DEBRIS VALFINAL

Let R be as in Proposition **??**, and let $\iota\colon R \to \hat{R}^\nu$ be the completion morphism. Let ν_1 be a valuation with which ν is composed, and denote by $\hat{\nu}_1$ its extension to $\hat{R}^\nu$ according to the first Corollary. Denote by $\overline{\nu}$ the residual valuation on $R/(m_{\nu_1}\cap R)$. After subsection **??**, we know that the $\overline{\nu}$-adic topology on $R/(m_{\nu_1}\cap R)$ is the quotient of the ν-adic topology of R. Even if R is complete for the ν-adic topology, the quotient $R/(m_{\nu_1}\cap R)$ is not complete for the $\overline{\nu}$-adic topology in general (see example 2) above). However, we have[1]

Proposition 0.1 *If the quotient ring $\hat{R}^\nu/(m_{\hat{\nu}_1}\cap \hat{R}^\nu)$ is complete for the $m_{\hat{\nu}}\cap \hat{R}^\nu/(m_{\hat{\nu}_1}\cap \hat{R}^\nu)$-adic topology, the morphism*

$$R/(m_{\nu_1}\cap R) \to \hat{R}^\nu/(m_{\hat{\nu}_1}\cap \hat{R}^\nu)$$

induced by ι is the completion morphism of the first ring with respect to the $\overline{\nu}$-adic topology, quotient of the ν-adic topology of R.

Proof Since R is nœtherian, the value semigroup Γ of ν on R is well ordered (Proposition **??**), so that the the ν-adic topology may be defined by a decreasing chain of ideals. Moreover, if δ_1 denotes the smallest non-zero element of $\Gamma_1 = \nu_1(R\setminus\{0\})$, which is also well ordered, the ideal $m_{\nu_1}\cap R$ is equal to $\{x\in R \mid \nu_1(x)\geq \delta_1\}$, so that its closure in $\hat{R}^\nu$ is $m_{\hat{\nu}_1}\cap \hat{R}^\nu$. Now we can apply Theorem 8.1, page 56 of [Ma], which states the equality we seek. □

For the next two propositions, given $R\subset R_\nu$, remember that there is a unique sequence of consecutive valuation rings

$$R_\nu \subset R_{\nu_1} \subset \cdots \subset R_{\nu_{h-1}}$$

ending at the ring of the valuation of height one ν_{h-1} with which ν is composed, and by the results of subsection **??** we can produce a sequence of consecutive specializations of the valuation ν having a center in R, of the form

$$R \subset R_{\nu_{-s}} \subset R_{\nu_{-s+1}} \subset \cdots \subset R_{\nu_{-1}} \subset R_\nu$$

and such that $R_{\nu_{-s}}$ dominates R.

Proposition 0.2 *Let R be a nœtherian equicharacteristic analytically irreducible local ring, and ν a valuation of its field of fractions, non negative on R. If, for all pairs of consecutive valuation rings $R_\mu\subset R_{\mu_1}$ appearing in the sequence of consecutive valuation rings which connects R_ν to $R_{\nu_{h-1}}$, the quotient $R/(m_{\mu_1}\cap R)$ is complete for the $m_\mu\cap R/(m_{\mu_1}\cap R)$-adic topology and R is complete for the $\mathbf{p}$-adic topology, where $\mathbf{p}$ is the center of the valuation of height one with which ν is composed, each pair $(R, m_{\nu_i}\cap R)$ is henselian. If the assumption of completeness can be extended to a sequence of specializations of ν ending at a valuation ring dominating R, the local ring (R,m) is henselian.*

[1] In the classical theory of the completion of valued fields, it is the ν-adic completions of the valuation rings which play a role, more or less explicitly. They are not nœtherian in general, and a result such as this Proposition is not true; I believe this complicates the study of *complétion par étages*; see [Ri].

