The phantom of transparency

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Per te, Peppe.

I will discuss the status of the implicit and the explicit in science, mostly in logic[1]. I will especially denunciate, expose a deep and pregnant unsaid of scientific activity : the subliminal idea that, beyond immediate perception, could exist a world, a layer of reading, completely intelligible, i.e., explicit and immediate. What I will call the fantasy (or phantom, as a matter of joke) of *transparency*.

Transparency has little to do with poetical ideas (the key of dreams, etc.). It is indeed a unidimensional *underside* of the universe, not always monstrous, but anyway grotesque. Think of this Axis of Evil supposedly responsible of all the misery of the world, or of these unbelievable *minority studies* which expose the carefully concealed truths : for *feminine studies* Shakespeare was a woman, for *african studies* he was an Arab, the Cheikh Zubayr !

Nevertheless, everything starts from a correct premise, to go beyond mere apparences ; but, to do so, one imagines an « other side of the mirror » whose delimitations are neat, precise, without the slightest ambiguity : the world is seen as a rebus of which it suffices to find the key. In the transparent world, everything is so immediate, legible, that one does no longer need to ask questions, i.e., no longer need to think. This putting in question of the very idea of question leads to the worse idiocies : if answers are so easy to access, is it because God amuses Himself with presenting us an encoded world

[1]Due to the theme, I will exceptionnally be slightly *ad hominem*, but not too much.

for the sole purpose of testing us ? Unless men are to blame, whose industry is devoted to dissimulation for unawovable reasons ; such a behaviour thus justifies the « question », the cognitive protocol in fashion in Guantanamo.

One must anyway admit that a question need not have answers, that it is not even bound to have some, since a great part of scientific activity consists, precisely, in seeking the *good* questions. Thus, the correspondence between planets and regular polyhedra, of which Kepler was so proud, is not even a wrong hypothesis, it is an absurd connection, which only deserves a shrug of the shoulders, a question that didn't deserved to be posed, to be compared to speculations linking the length of a ship with the age of its captain. Transparency stumbles on the questioning as to the interest of questions, next on the difficulty to find the answers to the supposedly good problems. Indeed, answers are, mostly, partial : a half-answer accompanied with a new question. The relation question/answer thus becomes an endless dialogue, an *explicitation* process ; it is in this process, which yields no definitive and totalising key, where the afterworld of apparences, i.e., knowledge, is to be found.

1 Logics of transparency

There is in logic a fantasy of transparency that can be summarised by a word : *semantics*. Before discussing the limitations of semantics, it is of interest to discuss bad logic, the logic of those who do not have the words, i.e., technical comptence : we only hear the music, i.e., this affirmation of a transparent world. One should also quote the logical entries of *Wikipedia*, usually written and rewritten by sectarians of transparency, but this material is too labile.

1.1 Abduction

« If $A \Rightarrow B$, it is because B needed A hence $B \Rightarrow A$ ». This amalgamation between causes and effects trips over the wire ; in logic as well as in other domains, e.g., in politics : take this Devedjian[2] explaining the misery of suburbs by... the misdeeds of left wing politicians. As a sort of wink to Giuseppe Longo, let us also mention the abductive dimension of the fantasy of DNA (genetic transparency), witness the « gene of pedophily » dear to a friend of the same Devedjian, Mr. Sarkozy. The unofficial model of abduction claim is Sherlock Holmes, with his warped, undoubtedly amusing, deductions : indeed, to analyse the ashes of a cigar and conclude that the criminal is 47,

[2]French politician, not quite a « red ».

back from India and limps from the left foot, is at least, unexpected. What Sherlock Holmes actually supposes, is a world transparent at the level of police, criminal activities, the key to this world being the science of ashes, a sort of necromancy, positive but just as absurd[3]. Metaphorically, this pseudo-science refers to this afterworld in which all questions are supposed to have received their answer. There are however question which have no room in this too polished (and policed) world, typically those of the form « is this problem well-posed ? ».

The search for possible causes is, however, an ancient and legitimate activity, albeit not a mode of reasoning : this would put apprences in command. Mathematics created a special category for those possible causes, in want of legitimation and, for that reason, in the limbs of reasoning : *conjectures*, interesting hypotheses, on which one attracts attention. The process of integration of a conjecture in the corpus is complex and by no means transits through an inversion of the sense of reasoning.

It should be observed that mathematical *induction* is close to abduction. Etymologically, induction is the reasoning by generalisation which, to avoid being abusive, must transit through the emission of conjectures. What one calls mathematical induction is an abduction which moves from possible causes to possible methods of constructions, see *infra* the development on categories. Mathematical induction is not, contrarily to abduction, a grotesque mistake of reasoning ; it is nevertheless, see *infra*, a form of transparency.

