (or surface) as measured by the number of "critical" lines of sight with its algebraic complexity as measured by the degree. Given an algebraic surface S of degree d in affine three-space and a point 0 outside it, the lines emanating from 0 and tangent to S touch S along an algebraic curve P. Taking a general hyperplane H through 0, we see that the number of tangents drawn from 0 to the curve $S \cap H$ is the number of points in $P \cap H$ and therefore is bounded by the degree of the algebraic curve P drawn on S. This algebraic curve is an example of a *polar curve*; it is the generalization of Apollonius' circles. Wallis' question was answered by

Poncelet, who saw (without writing a single equation; see [Pon, p. 361 and ff.]) that the natural setting for the problem which had been stated in the affine real domain by Wallis was the complex projective plane and that the number of tangents drawn from a point with projective coordinates $(\xi : \eta : \zeta)$ to the curve C with homogeneous equation $F(x, y, z) = 0$ is equal to the number of intersection points of C with a curve of degree $d-1$. The equation of this curve was written explicitely later by Plücker (see [Pl]):

$$\xi\frac{\partial F}{\partial x} + \eta\frac{\partial F}{\partial y} + \zeta\frac{\partial F}{\partial z} = 0. \tag{1}$$

This equation is obtained by *polarizing* the polynomial $F(x, y, z)$ with respect to $(\xi : \eta : \zeta)$, a terminology which comes from the study of conics; it is the method for obtaining a bilinear form from a quadratic form (in characteristic $\neq 2$).
It is the polar curve of C with respect to the point $(\xi : \eta : \zeta)$, or rather, in the terminology of [Te3], the projective curve associated to the *relative polar surface* of the map $(\mathbf{C}^3, 0) \to (\mathbf{C}^2, 0)$ given by $(F, \xi x + \eta y + \zeta z)$. The term emphasizes that it is attached to a morphism, unlike the polar varieties à la Todd, which in this case would be the points on the curve C where the tangent line contains the point $(\xi : \eta : \zeta)$. In any case, it is of degree $d - 1$, where d is the degree of the polynomial F and by Bézout's theorem, except in the case where the curve C is not reduced, i.e., has multiple components, the number of intersection points counted with multiplicities is exactly $d(d - 1)$. So we conclude with Poncelet that the number (counted with multiplicities) of points of the nonsingular curve C where the tangent goes through the point with coordinates $(\xi : \eta : \zeta)$ is equal to $d(d - 1)$. The equations written by Plücker shows that, as the point varies in the projective plane, the equations (1) describe a *linear system* of curves of degree $d-1$ in the plane, which cuts out a linear system of points on the curve C, of degree $d(d - 1)$. It is the most natural linear system after that which comes from the lines (hyperplanes)

$$\lambda x + \mu y + \nu z = 0, \quad (\lambda : \mu : \nu) \in \mathbf{P}^2$$

and comes from the linear system of points in the dual space $\check{\mathbf{P}}^2$ while the linear system (1) can be seen as coming from the linear system of lines in $\check{\mathbf{P}}^2$. The projective duality between $\mathbf{P}^2$ and the space $\check{\mathbf{P}}^2$ of lines in $\mathbf{P}^2$ exchanges the two linear systems. The *dual curve* $\check{C} \in \check{\mathbf{P}}^2$ is the closure of the set of points in $\check{\mathbf{P}}^2$ corresponding to lines in $\mathbf{P}^2$ which are tangent to C at a nonsingular point. Its degree is called the *class* of the curve C. It is the number of intersection points of $\check{C}$ with a general line of $\check{\mathbf{P}}^2$, and that, by construction, is the number of tangents to C passing through a given general point of $\mathbf{P}^2$.

In the theory of algebraic curves, there is an important formula called the Riemann-Hurwitz formula. Given an algebraic map $f : C \to C'$ between compact nonsingular complex algebraic curves, which is of degree $\deg f = d$ (meaning that for a general point $c' \in C'$, $f^{-1}(c')$ consists of d points, and is ramified at the points $x_i \in C,\ 1 \leq i \leq r$, which means that near x_i, in suitable local coordinates on C and C', the map f is of the form $t \mapsto t^{e_i+1}$ with $e_i \in \mathbf{N},\ \ e_i \geq 1$.

The integer e_i is the *ramification index* of f at x_i. Then we have the Riemann-Hurwitz formula relating the genus of C and the genus of C' via f and the ramification indices:

$$2g(C) - 2 = d(2g(C') - 2) + \sum_i e_i.$$

If we apply this formula to the case $C' = \mathbf{P}^1$, knowing that any compact algebraic curve is a finite ramified covering of $\mathbf{P}^1$, we find that we can calculate the genus of C from any linear system of points made of the fibers of a map $C \to \mathbf{P}^1$ if we know its degree and its singularities: we get

$$2g(C) = 2 - 2d + \sum e_i.$$

The ramification points x_i can be computed as the so-called *jacobian divisor* of the linear system, which consists of the singular points, properly counted, of the singular members of the linear system. In particular if C is a plane curve and the linear system is the system of its plane sections by lines through a general point $x = (\xi : \eta : \zeta)$ of $\mathbf{P}^2$, the map f is the projection from C to $\mathbf{P}^1$ from x; its degree is the degree m of C and its ramification points are exactly the points where the line from x is tangent to C. Since x is general, these are simple tangency points, so the e_i are equal to 1, and their number is equal to the class $\check{m}$ of C; the formula gives

$$2g(C) - 2 = -2m + \check{m}\ ,$$

thus giving for the genus an expression linear in the degree and the class.

This is the first example of the relation between the "characteristic classes" (in this case only the genus) and the polar classes; in this case the curve itself, of degree m and the degree of the polar locus, or apparent contour from x, in this case the class $\check{m}$. After deep work by Zeuthen, C. Segre, Severi, it was Todd who in three fundamental papers ([To1], [To2], [To3]) found the correct generalization of the formulas known for curves and surfaces.

More precisely, given a nonsingular d-dimensional variety V in the complex projective space $\mathbf{P}^{n-1}$, for a linear subspace $D \subset \mathbf{P}^{n-1}$ of dimension $n-d+k-3$, i.e., of codimension $d-k+2$, with $0 \leq k \leq d$, let us set

$$P_k(V; D) = \{v \in V / \dim(T_{V,v} \cap D) \geq k-1\}.$$

This is the *Polar variety* of V associated to D; if D is general, it is either empty or purely of codimension k in V. If $n = 3$, $d = 1$ and $k = 1$, we find the points of the projective plane curve V where the tangent lines go through the point $D \in \mathbf{P}^2$. A tangent hyperplane to V at a point v is a hyperplane containing the tangent space $T_{V,v}$. The polar variety $P_k(V, D)$ with respect to a general D of codimension $d-k+2$ consists of the points of V where a tangent hyperplane contains D, a condition which is equivalent to the dimension inequality. We see that this construction is a direct generalization of the apparent contour. The eye 0 is replaced by the linear subspace D!

Todd shows that (rational equivalence classes of) [2] the following formal linear combinations of varieties of codimension k, for $0 \leq k \leq d$:

$$V_k = \sum_{j=0}^{j=k} (-1)^j \binom{d-k+j+1}{j} P_{k-j}(V; D_{d-k+j+2}) \cap H_j,$$

where H_j is a linear subspace of codimension j and $D_{d-k+j+2}$ is of codimension $d-k+j+2$, are independent of all the choices made and of the embedding of V in a projective space, provided that the $D's$ and the H's have been chosen general enough. Our V_k are in fact Todd's V_{d-k}. The intersection numbers arising from intersecting these classes and hyperplanes in the right way to obtain numbers contain a wealth of numerical invariants, such as Euler characteristic and genus. Even the arithmetic genus, which is the generalization of the differential forms definition of the genus of a curve, can be computed. Around 1950 it was realized that the classes of Todd, which had also been considered independently by Eger, are nothing but the Chern classes of the tangent bundle of V.

On the other hand, the basic topological invariant of the variety V, its Euler-Poincaré characteristic (also called Euler characteristic for short) satisfies the equality:

$$\chi(V) = \deg V_d = \sum_{j=0}^{d} (-1)^j (j+1)(P_{d-j}(V).H_j), \tag{E}$$

where $P_{d-j}(V)$ is the polar variety of codimension $d-j$ with respect to a general D of codimension $j+2$, which is omitted from the notation, and $(a.b)$ denotes the intersection number in $\mathbf{P}^{n-1}$. In this case, since we intersect with a linear space of complementary dimension, $(P_{d-j}(V).H_j)$ is just the degree of the projective variety $P_{d-j}(V)$.

So Todd's results give a rather complete generalization of the genus formula for curves, both in its analytic and its topological aspects. This circle of ideas was considerably extended, in a cohomological framework, to generalized notions of genus and characteristic classes for nonsingular varieties; see [H] and [Po, Chapters 48,49]. Todd's construction was modernized and extended to the case of a singular projective variety by R. Piene (see [Pi]).

What we use here is a local form, introduced in [L-T1], of the polar varieties of Todd, adapted to the singular case and defined for any equidimensional and reduced germ of a complex analytic space. The case of a singular projective variety which we have just seen can be deemed to be the special case where our singularity is a cone.

We do not take classes in (Borel-Moore) homology or elsewhere because the loss of geometric information is too great, but instead look at "sufficiently general" polar varieties of a given dimension as geometric objects. The hope is that the equisingularity class (up to a Whitney equisingular deformation) of the general polar varieties of a germ is an analytic invariant.[3] What is known

[2] Which he invents for the occasion.

[3] Indeed, the statement at the end of remark 3.2 in [Te3] should be entitled "problem" and not "theorem".

is that the multiplicity is, and this is what we use below.

Since the stratification we build is determined by local conditions and is canonical, the stratifications defined in the open subsets of a covering of a complex analytic space X will automatically glue up. Therefore it suffices to study the stratifications locally assuming $X \subset \mathbf{C}^n$, as we do here. We emphasize that the result of the construction for X, unlike its tools, is independent of the embedding $X \subset \mathbf{C}^n$.

2 Limits of tangent spaces, the conormal space and the tangent cone.

To set the working grounds, let us fix a reduced and pure-dimensional germ of analytic subspace $(X, 0) \subset (\mathbf{C}^n, 0)$. That is, we are assuming that X is given to us by an ideal I of $\mathbf{C}\{z_1, \ldots, z_n\}$ generated say by $(f_1, \ldots, f_k)$, containing all analytic functions vanishing on X, and also that all the irreducible components of X, correponding to the minimal prime ideals of $\mathbf{C}\{z_1, \ldots, z_n\}$ which contain I, have the same dimension d.

By definition, a singular point of a complex analytic space is a point where the tangent space cannot be defined as usual. However, as a substitute, we can look at all limit positions of tangent spaces at nonsingular points tending to a given singular point.

Definition 2.1. *Given a closed d-dimensional analytic subset X in an open set of $\mathbf{C}^n$, a d-plane T of the Grassmannian $G(d, n)$ of d-dimensional vector subspaces of $\mathbf{C}^n$ is a limit at $x \in X$ of tangent spaces to the analytic space X if there exists a sequence $\{x_i\}$ of nonsingular points of X and a sequence of d-planes $\{T_i\}$ of $G(d, n)$ such that for all i, the d-plane T_i is the direction of the tangent space to X at x_i, the sequence $\{x_i\}$ converges to x and the sequence $\{T_i\}$ converges to T.*

How can we determine these limit positions? Recall that if X is an analytic space then $Sing\, X$, the set of singular points of X, is also an analytic space and the nonsingular part $X^0 = X \backslash Sing\, X$ is dense in X and has the structure of a complex manifold.

Let X be a representative of $(X, 0)$, consider the application (the Gauss map)

$$\begin{aligned} \gamma_{X^0} : X^0 &\longrightarrow G(d, n) \\ x &\longmapsto T_x X^0, \end{aligned}$$

where $T_x X^0$ denotes the direction in $G(d, n)$ of the tangent space to the manifold X^0 at the point x. Let NX be the closure of the graph of γ_{X^0} in $X \times G(d, n)$. It can be proved that NX is an analytic subspace of dimension d ([Whi], theorem 16.4).

Definition 2.2. *The morphism $\nu_X : NX \longrightarrow X$ induced by the first projection of $X \times G(d,n)$, is called the Semple-Nash Modification of X.*

$$\begin{tikzcd} & NX \subset X \times G(d,n) \arrow[dl, "\nu_X"'] \arrow[dr, "\gamma_X"] & \\ X & & G(d,n) \end{tikzcd}$$

It is an isomorphism over the nonsingular part of X and is proper since the Grassmannian is compact and the projection $X \times G(d,n) \to X$ is proper. It is therefore a proper birational map. It seems to have been first introduced by Semple (see the end of [Se]) who also asked whether iterating this construction would eventually resolve the singularities of X, and later rediscovered by Nash who asked the same question. It is still without answer except for curves.

The notation NX is justified by the fact that the Semple-Nash transformation is independent, up to a unique X-isomorphism, of the embedding of X into a nonsingular space. See [Te2, §2], where the abstract construction of the Semple-Nash modification is explained in terms of the Grothendieck Grassmannian associated to the module of differentials of X. The fiber $\nu_X^{-1}(0)$ is a closed algebraic subvariety of $G(d,n)$; set-theoretically, it is the set of limit positions of tangent spaces at points of X^0 tending to 0.
In [Hn], H. Hennings has announced a proof of the fact that if x is an isolated singular point of X, the dimension of $\nu_X^{-1}(x)$ is $\dim X - 1$, generalizing a result of A. Simis, K. Smith and B. Ulrich in [SSU].

The Semple-Nash modification, however, is a little difficult to handle because of the fact that the rich geometry of the Grassmannian entails somewhat cumbersone computations. There is a less intrinsic but more amenable way of encoding the limits of tangent spaces. The idea is to replace a tangent space to X^0 by the collection of all the hyperplanes of $\mathbf{C}^n$ which contain it. Tangent hyperplanes live in a projective space, namely the dual projective space $\check{\mathbf{P}}^{n-1}$, which is easier to deal with than the Grassmannian.

2.1 Some symplectic Geometry

In order to describe this set of tangent hyperplanes, we are going to use the language of symplectic geometry and lagrangian submanifolds. So let us start with a few definitions.

Let M be any n-dimensional manifold, and let ω be a de Rham 2-form on M, that is, for each $p \in M$, the map

$$\omega_p : T_pM \times T_pM \to \mathbf{R}$$

is skew-symmetric bilinear on the tangent space to M at p, and ω_p varies smoothly with p. We say that ω is **symplectic** if it is closed and ω_p is non-degenerate for all $p \in M$. A **symplectic manifold** is a pair (M, ω), where M

is a manifold and ω is a symplectic form.

Now, for any manifold M, its cotangent bundle T^*M has a canonical symplectic structure as follows. Let

$$\pi : T^*M \longrightarrow M$$
$$p = (x, \xi) \longmapsto x,$$

where $\xi \in T^*_x M$, be the natural projection. The **Liouville 1-form** α on T^*M may be defined pointwise by:

$$\alpha_p(v) = \xi\left((d\pi_p)v\right), \quad \text{for } v \in T_p(T^*M).$$

Note that $d\pi_p : T_p(T^*M) \to T_x M$, so that α is well defined. Then, the **canonical symplectic 2-form** ω on T^*M is defined as

$$\omega = -d\alpha.$$

And it is not hard to see, that if $(U, x_1, \ldots, x_n)$ is a coordinate chart for M with associated cotangent coordinates $(T^*U, x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$, then locally:

$$\omega = \sum_{i=1}^{n} dx_i \wedge d\xi_i.$$

Definition 2.3. *Let (M, ω) be a 2n-dimensional symplectic manifold. A submanifold Y of M is a **lagrangian submanifold** if, at each $p \in Y$, T_pY is a lagrangian subspace of T_pM, i.e., $\omega_p|_{T_pY} \equiv 0$ and $\dim T_pY = \frac{1}{2}\dim T_pM$. Equivalently, if $i : Y \hookrightarrow M$ is the inclusion map, then Y is **lagrangian** if and only if $i^*\omega = 0$ and $\dim Y = \frac{1}{2}\dim M$. If Y has singularities, we say that it is a **lagrangian subspace** of M if it is purely of dimension $\frac{1}{2}\dim M$ and the nonsingular part of the corresponding reduced subspace is a lagrangian submaniforld.*

Example 2.1. *The **zero section** of T^*M*

$$X := \{(x, \xi) \in T^*M | \xi = 0 \textit{ in } T^*_x M\}$$

is an n-dimensional lagrangian submanifold of T^*M.

2.2 Conormal space.

Let now $X \subset M$ be a possibly singular complex subspace of pure dimension d, and let as before $X^0 = X \backslash Sing\, X$, be the nonsingular part of X, so it is a submanifold of M.

Definition 2.4. *Set*

$$N^*_x X^0 = \{\xi \in T^*_x M | \xi(v) = 0, \ \forall v \in T_x X^0\};$$

*this means that the hyperplane $\{\xi = 0\}$ contains the tangent space to X^0 at x. The **conormal bundle** of X^0 is*

$$T^*_{X^0} M = \{(x, \xi) \in T^*M | x \in X^0, \ \xi \in N^*_x X^0\}.$$

Proposition 2.1. *Let* $i : T^*_{X^0}M \hookrightarrow T^*M$ *be the inclusion, and let* α *be the Liouville 1-form in* T^*M *as before. Then* $i^*\alpha = 0$. *In particular the conormal bundle* $T^*_{X^0}M$ *is a conic lagrangian submanifold of* T^*M, *and has dimension* n.

Proof.
See [CdS], Proposition 3.6. □

In the same context, we can define the **conormal space of X in M**, denoted as $T^*_X M$, as the closure of $T^*_{X^0}M$ in T^*M, with the **conormal map** $\kappa_X : T^*_X M \to X$, induced by the natural projection $\pi : T^*M \to M$. The conormal space may be singular, but it is of dimension n, and by proposition 2.1, α vanishes on every tangent vector at a nonsingular point, so it is by construction a lagrangian subspace of T^*M.

In words, the fiber of the conormal map $\kappa_X\colon T^*_X M \to X$ above a point $x \in X$ consists, if $x \in X^0$, of all the equations of hyperplanes tangent to X at x, in the sense that they contain the tangent space $T_x X^0$. If x is a singular point, the fiber consists of all equations of limits of hyperplane directions tangent at nonsingular points of X tending to x. In addition, if $x \in X^0$, the fiber $\kappa_X^{-1}(x)$ is isomorphic to $\mathbf{C}^{n-d}$, the space of linear forms on $\mathbf{C}^n$ vanishing on $T_x X^0$.

The fibers of κ_X are invariant under multiplication by an element of $\mathbf{C}^*$, and we can divide by the equivalence relation this defines. The idea is to remember only the defining forms up to homothety of tangent hyperplanes, and not a specific linear form defining it. That is, the conormal stable by vertical homotheties (a property also called **conical**), so we can "projectivize" it. Moreover, we can characterize those subvarieties of the cotangent space which are the conormal spaces of their images in M.

Proposition 2.2. (see [P]) *Let* M *be a nonsingular analytic variety of dimension* n *and let* L *be a closed conical irreducible analytic subvariety of* T^*M. *The following conditions are equivalent:*
1) The variety L *is the conormal space of its image in* M.
2) The Liouville 1-form α *vanishes on all tangent vectors to* L *at every nonsingular point of* L.
3) The symplectic 2-form $\omega = -\mathrm{d}\alpha$ *vanishes on every pair of tangent vectors to* L *at every nonsingular point of* L.

Since conormal varieties are conical, we may as well projectivize with respect to vertical homotheties of T^*M and work in $\mathbf{P}T^*M$, where it still makes sense to be lagrangian since α is homogeneous by definition[4].

Now, going back to our original problem we have $X \subset M = \mathbf{C}^n$, so $T^*M = \mathbf{C}^n \times \check{\mathbf{C}}^n$ and $\mathbf{P}T^*M = \mathbf{C}^n \times \check{\mathbf{P}}^{n-1}$. So we have the **(projective) conormal space** $\kappa_X : C(X) \to X$ with $C(X) \subset X \times \check{\mathbf{P}}^{n-1}$, where $C(X)$ denotes the projectivization of the conormal space $T^*_X M$. Note that we have not changed

[4] In symplectic geometry it is called **Legendrian** with respect to the natural contact structure on $\mathbf{P}T^*M$.

the name of the map κ_X after projectivizing since there is no ambiguity, and that the dimension of $C(X)$ is $n-1$, which shows immediately that it depends on the embedding of X in an affine space. We have the following result:

Proposition 2.3. *The (projective) conormal space $C(X)$ is a closed, reduced, complex analytic subspace of $X \times \check{\mathbf{P}}^{n-1}$ of dimension $n-1$. For any $x \in X$ the dimension of the fiber $\kappa_X^{-1}(x)$ is at most $n-2$.*

Proof.
These are classical facts. See [CdS] or [Te3], proposition 4.1, p. 379. □

Now we are going to describe the relation between the conormal space of $(X, 0) \subset (\mathbf{C}^n, 0)$ and its Semple-Nash modification. It is convenient here to use the notations of projective duality of linear spaces. Given a vector subspace $T \subset \mathbf{C}^n$ we denote by $\mathbf{P}T$ its projectivization, i.e., the image of $T \setminus \{0\}$ by the projection $\mathbf{C}^n \setminus \{0\} \to \mathbf{P}^{n-1}$ and by $\check{T} \subset \check{\mathbf{P}}^{n-1}$ the projective dual of $\mathbf{P}T \subset \mathbf{P}^{n-1}$, which is a $\mathbf{P}^{n-d-1} \subset \check{\mathbf{P}}^{n-1}$, the set of all hyperplanes H of $\mathbf{P}^{n-1}$ containing $\mathbf{P}T$. We denote by $\check{\Xi} \subset G(d, n) \times \check{\mathbf{P}}^{n-1}$ the cotautological $\mathbf{P}^{n-d-1}$-bundle over $G(d, n)$, that is $\check{\Xi} = \{(T, H) \mid T \in G(d, n),\ H \in \check{T} \subset \check{\mathbf{P}}^{n-1}\}$, and consider the intersection

$$\begin{tikzcd}
E := (X \times \check{\Xi}) \cap (NX \times \check{\mathbf{P}}^{n-1}) \arrow[r, hook] \arrow[d, "p_1"'] \arrow[dr, "p_2"] & X \times G(d, n) \times \check{\mathbf{P}}^{n-1} \arrow[d] \\
NX & X \times \check{\mathbf{P}}^{n-1}
\end{tikzcd}$$

and the morphism p_2 induced on E by the projection onto $X \times \check{\mathbf{P}}^{n-1}$. We then have the following:

Proposition 2.4. *Let $p_2 : E \to X \times \check{\mathbf{P}}^{n-1}$ be as before. The set-theoretical image $p_2(E)$ of the morphism p_2 coincides with the conormal space of X in $\mathbf{C}^n$*

$$C(X) \subset X \times \check{\mathbf{P}}^{n-1}.$$

.

Proof. By definition, the conormal space of X in $\mathbf{C}^n$ is an analytic space $C(X) \subset X \times \check{\mathbf{P}}^{n-1}$, together with a proper analytic map $\kappa_X : C(X) \to X$, where the fiber over a smooth point $x \in X^0$ is the set of tangent hyperplanes, that is the hyperplanes H containing the direction of the tangent space T_xX. That is, if we define $E^0 = \{(x, T_xX, H) \in E \,|\, x \in X^0, H \in \check{T}_xX\}$, then by construction $E^0 = p_1^{-1}(\nu^{-1}(X^0))$, and $p_2(E^0) = C(X^0)$. Since the morphism p_2 is proper, in particular it is closed, which finishes the proof. □

Corollary 2.1. *A hyperplane $H \in \check{\mathbf{P}}^{n-1}$ is a limit of tangent hyperplanes to X at 0, i.e., $H \in \kappa_X^{-1}(0)$, if and only if there exists a d-plane $(0, T) \in \nu_X^{-1}(0)$ such that $T \subset H$.*

Proof. Let $(0,T) \in \nu_X^{-1}(0)$ be a limit of tangent spaces to X at 0. By construction of E and proposition 2.4, every hyperplane H containing T is in the fiber $\kappa_X^{-1}(0)$, and so is a limit at 0 of tangent hyperplanes to X^0.
On the other hand, by construction, for any hyperplane $H \in \kappa_X^{-1}(0)$ there is a sequence of points $\{(x_i, H_i)\}_{i \in \mathbf{N}}$ in $\kappa_X^{-1}(X^0)$ converging to $p = (0, H)$. Since the map p_2 is surjective, by definition of E, we have a sequence $(x_i, T_i, H_i) \in E^0$ with $T_i = T_{x_i}X^0 \subset H_i$. By compactness of Grassmannians and projective spaces, this sequence has to converge, up to taking a subsequence, to (x, T, H) with T a limit at x of tangent spaces to X. Since inclusion is a closed condition, we have $T \subset H$. □

Corollary 2.2. *The morphism $p_1 : E \to NX$ is a locally analytically trivial fiber bundle with fiber $\mathbf{P}^{n-d-1}$.*

Proof. By definition of E, the fiber of the projection p_1 over a point $(x,T) \in NX$ is the set of all hyperplanes in $\mathbf{P}^{n-1}$ containing $\mathbf{P}T$. In fact, the tangent bundle T_{X^0}, lifted to NX by the isomorphism $NX^0 \simeq X^0$, extends to a fiber bundle over NX, called the Nash tangent bundle of X. It is the pull-back by γ_X of the tautological bundle of $G(d,n)$, and E is the total space of the $\mathbf{P}^{n-d-1}$-bundle of the projective duals of the projectivized fibers of the Nash bundle. □

By definition of E, the map p_2 is an isomorphism over $C(X^0)$. In general the fiber of p_2 over a point $(x,H) \in C(X)$ is the set of limit directions at x of tangent spaces to X that are contained in H. If X is a hypersurface, the conormal map coincides with the Semple-Nash modification. In general, while it follows from proposition 2.4 that the geometric structure of the inclusion $\kappa_X^{-1}(x) \subset \check{\mathbf{P}}^{n-1}$ determines the set of limit positions of tangent spaces, i.e., the fiber $\nu_X^{-1}(x)$ of the Semple-Nash modification, the correspondence is not so simple: by proposition 2.4 and its corollary, the points of $\nu_X^{-1}(x)$ correspond to some of the projective subspaces $\mathbf{P}^{n-d-1}$ of $\check{\mathbf{P}}^{n-1}$ contained in $\kappa_X^{-1}(x)$.

2.3 Conormal spaces and projective duality

Let us assume for a moment that $V \subset \mathbf{P}^{n-1}$ is a projective algebraic variety. In the spirit of last section, let us take $M = \mathbf{P}^{n-1}$ with coordinates $(x_0 : \ldots : x_{n-1})$, and consider the dual projective space $\check{\mathbf{P}}^{n-1}$ with coordinates $(\xi_0 : \ldots : \xi_{n-1})$; its points are the hyperplanes of $\mathbf{P}^{n-1}$ with equations $\sum_{i=0}^{n-1} \xi_i x_i = 0$.

Definition 2.5. *Define the **incidence variety** $I \subset \mathbf{P}^{n-1} \times \check{\mathbf{P}}^{n-1}$ as the set of points satisfying:*

$$\sum_{i=0}^{n-1} x_i \xi_i = 0,$$

where $(x_0 : \ldots : x_{n-1}; \xi_0 : \ldots : \xi_{n-1}) \in \mathbf{P}^{n-1} \times \check{\mathbf{P}}^{n-1}$

Lemma 2.1. (Kleiman; see [Kl2]) *The projectivized cotangent bundle of $\mathbf{P}^{n-1}$ is naturally isomorphic to I.*

Proof.
Let us first take a look at the cotangent bundle of $\mathbf{P}^{n-1}$:

$$\pi : T^*\mathbf{P}^{n-1} \longrightarrow \mathbf{P}^{n-1}.$$

Remember that the fiber $\pi^{-1}(x)$ over a point x in $\mathbf{P}^{n-1}$ is by definition isomorphic to $\check{\mathbf{C}}^{n-1}$, that is, the vector space of linear forms over $\mathbf{C}^{n-1}$. Recall that projectivizing the cotangent bundle means projectivizing the fibers, and so we get a map:

$$\Pi : \mathbf{P}T^*\mathbf{P}^{n-1} \longrightarrow \mathbf{P}^{n-1}$$

where the fiber is isomorphic to $\check{\mathbf{P}}^{n-2}$. So we can see a point of $\mathbf{P}T^*\mathbf{P}^{n-1}$ as a pair $(x,\chi) \in \mathbf{P}^{n-1} \times \check{\mathbf{P}}^{n-2}$. On the other hand, if we fix a point $(x) \in \mathbf{P}^{n-1}$, then the equation defining the incidence variety I, tells us that for a fixed (x), the set of points $((x),(\xi)) \in I$ is the set of hyperplanes of $\mathbf{P}^{n-1}$ that go through the point (x), which we know is isomorphic to $\check{\mathbf{P}}^{n-2}$.