Proof Notice first that by definition of restricted power series, for $m_{\nu_j} \cap R \subseteq m_{\nu_{j-1}} \cap R$, the natural map of polynomial rings extends to a map of rings of restricted power series

$$\big(R/(m_{\nu_j} \cap R)\big)\{X\} \to \big(R/(m_{\nu_{j-1}} \cap R)\big)\{X\}.$$

We prove that $(R,\ m_{\nu_i} \cap R)$ satisfies Hensel's Lemma in the sense of Bourbaki (*cf* [B3], Chap. 3, §4, No.3), which we now recall.

Let $F \in R\{X\}$ be a restricted power series, and assume that the image of F modulo $m_{\nu_i} \cap R$, denoted by $\overline{F}$, admits a factorization $\overline{F} = \overline{P}.\overline{Q}$, where $\overline{P} \in \big(R/(m_{\nu_i} \cap R)\big)[X]$ is a unitary polynomial, and $\overline{Q}$ a restricted power series $\overline{Q} \in \big(R/(m_{\nu_{j-1}} \cap R)\big)\{X\}$, with $\overline{P}$, $\overline{Q}$ strongly relatively prime. Then they lift in a unique way to a unitary polynomial $P \in R[X]$ and a restricted power series $Q \in R\{X\}$, strongly relatively prime, such that $F = P.Q$. If F is a polynomial, so is Q.

To prove this, note that the ideals $m_{\nu_k} \cap R$ are closed for the ν_k-adic topologies, which all coincide. By Theorem 1 of the quoted paragraph of [B3], the result is true for ν_{h-1}, which is a valuation of height one, since the elements of $m_{\nu_{h-1}} \cap R$ are topologically nilpotent as we saw in Remark 1) following Proposition **??**.
To prove the general case, it suffices to note that the statement is amenable to induction on the height and that if $\mathrm{h}(\nu_{j-1}) > 1$, there is a unique (up to isomorphism) valuation ν_j of R which is the immediate successor of ν_{j-1}. The valuation induced by ν_{j-1} on the quotient ring $R/(m_{\nu_j} \cap R)$ is of height at most one, according to Section **??**. Assuming that we have a decomposition in $\big(R/(m_{\nu_{j-1}} \cap R)\big)\{X\}$ as in Hensel's lemma, by using the case of height one, we can lift it to a decomposition in $\big(R/(m_{\nu_j} \cap R)\big)\{X\}$, and continue this until we have a decomposition in $\big(R/\mathbf{p}\big)\{X\}$, where $\mathbf{p}$ is the center of the valuation ν_{h-1} of height one with which ν is composed, and we are reduced to the case of height one. The last sentence is proved by applying this result to the valuation ring $R_{\nu_{-s}}$ which dominates (R, m). □

Proposition 0.3 *Let R be a nœtherian equicharacteristic analytically irreducible local ring, and ν a valuation of its field of fractions, non negative on R. If for all pairs of consecutive valuation rings $R_\mu \subset R_{\mu_1}$ appearing in a sequence of nested valuation rings which contains R_ν and connects a valuation ring dominating R to the ring of a valuation of height one the quotient $R/(m_{\mu_1} \cap R)$ is complete for the $m_\mu \cap R/(m_{\mu_1} \cap R)$-adic topology and R is complete for the $\mathbf{p}$-adic topology, where $\mathbf{p}$ is the center of the valuation of height one with which ν is composed, the local ring R admits a field of representatives.*

Proof We have a sequence of quotient rings

$$R/(m_{\nu_{h-1}} \cap R) \to \cdots \to R/(m_{\nu_{-s+1}} \cap R) \to R/(m_{\nu_{-s}} \cap R) = k_R,$$

where some of the maps may be isomorphisms. I need the following:

Lemma 0.4 *Let S be a nœtherian equicharacteristic local ring and I an ideal of S. Assume that the quotient S/I admits a field of representatives and that S is complete for the I-adic topology. Then S admits a field of representatives.*

Proof This follows from Cohen's Theorem as found in [Z-S], Vol. 2, Chap. VIII, §12 after making the following remark.