1.2 Non monotonic logics

Still under the heading « science for the half-wits », let us mention non monotonic « logics ». They belong in our discussion because of the fantasy of completeness, i.e., of the answer to all questions. Here, the slogan is *what is not provable is false* : one thus seeks a completion by adding unprovable statements. Every person with a minimum of logical culture knows that this completion (that would yield transparency) is fundamentally impossible, because of the undecidability of the halting problem, in other terms, of incompleteness, which has been rightly named : it denotes, not a want with respect to a preexisiting totality, but the fundamentally incomplete nature of the cognitive process[4].

[3]On the other hand, the same Sherlock Holmes boasts ignoring the rotation of Earth around the Sun : this is not part of « positive science ».

[4]To be put in relation with the unbounded operators of functional analysis, intrinsically and desperately partial.

1.3 Epistemic logic

The previous fiddlings attracted the cordial jealousy of epistemic logicians who consider their domain as the worst logic ever, a claim that can be grounded. Epistemic logic is an archipelago of rather afflictive abductive anectodes, the most famous of which being that of the 49 Baghdad cuckolds. In this story, the *Café du Commerce*[5] is admirative of the 48 iterations of the same idiocy, anyway gone flat after the first step, and that we shall transpose in Texas, between V (Vardi) and W (Bush) : they know that at least one of them is wearing the horns, moreover V, knowing that W is betrayed, cannot conclude ; but, since W does nor react either, V surmises that the situation is symmetrical and, subsequently, slays his supposedly inconstant spouse. This nonsense rests upon the idea of a perfect, immediate, transparent knowledge ; not too speak of the hidden assumption that the actors (at least V) are familiar with epistemic « logic ».

Of course, as soon as this transparency faints, for instance if one takes into account the intellectual limitations of W, one sees that V may have killed an innocent wife. Technically speaking, the slowness of W corresponds to the complexity of deduction, of algorithms and, *in fine* brings us back to undecidability. This explains why epistemic logic never succeeded outside this *Café du Commerce* where it flourishes : it contradicts the incompleteness theorem.

Transparency takes here the form « he who stays silent must have something to hide ». These ways of forcing the mute into talking have a rear taste of torture : one thinks of the « gégène[6] » de 1957, of the bathtub of the Gestapo, a.k.a. « waterboarding » in Guantanamo and also of the 1937 purges[7]. Epistemic logic is thus the derisive scientific counterpart of totalitarism.

1.4 Explicit mathematics

Still within bad logic, but in the upper category, let us consider the « explicit mathematics » of S. Feferman : it is a tentative bureaucratisation of science, an attempt just as exciting as a fiction by Leonid Brezhnev. But, rather than the mediocrity of the approach, we shall interrogate this extravagant association « mathematics + explicit » : this is indeed an *oxymoron*.

[5]In France, the metaphoric location of commonplace ideas.

[6]Torture by electricity practiced by the French paratroopers.

[7]Yezhov's *ante litteram* version of epistemic logic was organised along two types of questions : « Why don't you denounce this traitor ? » and, after the denounciation, « Why did you denounce that innocent, causing his death ? ».

Are mathematics, can they be, explicit ? Since they are an extreme of thought, there would therefore be an explicit thought. Coming back to the words : in « implicit », there is *imply* thus implication, logic or not ; what is implicit is what we can indirectly access to, i.e., through thought. On the other hand, « explicit » refer to explication, *explicitation* : explicit thus means direct access

As we just said, thought (and the major part of human activity) belongs in the implicit. Out of thought, one can mention this superb abstraction constituted by money, which evoluted through centuries from gold to paper. An explicit economy would be barter, W giving his wife to V in exchange of a cow. In the same way, an explicit mathematics would be a verification of the style $2 + 0 = 2$, of which any mathematician knows that it is not a real theorem. What is problematic here is not the extreme facility, it is the absence of any implicit contents ; *a contrario* $x + 0 = x$ has an implicit contents (one makes it explicit by providing a value for the variable, e.g., $x = 2$).

To take an analogy, there is the same difference between a verification and a theorem as between a table of logarithms and a pocket calculator : the table proposes us a long, but frozen, list of values, whereas the calculator posseses, at least in advance, no answer to the query. And, by the way, computer scientists, which are people of common sense, never dreamed of an « explicit computer », sort of monstrous telephone directory.

2 Semantics

This *newspeak* expression originally refers to a theory of signs, thus of meaning. Semantics turns out, in fact, to be a fantastic *machine à décerveler*[8] by obfuscation of the sense. This is because of its pretension at materialising this transparent world ; the failure of the project runs into intellectual skulduggery. Semantics rests upon the fantasy of a reduction to boolean truth values : obvious, since *one can answer any query* ! Observe that the other major dogma of current life « one can compare everything » is to found again in fuzzy logics (which brings us back to the indignities of the previous section) and also in the various aspects of tarskism, see *infra*.

2.1 From Frege to Tarski

The distinction sense/denotation, due to Frege, is to some extent, the noble version of the myth of transparency : sense refers to a denotation, ideal

[8]To remove the brain, from Alfred Jarry.

and definitive ; for instance, Venus for all the poetical descriptions « star of the shepherd, of the morning, the evening, etc. ». This dichotomy *a priori* excludes any link other than fantasmatic between the two aspects, the sense and its underside, the denotation : $A \Rightarrow A$ refers to a denotation which is, by definition, completly alien to us. It is thus impossible to understand how the slightest reasoning is possible : just like Zeno's arrow lingers on, one does not gather how or why the slightest cognitive act can legitimately be performed.