Now to explicitly define the map, take a chart $\mathbf{C}^{n-1} \times \left\{\check{\mathbf{C}}^{n-1} \setminus \{0\}\right\}$ of the manifold $T^*\mathbf{P}^{n-1} \setminus \{\text{zero section}\}$, where the $\mathbf{C}^{n-1}$ corresponds to a usual chart of $\mathbf{P}^{n-1}$ and $\check{\mathbf{C}}^{n-1}$ to its associated cotangent chart. Define the map:

$$\begin{aligned}
&\phi_i : \mathbf{C}^{n-1} \times \left\{\check{\mathbf{C}}^{n-1} \setminus \{0\}\right\} \longrightarrow \mathbf{P}^{n-2} \times \check{\mathbf{P}}^{n-2} \\
&(z_1,\ldots,z_{n-1};\xi_1,\ldots,\xi_{n-1}) \longmapsto \left(\varphi_i(z), (\xi_1 : \cdots : \xi_{i-1} : -\sum_{j=1}^{n-1*_i} z_j\xi_j : \xi_{i+1} : \cdots : \xi_{n-1})\right)
\end{aligned}$$

where $\varphi_i(z) = (z_1 : \cdots : z_{i-1} : 1 : z_{i+1} : \cdots : z_{n-1})$ and the star means that the index i is excluded from the sum.

An easy calculation shows that ϕ_i is injective, has its image in the incidence variety I and is well defined on the projectivization $\mathbf{C}^{n-1} \times \check{\mathbf{P}}^{n-2}$. It is also clear, that varying i from 1 to $n-1$ we can reach any point in I. Thus, all we need to check now is that the ϕ_j's paste together to define a map. For this, the important thing is to remember that if φ_i and φ_j are charts of a manifold, and $h := \varphi_j^{-1}\varphi_i = (h_1,\ldots,h_{n-1})$ then the change of coordinates in the associated cotangent charts $\tilde{\varphi}_i$ and $\tilde{\varphi}_j$ is given by:

$$\begin{tikzcd}
 & T^*M \arrow[dr, "\tilde{\varphi_j}^{-1}"] & \\
\mathbf{C}^{n-1} \times \check{\mathbf{C}}^{n-1} \arrow[ur, "\tilde{\varphi}_i"] \arrow[rr, "h"'] & & \mathbf{C}^{n-1} \times \check{\mathbf{C}}^{n-1}
\end{tikzcd}$$

$$(x_1,\ldots,x_{n-1};\xi_1,\ldots,\xi_{n-1}) \qquad \longmapsto \qquad (h(x);(Dh^{-1}|_x)^T(\xi))$$

$\square$

Now, by lemma 2.1 the incidence variety I inherits the Liouville 1-form $\alpha(:= \sum \xi_i dx_i$ locally) from its isomorphism with $\mathbf{P}T^*\mathbf{P}^{n-1}$. But, exchanging $\mathbf{P}^{n-1}$

and $\check{\mathbf{P}}^{n-1}$, I is also isomorphic to $\mathbf{P}T^*\check{\mathbf{P}}^{n-1}$ so it also inherits the 1-form $\check{\alpha}(:= \sum x_i d\xi_i$ locally).

Lemma 2.2. (Kleiman; see [Kl2]) *Let I be the incidence variety as above. Then $\alpha + \check{\alpha} = 0$ on I.*

Proof.
Note that if the polynomial $\sum_{i=0}^{n-1} x_i\xi_i$ defined a function on $\mathbf{P}^{n-1} \times \check{\mathbf{P}}^{n-1}$, we would obtain the result by differentiating it. The idea of the proof is basically the same, it involves identifying the polynomial $\sum_{i=0}^{n-1} x_i\xi_i$ with a section of the line bundle $p^*O_{\mathbf{P}^{n-1}}(1) \otimes \check{p}^*O_{\check{\mathbf{P}}^{n-1}}(1)$ over I, where p and $\check{p}$ are the natural projections of I to $\mathbf{P}^{n-1}$ and $\check{\mathbf{P}}^{n-1}$ respectively and $O_{\mathbf{P}^{n-1}}(1)$ denotes the canonical line bundle, introducing the appropriate flat connection on this bundle, and differentiating. $\square$

In particular, this lemma tells us that if at some point $z \in I$, we have that $\alpha = 0$, then $\check{\alpha} = 0$ too. Thus, if we have a closed conical irreducible analytic subvariety of $T^*\mathbf{P}^{n-1}$ as in proposition 2.2, then it is the conormal space of its image in $\mathbf{P}^{n-1}$ if and only if it is the conormal space of its image in $\check{\mathbf{P}}^{n-1}$. So we have $\mathbf{P}T^*_V\mathbf{P}^{n-1} \subset I \subset \mathbf{P}^{n-1} \times \check{\mathbf{P}}^{n-1}$ and the restriction of the two canonical projections:

$$
\begin{tikzcd}
 & \mathbf{P}T^*_V\mathbf{P}^{n-1} \subset I \arrow[dl, "p"'] \arrow[dr, "\check{p}"] & \\
V \subset \mathbf{P}^{n-1} & & \check{\mathbf{P}}^{n-1} \supset \check{V}
\end{tikzcd}
$$

Definition 2.6. *The **dual variety** $\check{V}$ of $V \subset \mathbf{P}^{n-1}$ is the image of $\mathbf{P}T^*_V\mathbf{P}^{n-1} \subset I$ in $\check{\mathbf{P}}^{n-1}$. So by construction $\check{V}$ is the closure in $\check{\mathbf{P}}^{n-1}$ of the set of hyperplanes tangent to V^0.*

Now, by symmetry, we immediately get that $\check{\check{V}} = V$. But what is more, we see that establishing a projective duality is equivalent to finding a lagrangian subvariety in I; its images in $\mathbf{P}^{n-1}$ and $\check{\mathbf{P}}^n$ are necessarily dual.

Let us assume now that $(X, 0) \subset (\mathbf{C}^n, 0)$ is a cone over a projective algebraic variety $V \subset \mathbf{P}^{n-1}$.

Lemma 2.3. *Let $x \in X^0$ be a nonsingular point of X. Then the tangent space T_xX^0, contains the line $\mathcal{L}$ joining x to the origin. Moreover, the tangent map at x to the projection $\pi\colon X \setminus \{0\} \to V$ induces an isomorphism $T_xX^0/\mathcal{L} \simeq T_{V,\pi(x)}$.*

Proof.
This is due to Euler's identity for a homogeneous polynomial of degree m:

$$m.f = \sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i}$$

and the fact that if $\{f_1, \ldots, f_r\}$ is a set of homogeneous polynomials defining X, then $T_x X^0$ is the kernel of the matrix:

$$\begin{pmatrix} df_1 \\ \cdot \\ \cdot \\ df_r \end{pmatrix}$$

representing the differentials df_i in the basis $dx_1, \ldots, dx_n$. $\square$

It is also important to note that at all nonsingular points x of X in the same generating line the tangent space to X^0 is constant since the partial derivatives are homogeneous again. By lemma 2.3, the quotient by the generating line is the tangent space to V at the point corresponding to the generating line.

So, $\mathbf{P}T^*_X\mathbf{C}^n$ has an image in $\check{\mathbf{P}}^{n-1}$ which is the projective dual of V.

$$\begin{tikzcd}[column sep=small] & \mathbf{P}T^*_V\mathbf{P}^{n-1} \arrow[dl] \arrow[dr] & & \mathbf{P}T^*_X\mathbf{C}^n \subset \check{\mathbf{P}}^{n-1} \times \mathbf{C}^n \arrow[dl] \arrow[dr] & \\ V \subset \mathbf{P}^{n-1} & & \check{\mathbf{P}}^{n-1} \supset \check{V} & & X \subset \mathbf{C}^n \end{tikzcd}$$

The fiber over 0 of $\mathbf{P}T^*_X\mathbf{C}^n \to X$ as subvariety of $\check{\mathbf{P}}^{n-1}$, is equal to $\check{V}$: it is the set of limit positions at 0 of hyperplanes tangent to X^0.
For more information on projective duality, in addition to Kleiman's papers one can consult [Tev].

A *relative* version of the conormal space and of projective duality will play an important role in these notes. Useful references are [HMS], [Kl2], [Te3]. The relative conormal space is used in particular to define the *relative polar varieties.*

Let $f : X \to S$ be a morphism of reduced analytic spaces, with purely d-dimensional fibers and such that there exists a closed nowhere dense analytic space such that the restriction to its complement X^0 in X :

$$f|_{X^0} : X^0 \longrightarrow S$$

has all its fibers smooth. They are manifolds of dimension $d = \dim X - \dim S$. Let us assume furthermore that the map f is induced, via a closed embedding $X \subset Z$ by a smooth map $F \colon Z \to S$, This means that locally on Z the map F is analytically isomorphic to the first projection $S \times \mathbf{C}^N \to S$. Locally on X, this is always the case because we can embed the graph of f, which lies in $X \times S$, into $\mathbf{C}^N \times$. Let us denote by $\pi_F \colon T^*(Z/S) \to Z$ the relative cotangent bundle of Z/S, which is a fiber bundle whose fiber over a point $z \in Z$ is the dual $T^*_x(Z/S)$ of the tangent vector space at z to the fiber $F^{-1}(F(z))$. For $x \in X^0$, denote by M_x the submanifold $f^{-1}(f(x)) \cap (X^0)$ of X^0. Using this submanifold we will build the **conormal space of X relative to f**, denoted by $T^*_{X/S}(Z/S)$, by setting

$$N^*_x M_x = \{\xi \in T^*_x(Z/S) | \xi(v) = 0,\ \forall v \in T_x M_x\}$$

and

$$T^*_{X^0/S}(Z/S) = \{(x,\xi) \in T^*(Z/S) | x \in X^0,\ \xi \in N^*_x M_x\},$$

and finally taking the closure of $T^*_{X^0/S}(Z/S)$ in $T^*(Z/S)$, which is a complex analytic space $T^*_{X/S}(Z/S)$ by general theorems (see [Re], [Ka]). Since X^0 is dense in X, this closure maps onto X by the natural projection $\pi_F \colon T^*(Z/S) \to Z$.

Now we can projectivize with respect to the homotheties on ξ, as in the case where S is a point we have seen above. We obtain the (projectivized) relative conormal space $C_f(X) \subset \mathbf{P}T^*(Z/S)$ (also denoted by $C(X/S)$), naturally endowed with a map

$$\kappa_f \colon C_f(X) \to X.$$

We can assume that locally the map f is the restriction of the first projection to $X \subset S \times U$, where U is open in $\mathbf{C}^n$. Then we have $T^*(S \times U/S) = S \times U \times \check{\mathbf{C}}^n$ and $\mathbf{P}T^*(S \times U/S) = S \times U \times \check{\mathbf{P}}^{n-1}$. This gives an inclusion $C_f(X) \subset X \times \check{\mathbf{P}}^{n-1}$ such that κ_f is the restriction of the first projection, and a point of $C_f(X)$ is a pair (x, H), where x is a point of X and H is a limit direction at x of hyperplanes of $\mathbf{C}^n$ tangent to the fibers of the map f at points of X^0. Of course, by taking for S a point we recover the classical case studied above.

Definition 2.7. *Given a smooth morphism $F \colon Z \to S$ as above, the projection to S of $Z = S \times U$, with U open in $\mathbf{C}^n$, we shall say that a reduced complex subspace $W \subset T^*(Z/S)$ is F-**lagrangian** (or S-**lagrangian** if there is no ambiguity on F) if the fibers of the composed map $q := (\pi_F \circ F)|W \colon W \to S$ are purely of dimension $n = \dim Z - \dim S$ and the differential ω_F of the relative Liouville differential form α_F on $\mathbf{C}^n \times \check{\mathbf{C}}^n$ vanishes on all pairs of tangent vectors at smooth points of the fibers of the map q.*

With this definition it is not difficult to verify that $T^*_{X/S}(Z/S)$ is F-lagrangian, and by abuse of language we will say the same of $C_f(X)$. But we have more:

Proposition 2.5. (Lê-Teissier, see [L-T2], proposition 1.2.6) *Let $F \colon Z \to S$ be a smooth complex analytic map with fibers of dimension n. Assume that S is reduced. Let $W \subset T^*(Z/S)$ be a reduced closed complex subspace and set as above $q = \pi_F \circ F|W \colon W \to S$. Assume that the dimension of the fibers of q over points of dense open analytic subsets U_i of the irreducible components S_i of S is n.*

1. *If the Liouville form on $T^*_{F^{-1}(s)} = (\pi_F \circ F)^{-1}(s)$ vanishes on the tangent vectors at smooth points of the fibers $q^{-1}(s)$ for $s \in U_i$ and all the fibers of q are of dimension n, the Liouville form vanishes on tangent vectors at smooth points of all fibers of q.*

2. *The following conditions are equivalent:*

   - *The subspace $W \subset T^*(Z/S)$ is F-lagrangian;*

- *The fibers of q, once reduced, are all purely of dimension n and there exists a dense open subset U of S such that for $s \in U$ the fiber $q^{-1}(s)$ is reduced and is a lagrangian subvariety of $(\pi_F \circ F)^{-1}(s)$;*

  If moreover W is homogeneous with respect to homotheties on $T^*(Z/S)$, these conditions are equivalent to:

- *All fibers of q, once reduced, are purely of dimension n and each irreducible component W_j of W is equal to $T^*_{X_j/S}(Z/S)$, where $X_j = \pi_F(W_j)$.*

Assuming that W is irreducible, the gist of this proposition is that if W is, generically over S, the relative conormal of its image in Z, and if the dimension of the fibers of q is constant, then W is everywhere the relative conormal of its image. This is essentially due to the fact that the vanishing of a differential form is a closed condition on a cotangent space. In section 4.4 we shall apply this, after projectivization with respect to homotheties on $T^*(Z/S)$, to give the lagrangian characterization of Whitney conditions.

2.4 Tangent cone

At the very beginning we mentioned how the limit of tangent spaces can be thought of as a substitute for the tangent space at singular points. However there is another common substitute for the missing tangent space, the **tangent cone**.

Let us start by the geometric definition. Let $X \subset \mathbf{C}^n$ be a representative of $(X, 0)$. The canonical projection $\mathbf{C}^n \setminus \{0\} \to \mathbf{P}^{n-1}$ induces the secant map

$$s_X \colon X \setminus \{0\} \to \mathbf{P}^{n-1}, \quad x \mapsto [0x].$$

Denote by E_0X the closure in $X \times \mathbf{P}^{n-1}$ of the graph of s_X. E_0X is an analytic subspace of dimension d, and the natural projection $e_0X \colon E_0X \to X$ induced by the first projection is called the **blowing up** of 0 in X. The fiber $e_0^{-1}(0)$ is a projective subvariety of $\mathbf{P}^{n-1}$ of dimension d-1, not necessarily reduced. (See [Whi1])

Definition 2.8. *The cone with vertex 0 in $\mathbf{C}^n$ corresponding to the subset $|e_0^{-1}(0)|$ of the projective space $\mathbf{P}^{n-1}$ is the set-theoretic **tangent cone**.*

The construction shows that, set-theoretically, $e_0^{-1}(0)$ is the set of limit directions of secant lines $0x$ for points $x \in X \setminus \{0\}$ tending to 0. This means more precisely that for each sequence $(x_i)_{i\in\mathbf{N}}$ of points of $X \setminus \{0\}$, tending to 0 as $i \to \infty$ we can, since $\mathbf{P}^{n-1}$ is compact, extract a subsequence such that the directions $[0x_i]$ of the secants $0x_i$ converge. The set of such limits is the underlying set of $e_0^{-1}(0)$. (See [Whi2, Theorem 5.8])

Now, on to the algebraic definition. Let $\mathcal{O} = \mathcal{O}_{X,0} = \mathbf{C}\{z_1, \dots, z_n\}/\langle f_1, \dots, f_k\rangle$ be the local algebra of X at 0 and let $\mathfrak{m} = \mathfrak{m}_{X,0}$ be its maximal ideal. There is a natural filtration of $\mathcal{O}_{X,0}$ by the powers of $\mathfrak{m}$:

$$\mathcal{O}_{X,0} \supset \mathfrak{m} \supset \cdots \supset \mathfrak{m}^i \supset \mathfrak{m}^{i+1} \supset \cdots,$$

which is *separated* in the sense that $\bigcap_{i=o}^{\infty} \mathfrak{m}^i = (0)$ because the ring $\mathcal{O}_{X,0}$ is nœtherian.

Definition 2.9. *We define the* ***associated graded ring of*** $\mathcal{O}$ ***with respect to*** $\mathfrak{m}$*, written* $\mathrm{gr}_{\mathfrak{m}} O$ *to be the graded ring*

$$\mathrm{gr}_{\mathfrak{m}}\mathcal{O} \colon = \bigoplus_{i \geq 0} \mathfrak{m}^i/\mathfrak{m}^{i+1},$$

where $\mathfrak{m}^0 = \mathcal{O}$.

Note that $\mathrm{gr}_{\mathfrak{m}}\mathcal{O}$ is generated as $\mathbf{C}-$algebra by $\mathfrak{m}/\mathfrak{m}^2$, which is a finite dimensional vector space. Thus, $\mathrm{gr}_{\mathfrak{m}}\mathcal{O}$ is a finitely generated $\mathbf{C}-$algebra, to which we can associate a complex analytic space Specan $\mathrm{gr}_{\mathfrak{m}}\mathcal{O}$. Moreover, since $\mathrm{gr}_{\mathfrak{m}}\mathcal{O}$ is graded and finitely generated in degree one, the associated affine variety Specan $\mathrm{gr}_{\mathfrak{m}}\mathcal{O}$ is a cone. (For more on Specan, see [He-Or, Appendix I, 3.4 and Appendix III 1.2] or additionally [Ka, p. 172])

Definition 2.10. *We define the* ***tangent cone*** $C_{X,0}$ *as the complex analytic space* Specan $(\mathrm{gr}_{\mathfrak{m}}\mathcal{O})$.

We have yet to establish the relation between the geometric and algebraic definitions of the tangent cone. In order to do that we will need to introduce the **specialization of X to its tangent cone**, which is a very interesting and important construction in its own right.

Take the representative $(X, 0)$ of the germ associated to the analytic algebra $\mathcal{O}$ from above. Now, the convergent power series $f_1, \dots, f_k$, define analytic functions in a small enough polycylinder around 0, $P(\alpha) := \{z \in \mathbf{C}^n \ : |z_i| < \alpha_i\}$. Suppose additionally that the initial forms of the f_i's generate the homogeneous ideal of initial forms of elements of $I =< f_1, \dots, f_k >$.
Let $f_i = f_{m_i}(z_1, \dots, z_n) + f_{m_{i+1}}(z_1, \dots, z_n) + f_{m_{i+2}}(z_1, \dots, z_n) + \dots$, and set

$$F_i := v^{-m_i} f(vz_1, \dots, vz_n) =$$

$$f_{m_i}(z_1, \dots, z_n) + v f_{m_{i+1}}(z_1, \dots, z_n) + v^2 f_{m_{i+2}}(z_1, \dots, z_n) + \dots \in \mathbf{C}[[v, z_1, \dots, z_n]].$$

Note that the series F_i, actually converge in the domain of $\mathbf{C}^n \times \mathbf{C}$ defined by the inequalities $|vz_i| < \alpha_i$ thus defining analytic functions on this open set. Take the analytic space $\mathfrak{X} \subset \mathbf{C}^n \times \mathbf{C}$ defined by the F_i's and the analytic map defined by the projection to the t-axis.

$$\begin{array}{c} \mathfrak{X} \subset \mathbf{C}^n \times \mathbf{C} \\ \Big\downarrow p \\ \mathbf{C} \end{array}$$

So now, we have a family of analytic spaces parametrized by an open subset of the complex line $\mathbf{C}$. Note, that for $v \neq 0$, the analytic space $p^{-1}(v)$ is isomorphic to X and in fact for $v = 1$ we recover exactly the representative of the germ $(X, 0)$ with which we started. But for $v = 0$, the analytic space $p^{-1}(0)$ is the closed analytic subspace of $\mathbf{C}^n$ defined by the homogeneous ideal generated by the initial forms of elements of I.

We need a short algebraic parenthesis in order to explain the relation between this ideal of initial forms and our definition of tangent cone (definition 2.10).

2.4.1 Graded Rings and Ideals of Initial Forms

Let R be a noetherian ring, and $I \subset J \subset R$ ideals such that

$$R \supset J \supset \cdots \supset J^i \supset J^{i+1} \supset \cdots \quad .$$

is a separated filtration in the sense that $\bigcap_{i=o}^{\infty} J^i = (0)$.

Take the quotient ring $A = R/I$, define the ideal $\tilde{J}_i := (J^i + I)/I \subset A$ and consider the induced filtration

$$A \supset \tilde{J} \supset \cdots \supset \tilde{J}_i \supset \tilde{J}_{i+1} \supset \cdots .$$

Note that in fact $\tilde{J}_i = \tilde{J}^i$.

Consider now the associated graded rings

$$\mathrm{gr}_J R = \bigoplus_{i=0}^{\infty} J^i/J^{i+1},$$

$$\mathrm{gr}_{\tilde{J}} A = \bigoplus_{i=0}^{\infty} \tilde{J}_i/\tilde{J}_{i+1}.$$

Definition 2.11. *Let $f \in I$, since $\bigcap_{i=o}^{\infty} J^i = (0)$, there exists a largest natural number k such that $f \in J^k$. Define the initial form of f with respect to J as*

$$\mathrm{in}_J f := f \pmod{J^{k+1}} \in \mathrm{gr}_J R.$$

*Using this define, the **ideal of initial forms of** I as the ideal of $\mathrm{gr}_J R$ generated by the initial forms of all the element of I.*

$$\mathrm{In}_J I := < in_J f >_{f \in I} \subset \mathrm{gr}_J R.$$

Lemma 2.4. *Using the notations defined above, the following sequence is exact:*

$$0 \longrightarrow \mathrm{In}_J I \hookrightarrow \mathrm{gr}_J R \xrightarrow{\phi} \mathrm{gr}_{\tilde{J}} A \longrightarrow 0$$

that is, $\mathrm{gr}_{\tilde{J}} A \cong \mathrm{gr}_J R/\mathrm{In}_J I$.

Proof.
First of all, note that

$$\tilde{J}_i/\tilde{J}_{i+1} \cong \frac{\frac{J^i+I}{I}}{\frac{J^{i+1}+I}{I}} \cong \frac{J^i+I}{J^{i+1}+I} \cong \frac{J^i}{I \cap J^i + J^{i+1}},$$

where the first isomorphism is just the definition, the second one is one of the classical isomorphism theorems and the last one comes from the surjective map $J^i \to \frac{J^i+I}{J^{i+1}+I}$ defined by $x \mapsto x + J^{i+1} + I$. This last map tells us that there are natural surjective morphisms:

$$\begin{aligned} \varphi_i : \frac{J^i}{J^{i+1}} &\longrightarrow \frac{\tilde{J}_i}{\tilde{J}_{i+1}} \cong \frac{J^i}{I \cap J^i + J^{i+1}}, \\ x + J^{i+1} &\longmapsto x + I \cap J^i + J^{i+1}, \end{aligned}$$

which we use to define the surjective graded morphism of graded rings $\phi : \mathrm{gr}_J R \to \mathrm{gr}_{\tilde{J}} A$. Now, all that is left to prove is that the kernel of ϕ is exactly $\mathrm{In}_J I$.

Let $f \in I$ be such that $in_J f = f + J^{k+1} \in J^k/J^{k+1}$, then

$$\phi(\mathrm{in}_J f) = \varphi_k(f + J^{k+1}) = f + I \cap J^k + J^{k+1} = 0.$$

because $f \in I \cap J^k$. Since by varying $f \in I$ we get a set of generators of the ideal $\mathrm{In}_J I$, we have $\mathrm{In}_J I \subset Ker\ \phi$.

To prove the other inclusion, let $g = \bigoplus \overline{g_k} \in Ker\ \phi$, where we use the notation $\overline{g_k} := g_k + J^{k+1} \in J^k/J^{k+1}$. Then, $\phi(g) = 0$ implies by homogeneity $\phi(\overline{g_k}) = \varphi_k(g_k + J^{k+1}) = 0$ for all k. Now, suppose $\overline{g_k} \neq 0$ then

$$\varphi(g_k + J^{k+1}) = g_k + I \cap J^k + J^{k+1} = 0$$

implies $g_k = f + h$, where $0 \neq f \in (I \cap J^k) \setminus J^{k+1}$ and h belongs to J^{k+1}. But, this means that $g_k \equiv f \pmod{J^{k+1}}$, which implies $g_k + J^{k+1} = in_J f$ and concludes the proof. □

Now to relate our definition of the tangent cone with the space we obtained in our previous description of the specialization, just note that in our case the roles of R and $J \subset R$ are played by the ring of convergent power series $\mathbf{C}\{z_1, \ldots, z_n\}$, and its maximal ideal $\mathfrak{m}$ respectively, while I corresponds to the ideal $< f_1, \ldots, f_k >$ defining the germ $(X, 0) \subset (\mathbf{C}^n, 0)$ and A to its analytic algebra $O_{X,0}$.

More importantly, the graded ring $\mathrm{gr}_{\mathfrak{m}} R$, with this choice of R, is naturally isomorphic to the ring of polynomials $\mathbf{C}[z_1, \ldots, z_n]$ in such a way that definition 2.11 coincides with the usual concept of initial form of a series and tells us that

$$\mathrm{gr}_{\mathfrak{m}} \mathcal{O}_{X,0} \cong \frac{\mathbf{C}[z_1, \ldots, z_n]}{\mathrm{In}_{\mathfrak{m}} I}.$$

We would like to point out that there is a canonical way of working out the specialization in the algebraic setting that, woefully, can't be translated word for word into the analytic case, but that, nonetheless takes us to a weaker statement. Suppose that X is an algebraic variety, that is, the f_i's are polynomials in $z_1, \dots, z_n$, and consider the *extended Rees Algebra*[5] of $\mathcal{O} = \mathcal{O}_{X,0}$ with respect to $\mathfrak{m}$. (See [Za, Appendix], or [Eis, section 6.5])

$$\mathcal{R} = \bigoplus_{i \in \mathbf{Z}} \mathfrak{m}^i v^{-i} \subset \mathcal{O}[v, v^{-1}],$$

where $\mathfrak{m}^i = \mathcal{O}$ for $i \leq 0$. Note that $\mathcal{R} \supset \mathcal{O}[v] \supset \mathbf{C}[v]$, in fact it is a finitely generated $\mathcal{O}-$algebra and consequently a finitely generated $\mathbf{C}$-algebra (See [Mat, p. 120-122]). Moreover:

Proposition 2.6. *Let $\mathcal{R}$ be the extended Rees algebra defined above. Then:*

i) The $\mathbf{C}[v]-$algebra $\mathcal{R}$ is torsion free.

ii) $\mathcal{R}$ is faithfully flat over $\mathbf{C}[v]$

iii) The map $\phi : \mathcal{R} \to \mathrm{gr}_{\mathfrak{m}}\mathcal{O}$ sending xv^{-i} to the image of x in $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ is well defined and induces an isomorphism $\mathcal{R}/(v \cdot \mathcal{R}) \simeq \mathrm{gr}_{\mathfrak{m}}\mathcal{O}$.

iv) For any $v_0 \in \mathbf{C} \setminus \{0\}$ the map $\mathcal{R} \to \mathcal{O}$, sending $xv^{-i} \mapsto xv_0^{-i}$ induces an isomorphism $\frac{\mathcal{R}}{(v-v_0)\cdot\mathcal{R}} \simeq \mathcal{O}$.