Let I be an ideal of S such that S/I admits a field of representatives. Let $\phi\colon S \to S/I$ be the natural map. Set $B = \phi^{-1}(k)$, where $k \subset S/I$ is a field of representatives. We have $I = m \cap B$, where m is the maximal ideal of S.
It follows from this that B is closed in S for the I-adic topology, so that it is complete for its $m_B = m \cap B$-adic topology, and by Cohen's Theorem it has a field of representatives, which is also a field of representatives for S. □

I apply the lemma to the maps

$$R/(m_{\nu_j} \cap R) \to R/(m_{\nu_{j-1}} \cap R).$$

Each of them satisfies the hypotheses of the Lemma since by subsection **??** the valuation induced by ν_{j-1} on $R/(m_{\nu_j} \cap R)$ is of height at most one, and so the residual topology coincides with the $(m_{\nu_{j-1}} \cap R)/(m_{\nu_j} \cap R)$-adic topology, for which this ring is complete by our assumption. So we can step by step lift k to a field of representatives of $R/(m_{\nu_{h-1}} \cap R)$. Since the valuation ν_{h-1} is of height one, and we assume that the ring R is complete for the $\mathbf{p}$-adic topology, where $\mathbf{p} = m_{\nu_{h-1}} \cap R$, we can apply the lemma one last time to lift the field of representatives to R. □

Question. Can one extend this result to the case where R is not equicharacteristic, to prove under similar assumptions the existence of an injection $W(k) \to R$ for a nœtherian analytically irreducible local ring R of unequal characteristic with a perfect residue field k and a valuation ν, where $W(k)$ is the ring of Witt vectors relative to the residue field k of S ([B3], Chap. IX, §§1,2, [Ei], Chap.7)?

Remark 0.5 In order to be able to apply Propositions 0.1, 0.2 and 0.3 the easiest way is to deal with a local ring which is complete with respect to the $\mathbf{p}$-adic topologies for primes ideals $\mathbf{p}$ which are the centers of the valuations with which ν is composed. Then R and its quotients by the centers of valuations with which ν is composed will be complete and separated with respect to the ν-adic topology and its images in these quotients. This will be the case if R is m-adically complete, by [Z-S], *loc.cit*, and then of course the two propositions above are not needed in view of Hensel's lemma and Cohen's theorem; they will be used elsewhere.

Suppose that there is a regular local ring $R_0 \subset R$ such that R is a finite R_0-algebra. Then the completion $\hat{R}_0^{m_0}$ is isomorphic to $k[[x_1, \ldots, x_d]] \subset \hat{R}^m$. Consider the restriction $\nu^{(0)}$ of ν to R_0, which is centered at m_0. If we can prove the proposition for $\nu^{(0)}$, obtaining an ideal $H^{(0)}$ of $\hat{R}^{m_0}$ and an extension $\hat{\nu}^{(0)}$ to $\hat{R}^{m_0}/H^{(0)}$, the completeness of $\hat{R}^{m_0}$ will ensure that we can extend the valuation $\hat{\nu}^{(0)}$ to a valuation $\hat{\nu}$ of $\hat{R}^m/H^{(0)}\hat{R}^m$. It is then not difficult to check the assertion on the graded rings.
I may therefore assume that R is regular, and consider the valuations with which ν is composed. The case of height one was settled above. Let us consider two successive valuations μ and μ_1 with which ν is composed, with centers $\mathbf{p}$ and $\mathbf{p}_1$ respectively, and denote by Ψ the kernel of the corresponding map $\lambda\colon \Phi_{i-1} \to \Phi_i$ of value groups. Assume that we have built an ideal $H_1 \subset \hat{R}^m$ and a pseudo-valuation $\hat{\mu}_1$ on $\hat{R}^m/H_1$ which extends μ_1, with the properties that H_1 is a prime ideal, $H_1 \cap R = (0)$ and

$$\mathrm{gr}_{\hat{\mu}_1} \hat{R}^m/H_1 = \mathrm{gr}_{\mu_1} R \otimes_{R/\mathbf{p}_1} \hat{R}^m/\mathbf{p}_1\hat{R}^m.$$