Even less inspired, Tarski defines the answer to the question as... the answer to the question : thus, the transparent universe would be but a pleonasm of the immediate universe. This is what expresses the notorious *vérité de la Palice*[9] « $A \wedge B$ is true when A is true and B is true ». Therefore the denotation of $A \Rightarrow A$ reduces to the implication between the denotation of A and the denotation of A, which means strictly nothing. The failure of this sort of explanation induces a forward flight : real transparency should be sought, beyond immediate transparency, in a « meta » — this fuel for frozen brains — defined as an iterated pleonasm even iterable in meta-meta, etc. and, eventually, transfinitely ! This theology of transparency is but one more obscurantism.

2.2 Kripke models

Compared with the previous idiocies, Kripke models almost look as a conceptual breaktrough. But, if the first encounter, with its perfume of parallel worlds causes a certain jubilation, this enthousiasm is soon soothed by the absolute sterility of the object : like the *violon tzigane* of Boby Lapointe, Kripke models are reserved to those who have no alternative[10].

The idea underlying this approach is that the potential (which is another name for the implicit) is the sum, the totality, of the possibilities. *Et voilà pourquoi votre fille est muette !* From the philosophical standpoint, did one ever hear something more ridiculous ? For instance, can one say that a 200 € bill is the catalogue of everything we can buy with it ? Even neglecting the variability of price, one should make room for discontinued merchandises, or those not yet produced ! No, a 200 € bill is a *question* whose answer belongs in its protocol of circulation : one can exchange it against a merchandise of nominal value 200 €, but also against two 100 € bills. The merchandise can, in turn, be partially implicit, witness this DVD reader which requires a disk to proceed. We discover on the way that the explicitation need not be total :

[9]French logician, the unquoted precursor of Tarski, typically « a quarter of hour before his death, he was still in life ».

[10]« — My son, there are two ways of playing violin, either you play in tune, or you play tsigan — I have no choice, Dad, it is why I play tsigan ».

it can be purely formal, or even partial ; in other terms, the implicit may refer, totally or partially, to other implicits.

Kripke models do crystallise this vision of the potential as a sum of possibilities, thence their paradoxical importance : although faulty, this idea is indeed difficult to refute, since of quasi universal implementation. Thus, (thanks, Brouwer !), a function will never be a graph, but an implicit structure, a construction, given, for instance, by a program : « give me an input, an argument n and I return you $F(n)$ ». It turns out that one can, nevertheles, « define » F through the associated graph $\{(n, F(n)); n \in \mathbb{N}\}$, which is, *stricto sensu*, a monstrous reduction, but what is also incredibly efficient. Thence the success of set theory and the concomitant washing up of Brouwer's ideas which became subjectivistic, « intensional » (after « meta », yet another swear word).

2.3 Categories

The categorical interpretation of logic (especially, intuitionistic logic) make questions appear as objects, answers as *morphisms*. Typically, the disjonction $A \vee B$ asks the question « A or B ? », whereas the morphisms inhabiting it[11] are proofs of A or proofs of B, thence answers to the question. Categories finally appear as the transparent world of morphisms : answers are combined through composition which is handled by categorical diagrams (thus commutative *a priori*, following an old joke). This means that, once entered in the realm of answers, everything is free of charge ; something else than equality should be introduced in order to say that composition has a cost : to paraphrase the dear Orwell, in a commutative diagram, one side is more commutative than the other. Composition is implemented by an algorithm, which is not transparency — which is but a fantasy —, but construction, a search, necessarily partial and faulty, of transparency.

Fundamentally, the categorical approach's weak point is *essentialism* : it presupposes the form (to which the expression *morphism* refers) thence cannot analyse it. This being said, the sort of transparency at work in categories is not trivial, contrarily to tarskian transparency ; the analysis of its limitations will eventually yield precious information.

2.4 Universal problems

Mathematical induction, the civilised form — since technically impeccable — of abduction, is expressed as the solution of a *universal problem* : in a cat-

[11] For purists, from the terminal object into it.

egory, one gives oneself constructors and this *induces* a destructor whose action amounts at inventoring of all *possible* construction means. The most familiar case is that of natural numbers, whose constructors are zero and the successor and whose destructor is the principle of recurrence. This idea is expedient, much nobler than Kripke models, but summary. For instance, defining integers as the solution of a universal problem makes them *ipso facto*, unique : the infinite, etymologically « unfinished », is thus reduced to its explicitation, which yields, in the case of natural numbers, this China Wall, the set $\{0, 1, 2, 3, \ldots\}$. This reduction turns out to be an aporia, exposed by Gödel's paradox (incompleteness).