Proof.
See the appendix written by the second author in [Za], [Eis, p. 171], or additionally [Bo] in the exercises for §6 of of Chap VIII. □

The proposition may be a little technical, but what it says is that the extended Rees algebra is a way of producing flat degenerations of a ring to its associated graded ring, since the inclusion morphism $\mathbf{C}[v] \hookrightarrow \mathcal{R}$ is flat. Now taking the space $\mathfrak{X}$ associated to $\mathcal{R}$ and the map $\mathfrak{X} \to \mathbf{C}$ associated to the inclusion $\mathbf{C}[v] \hookrightarrow \mathcal{R}$, we obtain a map:

$$\varphi : \mathfrak{X} \longrightarrow \mathbf{C}$$

such that

- φ is faithfully flat.
- $\varphi^{-1}(0)$ is the algebraic space associated to $gr_{\mathfrak{m}}O$, that is the tangent cone $C_{X,0}$
- The space $\varphi^{-1}(v_0)$, is isomorphic to X, for all $v_0 \neq 0$.

[5]This algebra was introduced for an ideal $I \subset R$ by D. Rees in [Rs] in the from $R[v, Iv^{-1}]$.

that is, we have produced a 1-parameter flat family of algebraic spaces specializing X to $C_{X,0}$.

As you can see, the problem when trying to translate this into the analytic case is first of all, that in general the best thing we can say is that the algebra $\mathcal{R}$ is finitely generated over $\mathcal{O}$, but not even essentially finitely generated over $\mathbf{C}$.

However, given any finitely generated algebra over an analytic algebra such as $\mathcal{O}$, there is a "smallest" analytic algebra which contains it (it means that any map from our algebra to an analytic algebra factors uniquely through this "analytization". The proof: our algebra is a quotient of a polynomial ring $\mathcal{O}[z_1, \ldots, z_s]$ by an ideal I; take the quotient of the corresponding convergent power series ring $\mathcal{O}\{z_1, \ldots, z_s\}$, which is an analytic algebra, by the ideal generated by I; it is again an analytic algebra. So we can use this, to translate our result, into a similar one which deals with germs of analytic spaces.

Taking the analytic algebra $\mathcal{R}^h$ associated to $\mathcal{R}$, and the analytic germ $\mathfrak{X}$ associated to $\mathcal{R}^h$, we have a germ of map induced by the inclusion $\mathbf{C}\{t\} \hookrightarrow \mathcal{R}^h$:

$$\varphi : (\mathfrak{X}, 0) \longrightarrow (\mathbf{D}, 0)$$

which preserves all the properties established in the algebraic case, that is:

- φ is faithfully flat.
- $\varphi^{-1}(0)$ is the germ of analytic space associated to $gr_{\mathfrak{m}}\mathcal{O}$, that is the tangent cone $C_{X,0}$
- $\varphi^{-1}(v_0)$ is a germ of analytic space isomorphic to $(X, 0)$, for all $v_0 \neq 0$.

that is, we have produced a 1-parameter flat family of germs of analytic spaces specializing $(X, 0)$ to $(C_{X,0}, 0)$. The way, this construction relates to our previous analytic construction is given by the next exercise.

Exercise 2.1.
1) Suppose $(X, 0)$ is a germ of hypersurface.
Then $\mathcal{O} = \mathbf{C}\{z_1, \ldots, z_n\}/ < f(z_1, \ldots, z_n) >$. Show that
$\mathcal{R}^h = \mathbf{C}\{v, z_1, \ldots, z_n\}/ < v^{-m} f(vz_1, \ldots, tz_n) >$.
Note that this makes sense since, as we saw above, writing

$$f = f_m(z_1, \ldots, z_n) + f_{m+1}(z_1, \ldots, z_n) + \ldots,$$

where f_k is an homogeneous polynomial of degree k, then:

$$v^{-m} f(vz_1, \ldots, vz_n) = f_m(z_1, \ldots, z_n) + v f_{m+1}(z_1, \ldots, z_n) + \ldots \in \mathbf{C}\{v, z_1, \ldots, z_n\}.$$

2)More generally, take $I \subset \mathbf{C}\{z_1, \ldots, z_n\}$ and choose generators f_i such that their initial forms f_{i,m_i} generate the ideal of all initial forms of elements of I. Then:

$$\mathcal{R}^h = \mathbf{C}\{v, z_1, \ldots, z_n\}/ < v^{-m_i} f_i(vz_1, \ldots, vz_n) > .$$

It is important to note that this computation implies that the biholomorphism $\mathbf{C}^n \times \mathbf{D}^* \to \mathbf{C}^n \times \mathbf{D}^*$, where $\mathbf{D}^*$ is the punctured disk, determined by $(z, v) \mapsto (vz, v)$ induces an isomorphism $\varphi^{-1}(\mathbf{D}^*) \simeq X \times \mathbf{D}^*$.

Finally, we can use this constructions to prove that our two definitions of the tangent cone are equivalent.

Proposition 2.7. *Let $|C_{X,0}|$ be the underlying set of the analytic space $C_{X,0}$. Then, generating lines in $|C_{X,0}|$ are the limit positions of secant lines $0x_i$ as $x_i \in X \setminus \{0\}$ tends to 0.*

Proof.
Since $\varphi : (\mathfrak{X}, 0) \to (\mathbf{D}, 0)$ is faithfully flat, the special fiber of the map φ is contained in the closure of $\varphi^{-1}(\mathbf{D}^*)$ (See teFi). The isomorphism $\varphi^{-1}(\mathbf{D}^*) \simeq X \times \mathbf{D}^*$ which we have just seen shows that for every point $\overline{x} \in \varphi^{-1}(0) = C_{X,0}$ there are sequences of points $(x_i, v_i) \in X \times \mathbf{D}^*$ tending to $\overline{x}$. Thus $\overline{x}$ is in the limit of secants $0x_i$.

□

So, we finally know that our two concepts of tangent cone coincide, at least set-theoretically. However, the tangent cone contains very little information on $(X, 0)$.

Example 2.2.
For all curves $y^2 - x^m$, $m \geq 3$, the tangent cone is $y^2 = 0$ and it is non-reduced.

2.5 Multiplicity

Nevertheless the analytic structure of $C_{X,0} = \mathrm{Specangr}_{\mathfrak{m}} O$ does carry some important piece of information on $(X, 0)$, its multiplicity.

For a hypersurface, $f = f_m(z_1, \ldots, z_n) + f_{m+1}(z_1, \ldots, z_n) + \ldots$, the multiplicity at 0 is just m=the degree of the initial polynomial. And, from the example above, its tangent cone is also a hypersurface with the same multiplicity at 0 in this sense. In general, it is more complicated.

Let $\mathcal{O}$ be the analytic algebra of X with maximal ideal $\mathfrak{m}$ as before. We have the following consequences of the fact that $\mathcal{O}$ is a noetherian $\mathbf{C}$-algebra:
a) For each $i \geq 0$, the quotient $\mathcal{O}/\mathfrak{m}^{i+1}$ is a finite dimensional vector space over $\mathbf{C}$, and the generating function:

$$\sum_{i \geq 0} (\dim_{\mathbf{C}} \mathcal{O}/\mathfrak{m}^{i+1}) T^i = \frac{Q(T)}{(1-T)^d}$$

is a rational function with numerator $Q(T) \in \mathbb{Z}[T]$, and $Q(1) \in \mathbb{N}$. See [A-M, p. 117-118], or [Bo, chap VIII].
b) For large enough i:

$$\dim_{\mathbf{C}} \mathcal{O}/m^{i+1} = e_{\mathfrak{m}}(\mathcal{O}) \frac{i^d}{d!} + \text{lower order terms},$$

and $e_{\mathfrak{m}}(\mathcal{O}) = Q(1)$ is called **the multiplicity of** X **at** 0, which we will denote by $m_0(X)$. See [Mat, §14].
c) A linear space L_ϵ of dimension $n-d$ at a sufficiently small distance $\epsilon > 0$ from the origin and of general direction has the property that in a neighborhood of the origin, if ϵ is small enough L_ϵ meets X transversally at nonsingular points of X and $e_{\mathfrak{m}}(\mathcal{O})$ points of this intersection $L_\epsilon \cap X$ tend to 0 with ϵ. See [He-Or, p. 510-555].
d) The multiplicity of X at 0 coincides with the multiplicity of $C_{X,0}$ at 0. This follows from the fact that the generating function defined above is the same for the $\mathbf{C}$-algebra $\mathcal{O}$ and for $\mathrm{gr}_{\mathfrak{m}}\mathcal{O}$. See [Bo, Chap. VIII, §7], and also [He-Or, Thm 5.2.1 & Cor.].

3 Normal Cone and Polar Varieties: the normal/conormal diagram

The normal cone is a generalization of the idea of tangent cone, where the point is replaced by a closed analytic subspace, say $Y \subset X$. If X and Y were nonsingular it would only be the normal bundle of Y in X. We will only consider the case where Y is a nonsingular subspace of dimension t.
However, we will take a global approach here. Let $(X, \mathcal{O}_X)$ be a reduced complex analytic space of dimension d and $Y \subset X$ a closed complex subspace defined by a coherent sheaf of ideals $J \subset \mathcal{O}_X$. It consists, for every open set $U \subset X$ of all elements of $\mathcal{O}_X(U)$ vanishing on $Y \cap U$, and as one can expect the structure sheaf $\mathcal{O}_Y$ is isomorphic to $(\mathcal{O}_X/J)|_Y$. Analogously to the case of the tangent cone, let us consider $gr_J\mathcal{O}_X$, but now as the associated sheaf of graded rings of $\mathcal{O}_X$ with respect to J:

$$\mathrm{gr}_J\mathcal{O}_X = \bigoplus_{i\geq 0} J^i/J^{i+1} = \mathcal{O}_X/J \oplus J/J^2 \oplus \cdots \ .$$

Definition 3.1. *We define the* ***normal cone*** $C_{X,Y}$ ***of*** X ***along*** Y*, as the complex analytic space* $\mathrm{Specan}_Y\ (gr_J\mathcal{O}_X)$*.*

Note that we have a canonical inclusion $\mathcal{O}_Y \hookrightarrow gr_J\mathcal{O}_X$, which gives $gr_J\mathcal{O}_X$ the structure of a locally finitely presented graded $\mathcal{O}_Y$-algebra and consequently, by the Specan construction, a canonical analytic projection $C_{X,Y} \overset{\Pi}{\to} Y$, in which the fibers are cones. The natural surjection $gr_J\mathcal{O}_X \to \mathcal{O}_X/J = \mathcal{O}_Y$ obtained by taking classes modulo the ideal $\bigoplus_{i\geq 1} J^i/J^{i+1}$ corresponds to an analytic section $Y \hookrightarrow C_{X,Y}$ of the map Π sending each point $y \in Y$ to the vertex of the cone $\Pi^{-1}(y)$.
To be more precise, note that the sheaf of graded $\mathcal{O}_X$-algebras $gr_J\mathcal{O}_X$ is a sheaf on X with support Y. Now if we take an open set U, where the ideal $J(U)$ is finitely generated and different from $\mathcal{O}_X(U)$, then we get

$$\mathrm{gr}_J\mathcal{O}_X(U) \cong \frac{\mathcal{O}_Y(U)\,[t_1,\ldots,t_r]}{\langle g_1,\ldots,g_s\rangle}.$$

where the g_i's are homogeneous polynomials.

Now, we would like to build, in analogy to the case of the tangent cone, the **Specialization of X to the normal cone of Y.**
Let us first take a look at it in the algebraic case, when we suppose that X is an algebraic variety, and $Y \subset X$ a closed algebraic subvariety defined by a coherent sheaf of ideals $J \subset \mathcal{O}_X$.

Again, analogously to the tangent cone and the Rees algebra technique. Consider the locally finitely presented sheaf of graded $\mathcal{O}_X$-algebras

$$\mathcal{R} = \bigoplus_{n \in \mathbf{Z}} J^n v^{-n} \subset \mathcal{O}_X[v, v^{-1}], \quad \text{where } J^n = \mathcal{O}_{X,0} \text{ for } n \leq 0.$$

Note that we have $\mathbf{C}[v] \subset \mathcal{O}_X[v] \subset \mathcal{R}$, where $\mathbf{C}$ denotes the constant sheaf, thus endowing $\mathcal{R}(X)$ with the structure of a $\mathbf{C}[v]$-algebra that results in an algebraic map

$$p : \mathrm{Spec}\mathcal{R} \longrightarrow \mathbf{C}$$

Moreover, the $\mathbf{C}[v]$-algebra $\mathcal{R}$ has all the analogous properties of proposition 2.6, which in turn gives the corresponding properties to p, defining a flat 1-parameter family of varieties such that:

1) The fiber over 0 is $\mathrm{Spec}(gr_J \mathcal{O}_X)$.

2) The general fiber is an algebraic space isomorphic to X.

that is, the map p gives a specialization of X to its normal cone along Y, namely $C_{X,Y}$.

Let us now look at the corresponding construction for germs of analytic spaces. Going back to the complex space $(X, \mathcal{O}_X)$, and the nonsingular subspace Y of dimension t, take a point $0 \in Y$, and a local embedding $(Y, 0) \subset (X, 0) \subset (\mathbf{C}^n, 0)$. Since Y is nonsingular we can assume it is linear, by choosing a sufficiently small representative of the germ $(X, 0)$ and adequate local coordinates on $\mathbf{C}^n$. Let $\mathcal{O}_{X,0} = \mathbf{C}\{z_1, \ldots, z_n\}/\langle f_1, \ldots, f_k\rangle$ be the analytic algebra of the germ, such that, $J = \langle z_1, \ldots, z_{n-t}\rangle$ is the ideal defining Y in X. Consider now the finitely generated $\mathcal{O}_{X,0}$-algebra:

$$\mathcal{R} = \bigoplus_{n \in \mathbf{Z}} J^n v^{-n}, \quad \text{where } J^n = \mathcal{O}_{X,0} \text{ for } n \leq 0.$$

So again, taking the analytic algebra $\mathcal{R}^h$ associated to $\mathcal{R}$, and the analytic germ Z associated to $\mathcal{R}^h$, we have a germ of map induced by the inclusion $\mathbf{C}\{v\} \hookrightarrow \mathcal{R}^h$:

$$p : (Z, 0) \longrightarrow (\mathbf{D}, 0)$$

which preserves all the properties established in the algebraic case, that is:

- $\cdot$ p is faithfully flat.
- $\cdot$ $p^{-1}(0)$ is the germ of analytic space associated to $\mathrm{gr}_J\mathcal{O}_{X,0}$, that is the germ of the normal cone $C_{X,Y}$
- $\cdot$ $p^{-1}(v)$ is a germ of analytic space isomorphic to $(X,0)$, for all $v \neq 0$.

that is, we have produced a 1-parameter flat family of germs of analytic spaces specializing $(X,0)$ to $(C_{X,Y},0)$.

Using this it can be shown that after choosing a local retraction $\rho : (X,0) \to (Y,0)$, the underlying set of $(C_{X,Y},0)$ can be identified with the set of limit positions of secant lines $x_i\rho(x_i)$ for $x_i \in X \setminus Y$ as x_i tends to $y \in Y$(For a proof of this, see [Hi]). We shall see more about this specialization in the global case a little later.

Retaking this germ approach, with $(Y,0) \subset (X,0) \subset \mathbf{C}^n$, and Y a linear subspace of dimension t we can now interpret definition 2.11 and lemma 2.4 in the following way. Using the notation of that section, let $R := \mathbf{C}\{z_1,\ldots,z_{n-t},y_1\ldots,y_t\}$, $J = \langle z_1,\ldots,z_{n-t}\rangle \subset R$ the ideal defining Y, $I = \langle f_1,\ldots,f_k\rangle \subset R$ the ideal defining X and $A = R/I = \mathcal{O}_{X,0}$. Then, the ring R/J is by definition $\mathcal{O}_{Y,0}$ which is isomorphic to $\mathbf{C}\{y_1,\ldots,y_t\}$, and its not hard to prove that

$$\mathrm{gr}_J R \cong \mathcal{O}_{Y,0}[z_1,\ldots,z_{n-t}].$$

More to the point, take an element $f \in I \subset R$, then we can write

$$f = \sum_{(\alpha,\beta)\in\mathbf{N}^t\times\mathbf{N}^{n-t}} c_{\alpha\beta} y^\alpha z^\beta.$$

Now define $\nu_y f = \min\,\{|\beta| \mid c_{\alpha\beta} \neq 0\}$ and one can prove that

$$\mathrm{in}_J f = \sum_{|\beta|=\nu_y f} c_{\alpha\beta} y^\alpha z^\beta,$$

which after rearranging the terms with respect to z gives us a polynomial in the variables z_k with coefficients in $\mathcal{O}_{Y,0}$, that is, an element of $gr_J R$. Note that this "polynomials" define analytic functions in $Y \times \mathbf{C}^{n-t} = \mathbf{C}^t \times \mathbf{C}^{n-t}$, and thus realize, by the Specan construction, the germ of the normal cone $(C_{X,Y},0)$ as a germ of analytic subspace of $(\mathbf{C}^n,0)$ with a canonical analytic map to $(Y,0)$. Let us clarify all this with an example.

Example 3.1. *Take $(X,0) \subset (\mathbf{C}^3,0)$ defined by $x^2 - y^2 z = 0$, otherwise known as Whitney's Umbrella. Then from what we have discussed we obtain:*

i) The tangent cone at 0, $C_{X,0} \subset \mathbf{C}^3$, is the analytic subspace defined by $x^2 = 0$.

ii) For $Y =$ z-axis, the normal cone along Y, $C_{X,Y} \subset \mathbf{C}^3$, is the analytic subspace defined by $x^2 - y^2 z = 0$, that is the entire space X.

iii) For $Y = y$-axis, the normal cone along Y, $C_{X,Y} \subset \mathbf{C}^3$, is the analytic subspace defined by $y^2z = 0$.

Proposition 3.1. (Hironaka, Teissier) *Given a t-dimensional closed nonsingular subspace $Y \subset X$ and a point $0 \in Y$ then for any local embedding $(Y, 0) \subset (X, 0) \subset \mathbf{C}^n$, the following conditions are equivalent:*

i) The multiplicity $m_y(X)$ of X at the points $y \in Y$ is locally constant on Y near 0.

ii) The dimension of the fibers of the maps $C_{X,Y} \to Y$ is locally constant on Y near 0.

iii) For every point $y \in Y$ there exists a dense open set of the Grassmanian of $(n-d+t)$-dimensional linear subspaces of $\mathbf{C}^n$ containing T_yY (the tangent space to Y at y) such that if W is a representative in an open $U \subset \mathbf{C}^n$ of a germ (W, y) at y of a nonsingular $(n - d + t)$-dimensional subspace of $\mathbf{C}^n$ containing Y and whose tangent space at y is in that open set, there exists an open neighborhood $B \subset U$ of y in $\mathbf{C}^n$ such that:

$$|W \cap X \cap B| = Y \cap B,$$

where $|Z|$ denotes the reduced space of Z.

Proof.
See [Hi], [He-Or, Appendix III, theorem 2.2.2], and for iii) see [Te3, Chapter I, 5.5]. The meaning of the last statement is that if the equality is not satisfied, there are t-dimensional components of the intersection $|W \cap X \cap B|$, distinct from $Y \cap B$, meeting Y at the point y; this must increase the multiplicity of X. □

Example 3.2. *let us look again at Whitney's Umbrella $(X, 0) \subset (\mathbf{C}^3, 0)$ defined by $x^2 - y^2z = 0$, and let Y be the y-axis.*

Taking W as the nonsingular 2-dimensional space defined by $z = ax$, gives

$$x(x - ay^2) = 0,$$

so that whenever $a \neq 0$, the intersection $W \cap X$ has two irreducible components: the y-axis, and the curve defined by the equations $x = ay^2$, $z = ax$.

We can do the same for $Y = z$-axis, and W defined by $y = ax$ which gives

$$x^2(1 - az^2),$$

which locally defines the z-axis.

In order to understand better the normal cone and proposition 3.1, we are going to introduce the blowing up of X along Y. As in the case of the tangent cone, let us start by a geometric description.

Let $(Y,0) \subset (X,0)$ be germ of nonsingular subspace of dimension t as before. We can choose a local analytic retraction $\rho : \mathbf{C}^n \to Y$, use it to define a map:

$$\begin{aligned} \phi : X \setminus Y &\longrightarrow \mathbf{P}^{n-1-t} \\ x &\longmapsto \text{direction of } \overline{x\rho(x)}, \end{aligned}$$

and then consider its graph in $(X \setminus Y) \times \mathbf{P}^{n-1-t}$. Note that, since we can assume Y is a linear subspace, in a suitable set of coordinates the map ρ is just the canonical linear projection $\rho : \mathbf{C}^n \to \mathbf{C}^t$. Moreover, the map ϕ maps $x = (x_1, \ldots, x_n) \in X \setminus Y \mapsto (x_{t+1} : \cdots : x_n) \in \mathbf{P}^{n-1-t}$.

Just as in the case of the blow-up of a point, the closure of the graph is a complex analytic space $E_Y X \subset X \times \mathbf{P}^{n-1-t}$, and the natural projection map $e_y := p \circ i$:

$$\begin{tikzcd} E_Y X \arrow[r, hook, "i"] \arrow[dr, "e_y"'] & X \times \mathbf{P}^{n-1-t} \arrow[d, "p"] \\ & X \end{tikzcd}$$

is proper. Moreover the map e_y induces an isomorphism $E_Y X \setminus e_y^{-1}(Y) \to X \setminus Y$. Note that if we take an open cover of the complex space X, consisting only of local models, we can do an analogous construction in each local model (see proof of 3.2) and then paste them all up to obtain a global blow-up. However there is an algebraic construction which will save us the effort of pasting by doing it all in one fell swoop. Let $J \subset O_X$ be the ideal defining $Y \subset X$ as before.

Definition 3.2. *The* ***Rees algebra****, or* ***blowup algebra*** *of J in $\mathcal{O}_X$ is the graded $\mathcal{O}_X$-algebra:*

$$P(J) = \bigoplus_{i \geq 0} J^i = \mathcal{O}_X \oplus J \oplus J^2 \oplus \cdots \ .$$

Note that $P(J)/JP(J) \cong \mathrm{gr}_J \mathcal{O}_X$, the associated graded ring of $\mathcal{O}_X$ with respect to J. Moreover, since J is locally finitely generated, $P(J)$ is a locally finitely presented graded $\mathcal{O}_X$-algebra, generated in degree 1, and as such, it has locally a presentation, for suitable open sets $U \subset X$:

$$\frac{\mathcal{O}_{X|U}[z_1, \ldots, z_{n-t}]}{(g_1, \ldots, g_m)} \cong P(J)|U,$$

where the g_i are homogenous polynomials in $z_1, \ldots, z_{n-t}$ with coefficients in $\mathcal{O}_X|U$.

Defining $\tilde{E}_Y X$ as the **projective analytic spectrum** of $P(J)$, $\tilde{E}_Y X = \mathrm{Projan} P(J)$ (see [He-Or, Appendix III, 1.2.8]), we can view this as defining a family of projective varieties parametrized by X, as a result of the $\mathcal{O}_X$-algebra structure.

$$\begin{array}{c}\tilde{E}_Y X \subset X \times \mathbf{P}^{n-1-t} \\ \downarrow \\ X\end{array}$$

To check that these two spaces are the same, its enough to check that they are the same locally, that is for each open set of an appropiate open cover of X, and that is where the next proposition comes into play:

Proposition 3.2. *Take a point $x \in X$ and a sufficiently small neighborhood $x \in U \subset X$ such that the ideal $J(U) \subset \mathcal{O}_X(U)$ is finitely generated. Then choosing a system of generators $J = \langle h_1, \ldots, h_s \rangle$ gives an embedding $E_Y X \subset X \times \mathbf{P}^{s-1}$ and an embedding $\tilde{E}_Y X \subset X \times \mathbf{P}^{s-1}$. Their images are equal.*

Proof.
Let $Y \subset X$ be the subspace defined by J, which in the following will mean $Y \cap U \subset U \subset X$ to avoid complicated and unnecessary notation, but always keeping in mind that we are working in a special open set U of X which allows us to use the finiteness conditions. Now consider the map:

$$\begin{aligned}\lambda : X \setminus Y &\longrightarrow \mathbf{P}^{s-1} \\ x \quad &\longmapsto (h_1(x) : \cdots : h_s(x)),\end{aligned}$$

and as before, let $E_Y X \subset X \times \mathbf{P}^{s-1}$ be the closure of the graph of λ.

On the other hand, consider the presentation $\mathcal{O}_X[z_1, \ldots, z_s]/(g_1, \ldots, g_m) \cong P(J)$ where the isomorphism is defined by $z_i \mapsto h_i$. Note that the $g_i's \in \mathcal{O}_X[z_1, \ldots, z_s], i = 1, \ldots, m$, generate the ideal of all homogeneous relations $g(h_1, \ldots, h_s) = 0,\ g \in \mathcal{O}_X[z_1, \ldots, z_s]$ which are exactly the equations for the closure of the graph. To see why this last statement is true, recall that:

$$\text{graph}(\lambda) = \{(x, z_1 : \cdots : z_s) \subset X \times \mathbf{P}^{s-1} | \, (z_1 : \cdots : z_s) = (h_1(x) : \cdots : h_s(x))\}.$$

and remember that the elements $g \in \langle g_1, \ldots, g_m \rangle$ are homogeneous polynomials in $z_1, \ldots, z_s$ with coefficients in $\mathcal{O}_X$, so they define analytic functions in $X \times \mathbf{C}^s$ such that the homogeneity in the z's allow us to look at their zeros in $X \times \mathbf{P}^{s-1}$. Moreover, if $(x, z_1 : \cdots : z_s) \in \text{graph}(\lambda)$, then $[z] = [h(x)]$ thus $g(z) = 0$, that is they are the equations defining the graph, and consequently its closure. □

Finally, to relate all this to the normal cone, note that in the map:

$$e_y : E_Y X \longrightarrow X$$

the inverse image of Y is the projective family associated to the family of cones

$$C_{X,Y} \longrightarrow Y$$

this is clear, set-theoretically, in the geometric description. In the algebraic description, it follows from the identity:

$$\bigoplus_{i\geq 0} J^i \bigotimes_{\mathcal{O}_X} \mathcal{O}_X/J \cong \bigoplus_{i\geq 0} J^i/J^{i+1} = \mathrm{gr}_J\mathcal{O}_X$$

and the fact that fiber product corresponds germ-wise to tensor product.

$$\begin{tikzcd} \mathcal{O}_X \arrow[r, hook] \arrow[d] & \bigoplus_{i\geq 0} J^i \\ \mathcal{O}_X/J & \end{tikzcd} \qquad \begin{tikzcd} X & E_Y X \arrow[l, "e_y"'] \\ Y \arrow[u, hook] & \end{tikzcd}$$

The real trick comes when, in the analytic setting, we want to build the specialization to the normal cone in a global scenario. We will describe a geometric construction for this. Consider the complex space $X \times \mathbf{C}$, and the closed nonsingular complex subspace $Y \times \{0\} \subset X \times \mathbf{C}$ defined by the coherent sheaf of ideals $\langle J, v\rangle$.

Let $Z \to X \times \mathbf{C}$ denote the blowup of $X \times \mathbf{C}$ along $Y \times \{0\}$. Since v is a generator of the ideal defining the blown-up subspace, there is an open set $U \subset Z$, where v generates the pullback of the ideal $\langle J, v\rangle \subset \mathcal{O}_{X\times\mathbf{C}}$, that is the ideal defining the exceptional divisor. One can verify that our old acquaintance, the sheaf of $\mathcal{O}_X$-algebras $\mathcal{R}$, can be identified with the sheaf of analytic functions, algebraic in v, over U. Moreover, the analytic map:

$$U \subset Z \longrightarrow X \times \mathbf{C} \to \mathbf{C}$$

is precisely the map which gives us the **specialization to the normal cone**.