Reasoning by induction on the dimension, and using the results of section **??** for rings of smaller dimension than R, we may assume embedded local uniformization for the image $\overline{\nu}$ of ν in the ring $R/\mathbf{p}_1$ by a toric map of R. Thus, after replacing R by a toric modification R' of R, which is still a regular local ring, we may assume that both R and $R/\mathbf{p}_1$ are regular. For the next argument we write R instead of R', for simplicity. The valuation $\overline{\mu}$, which is of height at most one, may be extended to the quotient of $\hat{R}^m/\mathbf{p}_1\hat{R}^m$ by an ideal $\overline{H} = \bigcap_{\psi\in\Psi_+}\mathcal{P}_\psi\hat{R}^m/\mathbf{p}_1\hat{R}^m$. We may for the same reason assume that $\overline{H}$ is generated by a regular sequence $(\overline{b}_1,\dots,\overline{b}_k)$ in $\hat{R}^m/\mathbf{p}_1^m\hat{R}^m$. Representatives $(b_1,\dots,b_k)$ in $\hat{R}^m$ generate an ideal $\tilde{H}\subset\hat{R}^m$. The ideal $\mathbf{p}_1\hat{R}^m$ is generated by $(x_1,\dots,x_c)$, with $x_i\in R$. Assume first that $c=1$. Then we have $\tilde{H}\cap R=(0)$. Indeed, let $a\in\tilde{H}\cap R$, say $a=\sum_{j=1}^k\lambda_j b_j$. The image of a in $R/\mathbf{p}_1$ is in $\overline{H}\cap(R/\mathbf{p}_1)=(0)$ so that $a\in\tilde{H}\cap\mathbf{p}_1\hat{R}^m$. The two ideals are transversal (their intersection is equal to their product) so that $a=\lambda x_1 a'$ with $a'\in\tilde{H}\cap R$; by iteration this shows that $a\in\bigcap_{i=1}^\infty x_1^i\hat{R}^m=(0)$. Using this argument inductively to lift the generators $(\overline{b}_1,\dots,\overline{b}_k)$ of $\overline{H}$ to the quotients $\hat{R}^m/(x_1,\dots,x_j)\hat{R}^m$ we finally get that the ideal $\tilde{H}$ satisfies $\tilde{H}\cap R=(0)$, and $\tilde{H}\subset\mathbf{p}\hat{R}^m$. Now we can extend μ to $\hat{R}^m/H=\hat{R}^m/H_1/\tilde{H}(\hat{R}^m/H_1)$ as follows: given an element $y\in(\hat{R}^m/H)\setminus\{0\}$, we can lift it to $\tilde{y}\in\hat{R}^m/H_1$ which has a pseudo-valuation $\hat{\mu}_1(\tilde{y})\in\Phi_i$. Since $\tilde{y}\notin\bigcap_{\psi\in\Psi_+}\mathcal{P}_\psi\hat{R}^m/H_1$, it follows from ([Z-S], Vol.II, Appendix 3, Lemma 4) that there is a $\phi\in\Phi_{i-1+}$ such that $\tilde{y}\notin\mathcal{P}_\phi\hat{R}^m/H_1$, and $\lambda(\phi)=\hat{\mu}_1(\tilde{y})$. By the argument invoked at the beginning, there is a largest $\eta\in\Phi_{i-1}$ such that $\tilde{y}\in\mathcal{P}_\eta$, and $\lambda(\eta)=\hat{\mu}_1(\tilde{y})$. Setting $\hat{\mu}(y)=\eta$ defines a pseudo-valuation $\hat{\mu}$ on $\hat{R}^m/H$ with value group Φ_{i-1}. We have $H\subset\mathbf{p}$ so the same argument as above shows that

$$\mathrm{gr}_{\hat{\mu}}\hat{R}^m/H=\mathrm{gr}_\mu R\otimes_{R/\mathbf{p}}(\hat{R}^m/\mathbf{p}\hat{R}^m)^{(\overline{\nu})}.$$