We arrived in a strange situation ; the reflexion on the infinite has been sacrificed on the altar of efficiency to the construction of expedient mathematical tools ; just as equal temperament sacrificed natural resonances to the exigencies of piano builders. For most utilisations, such compromises are reasonable, but there are cases where they are disastrous. Typically, the theory of algorithmic complexity cannot develop on such bases : indeed, an algorithm is an explicitation procedure ; how can we seriously speak of such a thing in a universe where answers (all answers) exist, long before the corresponding questions have been asked ?

3 From semantics to the cognitive onion

3.1 Genesis of the categorical interpretation

The progress of logical thought can be identified with a progressive liberation from the essentialist gangue. Essentialism, this morphological simplism, supposes the anteriority of the explicit over the implicit. This thomism works marvelously well in classical logic, but fails when one does stick to the knitting : since everything proceeds from the sky, one runs into arbitrariness, into sectarism : witness modal logics, by nature disposable and interchangeable.

Originally, logic is interested in unavoidable thruths, in the « laws of thought ». A logical formalism, as it can be found in laborious textbooks — and, *dixit* Kreisel apropos « the » Mendelsohn, popular for that very reason — looks like a list, not that far from a cooking recipe, but succeeds anyway in its task, that of codifying those universal truths.

Schönfinkel, as early as the the years 1920 and, later, Curry would eventually individuate the functional (in fact, algorithmic *ante litteram*) meaning of some of those axioms (and rules) : this is Curry's isomorphism, recentered in 1969 by Howard around the works of Gentzen, which established a functional reading of (intuitionistic) logic : a proof of $A \Rightarrow B$ is a function from

A to B.

Since the principles of logic are of a frightening generality, the search for spaces harbouring such functions turned out to be quite difficult. The only available solution — functions as set-theoretic graphs — being disqualified for questions of size (monstrous *cardinals*) or algorithmics (not computable) : a hammer to crush a fly, moreover antagonistic to the approach. One thus sought morphologic criteria in order not to embark « too many » functions, thence to construct *closed cartesian categories* (CCC) : those are indeed the exact category-theoretic formulation of intuitionistic logic.

3.2 Scott domains

Seeking a non trivial CCC (i.e., other than the category of sets, the improper category, if any) is a delicate mission. One is quickly led to restrict the search to topological spaces ; but anybody with vague notions of topology knows that a function space admits several topologies (e.g., simple vs. uniform convergence) and that, *for good reasons* : some logical operations require simple, others uniform convergence. Thence, the discovery by Scott, around 1969, of a topology making *all* logical operations continuous must be considered as real breakthrough, the mother of all ulterior developments. However, a deep gap separates *Scott domains* from « real » topology : it suffices to remark that on these badly behaved spaces, a separately continuous function $f(x, y)$ is continuous ! The mediocrity of this topology[12] should have alarmed the milieu ; integrism, the taste for final solutions (a synonym for « transparency ») coped in its way with the mismatch : it is usual topology which should be modified ! But let the dead bury their dead. . . .

The continuity of logical operations expresses, under a sophisticated form, the same transparency obsession, which takes here the form of a perfect control of logical complexity ; whereas the incompleteness theorem, which supposes functions of arbitrary complexity, cannot cope with continuity, unless one fiddles with topology. Let us put it bluntly : the non-continuity is the native, tangible, manifestation of incompleteness, of non-transparency[13].

[12]Never Hausdorff ; to be put together, at the other extremity of the spectrum, with extremely discontinuous topologies, never separable.

[13]Historically, incompleteness refutes the propensity of internalising everything, a monstrous idea which looks a too tight hernial bandage. This is why Gödel's paradox finds its way through artificious statements of the sort « I am not provable », i.e., « I mean strictly nothing » : it must « go out » and the result is not a beautiful thing to watch.

3.3 Coherent spaces

Scott's pseudo-topology belongs to the tradition of his master Tarski, which consists in favouring continuous increasing functions on complete lattices (remember his corny fixed point « theorem »), to which one can also link this other pupil of the same, Feferman, who (mis)treats ordinals by means of increasing functions commuting to suprema : here, the ultimate reference, the transparent world should be that of ordinals, to which one tries to reduce mathematical thought through laborious and stereotyped results. This school professes that everything is continuous, and since their topologies are but orders in disguise, that everything is comparable.

Let us give an example : one can endow a power space $\wp(X)$ with a topology à la Scott (the basic open sets are the $\mathcal{O}_a := \{A; a \subset A \subset X\}$, $a \subset X$ finite), corresponding to the order (of inclusion) topology. Thus, morphisms are those functions Φ such that :

(i) Φ sens $\wp(X)$ into $\wp(Y)$.

(ii) Φ is increasing.

(iii) Φ commutes with directed suprema.