We will end this section with a proposition that will prove to be very useful to prove theorem 3.7, and its generalization in the section of relative duality.

Proposition 3.3. *Let $X \subset \mathbf{C}^n$ be a reduced analytic subspace of dimension d and for $x \in X$, let $\varphi : \mathfrak{X} \to \mathbf{C}$ be the specialization of X to the tangent cone $C_{X,x}$. Let $\kappa = \kappa_X : T^*_X\mathbf{C}^n \to X$ denote the conormal space of X in $\mathbf{C}^n \times \check{\mathbf{C}}^n$. Then the relative conormal space*

$$q : T^*_{\mathfrak{X}}(\mathbf{C}^n \times \mathbf{C}/\mathbf{C}) \to \mathfrak{X} \to \mathbf{C}$$

is isomorphic, as an analytic space over $\mathbf{C}$, to the specialization space of $T^*_X\mathbf{C}^n$ to the normal cone $C_{T^*_X\mathbf{C}^n,\kappa^{-1}(x)}$ of $\kappa^{-1}(x)$ in $T^*_X\mathbf{C}^n$. In particular, the fibre $q^{-1}(0)$ is isomorphic to this normal cone.

Proof. We shall see a proof in a more general situation below in subsection 4.4. $\square$

Corollary 3.1. *The relative conormal space* $\kappa_\varphi : T^*_{\mathfrak{X}}(\mathbf{C}^n \times \mathbf{C})/\mathbf{C}) \to \mathfrak{X}$ *is* $\varphi-$*lagrangian.*

Proof. We will use the notation of the proof of proposition 4.5. From definition 2.7 we need to prove that every fiber $q^{-1}(s)$ is a lagrangian subvariety of $\{s\} \times \mathbf{C}^n \times \check{\mathbf{C}}^n$. But, by proposition 3.3 we know that for $s \neq 0$, the fiber $q^{-1}(s)$ is isomorphic to $T^*_X\mathbf{C}^n$ and so it is lagrangian. Thus, by proposition 2.5 all we need to prove is that the special fiber $q^{-1}(0)$ has the right dimension, which in this case is equal to n.
Proposition 3.3 also tells us that the fiber $q^{-1}(0)$ is isomorphic to the normal cone

$$C_{T^*_X\mathbf{C}^n, T^*_{\{x\}}\mathbf{C}^n \cap T^*_X\mathbf{C}^n} = C_{T^*_X\mathbf{C}^n, \kappa^{-1}(x)}.$$

Finally, since the projectivized normal cone $\mathbf{P}C_{T^*_X\mathbf{C}^n, T^*_{\{x\}}\mathbf{C}^n \cap T^*_X\mathbf{C}^n}$ is obtained as the exceptional divisor of the blowup of $T^*_X\mathbf{C}^n$ along $\kappa_X^{-1}(x)$, it has dimension $n-1$ and so the cone over this projective variety has dimension n, which finishes the proof. □

3.1 Local Polar Varieties

In this section we introduce the local polar varieties of a germ of a reduced equidimensional complex analytic space $(X, 0) \subset (\mathbf{C}^n, 0)$. The dimension of X is denoted by d and to make the comparison with the case of projective varieties V of dimension d mentioned in the introduction, we must think of X as the cone over V with vertex $0 \in \mathbf{C}^n$, which is of dimension $d+1$.

Local polar varieties were first constructed, using the Semple-Nash modification and all Schubert cycles of the Grassmannian, in [L-T1]. The description in terms of the conormal space of the local polar varieties used here, which correspond to special Schubert cycles, is a contribution of Henry-Merle which appears in [HM], [HMS], and [Te3]. We will in this subsection get a first glimpse into a special case of what will be called the normal-conormal diagram. Let us denote by $C(X)$ and E_0X respectively the conormal space of X and the blowup of 0 in X as before, then we have the diagram:

$$\begin{tikzcd}
E_0C(X) \arrow[r, "\hat{e}_0"] \arrow[dd, "\kappa'"] \arrow[ddr, "\xi"] & C(X) \arrow[r, hook] \arrow[dd, "\kappa"] \arrow[dr, "\lambda"] & X \times \check{\mathbf{P}}^{n-1} \arrow[d, "pr_2"] \\
 & & \check{\mathbf{P}}^{n-1} \\
E_0X \arrow[r, "e_0"'] & X &
\end{tikzcd}$$

where $E_0C(X)$ is the blowup of $\kappa^{-1}(0)$ in $C(X)$, and κ' is obtained from the universal property of the blowup, with respect to E_0X and the map ξ.
It is worth mentioning, that $E_0C(X)$ lives inside the fiber product $C(X) \times_X E_0X$ and can be described in the following way: take the inverse

image of $E_0X \setminus e_0^{-1}(0)$ in $C(X) \times_X E_0X$ and close it, thus obtaining κ' as the restriction of the second projection to this space.

Let $D_{d-k+1} \subset \mathbf{C}^n$ be a linear subspace of codimension $d-k+1$, for $0 \leq k \leq d-1$, and let $L^{d-k} \subset \check{\mathbf{P}}^{n-1}$ the dual space of D_{d-k+1}, that is the linear space of hyperplanes of $\mathbf{C}^n$ that contain D_{d-k+1}.

Proposition 3.4. *For a sufficiently general D_{d-k+1}, the image $\kappa(\lambda^{-1}(L^{d-k}))$ is the closure in X of the set of points of X^0 which are critical for the projection $\pi|_{X^0} : X^0 \to \mathbf{C}^{d-k+1}$, induced by the projection $\mathbf{C}^n \to \mathbf{C}^{d-k+1}$ with kernel $D_{d-k+1} = (L^{d-k})\check{}$.*

Proof.
Note that $x \in X^0$ is critical for π, iff the tangent map $d_x\pi : T_xX^0 \to \mathbf{C}^{d-k+1}$ is not onto, that is iff $\dim\ker d_x\pi \geq k$ since dim $T_xX^0 = d$, and $\ker d_x\pi = D_{d-k+1} \cap T_xX^0$.
Now, note that the conormal space $C(X^0)$ of the nonsingular part of X is equal to $\kappa^{-1}(X^0)$ so by definition:

$$\lambda^{-1}(L^{d-k}) \cap C(X^0) = \{(x, H) \in C(X) | x \in X^0,\ H \in L^{d-k},\ T_xX^0 \subset H\}$$

equivalently:

$$\lambda^{-1}(L^{d-k}) \cap C(X^0) = \{(x, H), \in C(X) | x \in X^0,\ H \in \check{D},\ H \in (T_xX^0)\check{}\ \}$$

thus $H \in \check{D} \cap (T_xX^0)\check{}$, and from the equation $\check{D} \cap (T_xX^0)\check{} = (D + T_xX^0)\check{}$ we deduce that the intersection is not empty iff $D + T_xX^0 \neq \mathbf{C}^n$, which implies that dim $D \cap T_xX^0 \geq k$, and consequently $\kappa(H) = x$ is a critical point.

Now, according to [Te3, Chapter IV,1.3], there exists an open dense set U_k in the grasmannian of $n-d+k-1$-planes of $\mathbf{C}^n$ such that if $D \in U_k$, the intersection $\lambda^{-1}(L^{d-k}) \cap C(X^0)$ is dense in $\lambda^{-1}(L^{d-k})$. So, for any $D \in U$, since κ is a proper map, and thus closed, we have that $\kappa(\lambda^{-1}(L^{d-k})) = \kappa\left(\overline{\lambda^{-1}(L^{d-k}) \cap C(X^0)}\right) = \overline{\kappa(\lambda^{-1}(L^{d-k}))}$, which finishes the proof. See [Te3, Chap. 4, 4.1.1] for a complete proof of a more general statement. □

Remark 3.1. *It is important to have in mind the following easily verifiable facts:*

a) As we have seen before, the fiber $\kappa^{-1}(x)$ over a regular point $x \in X^0$ in the (projectivized) conormal space $C(X)$ is of dimension $n-d-1$, so by semicontinuity of fiber dimension we have that dim $\kappa^{-1}(0) \geq n-d-1$.

b) The analytic set $\lambda^{-1}(L^{d-k})$ can be obtained by the intersection of $C(X)$ and $\mathbf{C}^n \times L^{d-k}$ in $\mathbf{C}^n \times \check{\mathbf{P}}^{n-1}$. However, the space $\mathbf{C}^n \times L^{d-k}$ is "linear", defined by $n-d+k-1$ linear equations, namely it is the intersection of this same number of "hyperplanes". Thus for a general L^{d-k}, this intersection is of pure dimension $n-1-n+d-k+1 = d-k$.

The proof of this is not immediate because we are working over an open neighborhood of a point $x \in X$, so we cannot assume that $C(X)$ is compact. However (see [Te3]) we can take a Whitney stratification of $C(X)$ such that the closed algebraic subset $\kappa^{-1}(0) \subset \check{\mathbf{P}}^{n-1}$ is a union of strata. Now by general transversality theorems in algebraic geometry (see [Kl]) a sufficiently general L^{d-k} will be transversal to all the strata of $\kappa^{-1}(0)$ in $\check{\mathbf{P}}^{n-1}$ and then because of the Whitney conditions $\mathbf{C}^n \times L^{d-k}$ will be transversal to all the strata of $C(X)$, which will imply in particular the statement on the dimension. Note that the existence *of Whitney stratifications does not depend on the existence of polar varieties. In [Te3] it is deduced from the idealistic Bertini theorem.*

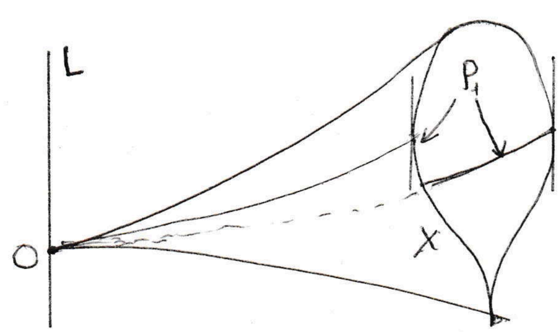
c) *The fact that $\lambda^{-1}(L^{d-k}) \cap C(X^0)$ is dense in $\lambda^{-1}(L^{d-k})$ means that if a limit of tangent hyperplanes at points of X^0 contains D_{d-k+1}, it is a limit of tangent hyperplanes which also contain D_{d-k+1}. This equality holds because transversal intersections preserve the boundary condition; see [Ch], [Te3, Remarque 4.2.3].*

d) *Note that for a fixed L^{d-k}, the germ $(P_k(X; L^{d-k}), 0)$ is empty if and only if the intersection $\kappa^{-1}(0) \cap \lambda^{-1}(L^{d-k})$ is empty. Now, from a) we know that dim $\kappa^{-1}(0) = n - d - 1 + r$ with $r \geq 0$. Thus, by the exact same argument as in b), this implies that the polar variety is not empty, that is dim $\kappa^{-1}(0) \cap \lambda^{-1}(L^{d-k}) \geq 0$, if and only if $r \geq k$.*

Definition 3.3. *With the notations and hypotheses of proposition 3.4, define for $0 \leq k \leq d-1$ the* ***local polar variety****.*

$$P_k(X; L^{d-k}) = \kappa(\lambda^{-1}(L^{d-k}))$$

A priori, we have just defined $P_k(X; L^{d-k})$ set-theoretically, however we have the following result, for which a proof can be found in [Te3, Chapter IV, 1.3.2].



Proposition 3.5. *The local polar variety $P_k(X;L^{d-k}) \subseteq X$ is a reduced closed analytic subspace of X, either of pure codimension k in X or empty.*

We have thus far defined a local polar variety that depends on both the choice of the embedding $(X,0) \subset (\mathbf{C}^n,0)$ and the choice of the linear space D_{d-k+1}. However, an important information we will extract from these polar varieties is their multiplicities at 0, and these numbers are analytic invariants provided the linear spaces used to define them are general enough. This generalizes the invariance of the degrees of Todd's polar loci which we saw in the introduction.

Proposition 3.6. *Let $(X,0) \subset (\mathbf{C}^n,0)$ be as before, then for every $0 \le k \le d-1$ and a sufficiently general linear space $D_{d-k+1} \subset \mathbf{C}^n$ the multiplicity of the polar variety $P_k(X;L^{d-k})$ at 0 depends only on the analytic type of $(X,0)$.*

Proof.
See [Te3, Chapter IV, Théorème 3.1]. □

This last result allows us to associate to any reduced, pure d-dimensional, analytic local algebra $O_{X,x}$ a sequence of d integers $(m_0,\dots,m_{d-1})$, where m_k is the multiplicity at x of the polar variety $P_k(X;L^{d-k})$ calculated from any given embedding $(X,x) \subset (\mathbf{C}^n,0)$, and a wise choice of D_{d-k+1}.

Remark 3.2. *Since for a linear space L^{d-k} to be "sufficiently general" means that it belongs to an open dense subset specified by certain conditions, we can just as well take a sufficiently general flag*

$$L^1 \subset L^2 \subset \cdots \subset L^{d-2} \subset L^{d-1} \subset L^d \subset \check{P}^{n-1}$$

which by definition of a polar variety and proposition 3.6, gives us a chain

$$P_{d-1}(X;L^1) \subset P_{d-2}(X;L^2) \subset \cdots \subset P_1(X;L^{d-1}) \subset P_0(X;L^d) = X,$$

of polar varieties, each with generic multiplicity at the origin. This implies that if the germ of a general polar variety $(P_k(X;L^{d-k}),0)$ is empty for a fixed k, then it will be empty for all $d-1 \ge l \ge k$. This fact can also be deduced from 3.1 d) by counting dimensions.

Definition 3.4. (Definition of polar varieties for singular projective varieties) *Let $V \subset \mathbf{P}^{n-1}$ be a reduced equidimensional projective variety of dimension d. Let $(X,0) \subset (\mathbf{C}^n,0)$ be the germ at 0 of the cone over V. The polar varieties $P_k(X,L^{d+1-k})$, $0 \le k \le d$ are cones because tangent spaces are constant along the generating lines (see lemma 2.3). The associated projective subvarieties of V are the polar varieties of V and are denoted by $P_k(V)$ or $P_k(V,L^{d+1-k})$ or $P_k(V,D_{d-k+2})$ with $L^{d+1-k} = (D_{d-k+2})\check{} \subset \check{\mathbf{P}}^{n-1}$.*

Indeed, if V is nonsingular this definition coincides with the definition of $P_k(V,D_{d-k+2})$ given in the introduction. It suffices to take the linear subspace $L^{d+1-k} \subset \check{\mathbf{P}}^{n-1}$ to be the dual of the subspace $D_{d-k+2} \subset \mathbf{P}^{n-1}$ of codimension $d-k+2$ which appears in that definition.

Example 3.3.
Let $X := y^2 - x^3 - t^2x^2 = 0 \subset \mathbf{C}^3$*, so dim* $X = 2$*, and thus* $k = 0, 1$*. An easy calculation shows that the singular locus of* X *is the* $t-$*axis, and* $m_0(X) = 2$*.*

Now, note that for $k = 0$*,* D_3 *is just the origin in* $\mathbf{C}^3$*, so the the projection*

$$\pi : X^0 \to \mathbf{C}^3$$

with kernel D_3 *is the restriction to* X^0 *of the identity map, which is of rank* 2 *and we get that the whole* X^0 *is the critical set of such a map. Thus,*

$$P_0(X, L^2) = X.$$

For $k = 1$*,* D_2 *is of dimension 1. So let us take for instance* $D_2 = y-$*axis, so we get the projection*

$$\pi : X^0 \to \mathbf{C}^2 \quad (x, y, t) \mapsto (x, t),$$

and we obtain that the set of critical points of the projection is given by

$$P_1(X, L^1) = \begin{cases} x = -t^2 \\ y = 0 \end{cases}$$

If we had taken for D_2 *the line* $t = 0,\ \alpha x + \beta y = 0$*, we would have found that the polar curve is a nonsingular component of the intersection of our surface with the surface* $2\alpha y = \beta x(3x + 2t^2)$*. For* $\alpha \neq 0$ *all these polar curves are tangent to the t-axis. As we shall see in the next subsection, this means that the t-axis is an "exceptional cone" in the tangent cone* $y^2 = 0$ *of our surface at the origin, and therefore all the* 2*-planes containing it are limits at the origin of tangent planes at nonsingular points of our surface.*

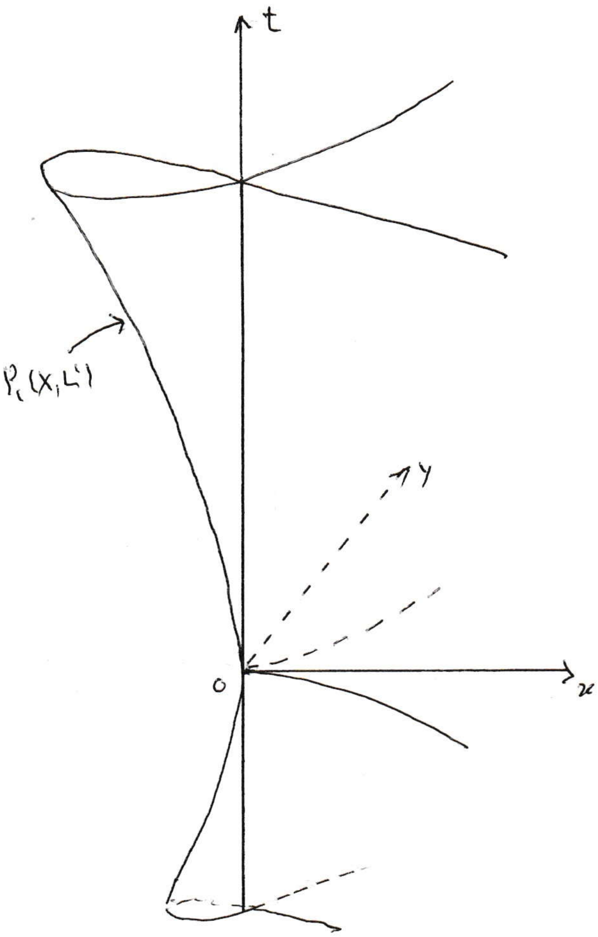
3.2 Limits of tangent spaces

We have talked before of the limits of tangent spaces, but with the help of the normal/conormal diagram and the polar varieties we will be able to describe the limits of tangent spaces to X at 0, assuming that $(X, 0)$ is reduced and purely d-dimensional. This method is based on Whitney's lemma and the two results which follow it:

Lemma 3.1. ***Whitney's lemma.-*** *Let* $(X, 0)$ *be a pure dimensional germ of analytic subspace, choose a representative* X *and let* $\{x_n\} \subset X^0$ *be a sequence of points tending to* 0*, such that*

$$\lim_{n\to\infty} [0x_n] = l \ \text{ and } \ \lim_{n\to\infty} T_{x_n}X = T,$$

then $l \subset T$*.*



This lemma originally appeared in [Whi1, Theorem 22.1], and you can also find a proof due to Hironaka in [L] and yet another below in assertion a) of theorem 3.7.

Theorem 3.7. (Lê-Teissier, see [L-T2])

I) *In the normal/conormal diagram*

$$\begin{tikzcd}
X \times \mathbf{P}^{n-1} \times \check{\mathbf{P}}^{n-1} & \supset & E_0C(X) \arrow[r, "\hat{e}_0"] \arrow[dd, "\kappa'"] \arrow[ddr, "\xi"] & C(X) \arrow[r, hook] \arrow[dd, "\kappa"] \arrow[dr, "\lambda"] & X \times \check{\mathbf{P}}^{n-1} \arrow[d, "pr_2"] \\
 & & & & \check{\mathbf{P}}^{n-1} \\
X \times \mathbf{P}^{n-1} & \supset & E_0X \arrow[r, "e_0"'] & X &
\end{tikzcd}$$

consider the irreducible components $\{D_\alpha\}$ *of* $D = |\xi^{-1}(0)|$. *Then:*

a) Each $D_\alpha \subset \mathbf{P}^{n-1} \times \check{\mathbf{P}}^{n-1}$ *is in fact contained in the incidence variety* $I \subset \mathbf{P}^{n-1} \times \check{\mathbf{P}}^{n-1}$.

b) Each D_α *is lagrangian in* I *and therefore establishes a projective duality of its images:*

$$\begin{tikzcd}
D_\alpha \arrow[r] \arrow[d] & W_\alpha \subset \check{\mathbf{P}}^{n-1} \\
V_\alpha \subset \mathbf{P}^{n-1} &
\end{tikzcd}$$

Note that, from commutativity of the diagram we obtain $\kappa^{-1}(0) = \bigcup_\alpha W_\alpha$, and $e_0^{-1}(0) = \bigcup_\alpha V_\alpha$. It is important to notice that these expressions are not necessarily the irreducible decompositions of $\kappa^{-1}(0)$ and $e_0^{-1}(0)$ respectively, since there may be repetitions. However, it is true that they contain the respective irreducible decompositions.

In particular, note that if $\dim V_{\alpha_0} = d-1$, then the cone $O(V_{\alpha_0}) \subset \mathbf{C}^n$ is an irreducible component of the tangent cone $C_{X,0}$ and its projective dual $W_{\alpha_0} = \check{V}_{\alpha_0} \subset \kappa^{-1}(0)$. That is, any tangent hyperplane to the tangent cone is a limit of tangent hyperplanes to X at 0. The converse is very far from true and we shall see more about this below.

II) *For any integer* k, $0 \le k \le d-1$ *and sufficiently general* $L^{d-k} \subset \check{\mathbf{P}}^{n-1}$ *the tangent cone* $C_{P_k(X,L),0}$ *of the polar variety* $P_k(X,L)$ *at the origin consists of:*

- *The union of the cones* $O(V_\alpha)$ *which are of dimension d-k (*$= \dim P_k(X,L)$*).*
- *The polar varieties of dimension d-k, for the projection p associated to L, of the cones* $O(V_\beta)$*, for dim* $O(V_\beta) = d-k+j$*, that is* $P_j(O(V_\beta), L)$*.*

Note that $P_k(X,L)$ is not unique, since it varies with L, but we are saying that their tangent cones have things in common. The V_α's are fixed, so the first part is the fixed part of $C_{P_k(X,L),0}$ because it is independent of L, the second part is the mobile part, since we are talking of polar varieties of certain cones, which by definition move with L.

Proof.
The proof of **I)**, which can be found in [L-T1], is essentially a strengthening of Whitney's lemma (lemma 3.1) using the normal/conormal diagram and the fact that the vanishing of a differential form (the symplectic form in our case) is a closed condition.
The proof of **II)**, also found in [L-T1], is somewhat easier to explain geometrically:
Using our normal/conormal diagram, remember that we can obtain the blowup $E_0(P_k(X,L))$ of the polar variety $P_k(X,L)$ by taking its strict transform under the morphism e_0, and as such we will get the projectivized tangent cone $\mathbf{P}C_{P_k(X,L),0}$ as the fiber over the origin.

The first step is to prove that set-theoretically the projectivized tangent cone can also be expressed as

$$|\mathbf{P}C_{P_k(X,L),0}| = \bigcup_\alpha \kappa'(\hat{e}_0^{-1}(\lambda^{-1}(L)\cap W_\alpha)) = \bigcup_\alpha \kappa'(D_\alpha \cap (\mathbf{P}^{n-1}\times L))$$

Now recall that the intersection $P_k(X,L)\cap X^0$ is dense in $P_k(X,L)$, so for any point $(0,[l]) \in \mathbf{P}C_{P_k(X,L),0}$ there exists a sequence of points $\{x_n\} \subset X^0$ such that the secants $[0x_n]$ converge to it. So, by definition of a polar variety, if $D_{d-k+1} = \check{L}$ and $T_n = T_{x_n}X^0$ then by 3.4 we know that dim $T_n \cap D_{d-k+1} \geq k$ which is a closed condition. In particular if T is a limit of tangent spaces obtained from the sequence $\{T_n\}$, then $T \cap D_{d-k+1} \geq k$ also. But if this is the case then, since the dimension of T is d, there exists a limit of tangent hyperplanes $H \in \kappa^{-1}(0)$ such that $T + D_{d-k+1} \subset H$ which is equivalent to $H \in \kappa^{-1}(0)\cap\lambda^{-1}(L) \neq \emptyset$. Consequently, the point $(0,[l],H) \in \bigcup_\alpha \hat{e}_0^{-1}(\lambda^{-1}(L)\cap W_\alpha)$, and so we have the inclusion:

$$|\mathbf{P}C_{P_k(X,L),0}| \subset \bigcup_\alpha \kappa'(\hat{e}_0^{-1}(\lambda^{-1}(L)\cap W_\alpha))$$

For the other inclusion, recall that $\lambda^{-1}(L)\setminus\kappa^{-1}(0)$ is dense in $\lambda^{-1}(L)$ and so $\hat{e}_0^{-1}(\lambda^{-1}(L))$ is equal set theoretically to the closure in $E_0C(X)$ of $\hat{e}_0^{-1}(\lambda^{-1}(L)\setminus \kappa^{-1}(0))$. Then for any point $(0,[l],H) \in \hat{e}_0^{-1}(\lambda^{-1}(L)\cap\kappa^{-1}(0))$ there exists a sequence $\{(x_n,[x_n],H_n)\}$ in $\hat{e}_0^{-1}(\lambda^{-1}(L)\setminus\kappa^{-1}(0))$ converging to it. Now by commutativity of the diagram, we get that the sequence $\{(x_n,H_n)\} \subset \lambda^{-1}(L)$ and as such the sequence of points $\{x_n\}$ lies in the polar variety $P_k(X,L)$. This implies in particular, that the sequence $\{(x_n,[0x_n])\}$ is contained in $e_0^{-1}(P_k(X,L)\setminus\{0\})$ and the point $(0,[l])$ is in the projectivized tangent cone $|\mathbf{P}C_{P_k(X,L),0}|$.

The second and final step of the proof is to use that from *a)* and *b)* we have that each $D_\alpha \subset I \subset \mathbf{P}^{n-1} \times \check{\mathbf{P}}^{n-1}$ is the conormal space of V_α in $\mathbf{P}^{n-1}$, with the restriction of κ' to D_α being its conormal morphism.

Note that D_α is of dimension $n-2$, and since all the maps involved are just projections, we can take the cones over the V_α's and proceed as in section 2.3. In this setting, we get that since L is sufficiently general, by proposition 3.4 and definition 3.3:

- For the D_α's corresponding to cones $O(V_\alpha)$ of dimension d-k (= dim $P_k(X,L)$), the intersection $D_\alpha \cap \mathbf{C}^n \times L$ is not empty and as such its image is a polar variety $P_0(O(V_\alpha),L) = O(V_\alpha)$.
- For the D_α's corresponding to cones $O(V_\alpha)$ of dimension d-k+j, the intersection $D_\alpha \cap \mathbf{C}^n \times L$ is either empty or of dimension $d-k$ and as such its image is a polar variety of dimension d-k,that is $P_j(O(V_\alpha),L)$.

You can find a proof of these results in [L-T1], [Te3] and [Te4]. □

So for any reduced and purely $d-$dimensional complex analytic germ $(X,0)$, we have a method to “compute” or rather describe, the set of limiting positions of tangent hyperplanes:

1) For all integers k, $0 \leq k \leq d-1$, compute the “general” polar varieties $P_k(X,L)$, leaving in the computation the coefficients of the equations of L as indeterminates. (Partial derivatives, jacobian minors and residual ideals with respect to the jacobian ideal);
2) Compute the tangent cones $C_{P_k(X,L),0}$. (Gröbner basis);
3) Sort out those irreducible components of the tangent cone of each $P_k(X,L)$ which are independent of L;
4) Take the projective duals of the corresponding projective varieties (Elimination).

We have noticed, that among the V_α's, there are those which are irreducible components of $\mathrm{Proj}C_{X,0}$ and those that don't reach the dimension.

Definition 3.5. *The cones $O(V_\alpha)$'s such that*

$$\dim V_\alpha < \dim \mathrm{Proj}C_{X,0}$$

are called exceptional cones.