The map $R\to\hat{R}^m/H$ induces the map

$$\mathrm{gr}_\mu R\to\mathrm{gr}_\mu R\otimes_{R/\mathbf{p}}1\subset\mathrm{gr}_\mu R\otimes_{R/\mathbf{p}}(\hat{R}^m/\mathbf{p}\hat{R}^m)^{(\overline{\nu})},$$

which shows that $H(\hat{R}^m/H_1)\bigcap R=(0)$. The point is that the ideals of the form $\mathbf{p}^N\mathcal{P}_{\phi_0}$ for some ϕ_0 with $\lambda(\phi_0)=\phi_1$ are cofinal in the ideals $(\mathcal{P}_\phi)_{\lambda(\phi)=\phi_1}$, as are the ideals $(\mathcal{P}_\psi\mathcal{P}_{\phi_0})_{\psi\in\Psi_+}$.
Following this inductive procedure, we continue until the valuation μ is our original valuation ν with center m. But now the graded ring $\mathrm{gr}_{\hat{\nu}}\hat{R}^m/H$ is equal to $\mathrm{gr}_\nu R$, so that it is an integral domain, and $\hat{\nu}$ is a valuation and not only a pseudo-valuation, so that the images in $\hat{R}^m/H$ of all the $\hat{\mu}$ are also valuations, those with which $\hat{\nu}$ is composed.
Now we remember that this was done in a toric modification (and localization at the point picked by ν) R' of R. Denote by $H'\subset\hat{R'}^{m'}$ the ideal built in the preceding argument. Denote by $\tau\colon\hat{R}^m\to\hat{R'}^{m'}$ the natural map of complete local rings, and set $H=\tau^{-1}(H')=H'\cap\hat{R}^m$. The valuation ν extends to a valuation $\hat{\nu}$ of $\hat{R}^m/H$, with group Φ and this proves the Proposition.

Remark 0.6 1) It would be preferable to show that one can reduce to the case where $\mathbf{p}_1$ is generated by a regular sequence by using the graded algebra of the valuation, and thus remove the assumption that R is finite over a regular local ring.

Let us set $K=\mathbf{q}(\infty)$ and note that we have by construction, since $\mathcal{P}_\phi\hat{R}_1\supset\mathcal{P}^+_{\phi_1}\hat{R}_1$ for all ϕ such that $\lambda(\phi)=\phi_1$

$$\frac{\bigcap_{\lambda(\phi)=\phi_1}\mathcal{P}_\phi\hat{R}_1}{\mathcal{P}^+_{\phi_1}\hat{R}_1}\subset(K\cap\mathcal{P}_{\phi_1}\hat{R}_1)\hat{R}_1/\mathcal{P}^+_{\phi_1}\hat{R}_1\subset\frac{(K+\mathcal{P}^+_{\phi_1})\cap\mathcal{P}_{\phi_1}\hat{R}_1}{\mathcal{P}^+_{\phi_1}\hat{R}_1}.$$

from this we deduce

$$\bigcap_{\lambda(\phi)=\phi_1}\mathcal{P}_\phi\hat{R}_1\subset(K+\mathcal{P}^+_{\phi_1})\cap\mathcal{P}_{\phi_1}\hat{R}_1+\mathcal{P}^+_{\phi_1}\hat{R}_1$$