It turns out that, not that far from those directed suprema dear to tarskians, one can find direct limits (i.e., directed inductive) ; the quasi-mechanical replacement of suprema with direct limits has extraordianry consequences, because of the intervention of the playmate of direct limits, the pull-back. Indeed, condider $\wp(X)$ as a category, with inclusions as morphisms ; it is a *degenerated* category, since there is at most one morphism from A into B ; nevertheless this case already does better than the so-called order topology. One takes as morphisms functors preserving direct limits and pull-backs, which yields (i) and (ii) as translation of « functor », (iii) as translation of preservation of direct limits, the preservation of pull-backs yielding :

(iv) $\Phi(A \cap B) = \Phi(A) \cap \Phi(B)$

This new property, called « stability » by Berry, with no analogue in topology, good or bad, is the origin of *coherent spaces* and *linear logic*. Indeed, naturally appears the notion of *linearity*, i.e., the preservation of all unions, directed or not, typically :

(v) $\Phi(A \cup B) = \Phi(A) \cup \Phi(B)$

Preservation of pull-backs thence yields :

(vi) $\Phi(A \setminus B) = \Phi(A) \setminus \Phi(B)$

In other terms, the possibility of *algebraising* the interpretation. Indeed, if one wants to extend the morphism Φ to algebraic combinations of sets, i.e., to functions from X to $\mathbb{Z}$ (or $\mathbb{R}, \mathbb{C}$), it is important to verify that Φ is already linear on the available combinations, which amounts at (iv) and (v). This linearisation can be pushed quite far : *coherent Banach spaces* (CBS), *quantum coherent spaces* (QCS). These interpretations interiorise the necessary non-continuity of logic. Thus, in CBS, a (non linear) function from A to B appears as an analytical function from the *open* ball of A to the *closed* ball of B ; the behaviour of an analytical function near its border being erratic, the composition of such functions is *a priori* impossible. Similarly, CBS dwell in finite dimension, due to the disappearance of the *trace* in infinite dimension, see *infra*.

3.4 Perfection and transparency

Linear logic reveals a *perfect* layer, corresponding to operations that one performs totally, once for all, see the *perfective* of slavic tongues. This layer is quite continuous (it involves finite dimensional spaces) and is thus compatible with a limited amount of transparency. At this layer appears *polarisation*, i.e., the dichotomy negative/positive.

It is indeed an old pragmatic distinction[14], reactivated by linear logic through the works of Andreoli : it is the second generation of logic programming, i.e., the computer science paradigm of *proof-search*. Thus, a program looks like a dialogue combining questions (negative) and answers (positive). This induces a dichotomy of logical primitives ; implication, universal quantification, are negative : for instance, $\forall n(A[n] \Rightarrow B[n])$ means « give me $n = N$ as well as $a[N]$ and I will return you $B[N]$ », provided one can « return » $B[N]$, i.e., B be positive. Which is the case if B is an intuitionistic disjonction $C \vee D$: to return $B[N]$ amounts at returning $C[N]$ (left) or returning $D[N]$ (right), say $D[N]$. This being done, depending on the polarity, positive or negative, of $D[N]$, one must either return more, or ask for fresh information, i.e., resume the questioning.

One must remark that the pure formal manipulation leading from $\forall n(A[n] \Rightarrow (C[n] \vee D[n]))$ to $C[N] \vee D[N]$ has no explicative value. By the way, I was able to perform it while ignoring everything about A, C, D, N, etc. Which means that this is only the priming of a process, an algorithm, of explicitation. This process admits numerous variants, all derived from *cut-elimination*, the celebrated *Hauptsatz* of Gentzen. What one will summarise in reminding that it is wrong that a proof of $C[N] \vee D[N]$ is either a proof

[14]At work in the *negative fragment* ($\Rightarrow, \wedge, \forall$) of intuitionistic logic.

of $C[N]$ or a proof of $D[N]$ (otherwise, what a masochism : enunciate $C \vee D$ when one has obtained D !). Those of that very form are the *cut-free* proofs ; since there are very few of them in nature (they are, if not explicit, the most explicit possible, hence not legible[15], one is led to content oneself with a partial elimination : one determines the first *bit* left/right of the disjunction before proceeding.

One therefore sees that explicitation presents itself through and interactive an dynamical form, in an intrinsically incomplete fashion. It takes the aspect of an alternation of polarities (negative = question, positive = answer). This idea is well captured by the metaphor of the *game*, an excellent idea of the same Gentzen, an idea later resumed — and massacred — by Lorenzen, but fortunately put again in honour in the years 1990. This being said, the game is but a metaphor, suffering from a want of mathematical depth and also of the supposed instantaneity of the answer, that can perhaps be neglected in the perfect case, but which is a faulty hypothsesis : there is a time of latency, corresponding to the algorithmic complexity of cut-elimination, i.e., of the explicitation process, whose metaphor is the slowness of W.

3.5 Ludics

Personnally, I prefer the image of a « cognitive onion », of which one strips off the successive skins[16], what is realised, at least partially, by *ludics*. The basic object, the *design* (= delogicalised proof) combines *actions* of alternating polarities : *negative* (questions) and *positive* (answers).

Everything resembling logic in its essentialist aspects, typically the *rule of the game*, which supposes a referee, hence new places of transparency, is expelled from ludics. Thus, *behaviours* are games whose rule is established by *consensus* between designs and counter-designs : everything is permitted, provided one reaches a conclusion (because one of the partners gives up).