Remark 3.3. *The dimension of $\kappa^{-1}(0)$ can be large for a singularity $(X,0)$ which has no exceptional cones. This is the case for example if X is the cone over a projective variety of dimension $d-1 < n-2$ in $\mathbf{P}^{n-1}$ whose dual is a hypersurface.*

So, now one may wonder if having no exceptional tangents makes X look like a cone. We wil give a partial answer to this question in section 6 in terms of the Whitney equisingularity along the axis of parameters of the flat family specializing X to its tangent cone.

4 Whitney Stratifications

Whitney had observed, as we can see from the statement of lemma 3.1, that "asymptotically" near 0 a germ $(X,0) \subset (\mathbf{C}^n,0)$ behaves like a cone with vertex 0, in the sense that for any sequence $(x_i)_{i\in\mathbf{N}}$ of nonsingular points of X tending to zero, the limit (up to restriction to a subsequence) of the tangent spaces $T_{x_i}X^0$ contains the limit of the secants $[0x_i]$. Suppose now, that we replace 0 by a nonsingular subspace $Y \subset X$, and we want to force X to "look like a cone with vertex Y".

Definition 4.1. *A cone with vertex Y is a space C equipped with a map*

$$\pi \colon C \longrightarrow Y$$

and homotheties in the fibers, i.e., a morphism $\eta \colon C \times \mathbf{C}^* \to C$ with $\pi \circ \eta = \pi \circ \mathrm{pr}_1$ inducing an action of the multiplicative group $\mathbf{C}^*$ in the fibers of π which has as fixed set the image of a section $\sigma \colon Y \to C$ of π; $\pi \circ \sigma = \mathrm{Id}_Y$ and $\sigma(Y) = \{c \in C | \eta(c,\lambda) = c \ \forall \lambda \in \mathbf{C}^*\}$.

Let us take a look at the basic example we have thus far constructed.

Example 4.1.
The reduced normal cone $|C_{X,Y}| \longrightarrow Y$, with the canonical analytic projection mentioned after definition 3.1.

Now, making an analogy with the opening statement of this section we can ask ourselves: What does it mean that "asymptotically" X is cone-like over Y? Well, here is Whitney's answer, again in terms of tangent spaces and secants:

4.1 Whitney's conditions

Let X be a reduced, pure dimensional analytic space of dimension d, let $Y \subset X$ be a nonsingular analytic subspace containing 0 of dimension t. Choose a local embedding $(X,0) \subset (\mathbf{C}^n,0)$ around 0, and a local retraction $\rho : (\mathbf{C}^n,0) \longrightarrow (Y,0)$. Note that, since Y is nonsingular we can assume it is an open subset of $\mathbf{C}^t$, $(X,0)$ is embedded in an open subset of $\mathbf{C}^t \times \mathbf{C}^{n-t}$ and the retraction ρ coincides with the first projection.

We say that X^0 satisfies Whitney's conditions along Y at 0 if for any sequence of pairs of points $\{(x_i,y_i)\}_{i\in\mathbf{N}} \subset X^0 \times Y$ tending to $(0,0)$ we have:

$$\lim_{i\to\infty}[x_iy_i] \subset \lim_{i\to\infty} T_{x_i}X,$$

where $[x_iy_i]$ denotes the line passing throught these two points. If we compare this to Whitney's lemma, it is just spreading out along Y the fact observed when $Y = \{0\}$.

In fact, Whitney stated 2 conditions, which together are equivalent to the above. Letting ρ be the aforementioned retraction, here we state the 2 conditions:

a) For any sequence $\{x_i\}_{i\in\mathbf{N}} \subset X^0$, tending to 0 we have:

$$T_0Y \subset \lim_{i\to\infty} T_{x_i}X.$$

b) For any sequence $\{x_i\}_{i\in\mathbf{N}} \subset X^0$, tending to 0 we have:

$$\lim_{i\to\infty}[x_i\rho(x_i)] \subset \lim_{i\to\infty} T_{x_i}X.$$

Whitney's conditions can also be characterized in terms of the conormal space and the normal /conormal diagram, as we will see later on.

Recall that our objective is to "stratify" X. What exactly do we mean by stratify, and how do Whitney conditions relate to this? What follows in this section consists mostly of material from [Lip] and [Te3].

Definition 4.2. *A **stratification** of X is a decomposition into a locally finite disjoint union $X = \bigcup X_\alpha$, of non-empty, connected, locally closed subvarietes called* strata*, satisfying:*

(i) Every stratum X_α is smooth (and therefore an analytic manifold).

(ii) For any stratum X_α, with closure $\overline{X_\alpha}$, both the boundary $\partial X_\alpha := \overline{X_\alpha} \setminus X_\alpha$ and the closure $\overline{X_\alpha}$ are closed analytic in X.

(iii) For any stratum X_α, the boundary $\partial X_\alpha := \overline{X_\alpha} \setminus X_\alpha$ is a union of strata.

Stratifications can be determined by *local stratyfing conditions* as follows. We consider conditions $C = C(W_1, W_2, x)$ defined for all $x \in X$ and all pairs $(W_1, x) \supset (W_2, x)$ of subgerms of (X, x) with (W_1, x) equidimensional and (W_2, x) smooth. For example, $C(W_1, W_2, x)$ could signify that the Whitney conditions hold at x.

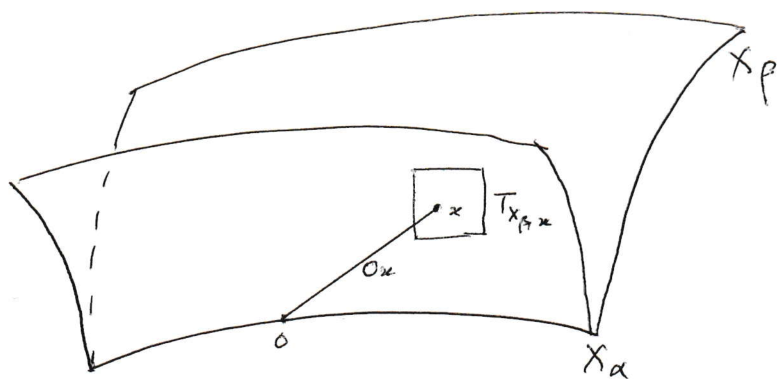
For such a C and for any subvarieties W_1, W_2 of X with W_1 closed and locally equidimensional, and W_2 locally closed, set

$$\begin{aligned} C(W_1, W_2) &:= \{x \in W_2 |\, W_2 \textit{ smooth at } x, \textit{ and if } x \in W_1 \textit{ then } (W_1, x) \supset (W_2, x) \textit{ and} \\ &\qquad C(W_1, W_2, x)\}, \\ \widetilde{C}(W_1, W_2) &:= W_2 \setminus C(W_1, W_2). \end{aligned}$$

The condition C is called *stratifying* if for any such W_1 and W_2, $\widetilde{C}(W_1, W_2)$ is contained in a nowhere dense closed analytic subset of W_2. In fact, it suffices that this be so whenever W_2 is smooth, connected, and contained in W_1.

Going back to our case, it is true that Whitney's conditions are stratifying. See [Whi1, Lemma 19.3, p. 540]. The key point is to prove, given $Y \subset X$ as in section 4.1, the set of points of Y where the pair (X^0, Y) satisfies Whitney's conditions contains the complement of a strict closed analytic subspace of Y. A proof of this different from Whitney's is given below as a consequence of Theorem 4.4.

Definition 4.3. *Let X be as above, then by a* ***Whitney Stratification*** *of X, we mean a stratification $X = \bigcup X_\alpha$, such that for any pair of strata X_β, X_α with $X_\alpha \subset \overline{X_\beta}$, the pair $(\overline{X_\beta}, X_\alpha)$ satisfies the Whitney conditions at every point $x \in X_\alpha$.*



4.2 Stratifications

We will now state two fundamental theorems concerning Whitney's conditions, the first of which was proved by Whitney himself and the second one by R. Thom and J. Mather. The proofs can be found in [Whi1], and [Ma] respectively.

Theorem 4.1. (Whitney) *Let M be a reduced complex analytic space and let $X \subset M$ be a locally closed analytic subspace of M. Then, there exists a Whitney stratification $M = \bigcup M_\alpha$ of M such that:*

i) X is a union of strata.

ii) If $M_\beta \cap \overline{M_\alpha} \neq \emptyset$ then $M_\beta \subset \overline{M_\alpha}$.

In fact, one can prove that *any stratifying condition gives rise to a locally finite stratification of any space X such that all pairs of strata satisfy the given condition.* See ([Lip, §2], [Te3, p. 478-480]).

Given a germ of t-dimensional nonsingular subspace $(Y, 0) \subset (\mathbf{C}^n, 0)$, by a (germ of) local retraction $\rho \colon (\mathbf{C}^n, 0) \to (Y, 0)$ we mean the first projection of a product decomposition $(\mathbf{C}^n, 0) \simeq (Y, 0) \times (\mathbf{C}^{n-t}, 0)$. By the implicit function theorem, such retractions always exist.

Theorem 4.2. (Thom-Mather)
Taking $M = X$ in the previous statement, let $X = \bigcup_\alpha X_\alpha$ be a Whitney stratification of X, let $x \in X$ and let $X_\alpha \subset X$ be the stratum that contains x. Then, for any local embedding $(X, x) \subset (\mathbf{C}^n, 0)$ and any local retraction $\rho : (\mathbf{C}^n, 0) \to (X_\alpha, x)$ and a real number $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$ there exists η_ϵ such that for any $0 < \eta < \eta_\epsilon$ there is a homeomorphism h

$$\begin{tikzcd}
\mathbf{B}(0,\epsilon) \cap \rho^{-1}(\mathbf{B}(0,\eta) \cap M_\alpha) \arrow[rr, "h"] \arrow[dr, "\rho"'] & & (\rho^{-1}(x) \cap \mathbf{B}(0,\epsilon)) \times (X_\alpha \cap \mathbf{B}(0,\eta)) \arrow[dl, "pr_2"] \\
& X_\alpha \cap \mathbf{B}(0,\eta) &
\end{tikzcd}$$

compatible with the retraction ρ, and inducing for each stratum X_β such that $\overline{X_\beta} \supset X_\alpha$ a homeomorphism

$$\overline{X_\beta} \cap \mathbf{B}(0,\epsilon) \cap \rho^{-1}(\mathbf{B}(0,\eta) \cap X_\alpha) \longrightarrow (\overline{X_\beta} \cap \rho^{-1}(x) \cap (\mathbf{B}(0,\epsilon)) \times (X_\alpha \cap \mathbf{B}(0,\eta))$$

where $\mathbf{B}(0,\epsilon)$ denotes the ball in $\mathbf{C}^n$ with center in the origin and radius ϵ.

In short, each $\overline{X_\beta}$, or if you prefer, the stratified set X, is locally topologically trivial along X_α at x. A natural question arises then, is the converse to the Thom-Mather theorem true? That is, does local topological triviality implies the Whitney conditions? The question was posed by the second author in [Te1] for families of hypersurfaces with isolated singulaities.

The answer is *NO*, in [BS] Briançon and Speder showed that the family of surface germs

$$z^5 + ty^6z + y^7x + x^{15} = 0$$

(each member, for small t, having an isolated singularity at the origin) is locally topologically trivial, but not Whitney.

We shall see below in section 5 that there is a converse, proved by Lê and Teissier (see [L-T3], and [T2]). Let us refer to the conclusion of the Thom-Mather theorem as condition (TT) (local topological triviality), so we can restate theorem 4.2 as: Whitney implies (TT). Let $X = \bigcup X_\alpha$ be a stratification of the complex analytic space X and let $d_\alpha = \dim X_\alpha$. We say that a stratification satisfies the condition $(TT)^*$ (local topological triviality for the general sections) if in addition to the condition (TT), for every $x \in X_\alpha$, there exists for every $k > \dim X_\alpha$ a Zariski open set Ω in $G(k - d_\alpha, n - d_\alpha)$ such that for any nonsingular space E containing X_α and such that $T_xE \in \Omega$, the (set-theoretic) intersection $\overline{X_\beta} \cap E$ satisfies (TT) for all X_β such that $\overline{X_\beta} \supset X_\alpha$.

Theorem 4.3. (Lê-Teissier)
For a stratification $X = \bigcup X_\alpha$ of a complex analytic space X, the following conditions are equivalent:

1) $X = \bigcup X_\alpha$ *is a Whitney stratification.*

2) $X = \bigcup X_\alpha$ *satifies condition* $(TT)^*$.

4.3 Whitney stratifications and polar varieties

We now have all the ingredients, so it is time to put them together. Let us fix a nonsingular subspace $Y \subset X$ through 0 of dimension t as before, recall that we are assuming X is a reduced, pure dimensional analytic space of dimension d. Let us recall the notations of section 3.1 and take a look at the normal/conormal diagram:

$$\begin{tikzcd} E_YC(X) \arrow[r, "\hat{e}_Y"] \arrow[d, "\kappa'"] \arrow[dr, "\xi"] & C(X) \arrow[d, "\kappa"] \\ E_YX \arrow[r, "e_Y"'] & Y \subset X \end{tikzcd}$$

Remember that $E_YC(X)$ is the blowup of $\kappa^{-1}(Y)$ in $C(X)$, and κ' is obtained from the universal property of the blowup, with respect to E_YX and the map ξ. Just as in the case where $Y = \{0\}$, it is worth mentioning that $E_YC(X)$ lives inside the fiber product $C(X) \times_X E_YX \subset X \times \mathbf{P}^{n-1-t} \times \check{\mathbf{P}}^{n-1}$ and can be described in the following way: take the inverse image of $E_YX \setminus e_Y^{-1}(Y)$ in $C(X) \times_X E_YX$ and close it, thus obtaining κ' as the restriction of the second projection to this space.

Looking at the definitions, it is not difficult to prove that, if we consider the divisor:

$$D = |\xi^{-1}(Y)| \subset E_YC(X), \quad D \subset Y \times \mathbf{P}^{n-1-t} \times \check{\mathbf{P}}^{n-1},$$

we have that, denoting by $\check{\mathbf{P}}^{n-1-t}$ the space of hyperplanes containing T_0Y:

- •) The pair (X, Y) satisfies Whitney's condition a) along Y if and only if we have the set theoretical equality $C(X) \cap C(Y) = \kappa^{-1}(Y)$. It satisfies Whitney's condition a) at 0 if and only if $\xi^{-1}(0) \subset \mathbf{P}^{n-1-t} \times \check{\mathbf{P}}^{n-1-t}$.

Indeed, note that we have the inclusion $C(X) \cap C(Y) \subset \kappa^{-1}(Y)$, so it all reduces to having the inclusion $\kappa^{-1}(Y) \subset C(Y)$, and since we have already seen that every limit of tangent hyperplanes H contains a limit of tangent spaces T, we are just saying that every limit of tangent hyperplanes to X at a point $y \in Y$, must be a tangent hyperplane to Y at y. Following this line of thought, satisfying condition a) at 0 is then equivalent to the inclusion $\kappa^{-1}(0) \subset \{0\} \times \check{\mathbf{P}}^{n-1-t}$ which implies $\xi^{-1}(0) \subset \mathbf{P}^{n-1-t} \times \check{\mathbf{P}}^{n-1-t}$.

- •) The pair (X^0, Y) satisfies Whitney's condition b) at 0 if and only if $\xi^{-1}(0)$ is contained in the incidence variety $I \subset \mathbf{P}^{n-1-t} \times \check{\mathbf{P}}^{n-1-t}$.

This is immediate from the relation between limits of tangent hyperplanes and limits of tangent spaces and the interpretation of $E_Y C(X)$ as the closure of the inverse image of $E_Y X \setminus e_Y^{-1}(Y)$ in $C(X) \times_X E_Y X$ since we are basically taking limits as $x \to Y$ of couples (l, H) where l is the direction in $\mathbf{P}^{n-1-t}$ of a secant line $[yx]$ with $x \in X^0 \setminus Y, y = \rho(x) \in Y$, where ρ is some local retraction of the ambient space to the nonsingular subspace Y, and H is a tangent hyperplane to X at x. So, in order to verify the Whitney conditions, it is important to control the geometry of the projection $D \to Y$ of the divisor $D \subset E_Y C(X)$.

Remark 4.1. *Although it is beyond the scope of these notes, we point out to the interested reader that there is an algebraic definition of the Whitney conditions for X^0 along $Y \subset X$ solely in terms of the ideals defining $C(X) \cap C(Y)$ and $\kappa^{-1}(Y)$ in $C(X)$. Indeed, the inclusion $C(X) \cap C(Y) \subset \kappa^{-1}(Y)$ follows from the fact that the sheaf of ideals $\mathcal{J}_{C(X)\cap C(Y)}$ defining $C(X) \cap C(Y)$ in $C(X)$ contains the sheaf of ideals $\mathcal{J}_{\kappa^{-1}(Y)}$ defining $\kappa^{-1}(Y)$, which is generated by the pull-back by κ of the equations of Y in X. What was said above means that condition a) is equivalent to the second inclusion in:*

$$\mathcal{J}_{\kappa^{-1}(Y)} \subseteq \mathcal{J}_{C(X)\cap C(Y)} \subseteq \sqrt{\mathcal{J}_{\kappa^{-1}(Y)}}.$$

It is proved in [L-T2], proposition 1.3.8 that having both Whitney conditions is equivalent to having the second inclusion in:

$$\mathcal{J}_{\kappa^{-1}(Y)} \subseteq \mathcal{J}_{C(X)\cap C(Y)} \subseteq \overline{\mathcal{J}_{\kappa^{-1}(Y)}}.$$

where the bar denotes the integral closure of the sheaf of ideals, which is contained in the radical and in general much closer to the ideal than the radical. The second inclusion is an algebraic expression of the fact that locally near every point of the common zero set the modules of local generators of the ideal $\mathcal{J}_{C(X)\cap C(Y)}$ are bounded, up to a multiplicative constant, by the supremum of the modules of generators of $\mathcal{J}_{\kappa^{-1}(Y)}$. See [Lej-Te]. We shall see more about it in section 6.

Definition 4.4. *Let $Y \subset X$ as before. Then we say that the local polar variety $P_k(X; L^{d-k})$ is equimultiple along Y at a point $x \in X \cap Y$ if the following two conditions are satisfied:*

(i) If $P_k(X; L^{d-k}) \neq \emptyset$ at x, then $P_k(X; L^{d-k}) \supset Y$ in a neighborhood of x.

(ii) The multiplicities $m_y(P_k(X; L^{d-k}))$ are locally constant for $y \in Y$ in a neighborhood of x.

Now we can state the main theorem of these notes, a complete proof of which can be found in [Te3, Chapter V, Thm 1.2, p. 455].

Theorem 4.4. (Teissier; see also [HM] for another proof and [Co], [CM] for extensions to subanalytic sets) *Given $0 \in Y \subset X$ as before, the following conditions are equivalent, where ξ is the diagonal map in the normal/conormal diagram above:*

1) The pair (X^0, Y) *satisfies Whitney's conditions at* 0.

2) The local polar varieties $P_k(X, L)$, $0 \leq k \leq d-1$, *are equimultiple along* Y *(at* 0*), for general* L.

3) $\dim \xi^{-1}(0) = n - 2 - t$.

Note that since $\dim \mathrm{D} = n-2$ condition 3) is open and the theorem implies that (X^0, Y) satisfies Whitney's conditions at 0 if and only if it satisfies Whitney's conditions in a neighborhood of 0.

Note also that by semicontinuity of fiber dimension, condition 3) is satisfied outside of a closed analytic subspace of Y, which shows that Whitney's is a stratifying condition.

Moreover, since a blowup does not lower dimension, the condition $\dim \xi^{-1}(0) = n-2-t$ implies $\dim \kappa^{-1}(0) \leq n-2-t$. So that, in particular $\kappa^{-1}(0) \not\supset \check{P}^{n-1-t}$, where $\check{P}^{n-1-t}$ denotes as before the space of hyperplanes containing T_0Y. This tells us that *a general hyperplane containing* T_0Y *is not a limit of tangent hyperplanes to* X. This fact is crucial in the proof that Whitney conditions are equivalent to the equimultiplicity of polar varieties since it allows the start of an inductive process. In the actual proof of [Te3], one reduces to the case where $\dim Y = 1$ and shows by a geometric argument that the Whitney conditions imply that the polar curve has to be empty, which gives a bound on the dimension of $\kappa^{-1}(0)$. Conversely, the equimultiplicity condition on polar varieties gives bounds on the dimension of $\kappa^{-1}(0)$ by implying the emptiness of the polar curve and on the dimension of $e_Y^{-1}(0)$ by Hironaka's result, hence a bound on the dimension of $\xi^{-1}(0)$.

It should be noted that Hironaka had proved in [Hi] that the Whitney conditions for X^0 along Y imply equimultiplicity of X along Y.

Finally, a consequence of the theorem is that *given a complex analytic space* X*, there is a unique minimal (coarsest) Whitney stratification*; any other Whitney stratification of X is obtained by adding strata inside the strata of the minimal one. A detailed explanation of how to construct this "canonical" Whitney stratification using theorem 4.4, and a proof that this is in fact the coarsest one can be found in [Te3, Chap. VI, § 3].

4.4 Relative Duality

There still is another result which can be expressed in terms of the relative conormal space and consequently of course, in terms of relative duality. We first need the:

Proposition 4.5. (Versions of this appear in [La], [Sa].) *Let $X \subset \mathbf{C}^n$ be a reduced analytic subspace of dimension d and let $Y \subset X$ be a nonsingular analytic proper subspace of dimension t. Let $\varphi : \mathfrak{X} \to \mathbf{C}$ be the specialization of X to the normal cone $C_{X,Y}$ of Y in X, and let $C(X), C(Y)$ denote the conormal spaces of X and Y respectively, in $\mathbf{C}^n \times \check{\mathbf{C}}^n$. Then the relative conormal space*

$$\kappa_\varphi \circ \varphi := q : C_\varphi(\mathfrak{X}) \to \mathfrak{X} \to \mathbf{C}$$

is isomorphic, as an analytic space over $\mathbf{C}$, to the specialization space of $C(X)$ to the normal cone $C_{C(X),C(Y)\cap C(X)}$ of $C(Y) \cap C(X)$ in $C(X)$. In particular, the fibre $q^{-1}(0)$ is isomorphic to this normal cone.

Proof.
Let $I \subset J$ be the coherent ideals of the structure sheaf of $\mathbf{C}^n$ that define the analytic subspaces X and Y respectively, and let $p : \mathfrak{D} \to \mathbf{C}$ be the specialization space of $C(X)$ to the normal cone of $C(Y) \cap C(X)$ in $C(X)$. Note that, in this context, both spaces $\mathfrak{D}$ and $C_\varphi(\mathfrak{X})$ are analytic subspaces of $\mathbf{C} \times \mathbf{C}^n \times \check{\mathbf{C}}^n$. Let us consider a local chart, in such a way that $Y \subset X \subset \mathbf{C}^n$, locally becomes $\mathbf{C}^t \subset X \subset \mathbf{C}^n$ with local associated coordinates:

$$(v, y_1 \ldots, y_t, z_{t+1}, \ldots, z_n, a_1, \ldots, a_t, b_{t+1}, \ldots, b_n)$$

in $\mathbf{C} \times \mathbf{C}^n \times \check{\mathbf{C}}^n$.

Let $J := \langle z_{t+1}, \ldots, z_n \rangle$ be the ideal defining Y in $\mathbf{C}^n$. One can verify that, just as in the case of the tangent cone (see exercise 2.1 b)), if $f_1, \ldots, f_r$, are local equations for X in $\mathbf{C}^n$ such that their initial forms $in_J f_i$ generate the ideal of $gr_J O_X$ defining the normal cone of X along Y. Then the equations $F_i := v^{-k_i} f_i(y, vz),\ i = 1, \ldots, r$, where $k_i = sup\{k | f_i \in J^k\}$ locally define the specialization space $\varphi : \mathfrak{X} \to \mathbf{C}$ of X to the normal cone $C_{X,Y}$. Furthermore, if you look closely at the equations, you will easily verify that the open set $\mathfrak{X} \setminus \varphi^{-1}(0)$ is isomorphic over $\mathbf{C}^*$ to $\mathbf{C}^* \times X$, via the morphism Φ defined by the map $(v, y, z) \mapsto (v, y, vz)$.

We can now consider the relative conormal space,

$$q : C_\varphi(\mathfrak{X}) \to \mathfrak{X} \to \mathbf{C},$$

and thanks to the fact that $\mathfrak{X} \setminus \varphi^{-1}(0)$ is an open subset with fibers $\mathfrak{X}(v)$ isomorphic to X, the previous isomorphism Φ implies that $C_\varphi(\mathfrak{X}) \setminus q^{-1}(0)$ is isomorphic over $\mathbf{C}^*$ to $\mathbf{C}^* \times C(X)$.

On the other hand, note that, since $J = \langle z_{t+1}, \ldots, z_n \rangle$ in $\mathcal{O}_{\mathbf{C}^n}$, then the conormal space $C(Y)$ is defined in $\mathbf{C}^n \times \check{\mathbf{P}}^{n-1}$ by the sheaf of ideals J^C generated (in $\mathcal{O}_{\mathbf{C}^n \times \check{\mathbf{C}}^n}$) by $(z_{t+1}, \ldots, z_n, a_1, \ldots, a_t)$.
Thus, if we chose local generators $(g_1, \ldots, g_s)$ for the sheaf of ideals defining $C(X) \subset \mathbf{C}^n \times \check{\mathbf{P}}^{n-1}$, whose $J^C \mathcal{O}_{C(X)}$-initial forms generate the initial ideal,

the equations $G_i(v,y,z,a,b) = v^{-l_i} g_i(v,y,vz,va,b)$ locally define a subspace $\mathbf{D} \subset \mathbf{C} \times \mathbf{C}^n \times \check{\mathbf{P}}^{n-1}$ with a faithfully flat projection $\mathbf{D} \xrightarrow{p} \mathbf{C}$, where the fiber $p^{-1}(0)$ is the normal cone $C_{C(X),C(Y)\cap C(X)}$. Note that in this case the l_i's are defined with respect to the ideal $J^C\mathcal{O}_{C(X)}$.
The open set $\mathbf{D} \setminus p^{-1}(0)$ is isomorphic to $\mathbf{C}^* \times C(X)$ via the morphism defined by $(v,y,z,a,b) \mapsto (v,y,vz,va,b)$.

This last morphism is a morphism of the ambient space to itself over $\mathbf{C}$

$$\begin{aligned} \psi : \mathbf{C} \times \mathbf{C}^n \times \check{\mathbf{C}}^n &\longrightarrow \mathbf{C} \times \mathbf{C}^n \times \check{\mathbf{C}}^n \\ (v,y,z,a,b) &\longmapsto (v,y,vz,va,b) \end{aligned}$$

which turns out to be an isomorphism when restricted to the open dense set $\mathbf{C}^* \times \mathbf{C}^n \times \check{\mathbf{C}}^n$. So, if we take the analytic subspace $\mathbf{C}^* \times C(X)$ in the image, as a result of what we just said, we have the equality $\psi^{-1}(\mathbf{C}^* \times C(X)) = \mathbf{D} \setminus p^{-1}(0)$.

Finally, recall that both morphisms defining q, are induced by the natural projections

$$\mathbf{C} \times \mathbf{C}^n \times \check{\mathbf{C}}^n \to \mathbf{C} \times \mathbf{C}^n \to \mathbf{C},$$

and therefore we have a commutative diagram:

$$\begin{tikzcd}
C_\varphi(\mathfrak{X}) \arrow[r, hook] \arrow[d, "\kappa_\varphi"'] \arrow[ddr, bend right=60, "q"'] & \mathbf{C} \times \mathbf{C}^n \times \check{\mathbf{C}}^n \arrow[r, "\psi"] \arrow[d, "\pi"] & \mathbf{C} \times \mathbf{C}^n \times \check{\mathbf{C}}^n \arrow[d, "\pi"] \\
\mathfrak{X} \arrow[r, hook] \arrow[dr, "\varphi"'] & \mathbf{C} \times \mathbf{C}^n \arrow[r, "\phi"] \arrow[d] & \mathbf{C} \times \mathbf{C}^n \arrow[dl] \\
& \mathbf{C} &
\end{tikzcd}$$

To finish the proof, it is enough to check that the image by ψ of $C_\varphi(\mathfrak{X}) \setminus q^{-1}(0)$ is equal to $\mathbf{C}^* \times C(X)$, since we already know that $\psi^{-1}(\mathbf{C}^* \times C(X)) = \mathbf{D} \setminus p^{-1}(0)$ and so we will find an open dense set common to both spaces, which are faithfully flat over $\mathbf{C}$, and consequently the closures will be equal.