By the computation we made above, the image in $\hat{R}_1/K$ of $\mathcal{P}_{\phi_1}\hat{R}_1$ is $\bigcap_{\lambda(\phi)=\phi_1}\mathcal{P}_\phi\hat{R}_1+\mathcal{P}^+_{\phi_1}\hat{R}_1$, so that if we pass to the quotient $\hat{R}_0=\hat{R}_1/K$ we get

$$\bigcap_{\lambda(\phi)=\phi_1}\mathcal{P}_\phi\hat{R}_0/\mathcal{P}^+_{\phi_1}\hat{R}_0=(0).$$

From this follows, by Lemma 0.14

$$\frac{\bigcap_{\lambda(\phi)=\phi_1}\mathcal{P}_\phi\hat{R}_1/H}{\mathcal{P}^+_{\phi_1}\hat{R}_1/H}=(0).$$

Remark also that we have $\frac{\hat{R}_1/H}{\mathbf{p}_1\hat{R}_1/H}=\frac{\hat{R}_1/\mathbf{p}_1\hat{R}_1}{\overline{H}}=\hat{\overline{R}}_1/\overline{H}$.

Let us now consider the ring $\hat{\overline{R}}_0=\hat{\overline{R}}_1/\overline{H}$ and the $\hat{\overline{R}}_0$-modules $N_\phi:=\frac{\mathcal{P}_\phi\hat{\overline{R}}_0}{\mathcal{P}^+_{\phi_1}\hat{\overline{R}}_0}\subset M:=\frac{\mathcal{P}_{\phi_1}\hat{\overline{R}}_0}{\mathcal{P}^+_{\phi_1}\hat{\overline{R}}_0}$; we have just seen that $\bigcap_{\lambda(\phi)=\phi_1}N_\phi=(0)$. It follows from lemma 0.19 that we have $(N_\phi:M)=\mathcal{P}_{\psi(\phi)}\hat{\overline{R}}_0\subset\mathbf{p}\hat{\overline{R}}_0$.
Since the ring R is assumed to be excellent, the ideal $\mathbf{p}\hat{R}^m$ is the intersection of prime ideals of the same height, and therefore the same is true for $\mathbf{p}\hat{\overline{R}}_0$ since $H\subset\mathbf{p}\hat{R}_1$. (see [EGA], §7, scholie 7.8.3, (x), p. 216). Let us consider the multiplicative subset T of $\hat{\overline{R}}_0$ which is the complement of the union of the prime ideal of $\mathbf{p}\hat{\overline{R}}_0$. The rings $(\hat{\overline{R}}_0)_T/\mathbf{p}^n(\hat{\overline{R}}_0)_T$ are artinian. Moreover, since $(N_\phi:M)\subset\mathbf{p}\hat{\overline{R}}_0$, we have that $\bigcap_{\lambda(\phi)=\phi_1}(N_\phi\otimes_{\hat{\overline{R}}_0}(\hat{\overline{R}}_0)_T)=(0)$.
Now we can apply the linear compactness of complete filtered modules (see [B3], Chap III, §2, No. 7), or generalized Chevalley theorem. It gives us that for every integer n there exists $\phi\in\lambda^{-1}(\phi_1)$ such that $N_\phi\otimes_{\hat{\overline{R}}_0}(\hat{\overline{R}}_0)_T\subset\mathbf{p}^nM\otimes_{\hat{\overline{R}}_0}(\hat{\overline{R}}_0)_T$.
Finally we can deduce from this the