This being said, ludics stumbles on a point : its space of questions/answers (the *actions*) is pre-constituted. Which permits the description of the onion through all ways of peeling it. This set of processes is an ultimate possibility of transparency, of semantics, of which we must must find the *blind spot*.

[15]If a lemma is used three times in a proof, its cut-free version will require three independent subproofs.

[16]As in *Le retour du divin* by Audiberti : the heroin Martine strips her beloved Ambroise from his coat, revealing a second coat, etc. ; of the beautiful Ambroise, Martine will eventually hug but the successive coats.

4 Negation

4.1 Coherent spaces

Linear negation is the exchange question/answer. It originates from a natural operation of *coherent spaces* that I will justify in terms of logical onion. Let us consider complete cognitive processes (sequences of questions/answers) of a certain type : a proof will be interpreted as the set of all sequences that can be associated with it. If two such sequences differ, they bifurcate *negatively* (i.e., on different questions), what one will note $x \frown x'$; this yields :

DÉFINITION 1 (COHERENT SPACES)
A coherent space *is the pair* $X = (|X|, \supset\!\!\subset_X)$ *of a set (the* web $|X|$*) and a reflexive and symmetric binary relation on* $|X|$*, the* coherence. *One notes* $\frown_X$ *strict coherence (between distinct elements). A* clique *of* X*, notation* $a \sqsubset X$*, is a subset of* $|X|$ *made of pairwise coherent elements.*

I content myself with linear negation :

DÉFINITION 2 (NEGATION)
If $X = (|X|, \supset\!\!\subset_X)$ *is a coherent space, its* linear negation *is the space* $\sim X = (|X|, \supset\!\!\subset_{\sim X})$ *: same web, but opposite coherence, i.e.,* $x \frown_{\sim X} y$ *iff* $x \not\supset\!\!\subset_X y$.

Linear negation corresponds to the exchange question/answer since, if $x \frown x'$ means that x, x' bifurcate negatively, $x \smile x'$ (i.e., $x \not\supset\!\!\subset x'$) means that x, x' bifurcate positively (i.e., on answers).

The next property comes from *stability*, see *supra* :

PROPOSITION 1
A clique and an anti-clique (i.e., clique in the negation) have at most one point in common, i.e., $\sharp(a \cap b) \leq 1$.

Intuitively, two distinct sequences must bifurcate either negatively or positively. By the way, one can dispense with the coherence relation by introducing *polarity* :

DÉFINITION 3 (POLARITY)
$a, b \subset |X|$ *are* polar*, notation* $a \,\rotatebox[origin=c]{180}{$\curlywedge$}\, b$ *when* $\sharp(a \cap b) \leq 1$.

and redefine coherent spaces in an *existentialist* fashion :

DÉFINITION 4 (COHERENT SPACES)
A coherent space *of web* $|X|$ *is a set of subsets of* $|X|$ *equal to its bipolar.*

This definition is equivalent to the previous ; thus, $x \frown y$ iff $\{x, y\} \sqsubset X$.

Linear implication, fundamental connective of linear logic, corresponds to the formation, from $X = (|X|, \frown_X)$ and $Y = (|Y|, \frown_Y)$ of a new coherent space, $X \multimap Y = (|X \multimap Y|, \frown_{X \multimap Y})$:

$$|X \multimap Y| \quad := \quad |X| \times |Y| \tag{1}$$
$$(x, x') \frown_{X \multimap Y} (y, y') \quad :\Leftrightarrow \quad x \frown_X x' \;\Rightarrow\; y \frown_Y y' \tag{2}$$

This definition is justified by the following (easy) theorem :

Théorème 1 (Linear implication)
The cliques of $X \multimap Y$ are in bijection with the linear *functions from X into Y, i.e., those functions Φ from the cliques of X to the cliques of Y preserving arbitrary unions and stable. The correspondence is given by ($a \sqsubset X$, $b \sqsubset \sim Y$) :*

$$\sharp([\Phi]a \cap b) = \sharp(\Phi \cap (a \times b)) \tag{3}$$

Which is the starting point of linear logic. Remark that the identity function from X into X is represented by the clique $\Delta_X := (x, x); x \in |X|$ of $X \multimap X$; indeed :

$$\sharp(\Delta_X \cap (a \times b)) = \sharp(a \cap b) \tag{4}$$

thence $[\Delta_X]a = a$.

4.2 Quantum coherent spaces

The notion of *point* of a coherent space supposes a preconstitution of the space of questions/answers, i.e., the survival of an architecture subject/object, albeit seriously amended. Further developments were concerned with the dissolution of the notion of point, which should exist but in relation with the choice of a subject ; this accepted *subjectivity* is the only rampart against sujectivism.

The first step (PCS, *probabilistic* coherent spaces) does not quite put points into question : in a clique, points are given a real *mass* $f(x) \in [0, 1]$. In what follows we assume that the webs $|X|, |Y|, \ldots$ are finite :

(i) A *probabilistic clique* (pc) of web $|X|$ is a function $f : |X| \mapsto [0, 1]$.