Let $(y,z) \in X$ be a smooth point, then the vectors

$$\nabla f_i(y,z) := \left(\frac{\partial f_i}{\partial y_1}(y,z), \cdots, \frac{\partial f_i}{\partial y_t}(y,z), \frac{\partial f_i}{\partial z_{t+1}}(y,z), \cdots, \frac{\partial f_i}{\partial z_n}(y,z) \right),$$

representing the 1-forms df_i in the basis dy_j, dz_i , generate the linear subspace of $\check{\mathbf{C}}^n$ encoding all the 1-forms that vanish on the tangent space $T_{(y,z)}X^0$, i.e. the fiber over the point (y,z) in $C(X)$. Analogously, let $(v,y,z) \in \mathfrak{X}$ be a smooth point in $\mathfrak{X} \setminus \varphi^{-1}(0)$, then the vectors

$$\nabla F_i(v,y,z) := \left(\frac{\partial F_i}{\partial y_1}(v,y,z), \cdots, \frac{\partial F_i}{\partial y_t}(v,y,z), \frac{\partial F_i}{\partial z_{t+1}}(v,y,z), \cdots, \frac{\partial F_i}{\partial z_n}(v,y,z) \right)$$

generate the linear subspace of $\check{\mathbf{C}}^n$ encoding of all the 1-forms that vanish on the tangent space $T_{(v,y,z)}\mathfrak{X}(v)^0$, i.e. the fiber over the point (v,y,z) in $C_\varphi(\mathfrak{X})$. But, according to our choice of (v,y,z), we know that $\phi((v,y,z)) = (v,y,vz)$ is a smooth point of $\mathbf{C}^* \times X$ and in particular (y,vz) is a smooth point of X. Moreover, notice that:

$$\begin{aligned}\frac{\partial F_i}{\partial y_j}(v,y,z) &= v^{-n_i}\frac{\partial f_i}{\partial y_j}(y,vz)\\ \frac{\partial F_i}{\partial z_k}(v,y,z) &= v^{-n_i+1}\frac{\partial f_i}{\partial z_k}(y,vz)\end{aligned}$$

and therefore the image of the corresponding point

$$\begin{aligned}\psi(v,y,z,\frac{\partial F_i}{\partial y_j}(v,y,z),\frac{\partial F_i}{\partial z_k}(v,y,z)) &= (v,y,vz,v^{-n_i+1}\frac{\partial f_i}{\partial y_j}(y,vz),v^{-n_i+1}\frac{\partial f_i}{\partial z_k}(y,vz))\\ &= (v,y,vz,v^{-n_i+1}\nabla f_i(y,vz))\end{aligned}$$

is actually a point in $\mathbf{C}^*\times C(X)$. Since $v \neq 0$, the $v^{-n_i+1}\nabla f_i(y,vz)$ also generate the fiber over the point $(v,y,vz) \in \mathbf{C}^* \times X$ by the map $\mathbf{C}^* \times C(X) \to \mathbf{C}^* \times X$ induced by κ_φ and the isomorphism $\varphi^{-1}(\mathbf{C}^*) \simeq \mathbf{C}^* \times X$, which implies that ψ sends $C_\varphi(X) \setminus q^{-1}(0)$ onto $\mathbf{C}^* \times C(X)$. ☐

Going back to our normal-conormal diagram:

$$\begin{tikzcd} E_Y C(X) \arrow[r, "\hat{e}_Y"] \arrow[d, "\kappa'"] \arrow[dr, "\xi"] & C(X) \arrow[d, "\kappa"] \\ E_Y X \arrow[r, "e_Y"'] & Y \subset X \end{tikzcd}$$

Consider the irreducible components $D_\alpha \subset Y \times \mathbf{P}^{n-1-t} \times \check{\mathbf{P}}^{n-1}$ of $D = |\xi^{-1}(Y)|$, that is $D = \bigcup D_\alpha$, and its images:

$$\begin{aligned}V_\alpha &= \kappa'(D_\alpha) \subset Y \times \mathbf{P}^{n-1-t},\\ W_\alpha &= \hat{e}_Y(D_\alpha) \subset Y \times \check{\mathbf{P}}^{n-1}.\end{aligned}$$

Then we have:

Theorem 4.6. (Lê-Teissier, see [L-T2, Thm 2.1.1]) *The equivalent statements of theorem 4.4 are also equivalent to:*
For each α, the irreducible divisor D_α is the relative conormal space of its image $V_\alpha \subset C_{X,Y} \subset Y \times \mathbf{C}^{n-t}$ with respect to the canonical analytic projection $Y \times \mathbf{C}^{n-t} \to Y$ restricted to V_α, and all the fibers of the restriction $\xi : D_\alpha \to Y$ have the same dimension near 0.

In particular, Whitney's conditions are equivalent to the equidimensionality of the fibers of the map $D_\alpha \to Y$, plus the fact that each D_α is contained in $Y \times \mathbf{P}^{n-1-t} \times \check{\mathbf{P}}^{n-1-t}$, where $\check{\mathbf{P}}^{n-1-t}$ is the space of hyperplanes containing the tangent space $T_{Y,0}$, and the contact form on $\mathbf{P}^{n-1-t} \times \check{\mathbf{P}}^{n-1-t}$ vanishes on the smooth points of $D_\alpha(y)$ for $y \in Y$. This means that each D_α is Y-lagrangian and is equivalent to a relative (or fiberwise) duality:

$$\begin{tikzcd} D_\alpha \arrow[r] \arrow[d] & W_\alpha = Y-\text{dual of } V_\alpha \subset Y \times \check{\mathbf{P}}^{n-1-t} \\ Y \times \mathbf{P}^{n-1-t} \supset V_\alpha & \end{tikzcd}$$

The proof uses that the Whitney conditions are stratifying, and that theorem 4.4 and the result of remark 4.1 imply[6] that D_α is the conormal of its image over a dense open set of Y, and the condition $\dim \xi^{-1}(0) = n-2-t$ then gives exactly what is needed, in view of proposition 2.5, for D_α to be Y-lagrangian. Finally, we want to state another result relating Whitney's conditions to the dimension of the fibers of some related maps. A complete proof of this result can be found in [L-T2, Prop. 2.1.5 and Cor.2.2.4.1].

Corollary 4.1. *Using the notations above we have:*

1) *The pair (X^0, Y) satisfies Whitney's conditions at* 0 *is and only if for each α the dimension of the fibers of the projection $W_\alpha \to Y$ is locally constant near* 0.

2) *The pair (X^0, Y) satisfies Whitney's conditions at* 0 *is and only if for each α the dimension of the fibers of the projection $V_\alpha \to Y$ is locally constant near* 0.

Remark 4.2. *The fact that the Whitney conditions, whose original definition translates as the fact that $\xi^{-1}(Y)$ is in $Y \times \mathbf{P}^{n-1-t} \times \check{\mathbf{P}}^{n-1-t}$ and not just $Y \times \mathbf{P}^{n-1-t} \times \check{\mathbf{P}}^{n-1}$ (condition a) and moreover lies in the product $Y \times I$ of Y with the incidence variety $I \subset \mathbf{P}^{n-1-t} \times \check{\mathbf{P}}^{n-1-t}$ (condition b)), are in fact of a lagrangian, or legendrian, nature, explains their stability by general sections (by nonsingular subspaces containing Y) and linear projections.*

Problem: The fact that the Whitney conditions are of an algebraic nature, since they can be translated as an equimultiplicity condition for polar varieties by theorem 4.4 leads to the following question: given a germ $(X,x) \subset (\mathbf{C}^n, 0)$ of a reduced complex analytic space, endowed with its minimal Whitney stratification, does there exist a germ $(\mathcal{Y}, 0) \subset (\mathbf{C}^N, 0)$ of an *algebraic* variety and a germ $(\mathcal{H}, 0) \subset (\mathbf{C}^N, 0)$ of a nonsingular *analytic variety* transversal to the stratum of 0 in the minimal Whitney stratification of $(\mathcal{Y}, 0)$ such that $(X, 0)$ with its minimal Whitney stratification is analytically isomorphic to the intersection of $(\mathcal{Y}, 0)$, with its minimal Whitney stratification, with $(\mathcal{H}, 0)$ in $(\mathbf{C}^N, 0)$?

[6]The proof of this in [L-T2] uses a lemma, p.559, whose proof is incorrect, but easy to correct.

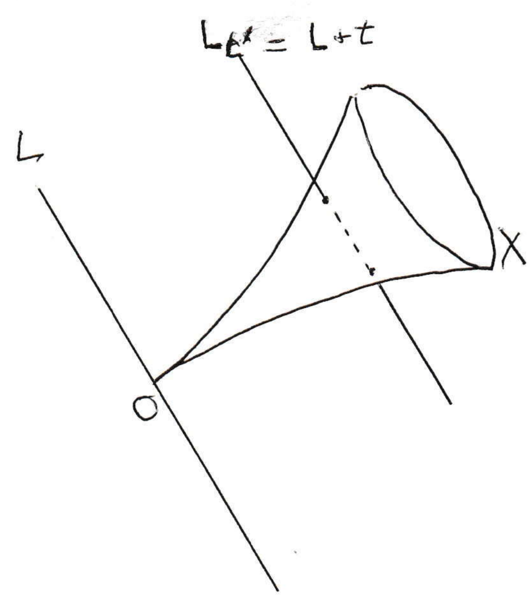
5 Whitney stratifications and the local total topological type

Warning *In this section and section 7, we modify the notation for polar varieties; the general linear space defining each polar variety becomes implicit, while the point at which the polar variety is defined appears in the notation $P_k(X,x)$.*

We have seen how to associate to a reduced equidimensional germ (X,x) of a d-dimensional complex analytic space a generalized multiplicity (recall that $(X,x) = P_0(X,x)$):

$$(X,x) \mapsto (m_x(X,x), m_x(P_1(X,x)), \ldots, m_x(P_{d-1}(X,x)))\,.$$

We know that the multiplicity $m_x(X)$ of a reduced germ (X,x) of a d-dimensional complex analytic space has a geometric interpretation as follows: given a local embedding $(X,x) \subset (\mathbf{C}^n, 0)$ there is a dense Zariski open set U of the Grassmannian of $n-d$-dimensional linear subspaces $L \subset \mathbf{C}^n$ such that for $L \in U$, with equation $\ell(z) = 0$, there exists $\epsilon > 0$ and $\eta(\epsilon, \ell) > 0$ such that the affine linear space $L_{t'} = \ell^{-1}(t')$ intersects X transversally in $m_x(X)$ points inside the ball $\mathbf{B}(0,\epsilon)$ whenever $0 < |t'| < \eta(\epsilon, \ell)$. Taking $t \in \mathbf{B}(0,\epsilon)$ such that $\ell(t) = t'$, we can write $L_{t'}$ as $L + t$.



We may ask whether there is a similar interpretation of the other polar multiplicities in terms of the local geometry of $(X,x) \subset (\mathbf{C}^n,0)$. The idea, as in many other instances in geometry, is to generalize the number of intersection points $\mathrm{card}\{L_t \cap X\}$ by the Euler-Poincaré characteristic $\chi(L_t \cap X)$ when the dimension of the intersection is > 0 because the dimension of L_t is $> n-d$.

Proposition 5.1. (Lê-Teissier, see [L-T3]) *Let $X = \bigcup_\alpha X_\alpha$ be a Whitney stratified complex analytic set. Given $x \in X_\alpha$, choose a local embedding $(X,x) \subset (\mathbf{C}^n,0)$. Set $d_\alpha = \dim X_\alpha$. For each $n > i > d_\alpha$ there exists a Zariski open dense subset $W_{\alpha,i}$ in the Grassmannian $G(n-i,n)$ and for each $L_i \in W_{\alpha,i}$ a semi-analytic subset E_{L_i} of the first quadrant of $\mathbf{R}^2$, of the form $\{(\epsilon,\eta) | 0 < \epsilon < \epsilon_0, 0 < \eta < \phi(\epsilon)\}$ with $\phi(\epsilon)$ a certain Puiseux series in ϵ, such that the homotopy type of the intersection $X \cap (L_i + t) \cap \mathbf{B}(0,\epsilon)$ for $t \in \mathbf{C}^n$ is independent of $L_i \in W_{\alpha,i}$ and (ϵ,t) provided that $(\epsilon,|t|) \in E_{L_i}$. Moreover, this homotopy type depends only on the stratified set X and not on the choice of $x \in X_\alpha$ or the local embedding. In particular the Euler-Poincaré characteristics $\chi_i(X,X_\alpha)$ of these homotopy types are invariants of the stratified analytic set X.*

Definition 5.1. *The Euler-Poincaré characteristics $\chi_i(X,X_\alpha)$ are called the local vanishing Euler-Poincaré characteristics of X along X_α.*

Corollary 5.1. (Kashiwara; see [K1], [K2]) *The Euler-Poincaré characteristics $\chi(X,X_\alpha)$ of the corresponding homotopy types when $i = d_\alpha + 1$ depend only on the stratified set X and the stratum X_α.*

The invariants $\chi(X,X_\alpha)$ appeared for the first time in [K1], in connexion with Kashiwara's index theorem for maximally overdetermined systems of linear differential equations.

Example 5.1.

- Let d be the dimension of X. Taking $X_\alpha = \{x\}$, which is permissible by Whitney's lemma (lemma 3.1), and $i = d$ gives $\chi_d(X,\{x\}) = m_x(X)$, as we saw above.
- Assume that $(X,x) \subset (\mathbf{C}^{d+1},0)$ is a hypersurface with isolated singularity at the point x (taken as origin in $\mathbf{C}^{d+1}$), defined by $f(z_1,\ldots,z_{d+1}) = 0$. By Whitney's lemma (lemma 3.1), in a sufficiently small neighborhood of x, the minimal Whitney stratification (see the end of section 4.3) is $(X \setminus \{x\}) \cup \{x\}$, and we have

  $$\chi_i(X,\{x\}) = 1 + (-1)^{d-i}\mu^{(d+1-i)}(X,x), \qquad (*)$$

  where $\mu^{(k)}(X,x)$ is the Milnor number of the restriction of the function f to a general linear space of dimension k through x.

  Let us recall that the Milnor number $\mu^{(d+1)}(X,x)$ of an isolated singularity of hypersurface as above is defined algebraically as the multiplicity in $\mathbf{C}\{z_1,\ldots,z_{d+1}\}$ of the jacobian ideal $j(f) = (\frac{\partial f}{\partial z_1},\ldots,\frac{\partial f}{\partial z_{d+1}})$, which is also the dimension of the $\mathbf{C}$-vector space $\frac{\mathbf{C}\{z_1,\ldots,z_{d+1}\}}{j(f)}$ since in this case the partial derivatives form a regular sequence. Topologically it is defined by the fact that for $0 < |\lambda| << \epsilon << 1$ the *Milnor fiber* $f^{-1}(\lambda) \cap \mathbf{B}(0,\epsilon)$ has the homotopy type of a bouquet of $\mu^{(d+1)}(X,x)$ spheres of dimension d. In fact this is true of any *smoothing* of (X,x) that is, any nonsingular fiber in an analytic family $F(v,z_1,\ldots,z_{d+1}) = 0$ with $F(0,z_1,\ldots,z_{d+1}) = f(z_1,\ldots,z_{d+1})$, within a ball $\mathbf{B}(0,\epsilon)$ and for $0 < |v| << \epsilon << 1$. This is a consequence of the fact that the basis of the miniversal deformation of an isolated singularity of hypersurface (or more generally, complete intersection) is nonsingular, and thus irreducible, and the smooth fibers are the fibers of a locally trivial fibration over the (connected) complement of the discriminant; see [Te2, §4]. Since $f^{-1}(0) \cap \mathbf{B}(0,\epsilon)$ is contractible the Milnor fiber has $\mu^{(d+1)}(X,x)$ *vanishing cycles* of dimension d. For all this, see [Mi]. The Milnor number of the restriction of the function f to a general i-dimensional linear space through 0 is well defined and does not depend on the choice of the embedding $(X,x) \subset (\mathbf{C}^{d+1},0)$ or the general linear space in $\mathbf{C}^{d+1}$ but only on the analytic algebra $\mathcal{O}_{X,x}$. It is denoted by $\mu^{(i)}(X,x)$; by convention, $\mu^{(0)}(X,x) = 1$. See [Te1].

Let us now prove the equality $(*)$. By the results of [Te1], it suffices to prove the equality for $i = 1$. Then, we know by proposition 2.3 that a general hyperplane L_1 through x is not a limit of tangent hyperplanes to X at nonsingular points. Thus, if $0 < |t| << \epsilon$, the intersection $X \cap (L_1 + t) \cap \mathbf{B}(0,\epsilon)$ is nonsingular because it is a transversal intersection of nonsingular varieties. For the same reason, the intersection $L_1 \cap X \cap \mathbf{B}(0,\epsilon)$ is nonsingular outside of the origin. Choosing coordinates so that L_1 is given by $z_1 = 0$, we see that the intersections with a sufficiently small ball $\mathbf{B}(0,\epsilon)$ around x of $f(t,z_2,\ldots,z_{d+1}) = 0$ and $f(0,z_2,\ldots,z_{d+1}) = \lambda$, for small $|t|$, $|\lambda|$, are two smoothings of the hypersurface with isolated singularity $f(0,z_2,\ldots,z_{d+1}) = 0$. They are therefore diffeomorphic and thus have the same Euler characteristic. The first one is our $\chi_1(X,\{x\})$ and the second one is the Euler characteristic of a Milnor fiber of $f(0,z_2,\ldots,z_{d+1})$, which is $1 + (-1)^{d-1}\mu^{(d)}(X,x)$ in view of the bouquet description recalled above.

It is known from [L-T1, 4.1.8] (see also just after theorem 5.2 below) that the image of a general polar variety $P_k(X,x)$ by the projection $p\colon (\mathbf{C}^n,0) \to (\mathbf{C}^{d-k+1},0)$ which defines it has at the point $p(x)$ the same multiplicity as $P_k(X,x)$ at x. This is because for a general projection p the kernel of p is transversal to the tangent cone $C_{P_k(X,x),x}$ of the corresponding polar variety. Using this in the case of isolated singularities of hypersurfaces, it is known from ([Te1, Chap. II, proposition 1.2 and cor. 1.4] or [Te7, corollary p.610] that the multiplicities of the polar varieties are computed from the $\mu^{(k)}(X,x)$; we have the equalities

$$m_x(P_k(X,x)) = \mu^{(k+1)}(X,x) + \mu^{(k)}(X,x).$$

At this point it is important to note that the equality $m_x(P_{d-1}(X,x)) =$

$\mu^{(d)}(X,x)+\mu^{(d-1)}(X,x)$ which, by what we have just seen, implies the equality

$$\chi_1(X,\{x\})-\chi_2(X,\{x\})=(-1)^{d-1}m_x(P_{d-1}(X,x)),$$

implies the general formula

$$\chi_{d-k}(X,\{x\})-\chi_{d-k+1}(X,\{x\})=(-1)^k m_x(P_k(X,x)),$$

simply because an affine space $L_{d-k}+t$ can be viewed as the intersection of an L_1+t for a general L_1 with a general vector subspace L_{d-k-1} of codimension $d-k-1$ through the point x taken as origin of $\mathbf{C}^n$, and

$$m_x(P_k(X,x))=m_x(P_k(X,x)\cap L_{d-k-1})=m_x(P_k(X\cap L_{d-k-1}),x)).$$

The first equality follows from general results on multiplicities since L_{d-k-1} is general, and the second from general results on local polar varieties found in ([Te3, 5.4], [L-T1, 4.18]). This sort of argument is used repeatedly in the proofs.

The formula for a general stratified set is the following:

Theorem 5.2. (Lê-Teissier, see [L-T1, théorème 6.1.9], [L-T3, 4.11]) *With the conventions just stated, and for any Whitney stratified complex analytic set $X=\bigcup_\alpha X_\alpha\subset\mathbf{C}^n$, we have for $x\in X_\alpha$ the equality*

$$\chi_{d_\alpha+1}(X,X_\alpha)-\chi_{d_\alpha+2}(X,X_\alpha)=\\ \sum_{\beta\neq\alpha}(-1)^{d_\beta-d_\alpha-1}m_x(P_{d_\beta-d_\alpha-1}(\overline{X_\beta},x))(1-\chi_{d_\beta+1}(X,X_\beta)),$$

where it is understood that $m_x(P_{d_\beta-d_\alpha-1}(\overline{X_\beta},x))=0$ if $x\notin P_{d_\beta-d_\alpha-1}(\overline{X_\beta},x)$.

The main ingredients of the proof are Morse theory and the transversality theorem already mentioned above which states that the kernel of the projection defining a polar variety $P_k(X,L)$ is transversal to the tangent cone $C_{P_k(X.L),0}$ at the origin provided that the projection is general enough. Thus, the image of that polar variety by this projection, a hypersurface called the polar image, has the same multiplicity as the polar variety (see [L-T1], 4.1.8).
This is useful because one considers the intersections $X\cap(L_i+t)\cap\mathbf{B}(0,\epsilon)$ as intersections with $X\cap\mathbf{B}(0,\epsilon)$ of the fibers of linear projections $\mathbf{C}^n\to\mathbf{C}^i$ over a "general" point close to the image of the point $x\in X_\alpha$. Because we are in complex analytic geometry the variations of Euler-Poincaré characteristics can be computed as the number of intersection points of a general line with the polar image, which is its multiplicity. In addition, the existence of fundamental systems of good neighborhoods of a point of $\mathbf{C}^n$ relative to a Whitney stratification plays an important role.

Let us now go back to the definitions of stratifications and stratifying conditions (see definition 4.2). Given a complex analytic stratification $X=\bigcup_\alpha X_\alpha$ of a complex analytic space, we can consider the following incidence conditions:

1. "Punctual Whitney conditions", the incidence condition $\hat{W}_x(X_\alpha, X_\beta)$: For any α, any point $x \in X_\alpha$, any stratum X_β such that $\overline{X_\beta}$ contains x and any local embedding $(X, x) \subset (\mathbf{C}^n, 0)$, the pair of strata (X_β, x_α) satisfies the Whitney conditions at x.

2. "Local Whitney conditions", the incidence condition $W_x(X_\alpha, X_\beta)$: same as above except that the Whitney conditions must be satisfied at every point of some open neighborhood of x in X_α.

3. "(Local Whitney conditions)*": For each α, for every $x \in X_\alpha$ and every local embedding $(X, x) \subset (\mathbf{C}^n, 0)$, for every $i \leq n - d_\alpha$ there exists a dense Zariski open set U_i of the Grassmannian $G(n - i - d_\alpha, n - d_\alpha)$ of linear spaces of codimension i of $\mathbf{C}^n$ containing the tangent space $T_x X_\alpha$ such that for every germ of nonsingular subspace $(H_i, x) \subset (\mathbf{C}^n, 0)$ of codimension i containing (X_α, x) and such that $T_x H_i \in U_i$, we have $W_x(X_\alpha, X_\beta \cap H_i)$.

4. "Local Topological equisingularity", the incidence condition $(TT)_x$: For any α, any point $x \in X_\alpha$, any stratum X_β such that $\overline{X_\beta}$ contains x and any local embedding $(X, x) \subset (\mathbf{C}^n, 0)$, there exist germs of retractions $\rho\colon (\mathbf{C}^n, 0) \to (X_\alpha, x)$ and positive real numbers ϵ_0 such that for all ϵ, $0 < \epsilon \leq \epsilon_0$ there exists η_ϵ such that for all η, $0 < \eta \leq \eta_\epsilon$, there is an homeomorphism $\mathbf{B}(0, \epsilon) \cap \rho^{-1}(\mathbf{B}(0, \eta) \cap X_\alpha) \simeq (\rho^{-1}(x) \cap \mathbf{B}(0, \epsilon)) \times (\mathbf{B}(0, \eta) \cap X_\alpha)$ which is compatible with the retraction ρ and the projection to $\mathbf{B}(0, \eta) \cap X_\alpha$ and, for each stratum X_β such that $\overline{X_\beta}$ contains x, induces an homeomorphism:

   $$\overline{X_\beta} \cap \mathbf{B}(0, \epsilon) \cap \rho^{-1}(\mathbf{B}(0, \eta) \cap X_\alpha) \simeq (\overline{X_\beta} \cap \rho^{-1}(x) \cap \mathbf{B}(0, \epsilon)) \times (\mathbf{B}(0, \eta) \cap X_\alpha).$$

   This embedded local topological triviality, meaning that locally around x each $\overline{X_\beta}$ is topologically a product of the nonsingular X_α by the fiber $\rho^{-1}(x)$, in a way which is induced by a topological product structure of the ambient space, will be denoted by $TT_x(X_\alpha, X_\beta)$ for each specified $\overline{X_\beta}$.

5. "(Local Topological equisingularity)*", the incidence condition $(TT^*)_x$: For each α, for every $x \in X_\alpha$ and every local embedding $(X, x) \subset (\mathbf{C}^n, 0)$, for every $i \leq n - d_\alpha$ there exists a dense Zariski open set U_i of the Grassmannian $G(n - i - d_\alpha, n - d_\alpha)$ of linear spaces of codimension i of $\mathbf{C}^n$ containing the tangent space $T_x X_\alpha$ such that for every germ of nonsingular subspace $(H_i, x) \subset (\mathbf{C}^n, 0)$ of codimension i containing (X_α, x) and such that $T_x H_i \in U_i$, we have $TT_x(X_\alpha, X_\beta \cap H_i)$.

6. "χ^* constant": For each α, for every $x \in X_\alpha$, every stratum X_β such that $X_\alpha \subset \overline{X_\beta}$, and every local embedding $(X, x) \subset (\mathbf{C}^n, 0)$, the map which to every point $y \in X_\alpha$ in a neighborhood of x associates the sequence $\chi^*(\overline{X_\beta}, y) = (\chi_1(\overline{X_\beta}, \{y\}), \ldots, \chi_{n-d_\beta}(\overline{X_\beta}, \{y\}))$ is constant on X_α in a neighborhood of x. Recall that $\chi_i(\overline{X_\beta}, \{y\}))$ is the Euler characteristic of the intersection, within a small ball $\mathbf{B}(0, \epsilon)$ around y in $\mathbf{C}^n$, of $\overline{X_\beta}$ with an affine subspace of codimension i of the form $L_i + t$, where L_i is a vector

subspace of codimension i of general direction and $0 < |t| < \eta$ for a small enough η, depending on ϵ.

7. "M^* constant": For each α, for every $x \in X_\alpha$, for every stratum X_β such that $X_\alpha \subset \overline{X_\beta}$, and every local embedding $(X, x) \subset (\mathbf{C}^n, 0)$, the map which to every point $y \in X_\alpha$ in a neighborhood of x associates the sequence

$$M^*(\overline{X_\beta}, y) = \big(m_y(\overline{X_\beta}), m_y(P_1(\overline{X_\beta}, y)), \ldots, m_y(P_{d_\beta - 1}(\overline{X_\beta}, y))\big) \in \mathbf{N}^{d_\beta}$$

   is constant in a neigborhood of x.

   This condition is equivalent to saying that the polar varieties $P_k(\overline{X_\beta}, x)$ which are not empty contain X_α and are locally around x equimultiple along X_α.

The main theorem of [L-T3] (Théorème 5.3.1) is that for a stratification in the sense of definition 4.2 all these conditions are equivalent. Theorem 5.2, which relates the multiplicities of polar varieties with local topological invariants, plays a key role in the proof.

Recall that we saw in subsection 4.2 the result of Thom-Mather (see [Ma]) that Whitney stratifications have the property of local topological equisingularity defined above. We also mentioned that he converse is known to be false since Briançon-Speder gave an example in [BS]. The result just mentioned provides among other things the correct converse.

6 Specialization to the Tangent Cone and Whitney equisingularity

Let us now re-examine the question of how much does a germ of singularity $(X, 0)$ without exceptional cones resembles a cone. The obvious choice is to compare it with its tangent cone $C_{X,0}$, assuming that it is reduced, and we can rephrase the question by asking does the absence of exceptional cones implies that $(X, 0)$ is Whitney-equisingular to its tangent cone?