Lemma 0.7 *Keeping the notations just introduced, let $R\to R'$ be a rational toric extension subordinate to ν associated to a finite family $(\xi_j)_{j\in F}$ of elements such that $\nu(\xi_j)=0$ for $j\in F$. The equality*

$$\mathcal{P}_\phi(R')\hat{R}'^{m'}\cap\hat{R}^m=\mathcal{P}_\phi(R)\hat{R}^m$$

holds for every $\phi\in\Gamma$ and in particular, setting $H'=\bigcap_{\phi\in\Phi_+}\mathcal{P}_\phi(R')\hat{R}'^{m'}$, we have via the natural inclusion $\hat{R}^m\subset\hat{R}'^{m'}$,

$$H'\cap\hat{R}^m=H.$$

Proof The inclusion $\mathcal{P}_\phi(R)\hat{R}^m \subset \mathcal{P}_\phi(R')\hat{R}'^{m'} \cap \hat{R}^m$ is clear. Let us prove the inclusions $\mathcal{P}_\phi(R')\hat{R}'^{m'} \cap \hat{R}^m \subset \mathcal{P}_\phi(R)\hat{R}^m$. In view of our assumption, denoting by $\mathbf{p}'$ the center of ν in R', we have a natural birational inclusion $R/\mathbf{p} \subset R'/\mathbf{p}'$. Moreover, the map of $R/\mathbf{p}$-modules

$$\mathcal{P}_\phi(R)/\mathcal{P}^+_\phi(R) \to \mathcal{P}_\phi(R')/\mathcal{P}^+_\phi(R')$$

is injective. Since the m'-adic topology of $R'/\mathbf{p}'$ induces the m-adic topology on $R/\mathbf{p}$, the map of completions

$$\mathcal{P}_\phi(R)\hat{R}^m/\mathcal{P}^+_\phi(R)\hat{R}^m \to \mathcal{P}_\phi(R')\hat{R}^{m'}/\mathcal{P}^+_\phi(R')\hat{R}^{m'}$$

is also injective. If an element x of $\mathcal{P}_\phi(R')\hat{R}'^{m'} \cap \hat{R}^m$ belonged to $\mathcal{P}_{\phi_1}(R) \setminus \mathcal{P}^+_{\phi_1}(R)$ for $\phi_1 < \phi$, it would not belong to $\mathcal{P}^+_{\phi_1}(R')\hat{R}'^{m'}$, which contains $\mathcal{P}_\phi(R')\hat{R}'^{m'}$, and this is a contradiction. □

Definition 0.8 Assertion TLU(d) (Toric local uniformization in dimension $\leq d$) is the following: For every local nœtherian excellent equicharacteristic integral domain R of dimension $\leq d$, given a rational valuation ν on R, there is a rational toric extension $R \subset R' \subset R_\nu$ such that R' is a regular local ring , ν extends to a valuation $\hat{\nu}$ of a regular quotient $\hat{R'}^{m'}/H'$ of the m'-adic completion of R' in such a way that the induced map of graded rings

$$\mathrm{gr}_\nu R' \to \mathrm{gr}_{\hat{\nu}} \hat{R'}^{m'}/H'$$

is an isomorphism; in particular, $H' \cap R' = (0)$

2) One can slightly generalize the concept of a rational valuation as follows

Definition 0.9 The valuation ν is said to be generically rational if the valuation induced by ν on the local ring $R_{\mathbf{p}}$ is rational, where $\mathbf{p}$ denotes the center of ν in R.

The following result helps to explain the presence of binomial relations in certain non-rational cases.

Proposition 0.10 *Let $R \subset R_{\nu_1}$ be a generically rational valuation of R, and let $(\overline{\eta}_j)_{j\in J}$ be a set of generators of the $R/\mathbf{p}_1$-algebra $\mathrm{gr}_{\nu_1} R$. Then the map of $R/\mathbf{p}_1$-algebras sending W_j to $\overline{\eta}_j$ induces a presentation*

$$\mathrm{gr}_{\nu_1} R = R/\mathbf{p}_1[(W_j)_{j\in J}]/\big((\lambda_m W^m - \lambda_n W^n)_{(m,n)\in E}\big)\ , \qquad \lambda_m, \lambda_n \in R/\mathbf{p}_1 \setminus \{0\}.$$