(ii) If f, g are two pc of $|X|$, one sets $\ll f \mid g \gg := \sum_{x \in |X|} f(x)g(x)$; f, g are *polar*, notation $f \perp\!\!\!\sim g$, when $\ll f \mid g \gg \leq 1$.

(iii) A PCS of web $|X|$ is a set of pc equal to its bipolar.

One can develop without surprise the analogue of coherent spaces ; (3) thus becomes :

$$\ll \Phi(a) \mid b \gg \, = \, \ll \Phi \mid a \times b \gg \tag{5}$$

and, if Δ_X is the characteristic function of the diagonal, (4) becomes :

$$\ll \Delta_X \mid a \times b \gg \, = \, \ll a \mid b \gg \tag{6}$$

Considering the complex vector space $\mathbb{C}^{|X|}$, one can view a pc f as a diagonal matrix M_f with coefficients in $[0, 1]$; then $\ll f \mid g \gg$ can be expressed as the trace $\mathrm{tr}(M_f M_g)$. One gets quantum coherent spaces (QCS) by pulling the ladder off, i.e., the base of the vector space. If E is a finite dimensional complex Hilbert space, one can resume the refrain :

(i) A *quantum clique* (qc) of web E is a *hermitian* (self-adjoint) operator on E.

(ii) If u, v are two cq of E, one defines $\ll u \mid v \gg := \mathrm{tr}(uv)$; u, v are *polar*, notation $u \underset{\sim}{\downarrow} v$, when $0 \leq \, \ll u \mid v \gg \, \leq 1$.

(iii) A QCS of web $|X|$ is a set of qc equal to its bipolar.

In this framework, (3) becomes, without surprise :

$$\mathrm{tr}(([\Phi]u)v) = \mathrm{tr}(\Phi(u \times v)) \tag{7}$$

whereas (4) translates as :

$$\mathrm{tr}(\sigma_E(u \otimes v)) = \mathrm{tr}(uv) \tag{8}$$

where σ_E is the exchange, the « flip », of $E \otimes E$:

$$\sigma_E(x \otimes y) := y \otimes x \tag{9}$$

W.r.t. a base $\mathcal{B}$, one can also define the « diagonal » $\Delta_{\mathcal{B}}$ by :

$$\Delta_{\mathcal{B}}(\mathbf{b} \otimes \mathbf{b}') := \delta_{\mathbf{b}\mathbf{b}'} \quad (\mathbf{b}, \mathbf{b}' \in \mathcal{B}) \tag{10}$$

however, the equation :

$$\mathrm{tr}(\Delta_{\mathcal{B}}(u \otimes v)) = \mathrm{tr}(uv) \tag{11}$$

is valid but when u, v are diagonal w.r.t. $\mathcal{B}$. One can see (11) as a subjective version of (8) ; indeed, the constitution of the subject can be assimilated to the choice of a base. Note that the usual category-theoretic approach (universal problems, etc.) produces but the analogue of (11) ; (8) therefore belongs in the blind spot of categories. Thus, « η-expansion », which is a formulation of the universal problem, possesses no *convincing* category-theoretic refutation ; QCS can naturally tell the difference between the natural identity (8) and the abductive, reconstituted, identity (11). Concretely, (8) recopies without asking questions, whereas (11) is a police protocol : « Either you come from $\mathbf{b_1}$ and, subsequently, I return you $\mathbf{b_1}$; otherwise you come from $\mathbf{b_2}\ldots$ ». This distinction is to found again in *quantum computing* with the difficulty to duplicate an input without measuring it, i.e., to pass from (8) to (11). Indeed, measurement is fundamentally the reduction to the diagonal form w.r.t. a *distinguished* base ; distinguished by what ? Not by a « what », by a « who » : the subject.

4.3 Geometry of interaction

QCS, and it is their problem, are confined to finite dimension ; indeed, trace no longer exists, at least in a naïve way, in infinite dimension[17]. One can however proceed with the help of a logarithmic shrinking of the concepts : the tensor product becomes a direct sum, the trace becomes a determinant. Behind this, an easy finite dimensional identity involving the *exterior algebra* :

$$\mathrm{tr}(\Lambda u) = \det(1+u) \tag{12}$$

that it is important to explain, since it palys a central role in the discussion :

- The determinant of the matrix (m_{ij}) can be written as an alternated sum $\sum_{\sigma\in\mathfrak{S}(n)}(-1)^\sigma m_{1\sigma(1)}\ldots m_{n\sigma(n)}$. The summand are *all* the total circuits $\{1,\ldots,n\}$ balanced with the transition coefficients m_{ij} and the *signatures* $(-1)^\sigma$. The number $\det(1+u)$ therefore quantifies the *partial* circuits.

- The exterior algebra is, precisely, the space of those partial circuits ; thence (12).

It happens that Λ, the *Fock space* diverges in infinite dimension : indeed, it becomes a $\mathbf{I}_\infty$ algebra, traceless. On the other hand, the determinant can still be defined from the trace, see (13) *infra*.