To be more precise, let $(X, 0) \subset (\mathbf{C}^n, 0)$ be a reduced germ of an analytic singularity of pure dimension d, and let $\varphi : (\mathfrak{X}, 0) \to (\mathbf{C}, 0)$ denote the specialization of X to its tangent cone $C_{X,0}$. Let $\mathfrak{X}^0$ denote the open set of smooth points of $\mathfrak{X}$, and let Y denote the smooth subspace $0 \times \mathbf{C} \subset \mathfrak{X}$. Our aim is to study the equisingularity of $\mathfrak{X}$ along Y. More precisely, we want to determine whether the absence of exceptional cones will allow us to construct a Whitney stratification of $\mathfrak{X}$ in which the parameter axis Y is a stratum.

The first result in this direction was obtained by Lê and Teissier in [L-T4, Thm 2.2.1] and says that for a surface $(S, 0) \subset (\mathbf{C}^3, 0)$ with a reduced tangent cone the absence of exceptional cones is equivalent to $\{\mathfrak{X}^0, \mathrm{Sing}\mathfrak{X} \setminus Y, Y\}$ being a Whitney stratification of $\mathfrak{X}$. In particular $(S, 0)$ is Whitney equisingular to its tangent cone $(C_{S,0}, 0)$.

In the general case, we only have a partial answer which we will now describe. The first step to find out if such a stratification is possible, is to verify that the pair $(\mathfrak{X}^0, Y)$ satisfies Whitney's conditions. Since $\mathfrak{X} \setminus \mathfrak{X}(0)$ is isomorphic to the product $\mathbf{C}^* \times X$, Whitney's conditions are automatically verified everywhere in $\{0\} \times \mathbf{C}$, with the possible exception of the origin.

Theorem 6.1. *[Gi, Thm 8.11] Let $(X, 0)$ be a reduced and equidimensional germ of a complex analytic singularity, and suppose that its tangent cone $C_{X,0}$ is reduced. Then the following statements are equivalent:*

1. *The germ $(X, 0)$ does not have exceptional cones.*
2. *The pair $(\mathfrak{X}^0, Y)$ satisfies Whitney's condition a) at the origin.*
3. *The pair $(\mathfrak{X}^0, Y)$ satisfies Whitney's conditions a) and b) at the origin.*
4. *The germ $(\mathfrak{X}, 0)$ does not have exceptional cones.*

We would like to explain a little how one goes about proving this result. To begin with, we know that Whitney's condition b) is stronger than the condition a). However the equivalence of statements 2) and 3) tells us that in this case, for the pair of strata $(\mathfrak{X}^0, Y)$ at the origin, they are equivalent, and it is the special geometry of $\mathfrak{X}$ that plays a crucial role in this result.

Proposition 6.2. *[Gi, Proposition 6.1] If the pair $(\mathfrak{X}^0, Y)$ satisfies Whitney's condition a) at the origin, then it also satisfies Whitney's condition b) at the origin.*

Remark 6.1.

1. *For any point $y \in Y$, the tangent cone $C_{\mathfrak{X},y}$ is isomorphic to $C_{X,0} \times Y$, and the isomorphism is uniquely determined once we have chosen a set of coordinates. The reason is that for any $f(z)$ vanishing on $(X, 0)$, the function $F(z, v) = v^{-m} f(vz) = f_m + v f_{m+1} + v^2 f_{m+2} + \ldots$, vanishes in $(\mathfrak{X}, 0)$ and so for any point $y = (\underline{0}, v_0)$ the initial form of $F(z, v + v_0)$ in $\mathbf{C}\{z_1, \ldots, z_n, v\}$ is equal to the initial form of f at 0. That is $in_{(0,v_0)} F = in_0 f$.*
2. *The projectivized normal cone $\mathbf{P}C_{\mathfrak{X},Y}$ is isomorphic to $Y \times \mathbf{P}C_{X,0}$. This can be seen from the equations used to define $\mathfrak{X}$ (section 2, exercise 2.1), where the initial form of F_i with respect to Y, is equal to the initial form of f_i at the origin. That is $in_Y F_i = in_0 f_i$.*
3. *There exists a natural morphism $\omega : E_Y \mathfrak{X} \to E_0 X$, making the following*

diagram commute:

$$\begin{tikzcd} E_Y\mathfrak{X} \arrow[r, "\omega"] \arrow[d, "e_Y"'] & E_0X \arrow[d, "e_o"] \\ \mathfrak{X} \arrow[r, "\phi"'] & X \end{tikzcd}$$

Moreover, when restricted to the exceptional divisor $e_Y^{-1}(Y) = \mathbf{P}C_{\mathfrak{X},Y}$ *it induces the natural map* $\mathbf{P}C_{\mathfrak{X},Y} = Y \times \mathbf{P}C_{X,0} \to \mathbf{P}C_{X,0}$.
Algebraically, this results from the universal property of the blowup E_0X *and the following diagram:*

$$\begin{tikzcd} E_Y\mathfrak{X} \arrow[d, "e_Y"'] & E_0X \arrow[d, "e_o"] \\ \mathfrak{X} \arrow[r, "\phi"'] & X \end{tikzcd}$$

Note that, for the diagram to be commutative the morphism ω *must map the point* $((v, z), [z]) \in E_Y\mathfrak{X}\backslash\{Y \times \mathbf{P}^{n-1}\} \subset \mathfrak{X}\times\mathbf{P}^{n-1}$ *to the point* $((vz), [z])$ *in* $E_0X \subset X \times \mathbf{P}^{n-1}$.

Now we can proceed to the proof of 6.2.

Proof. (*of Proposition 6.2*)
We want to prove that the pair $(\mathfrak{X}^0, Y)$ satisfies Whitney's condition b) at the origin. We are assuming that it already satisfies condition a), so in particular we have that $\zeta^{-1}(0)$ is contained in $\{0\} \times \mathbf{P}^{n-1} \times \check{\mathbf{P}}^{n-1}$, but by the remarks made at the beginning of section 4.3 it suffices to prove that any point $(0, l, H) \in \zeta^{-1}(0)$ is contained in the incidence variety $I \subset \{0\} \times \mathbf{P}^{n-1} \times \check{\mathbf{P}}^{n-1}$. This is done by considering the normal/conormal diagram of $\mathfrak{X}$ augmented by the map $\omega : E_Y\mathfrak{X} \to E_0X$ of the remarks above and the map $\psi : C(\mathfrak{X}) \to C(X) \times \mathbf{C}$ defined by $((z_1, \ldots, z_n, v), (a_1 : \ldots : a_n : b)) \mapsto ((vz_1, \ldots, vz_n), (a_1 : \ldots : a_n), v)$

$$\begin{tikzcd} E_YC(\mathfrak{X}) \arrow[r, "\hat{e}_Y"] \arrow[d, "\kappa'_{\mathfrak{X}}"'] \arrow[dr, "\zeta"] & C(\mathfrak{X}) \arrow[r, "\psi"] \arrow[d, "\kappa_{\mathfrak{X}}"] & C(X) \times \mathbf{C} \\ E_Y\mathfrak{X} \arrow[r, "e_Y"'] \arrow[d, "\omega"'] & \mathfrak{X} & \\ E_0X & & \end{tikzcd}$$

By construction, there is a sequence (z_m, v_m, l_m, H_m) in $E_YC(\mathfrak{X}) \hookrightarrow C(\mathfrak{X}) \times_{\mathfrak{X}} E_Y\mathfrak{X}$ tending to $(0, l, H)$, where (z_m, v_m) is not in Y. Through $\kappa'_{\mathfrak{X}}$, we obtain a sequence (z_m, v_m, l_m) in $E_Y\mathfrak{X}$ tending to $(0, l)$, and through $\hat{e}_Y$ a sequence (z_m, v_m, H_m) tending to $(0, H)$ in $C(\mathfrak{X})$.

In this case the condition $a)$ means that $b = 0$ and so through ψ we obtain the sequence $(t_m z_m, \widetilde{H}_m)$ tending to $(0, \widetilde{H})$ in $C(X)$. Analogously, both the sequence $(v_m z_m, l_m)$ obtained through the map ω and its limit $(0, l)$ are in $E_0 X$. Finally, Whitney's lemma 3.1 tells us that in this situation we have that $l \subset \widetilde{H}$ and so the point $(0, l, H)$ is in the incidence variety. □

Lemma 6.1. *[Gi, Lemma 6.4] If the tangent cone $C_{X,0}$ is reduced and the pair $(\mathfrak{X}^0, Y)$ satisfies Whitney's condition a) then the germ $(X, 0)$ does not have exceptional cones.*

Proof. Since $(\mathfrak{X}^0, Y)$ satisfies Whitney's condition a), by proposition 6.2 it also satisfies Whitney's condition b). Recall that the auréole of $(\mathfrak{X}, 0)$ along Y is a collection $\{V_\alpha\}$ of subcones of the normal cone $C_{\mathfrak{X},Y}$ whose projective duals determine the set of limits of tangent hyperplanes to $\mathfrak{X}$ at the points of Y in the case that the pair $(\mathfrak{X}^0, Y)$ satisfies Whitney conditions a) and b) at every point of Y (S 2ee [L-T2, Thm. 2.1.1, Corollary 2.1.2, p. 559-561]). Among the V_α there are the irreducible components of $|C_{\mathfrak{X},Y}|$. Moreover:

1. By Remark 6.1 we have that $C_{\mathfrak{X},Y} = Y \times C_{X,0}$ so its irreducible components are of the form $Y \times \widetilde{V}_\beta$ where $\widetilde{V}_\beta$ is an irreducible component of $|C_{X,0}|$.
2. For each α the projection $V_\alpha \to Y$ is surjective and all the fibers are of the same dimension. (See [L-T2], Proposition 2.2.4.2, p. 570)
3. The hyperplane H corresponding to the point $(0 : 0 : \cdots : 0 : 1) \in \check{\mathbf{P}}^{n+1}$, which is $v = 0$, is transversal to $(\mathfrak{X}, 0)$ by hypothesis, and so by [L-T2, Thm. 2.3.2, p. 572] the collection $\{V_\alpha \cap H\}$ is the auréole of $\mathfrak{X} \cap H$ along $Y \cap H$.

Notice that $(\mathfrak{X} \cap H, Y \cap H)$ is equal to $(\mathfrak{X}(0), 0)$, which is isomorphic to the tangent cone $(C_{X,0}, 0)$ and therefore does not have exceptional cones. This means that for each α, either $V_\alpha \cap H$ is an irreducible component of $C_{X,0}$ or it is empty. But the intersection can't be empty because the projections $V_\alpha \to Y$ are surjective. Finally since all the fibers of the projection are of the same dimension, the V_α's are only the irreducible components of $C_{\mathfrak{X},Y}$.
But this means that if we define the affine hyperplane H_v as the hyperplane with the same direction as H and passing through the point $y = (0, v) \in Y$ for v small enough; H_v is transversal to $(\mathfrak{X}, y)$. So we have again that the collection $\{V_\alpha \cap H_v\}$ is the auréole of $\mathfrak{X} \cap H_v$ along $Y \cap H_v$, that is, the auréole of $(X, 0)$, so it does not have exceptional cones. □

At this point it is not too hard to prove the equivalence of statements 3) and 4) of theorem 6.1, namely that the pair $(\mathfrak{X}^0, Y)$ satisfies both Whitney conditions at the origin if and only if the germ $(\mathfrak{X}, 0)$ does not have exceptional cones (See [Gi, Proposition 6.5]).

The idea is that on the one hand we have that the Whitney conditions imply that $(X,0)$ has no exceptional cones and $b=0$, but this means that the map $\psi : C(\mathfrak{X}) \to C(X)$ $((z,v),[a:b]) \mapsto ((vz),[a])$ is defined everywhere. Thus, the set of limits of tangent hyperplanes to $(\mathfrak{X},0)$ is just the dual of the tangent cone. On the other hand since $C_{\mathfrak{X},0} = C_{X,0} \times \mathbf{C}$ the absence of exceptional cones implies $b=0$ which is equivalent to Whitney's condition a).

The key idea to prove Whitney's condition a) starting from the assumption that $(X,0)$ is without exceptional cones is to use its algebraic characterization given by the second author in [Te1] for the case of hypersurfaces and subsequently generalized by Gaffney in [Ga1] in terms of integral dependence of modules. To give an idea of how it is done let us look at the hypersurface case.

If $(X,0) \subset (\mathbf{C}^n,0)$ is a hypersurface then $(\mathfrak{X},0) \subset (\mathbf{C}^{n+1},0)$ is also a hypersurface say defined by $F \in \mathbf{C}\{z_1,\dots,z_n,v\}$. Note that in this case the conormal space $C(\mathfrak{X})$ coincides with the Semple-Nash modification and thus every arc

$$\gamma : (\mathbf{C}, \mathbf{C}\setminus\{0\}, 0) \to (\mathfrak{X}, \mathfrak{X}^0, 0)$$

lifts uniquelyto an arc

$$\tilde{\gamma} : (\mathbf{C}, \mathbf{C}\setminus\{0\}, 0) \to (C(\mathfrak{X}), C(\mathfrak{X}^0), (0,T))$$

given by

$$\tau \mapsto \left(\gamma(\tau), T_{\gamma(\tau)}\mathfrak{X} := \left(\frac{\partial F}{\partial z_0}(\gamma(\tau)) : \dots : \frac{\partial F}{\partial z_n}(\gamma(\tau)) : \frac{\partial F}{\partial v}(\gamma(\tau))\right)\right),$$

and so the vertical hyperplane $\{v=0\}$, or $(0:\dots:0:1)$ in projective coordinates, is not a limit of tangent spaces to $\mathfrak{X}$ at 0 if and only if $\frac{\partial F}{\partial v}$ tends to zero at least as fast as the slowest of the other partials, that is

$$\text{order}\,\frac{\partial F}{\partial v}(\gamma(\tau)) \geq \min_j \left\{\text{order}\frac{\partial F}{\partial z_j}(\gamma(\tau))\right\}$$

The point is that this is equivalent to $\frac{\partial F}{\partial v}$ being integrally dependent on the relative jacobian ideal $J_\varphi := \left\langle \frac{\partial F}{\partial z_j} \right\rangle$ in the local ring $O_{\mathfrak{X},0}$ as proved by Lejeune-Jalabert and the second author in ([Lej-Te], Thm 2.1). True, this is not precisely what we want, but it is very close because the pair $(\mathfrak{X}^0, Y)$ satisfies Whitney's condition a) at the origin if and only if $\frac{\partial F}{\partial v}$ tends to zero faster than the slowest of the other partials, that is :

$$\text{order}\,\frac{\partial F}{\partial v}(\gamma(\tau)) > \min_j \left\{\text{order}\frac{\partial F}{\partial z_j}(\gamma(\tau))\right\}$$

and according to the definition of strict dependence stated by Gaffney and Kleiman in ([Ga-Kl], [Section 3, 555]) , not only for ideals but more generally for modules, this is what it means for $\frac{\partial F}{\partial v}$ to be strictly dependent on the relative jacobian ideal J_φ in $O_{\mathfrak{X},0}$.

As for the proof, note that we already have the pair $(\mathfrak{X}^0, Y)$ satisfies Whitney's conditions at every point $y \in Y \setminus \{0\}$, that is, $\frac{\partial F}{\partial v}$ is strictly dependent on the relative jacobian ideal J_φ in $O_{\mathfrak{X},y}$ at all these points. That this condition carries over to the origin can be determined by the principle of specialization of integral dependence (see [Te6], Appendice 1, [Te3], Chap.1, §5, [Ga-Kl]) which in this case amounts to proving that the exceptional divisor E of the normalized blowup of $\mathfrak{X}$ along the ideal J_φ does not have irreducible components over the special fiber $\mathfrak{X}(0) := \varphi^{-1}(0)$. Fortunately, this normalized blowup is isomorphic to a space we know, namely the normalization of the relative conormal space $C_\varphi(\mathfrak{X})$ of 4.5:

$$\widetilde{\kappa_\varphi} : \widetilde{C_\varphi(\mathfrak{X})} \to \mathfrak{X},$$

and we are able to use the absence of exceptional cones to prove that E has the desired property.
This ends our sketch of proof of theorem 6.1.

Corollary 6.1. *[Gi, Corollaries 8.14 and 8.15]*
Let $(X, 0)$ satisfy the hypothesis of Theorem 6.1.

- *If $(X, 0)$ has an isolated singularity and its tangent cone is a complete intersection singularity, then the absence of exceptional cones implies that $C_{X,0}$ has an isolated singularity and $\{\mathfrak{X} \setminus Y, Y\}$ is a Whitney stratification of $\mathfrak{X}$.*
- *If the tangent cone $(C_{X,0}, 0)$ has an isolated singularity at the origin, then $(X, 0)$ has an isolated singularity and $\{\mathfrak{X} \setminus Y, Y\}$ is a Whitney stratification of $\mathfrak{X}$.*

We have verified that the absence of exceptional cones allows us to start building a Whitney stratification of $\mathfrak{X}$ having Y as a stratum. The question now is how to continue. We can prove ([Gi, Proposition 8.13]) that in the complete intersection case, the singular locus of $\mathfrak{X}$ coincides with the specialization space Z of $|\mathrm{Sing}\, X|$ to its tangent cone.

Suppose now that the germ $(|\mathrm{Sing}\, X|, 0)$ has a reduced tangent cone; then a stratum $\mathfrak{X}_\lambda$ containing a dense open set of Z will satisfy Whitney's conditions along Y if and only if the germ $(|\mathrm{Sing}\, X|, 0)$ does not have exceptional cones.

In view of this it seems reasonable to start by assuming the existence of a Whitney stratification $\{X_\lambda\}$ of $(X, 0)$ such that for every λ the germ $(\overline{X_\lambda}, 0)$ has a reduced tangent cone and no exceptional cones. In this case, the specialization space $\mathfrak{X}_\lambda$ of $(\overline{X_\lambda}, 0)$ is canonically embedded as a subspace of $\mathfrak{X}$, and the partition of $\mathfrak{X}$ associated to the filtration given by the $\mathfrak{X}_\lambda$ is a good place to start looking for the desired Whitney stratification of $\mathfrak{X}$ but this is to our knowledge, still an open problem. A precise formulation is the following:

Question1: 1) Let $(X,0)$ be a germ of reduced equidimensional complex analytic space, and let $X=\bigcup_{\lambda\in L}X_\lambda$ be the minimal (section 4.3) Whitney stratification of a small representative of $(X,0)$. Is it true that the following conditions are equivalent?

- The tangent cones $C_{\overline{X}_\lambda,0}$ are reduced and the $(\overline{X_\lambda},0)$ have no exceptional cones, for $\lambda\in L$.
- The specialization spaces $(\mathfrak{X}_\lambda)_{\lambda\in L}$ are the closures of the strata of the minimal Whitney stratification of $\mathfrak{X}$. If $\{0\}$ is a stratum in X, we understand its specialization space to be $Y=\{0\}\times\mathbf{C}\subset\mathfrak{X}$. Indeed, in this case the algebra $\mathcal{R}$ of proposition 2.6 is $k[v]$.

If that is the case, for a sufficiently small representative $\mathfrak{X}$ of $(\mathfrak{X},0)$, the spaces $(X,0)$ and $(C_{X,0},0)$ are isomorphic to the germs, at $(0,v_0)$ and $(0,0)$ respectively, of two transversal sections, $v=0$ and $v=v_0\neq 0$ of a Whitney stratification of $\mathfrak{X}\subset\mathbf{C}^n\times\mathbf{C}$, and so are Whitney-equisingular. Conversely, if Y is a stratum of a Whitney stratification of $\mathfrak{X}$, it is contained in a stratum of the minimal Whitney stratification of $\mathfrak{X}$, whose strata are the specialization spaces $\mathfrak{X}_\lambda$ of the strata X_λ of the minimal Whitney stratification of X. It follows from theorem 6.1 that the $\overline{X_\lambda}$ have a reduced tangent cone and no exceptional cones.

2) Given an algebraic cone C, reduced or not, which systems of irreducible closed subcones can be obtained as exceptional tangents for some complex analytic deformation of C having C as tangent cone?

7 Polar varieties, Whitney stratifications, and projective duality

See the warning concerning notation at the beginning of section 5. In this section we go back and forth between a projective variety $V\subset\mathbf{P}^{n-1}$, the germ $(X,0)$ at 0 of the cone $X\subset\mathbf{C}^n$ over V, and the germ (V,v) of V at a point $v\in V$, so that we also use the notations of section 3.1. Note that $P_k(V)$, $0\leq k\leq d$ denotes the polar varieties in the sense of definition 3.4.

The formula of theorem 5.2 can be applied to the special singular point which is the vertex 0 of the cone X in $\mathbf{C}^n$ over a projective variety V of dimension d in $\mathbf{P}^{n-1}$, which we assume not to be contained in a hyperplane. The dual variety $\check{V}$ of V was defined in subsection 2.3. Remember that every complex analytic space, and in particular V, has a minimal Whitney stratification (see the end of section 4.3). We shall use the following facts, with the notations of proposition 5.1:

Proposition 7.1. (Compare with [Te4])

1. *If $V=\bigcup V_\alpha$ is a Whitney stratification of V, denoting by X_α the cone over V_α, we have that $X=\{0\}\bigcup X_\alpha^*$, where $X_\alpha^*=X_\alpha\setminus\{0\}$, is a Whitney stratification of X. It may be that (V_α) is the minimal Whitney stratification of V but $\{0\}\bigcup X_\alpha^*$ is not minimal, if V is a cone.*

2. *If $L_i + t$ is an i-codimensional affine space in $\mathbf{C}^n$ it can be written as $L_{i-1} \cap (L_1 + t)$ with vector subspaces L_i and for general directions of L_i we have, denoting by $\mathbf{B}(0, \epsilon)$ the closed ball with center 0 and radius ϵ, for small ϵ and $0 < |t| << \epsilon$:*

   $$\chi_i(X, \{0\}) := \chi(X \cap (L_i + t) \cap \mathbf{B}(0, \epsilon)) = \chi(V \cap H_{i-1}) - \chi(V \cap H_{i-1} \cap H_1),$$

   where $H_i = \mathbf{P}L_i \subset \mathbf{P}^{n-1}$.

3. *For every stratum X^*_α of X, we have the equalities $\chi_i(X, X^*_\alpha) = \chi_i(V, V_\alpha)$.*

4. *If the dual $\check{V} \subset \check{\mathbf{P}}^{n-1}$ is a hypersurface, its degree is equal to $m_0(P_d(X, 0))$, which is the number of critical points of the restriction to V of a general linear projection $\mathbf{P}^{n-1} \setminus L_2 \to \mathbf{P}^1$.*

Note that we will apply statements 2) and 3) not only to the cone X but also to the cones $\overline{X_\beta}$ over the closed strata $\overline{V_\beta}$.

Proof. The first statement follows from the product structure of the cones along their generating lines outside of the origin, and the fact that $\overline{V_\beta} \times \mathbf{C}$ satisfies the Whitney conditions along $V_\alpha \times \mathbf{C}$ at a point $(x, \lambda) \in V_\alpha \times \mathbf{C}^*$ if and only if $\overline{V_\beta}$ satisfies those conditions along V_α at the point x.

To prove the second one, we first remark that it suffices to prove the result for $i = 1$ since we can then apply it to $X \cap L_{i-1}$. Assuming that $i = 1$ we may consider the minimal Whitney stratification of V and by an appropriate choice of coordinates assume that the hyperplane $z_1 = 0$ is transversal to the strata. Then, we use an argument very similar to the proof of the existence of fundamental systems of good neighborhoods in [L-T3]. In $\mathbf{P}^{n-1}$ with homogeneous coordinates $(z_1 : \ldots : z_n)$, we choose the affine chart $\mathbf{A}^{n-1} \simeq \mathbf{C}^{n-1} \subset \mathbf{P}^{n-1}$ defined by $z_1 \neq 0$. The distance function to $0 \in \mathbf{A}^{n-1}$ is real analytic on the strata of V.
Let us denote by $\mathbf{D}(0, R)$ the ball centered at 0 and with radius R in $\mathbf{A}^{n-1}$. By Bertini-Sard's theorem and Thom's isotopy theorem, we obtain that there exits a radius R_0, the largest critical value of the distance function to the origin restricted to the strata of V, such that the homotopy type of $V \cap \mathbf{D}(0, R)$ is constant for $R > R_0$ and equal to that of $V \setminus V \cap H$, where H is the hyperplane $z_1 = 0$. Thus, $\chi(V \setminus V \cap H) = \chi(V) - \chi(V \cap H) = \chi(V \cap \mathbf{D}(0, R))$. In fact, by the proof of the Thom-Mather theorem, the intersection $V \cap \mathbf{D}(0, R)$ is then a deformation retract of $V \setminus V \cap H$.
Since all that is required from our hyperplane $z_1 = 0$ is that it should be transversal to the strata of V, we may assume that the hyperplane L_1 is defined by $z_1 = 0$. Given $t \neq 0$ and ϵ, the application $(z_1 : \ldots : z_n) \mapsto (t, t\frac{z_2}{z_1}, \ldots, t\frac{z_n}{z_1})$ maps isomorphically $V \cap \mathbf{D}(0, R)$ onto $X \cap (L_1 + t) \cap \mathbf{B}(0, \epsilon)$ if $R = \frac{\epsilon}{|t|}$.

The third statement follows from the fact that locally at any point of X^*_α, the cone X, together with its stratification, is the product of V, together with its stratification, by the generating line through x of the cone, and product by a disk does not change the Euler characteristic.

Finally, we saw in lemma 2.3 that the fiber $\kappa^{-1}(0)$ of the conormal map $\kappa\colon C(X) \to X$ is the dual variety $\check{V}$. The last statement then follows from the very definition of polar varieties. Indeed, given a general line L^1 in $\check{\mathbf{P}}^{n-1}$, the corresponding polar curve in X is the cone over the points of V where a tangent hyperplane belongs to the pencil L^1; it is a finite union of lines and its multiplicity is the number of these lines, which is the number of corresponding points of V.

$\square$

Using proposition 7.1, we can rewrite in this case the formula of theorem 5.2 as a generalized Plücker formula for any d-dimensional projective variety $V \subset \mathbf{P}^{n-1}$ whose dual is a hypersurface:

Proposition 7.2. (Teissier, see [Te4]) *Given the projective variety $V \subset \mathbf{P}^{n-1}$ equipped with a Whitney stratification $V = \bigcup_{\alpha \in A} V_\alpha$, denote by d_α the dimension of V_α. We have, if the projective dual $\check{V}$ is a hypersurface in $\check{\mathbf{P}}^{n-1}$:*

$$(-1)^d \mathrm{deg}\check{V} =$$

$$\chi(V) - 2\chi(V \cap H_1) + \chi(V \cap H_2) - \sum_{d_\alpha < d} (-1)^{d_\alpha} \mathrm{deg}_{n-2} P_{d_\alpha}(\overline{V_\alpha})(1 - \chi_{d_\alpha+1}(V, V_\alpha)),$$

where H_1, H_2 denote general linear subspaces of $\mathbf{P}^{n-1}$ of codimension 1 and 2 respectively, $\mathrm{deg}_{n-2} P_{d_\alpha}(\overline{V_\alpha})$ is the number of nonsingular critical points of a general linear projection $\overline{V_\alpha} \to \mathbf{P}^1$, which is the degree of $\check{\overline{V_\alpha}}$ if it is a hypersurface and is set equal to zero otherwise. It is equal to 1 if $d_\alpha = 0$.

Here we remark that if (V_α) is the minimal Whitney stratification of the projective variety $V \subset \mathbf{P}^{n-1}$, and H is a general hyperplane in $\mathbf{P}^{n-1}$, the $V_\alpha \cap H$ that are not empty constitute the minimal Whitney stratification of $V \cap H$; see [Te3, lemma 4.2.2].