Here one fingers the mistake made by the advocates of transparency, since the simplistic Kripke models up to the most elaborated category-theoretic interpretations : the dialogue between questions and answers has been replaced with the space of their interactions ; if one can hardly oppose this in finite

[17]And even the type $\mathbf{II}_1$ algebras, which possess nevertheless a trace, will attribute a null trace to $\sigma_E(u\otimes v)$: without an identity function, the theory has a difficult start !

dimension, this reduction of the potential to the *list of possibles* diverges in infinite dimension : it is no longer a heresy, it is an impossibility.

Let us decline a last time the refrain of coherent spaces, this time with a logarithmic seasoning : this is *geometry of interaction* (GoI). One places oneself in a *finite* von Neumann algebra, typically the *hyperfinite factor* $\mathcal{R}_0$. Such an algebra possesses a trace, which enables one to define the *determinant* $\det(1-u)$ when the *spectral ratio* $\varrho(u)$ is < 1, by means of :

$$\det(1-u) := e^{\mathrm{tr}(u+u^2/2+u^3/3+\dots)} \tag{13}$$

What plays the role of the web, is the *carrier*, a projection $\pi \in \mathcal{R}_0$.

(i) The analogue of a clique, a *design* of carrier π, is a pair (α, A), $\alpha \in \mathbb{C}, A \in \mathcal{R}_0$ with $\pi A \pi = A$.

(ii) $\ll \alpha, A \mid \beta, B \gg$ is only defined when $\varrho(uv) < 1$, in which case $\ll \alpha, A \mid \beta, B \gg := \alpha\beta \det(1 - AB)$; (α, A) et (β, B) are *polar*, notation $(\alpha, A) \underset{\sim}{\downarrow} (\beta, B)$ when $\ll \alpha, A \mid \beta, B \gg \neq 1$.

(iii) A *behaviour* of carrier π is a set of designs of carrier π equal to its bipolar.

In this framework, (3) becomes, when (α, A) and (β, B) have orthogonal carriers π, π' (i.e., when $\pi\pi' = 0$) :

$$\ll [f, \Phi](\alpha, A) \mid \beta, B \gg = \ll f, \Phi \mid \alpha\beta, A + B \gg \tag{14}$$

with, as particular case :

$$\ll 1, \tau \mid \alpha\beta, A + B \gg = \ll \alpha, \theta A \theta^* \mid \beta, B \gg \tag{15}$$

where τ is the additive « flip » induced by a partial isometry θ from π onto π' :

$$\tau(x + y) := \theta^*(y) + \theta(x) \tag{16}$$

There is a real surprise, since the application $[f, \Phi](\alpha, A)$ is defined by :

$$[f, \Phi](\alpha, A) := (\phi\alpha \det(1 - \Phi A), \pi'\Phi(1 - A\Phi)^{-1}\pi') \tag{17}$$

The « operator » part corresponds to the solution of an *retroaction equation*, whereas the scalar part makes appear the *introspective* coefficient $\det(1-\Phi A)$ de (14) ; this coefficient didn't occur in (15), simply because $\det(1-\tau A) = 1$. In terms of circuits, one can see this coefficient as a *heating* quantifying those partial circuits avoiding π' ; in more traditional terms, one can see it as quantifying logical correctness : the respect of logical rules indeed guarantees the *nilpotency* of ΦA thence that $\det(1 - \Phi A) = 1$. This quantification of logical correctness is, obviously, a *truth value.*

4.4 Truth in becoming

A design is made of a truth value α and an operator A. In the absence of the second component, polarity collapses to :

$$\ll \alpha \mid \beta \gg = \alpha\beta \tag{18}$$

The polarity $\alpha\beta \neq 1$ (1 corresponding to « true ») could serve as base, what would be a bad joke of God, to another fuzzy « logic », which belongs in the same waste paper basket as the previous ones. What is problematic is not the idea of going beyond the boolean truth values $\{0, 1\}$, it is the fact of confining one to this static, dead, domain of truth.

A design is the sketch α of a truth value together with a process A for developing it, which supposes an interaction with other processes. The complete development demands a *counter-design* (β, B) and leads to $\alpha\beta \det(1 - AB)$; it is then a matter of *total* explicitation, relative to (β, B). But most explicitations are partial, thus (17) which partially explicits (ϕ, Φ) of carrier $\pi + \pi'$ by means of (α, A) whose carrier π still leaves things in the vague (what will happen in π'). Equation (14) links the partial explicitation to its becoming, the definitive and complete explicitation which never really occurs in logic, for reasons of consistency.

All this to say that the idea of truth value can recover part of the place that it usurpated in the old style of foundations. Simply, it splits into a part already computed (α) and a part in becoming (A) which can *in no way* be reduced to a collection of possible futures.

It only remains to revisit ludics, i.e., the cognitive onion, in the light of GoI, to get rid, once for all, I hope, of those unlikely *X rays of knowledge*.

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