This formula, in the special case where V is nonsingular, already appears in [Kl4, formula (IV, 72)]. It is *a priori* different in general from the very nice generalized Plücker formula given by Ernström in [Er], which also generalizes the formula (IV, 72) to the singular case, even when the dual variety is not a hypersurface:

Theorem 7.3. (Ernström, see [Er]) *Let $V \subset \mathbf{P}^{n-1}$ be a projective variety and let k be the codimension in $\check{\mathbf{P}}^{n-1}$ of the dual variey $\check{V}$. We have the following equality:*

$$(-1)^d \mathrm{deg}\check{V} = k\chi(V, \mathrm{Eu}_V) - (k+1)\chi(V \cap H_1, \mathrm{Eu}_{V\cap H_1}) + \chi(V \cap H_{k+1}, \mathrm{Eu}_{V \cap H_{k+1}}),$$

where the H_i are general linear subspaces of $\mathbf{P}^{n-1}$ of codimension i and $\chi(V, \mathrm{Eu}_V)$ is a certain linear combination with coefficients in $\mathbf{Z}$ of Euler characteristics of subvarieties of V, which is built using the properties of the local Euler obstruction $\mathrm{Eu}(V, v) \in \mathbf{Z}$ associated to any point v of V, especially that it is constructible i.e., constant on constructible subvarieties of V.

The local Euler obstruction is a local invariant of singularities which plays an important role in the theory of Chern classes for singular varieties, due to M-H. Schwartz and R. MacPherson (see [Br]). Its definition is outside of the scope of these notes but we shall give an expression for it in terms of multiplicities of polar varieties below.

Coming back to our formula, if $\check{V}$ is not a hypersurface, the polar curve $P_{d-1}(X, L)$ is empty, but the degree of $\check{V}$ is still the multiplicity at the origin of a polar variety of the cone X over V. We shall come back to this below.

The case where V has isolated singularities

Let us first treat the hypersurface case. Let $f(z_1, \dots, z_n)$ be a homogeneous polynomial of degree m defining a hypersurface $V \subset \mathbf{P}^{n-1}$ with isolated singularities, which is irreducible if $n > 3$. The degree of $\check{V}$ is the number of points of V where the tangent hyperplane contains a given general linear subspace L of codimension 2 in $\mathbf{P}^{n-1}$. By Bertini's theorem we can deform V into a nonsingular hypersurface V' of the same degree, by considering the hypersurface defined by $F_{v_0} = f(z_1, \dots, z_n) + v_0 z_1^m = 0$, where the open set $z_1 \neq 0$ contains all the singular points of V and v_0 is small and non zero.
Taking coordinates such that L is defined by $z_1 = z_2 = 0$, the class of V' is computed as the number of intersection points of V' with the curve of $\mathbf{P}^{n-1}$ defined by the equations $\frac{\partial F_{v_0}}{\partial z_3} = \cdots = \frac{\partial F_{v_0}}{\partial z_n} = 0$, which express that the tangent hyperplane to V' at the point of intersection contains L. This is the *relative polar curve* of [Te3]. For general z_1 this is a complete intersection and Bézout's theorem combined with proposition 7.2 gives

$$\deg \check{V}' = (-1)^{n-2}(\chi(V') - 2\chi(V' \cap H_1) + \chi(V' \cap H_2)) = m(m-1)^{n-2}.$$

Now as V' degenerates to V when $v_0 \to 0$, by what we saw in example 5.1, the topology changes only by $\mu^{(n-1)}(V, x_i)$ vanishing cycles in dimension $n-2$ attached to each of the isolated singular points $x_i \in V$ (example 5.1). This gives $\chi(V) = \chi(V') - \sum_i (-1)^{n-2}\mu^{(n-1)}(V, x_i)$.
We have $\chi(V \cap H_1) = \chi(V' \cap H_1)$ and $\chi(V \cap H_2) = \chi(V' \cap H_2)$ since H_1 and H_2, being general, miss the singular points and are transversal to V and V' so that $V' \cap H_1$ (resp. $V' \cap H_2$) is diffeomorphic to $V \cap H_1$ (resp. $V \cap H_2$). It follows from a theorem of Ehresmann that all nonsingular projective hypersurfaces of the same degree are diffeomeorphic.
We could have taken H_2 general in H_1 and H_1 to be $z_1 = 0$, and then $V \cap H_1 = V' \cap H_1,\ V \cap H_2 = V' \cap H_2$.

On the other hand, in our formula the Whitney strata of dimension $< n-2$ are the $\{x_i\}$ so all the $d_{\{x_i\}}$ are equal to zero while the $\chi_1(V, \{x_i\})$ are equal to $1 + (-1)^{n-3}\mu^{(n-2)}(V, x_i)$, corresponding to the Milnor number of a generic hyperplane section of V through x_i, as we saw in example 5.1.

Substituting all this in our formula of proposition 7.2 gives:

$$\begin{aligned}&(-1)^{n-2}\deg\check{V} = \\ &(-1)^{n-2}m(m-1)^{n-2} - \sum_i (-1)^{n-2}\mu^{(n-1)}(V,x_i) \\ &-\sum_i (1-(1+(-1)^{n-3}\mu^{(n-2)}(V,x_i)))\end{aligned}$$

Simplifying and rearranging gives:

$$\deg\check{V} = m(m-1)^{n-2} - \sum_i (\mu^{(n-1)}(V,x_i) + \mu^{(n-2)}(V,x_i)).$$

This formula was previously established in [Te6] by algebraic methods, based on the fact that the multiplicity in the ring $\mathcal{O}_{V,x_i}$ of the jacobian ideal is equal to $\mu^{(n-1)}(V,x_i)+\mu^{(n-2)}(V,x_i)$; see [Pi] for a proof in terms of characteristic classes, closer to the approach of Todd. This multiplicity is the intersection multiplicity of the relative polar curve with the hypersurface, which counts the number of intersection points of the polar curve with a Milnor fiber of the hypersurface singularity.
This shows that the "diminution of class" due to the singularity is the number of tangent hyperplanes containing L which are "absorbed" by the singularity; see [Te6].
Yet another proof, relating the degree of the dual variety to integrals of curvature and based on the relationship between (relative) polar curves and integrals of curvature brought to light by Langevin in [Lan], can be found in [Gr, §5].

The same method works for complete intersections with isolated singularities, since they can also be smoothed in the same way, and the generalized Milnor numbers also behave in a similar way. Using the results of Navarro Aznar in [N] on the computation of the Euler characteristics of nonsingular complete intersections and the results of Lê in [L] on the computation of Milnor numbers of complete intersections, as well as a direct generalization of the small trick of example 5.1 for the computation of $\chi_1(X,\{x\})$ (see also [Ga2]), one can produce a topological expression for the degrees of the duals of complete intersections with isolated singularities, in terms of the degrees of the equations and generalized Milnor numbers. We leave this to the reader as an interesting exercise. The answer, obtained by a different method, can be found in [Kl3] and a proof inspired by [Er] can also be found in [M-T]. The correction term coming from the singularities has the same form as in the hypersurface case.

In the general isolated singularities case (see [Kl3]), both the computation of Euler characteristics and the topological interpretation of local invariants at the singularities offer new challenges.

Conclusion: Given a projective variety V of dimension d endowed with its minimal Whitney stratification $V = \bigcup_{\alpha\in A} V_\alpha$, we can write the formula of proposition 7.2 as follows:

$$(-1)^d \deg \check{V} =$$
$$\chi(V) - 2\chi(V \cap H_1) + \chi(V \cap H_2) - \sum_{d_\alpha < d} (-1)^{d_\alpha} \deg_{n-2} \check{\overline{V_\alpha}} (1 - \chi_{d_\alpha+1}(V, V_\alpha)),$$

where we agree that $\deg_{n-2} \check{\overline{V_\alpha}} = \deg \check{\overline{V_\alpha}}$ if $\dim \check{V}_\alpha = n-2$, and is 0 if $\dim \check{V}_\alpha < n-2$. Then we see by induction on the dimension that:

Proposition 7.4. *The degree of the dual variety, when it is a hypersurface, ultimately depends on the Euler characteristics of the $\overline{V_\alpha}$ (or the V_α, since it amounts to the same by additivity of the Euler characteristic) and their general linear sections, and the local vanishing Euler characteristics $\chi_i(\overline{V_\beta}, V_\alpha)$.*

Problem: Given V as above with a defining homogeneous ideal, describe an algebraic method to produce an ideal defining the union of V and the duals of the other strata of the minimal Whitney stratification of the dual $\check{V}$.
For example, the dual of a general plane algebraic curve has only cusps and double points as singularities. The construction described above adds to the curve all its "remarkable tangents", namely its double tangents and inflexion tangents.

Using the properties of polar varieties and theorem 5.2 one can prove a similar formula in the case where the dual $\check{V}$ is not a hypersurface, and thus extend proposition 7.4 to all projective varieties. The degree of $\check{V}$ is then the multiplicity at the origin of the smallest polar variety of the cone X over V which is not empty. Now one uses the equalities

$$m_x(P_k(X, x)) = m_x(P_k(X, x) \cap L_{d-k-1}) = m_x(P_k(X \cap L_{d-k-1}), x)),$$

which we have seen before theorem 5.2. They tell us that the degree of $\check{V}$ is the degree of the dual of the intersection of V with a linear space of the appropriate dimension for this dual to be a hypersurface.

More precisely, when H is general hyperplane in $\mathbf{P}^{n-1}$, the following facts are consequences of the elementary properties of projective duality, remark 3.1, c), and the property that tangent spaces are constant along the generating lines of a cone (see lemma 2.3):

- If $\check{V}$ is a hypersurface, the dual of $V \cap H$ is the cone with vertex $\check{H}$ over the polar variety $P_1(\check{V}, \check{H})$, the closure in $\check{V}$ of the critical locus of the restriction to $\check{V}^0$ of the projection $\pi\colon \check{\mathbf{P}}^{n-1} \to \check{\mathbf{P}}^{n-2}$ from the point $\check{H} \in \check{\mathbf{P}}^{n-1}$. Since we assume that V is not contained in a hyperplane, the degree of the hypersurface $\check{V}$ is ≥ 2, hence this critical locus is of dimension $n-3$ and the dual of $V \cap H$ is a hypersurface. In appropriate coordinates its equation is a factor of the discriminant of the equation of $\check{V}$.
- Otherwise, the dual of $V \cap H$ is the cone with vertex $\check{H}$ over $\check{V}$, i.e., the join in $\check{\mathbf{P}}^{n-1}$ of $\check{V}$ and the point $\check{H}$.

Although they were suggested to us by the desire to extend Proposition 7.4 to the general case, these statements are not new. The authors are grateful to Steve Kleiman for providing the following references: for the first statement, [Wa2, Lemma d, p.5], and for the second one [HK, Thm. (4.10(a)), p.164]. On may also consult [Kl5].

Assuming that H is general, it is transversal to the stratum V^0 and to verify these statements one may consider only what happens at nonsingular points of $V \cap H$, which are dense in $V \cap H$. At those points, the space of hyperplanes containing the tangent space $T_{V,v}$ is of codimension one in the space of hyperplanes containing $T_{V\cap H,v}$ and does not contain the point $\check{H}$ since H is general. Any hyperplane containing $T_{V\cap H,v}$ and distinct from H determines with H a pencil. Because of the codimension one, the line in $\check{\mathbf{P}}^{n-1}$ representing this pencil must contain a point representing a hyperplane tangent to V at v. The closure in $\check{\mathbf{P}}^{n-1}$ of the union of the lines representing such pencils is the dual of $V \cap H$. It is a cone with vertex $\check{H}$ and because tangent spaces are constant along generating lines of cones, a tangent hyperplane to this cone must be tangent to $\check{V}$.
If $\dim\check{V} = n-3$ this cone is a hypersurface in $\check{\mathbf{P}}^{n-1}$. Otherwise we repeat the operation by intersecting $V \cap H$ with a new general hyperplane, and so on; we need to repeat this as many times as the *dual defect* $\delta(V) = \mathrm{codim}_{\check{\mathbf{P}}^{n-1}}\check{V} - 1$. We can then apply proposition 7.4 to $V \cap H_{\delta(V)}$ because its dual is a hypersurface. The cone $\check{H} * \check{V}$ from a point $\check{H}$ in $\check{\mathbf{P}}^{n-1}$ on a projective variety $\check{V}$ has the same degree as $\check{V}$. To see this, remember that the degree is the number of points of intersection with a general (transversal) linear space of complementary dimension. If a general linear space L of codimension $\dim\check{V}+1$ intersects $\check{H} * \check{V}$ transversally in m points, the cone $\check{H} * L$ will intersect transversally $\check{V}$ in m points. Thus, the iterated cone construction does not change the degree so that the degree of the dual of $V \cap H_{\delta(V)}$ is the degree of $\check{V}$.
The smallest non empty polar variety of the cone $(X,0)$ over V is $P_{d-\delta(V)}(X,0)$.

This suffices to show that proposition 7.4 is valid in general. *Obtaining a precise formula for the degree of $\check{V}$ in the general case is reduced to the computation of Euler-Poincaré characteristics and local vanishing Euler-Poincaré characteristics of general linear sections of V and of the strata of its minimal Whitney stratification.*

It would be interesting to compare this with the viewpoints of [Pi], [Er], and [A]. The comparison with [Er] would hinge on the following two facts:

- By corollary 5.1.2 of [L-T1] we have at every point $v \in V$ the equality

$$\mathrm{Eu}(V,v) = \sum_{k=0}^{d-1} (-1)^k m_v(P_k(V,v)).$$

- As an alternating sum of multiplicities of polar varieties, in view of theorem 4.4, the Euler obstruction is constant along the strata of a Whitney stratification.

Indeed, if we expand the formula written above proposition 7.4 in terms of the Euler characteristics of the strata V_α and their general linear sections, and then remove the symbols χ in front of them, we obtain a linear combination of the V_α and their sections, with coefficients depending on the local vanishing Euler-Poincaré characteristics along the V_α, which has the property that taking formally the Euler characteristic gives $(-1)^d \deg \check{V}$. Redistributing the terms using theorem 5.2 should then give Ernström's theorem. We leave this as a problem for the reader. Another interesting problem is to work out in the same way formulas for the other polar classes, or ranks (see [Pi, §2]).

Finally, by proposition 7.1 the formula of theorem 5.2 appears in a new light, as containing an extension to the case of non-conical singularities of the generalized Plücker formulas of projective geometry. Interesting connections between the material presented here and the theory of characteristic classes for singular varieties are presented in [Br].

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References

[A] P. Aluffi, *Projective duality and a Chern-Mather involution*, to appear. ArXiv:1601.05427v2.

[A-M] M. Atiyah & I. MacDonald, *Introduction to Commutative Algebra*, Addison-Wesley, 1969.

[Bo] N. Bourbaki, *Elements de Mathématique. Algèbre Commutative, Chap. VIII-IX*, Masson, 1983.

[Br] J-P. Brasselet, *The Schwartz classes of complex analytic singular varieties*, in: Singularity Theory, D. Chéniot, N. Dutertre, C. Murolo, A. Pichon, D. Trotman, Editors, World Scientific, 2007, 3-32.

[BS] J. Briançon et J-P. Speder, *La trivialité topologique n'implique pas les conditions de Whitney*, C.R. Acad. Sci. Paris, Sér A-B 280 (1975), no. 6, A365-A367.

[CdS] Ana Cannas da Silva, *Lectures on Symplectic Geometry*, Springer Lecture Notes in Mathematics No. 1764, 2001.

[Ch] D. Cheniot, *Sur les sections transversales d'un ensemble stratifié,* C.R. Acad.Sci., Paris., t. 275, 1972, 915-916.

[Co] G. Comte, *Equisingularité réelle, nombres de Lelong, et images polaires*, Annales Sci. Éc. Norm. Sup. (6) 33, (2000), no. 6, 757-788.

[CM] G. Comte et M. Merle, *Équisingularité réelle. II. Invariants locaux et conditions de régularité*, Ann. Sci. Éc. Norm. Sup. (4) 41 (2008), no. 2, 221-269.

[Eis] D. Eisenbud, *Commutative Algebra with a view toward Algebraic Geometry*, Springer, 1999.

[Er] L. Ernström *A Plücker formula for singular projective varieties*, Communications in Algebra, 25:9, (1997), 2897-2901, DOI: 10.1080/00927879708826029

[Fi] Gerd Fischer, *Complex Analytic Geometry*, Lecture Notes in Mathematics 538, Springer-Verlag, 1976.

[Ga1] T. Gaffney, *Integral closure of modules and Whitney equisingularity*, Invent. Math. 107, no 2, 1992, 301-322.

[Ga2] T. Gaffney, *Multiplicities and equisingularity of ICIS germs*, Invent. Math., 123, 1996, 209-220.

[Ga-Kl] T. Gaffney and S.L. Kleiman, *Specialization of integral dependence for modules*, Invent. Math. 137, no 3, 1999, 541-574.

[Gi] A. Giles Flores, *Specialization to the tangent cone and Whitney equisingularity*, Bulletin de la Société Mathématique de France, Vol 141, no 2, 2013, 299-342.

[Gr] P. A. Griffiths. *Complex differential and integral geometry and curvature integrals associated to singularities of complex analytic varieties*, Duke Math. J., Vol. 45, no. 3, September 1978, 427-512.

[H] F. Hirzebruch, *Topological methods in Algebraic Geometry*, Springer 1978, reprinted in Springer Classics, 1995.

[HK] [3] A. Hefez and S. Kleiman, *Notes on the duality of projective varieties*, Geometry today (Rome, 1984), Progr. Math., 60, Birkhuser Boston, Boston, MA, 1985, 143-183.

[HM] J-P. Henry and M. Merle, *Limites de normales, conditions de Whitney et éclatements d'Hironaka*, Proc. Symposia in Pure Math., No. 40, Vol. 1, Arcata 1981, A.M.S. 1983, 575-584.

[HMS] J. P. Henry, M. Merle et C. Sabbah, *Sur la condition de Thom stricte pour un morphisme analytique complexe*, Ann. Sci. École Norm. Sup. 17 (1984).

[He-Or] M. Herrmann, S. Ikeda, and U. Orbanz, *Equimultiplicity and Blowing up*, Springer-Verlag, 1988.

[Hi] H. Hironaka, *Normal cones in analytic Whitney Stratifications*, Publ. Math. I.H.E.S. No. 36, P.U.F. 1970.

[Hn] A. Hennings, *Fibre dimension of the Nash transformation*, arXiv:1410.8449v1 [math.AG]

[K1] M. Kashiwara, *Index theorem for a maximally overdetermined system of linear differential equations*, Proc. Japan Acad. 49 (1973), 803-804.

[K2] M. Kashiwara, *Systems of microdifferential equations*, Based on lecture notes by Teresa Monteiro Fernandes translated from the French. With an introduction by Jean-Luc Brylinski. Progress in Mathematics, 34. Birkhäuser Boston, Inc., Boston, MA, 1983. xv+159 pp. ISBN: 0-8176-3138-0.

[Ka] L. Kaup and B. Kaup, *Holomorphic Functions of Several Variables*, W. de Gruyter & Co., 1983.

[Kl] S. Kleiman, *The transversality of a general translate*, Compositio Math., 28, 1974, 287-297.

[Kl2] S. Kleiman, *About the conormal scheme*, in: CIME, Arcireale, Springer Lecture Notes No. 1092 (1984).

[Kl3] S. Kleiman, *A generalized Teissier-Plücker formula*, in: Classification of algebraic varieties (L'Aquila, 1992, volume 162 of Contemp. Math., pages 249-260. Amer. Math. Soc., Providence, RI, 1994.

[Kl4] S. Kleiman, *The enumerative theory of Singularities*, in: Real and complex singularities, Nordic Summer School, Oslo 1976, Sijthoff and Noordhoff 1977, 297-396.

[Kl5] S. Kleiman, *Tangency and duality*, Proceedings of the 1984 Vancouver conference in algebraic geometry, 163-225, CMS Conf. Proc., 6, Amer. Math. Soc., Providence, RI, 1986.

[L] Lê Dũng Tráng, *Calcul du nombre de Milnor d'une singularité isolée d'intersection complète*. Funkt. Analiz i iego Pril., 8, 1974, 45-49.

[Lan] R. Langevin, *Courbure et singularités complexes*, Comment. Math. Helvetici, 54 (1979), 6-16.

[La] G. Laumon, *Transformations canoniques et spécialisation pour les $\mathcal{D}$-modules filtrés*, in: Differential systems and singularities (Luminy, 1983). Astrisque No. 130 (1985), 56-129.

[L-T1] Lê D.T. and B. Teissier, *Variétés polaires locales et classes de Chern des variétés singulières* , Annals of Math. 114, 1981, 457-491.

[L-T2] Lê D.T. et B. Teissier, *Limites d'espaces tangents en géométrie analytique*, Comm. Math. Helv., **63**, 1988, 540-578.

[L-T3] Lê D.T. and B. Teissier, *Cycles évanescents, sections planes, et conditions de Whitney II*, in *Singularities*, Proc. Sympos. Pure Math., 40, 1983, part 2, 65-103.

[L-T4] Lê D.T. and B. Teissier, *Sur la géométrie des surfaces complexes, I. Tangentes exceptionelles*, in American Journal of Math., 101, No. 2, (Apr., 1979), 420-452.

[Lê] Lê D.T., *Limites d'espaces tangents sur les surfaces*, Nova Acta Leopoldina NF 52 Nr. 240, 1981, 119-137.

[Lej-Te] M. Lejeune-Jalabert et B. Teissier, *Clôture intégrale des idéaux et équisingularité, avec un appendice de J-J. Risler*, Annales de la Faculté des Sciences de Toulouse, XVII, no 4, 2008, 781-859.

[Lip] J. Lipman, *Equisingularity and simultaneous resolution of singularities*, in *Resolution of Singularities:a research textbook in tribute to Oscar Zariski*, Progress in math. v. 181, Birkhäuser, Basel, 2000, p. 485-505.

[Loj] S. Łojasiewicz, *Introduction to Complex Analytic Geometry*, Birkhauser Verlag, 1991.

[Ma] J. Mather, *Stratifications and Mappings*, in: *Dynamical Systems*, Academic Press, New York, 1973, 195-232.

[M-T] Y. Matsui and K. Takeuchi, *Generalized Plücker-Teissier-Kleiman formulas for varieties with arbitrary dual defect*, in: Real and complex singularities, World Sci. Publ., Hackensack, NJ, 2007, 248-270.

[Mat] H. Matsumura, *Commutative Ring Theory*, Cambridge University Press, 2000.

[Mi] J. Milnor, *Singular points of complex hypersurfaces*, Princeton University Press, 1968.

[N] V. Navarro Aznar, *On the Chern Classes and the Euler Characteristic for Nonsingular Complete Intersections*, Proceedings of the American Mathematical Society, Vol. 78, No. 1 (Jan., 1980), 143 -148.

[P] F. Pham, *Singularités des systèmes différentiels de Gauss-Manin*, with contributions by Lo Kam Chan, Philippe Maisonobe and Jean-Étienne Rombaldi. Progress in Mathematics, 2. Birkhuser, Boston, Mass., 1979.

[Pi] R. Piene, *Polar classes of singular varieties*, Annales scientifiques de l'É.N.S., 4è série, tome 11, No.2 (1978), 247-276.

[Pi2] R. Piene, *Polar varieties revisited*, in: Computer Algebra and Polynomials, Springer LNCS 8942, 2015, 139-150.

[Pl] J. Plücker, *Theorie der algebraischen Kurven*, Bonn, 1839.

[Pon] J-V. Poncelet, *Traité des propriétés projectives ds figures, ouvrage utile à ceux qui s'occupent des applications de la Géométrie descriptive et d'opérations géométriques sur le terrain. Tome second. Deuxième édition, revue par l'auteur et augmentée de sections et d'annotations nouvelles et jusqu'ici inédites.*, Gauthier-Villars, Paris 1866. Available at http://gallica.bnf.fr/ark:/12148/bpt6k5484980j

[Po] P. Popescu Pampu, *What is the genus?*, Springer Lecture Notes in Mathematics No. 2162, Springer 2016.

[Re] R. Remmert, *Holomorphe und meromorphe abbildungen complexe raüme*, Math. Annalen, 133 (1957), 328-370.

[Rs] D. Rees, ***a**-transforms of local rings and a theorem on multiplicities of ideals*, Math. Proc. Camb. Phil. Soc., **57**, (1961), 8-17.

[Sa] C. Sabbah, *Quelques remarques sur la géométrie des espaces conormaux*, Astérisque No. 131, S.M.F. 1985.

[Se] J.G. Semple, *Some investigations in the geometry of curve and surface elements*, Journ. London Math. Soc., (**3**), 4, 1954.

[SSU] A. Simis, K. Smith and B. Ulrich, *An algebraic proof of Zak's inequality for the dimension of the Gauss image*, Math. Z., 241 (2002), no. 4, 871-881.

[Te1] B. Teissier, *Cycles évanescents, sections planes et conditions de Whitney*, Singularités à Cargèse, Astérisque 7-8, 1974, 282-362. Available at: webusers.imj-prg.fr/ bernard.teissier/

[Te2] B. Teissier, *The hunting of invariants in the geometry of discriminants*, in: *Real and Complex singularities, Oslo 1976*, Per Holm Ed., Sijthoff & Noordhoff 1977. Available at: webusers.imj-prg.fr/ bernard.teissier/

[Te3] B. Teissier, *Variétés Polaires 2: Multiplicités polaires, sections planes, et conditions de Whitney*, in: *Actes de la conférence de géométrie algébrique à La Rábida 1981*, Springer Lecture Notes No. 961, p. 314-491. Available at: webusers.imj-prg.fr/ bernard.teissier/

[Te4] B. Teissier *Sur la classification des singularités des espaces analytiques complexes*, in: *Proceedings of the International Congress of Mathematicians, 1983, Warszawa.* Available at: webusers.imj-prg.fr/ bernard.teissier/

[Te5] B. Teissier, *Apparent contours from Monge to Todd*, in: 1830–1930: a century of geometry (Paris, 1989), 55–62, Lecture Notes in Phys., 402, Springer, Berlin, 1992. Available at: webusers.imj-prg.fr/ bernard.teissier/

[Te6] B. Teissier. *Résolution simultanée, II*, in: M. Demazure, H. C. Pinkham, and B. Teissier, editors, Séminaire sur les Singularités des Surfaces, 1976-77, volume 777 of Springer Lecture Notes in Mathematics. Springer, Berlin, 1980. Available at: webusers.imj-prg.fr/ bernard.teissier/

[Te7] B. Teissier, *Introduction to equisingularity problems*, in: Proceedings of A.M.S. Symposia in Pure Mathematics, Algebraic Geometry Arcata 1974, Vol. 29, 1975, 593-632. Available at: webusers.imj-prg.fr/ bernard.teissier/

[Tev] E. Tevelev, *Projective duality and homogeneous spaces*, Encyclopaedia Math. Sci., 133, Springer, Berlin, 2005.

[Th] R. Thom, *Ensembles et morphismes stratifiés*, Bull. Amer. Math. Soc.,**75**, 1969, 240-284.

[To1] J.A. Todd, *The geometrical invariants of algebraic loci*, Proc. London Mat. Soc , **4**3, 1937, 127-138.

[To2] J.A. Todd, *The arithmetical invariants of algebraic loci*, Proc. London Mat. Soc. **4**3, 1937, 190-225.

[To3] J.A. Todd, *The geometrical invariants of algebraic loci (second paper)*, Proc. London Mat. Soc. **4**5, 1939, 410-424.

[Wa1] A.H. Wallace,*Tangency and duality over arbitrary fields*. Proc. London Math. Soc. (3) 6 (1956), 321-342.

[Wa2] A.H. Wallace, *Homology theory on algebraic varieties*, International Series of Monographs on Pure and Applied Mathematics. Vol. 6 Pergamon Press, New York-London-Paris-Los Angeles 1958 viii+115 pp.

[Whi1] H. Whitney, *Tangents to an analytic variety*, Annals of Math., **81**, (1964), 496-549.

[Whi2] H. Whitney, *Local properties of analytic varieties, in: Differential and Combinatorial Topology*, Princeton Univ. Press, 1965.

[Za] O. Zariski, *Le problème des modules pour les branches planes, avec un appendice de B. Teissier*, Paris Hermann, 1986. ISBN: 2-7056-6036-4. English translation by Ben Lichtin, A.M.S., University Lecture Series, 39 (2006). Appendix available at: webusers.imj-prg.fr/ bernard.teissier/

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