Proof By our hypothesis, the valuation induced by ν_1 on $R_{\mathbf{p}_1}$ is rational. On the other hand we have $\mathcal{P}_{\phi_1}(R_{\mathbf{p}_1}) = \mathcal{P}_{\phi_1}(R)R_{\mathbf{p}_1}$ and, setting $\kappa(\mathbf{p}_1) = R_{\mathbf{p}_1}/\mathbf{p}_1 R_{\mathbf{p}_1}$,

$$\mathrm{gr}_{\nu_1} R_{\mathbf{p}_1} = \mathrm{gr}_{\nu_1} R \otimes_{R/\mathbf{p}_1} \kappa(\mathbf{p}_1).$$

The elements $\overline{\eta}_j \otimes 1$ generate the $\kappa(\mathbf{p}_1)$-algebra $\mathrm{gr}_{\nu_1} R_{\mathbf{p}_1}$, so by Proposition **??** it is of the form $\kappa(\mathbf{p}_1)[(W_j)_{j\in J}]/(W^m - \lambda_{mn} W^n)$, where $\lambda_{mn} \in \kappa(\mathbf{p}_1)^*$. If we remember that each $\mathcal{P}_{\phi_1}/\mathcal{P}^+_{\phi_1}$ is a torsion-free $R/\mathbf{p}_1$-module the result follows from the flatness of $\kappa(\mathbf{p}_1)$ over $R/\mathbf{p}_1$, writing $\lambda_{mn} = \lambda_m^{-1}\lambda_n$ with $\lambda_m,\ \lambda_n \in R/\mathbf{p}_1 \setminus \{0\}$. □

0.1 More speculation. Keeping the notations of the previous subsection and of subsection **??**, remark that if the images of the $(\xi_i)_{i\in F}$ generate the maximal ideal of $\hat{R}^{(\nu)}$, the $(\xi_i)_{i\in F}$ must generate the maximal ideal of R. Let us denote by $\tilde{\nu}$ the valuation of R which is monomial in the variables $(\xi_i)_{i\in F}$ and satisfies $\tilde{\nu}(\xi_i) = \nu(\xi_i)$ for $i \in F$. The valuation $\tilde{\nu}$ extends as a monomial valuation to $\hat{R}^{(\nu)}$ and a toric map which uniformizes as above the valuation $\tilde{\nu}$ via a uniformisation of its extension to $\hat{R}^{(\nu)}$ also uniformizes ν. Now the set of rational valuations which are uniformized by a given process is open in the patch topology of the Zariski-Riemann manifold. By the compactness, it would follow that every rational valuation is resolved by a toric map $\tilde{\nu}$

First, we have in any basic extension $\hat{\overline{\nu}}(\overline{\lambda}) < \infty$ unless $\lambda \in \mathbf{p}_{h-1}{}'(\hat{R}^m/H)'$, and in that case, after some calculation, we see that $x \in \hat{\mathcal{P}}^+_{\phi_{h-1}}$. Let us show that any element of $\hat{\mathcal{P}}_{\phi_{h-1}}/\hat{\mathcal{P}}^+_{\phi_{h-1}}$ is in one of the $\hat{\mathcal{P}}_{\phi}/\hat{P}_{\phi_{h-1}}$. Let us write $x = \sum_{i=1}^r c_i e_i$ a representative of such an element. Since all the e_i have the same ν_{h-1}-valuation, by considering $\overline{R} \to \overline{R}^{(\overline{\nu})} = \overline{R}^m/\overline{H}$

In the proof, I consider the valuations $(\nu_i)_{0\leq i\leq h-1}$ with which the rational valuation ν is composed, with $\nu_0 = \nu$ and ν_{h-1} of height one. The idea is to build a sequence of ideals H_i and extensions of the ν_i by loose valuations of the $\hat{R}^m/H_i$, by descending induction on i.

Proposition 0.11 *Let (R, ν) be a valued nœtherian local integral domain and H an ideal of R. Let us assume that H is closed in R for the ν-adic topology, that is*

$$\bigcap_{\phi\in\Phi_+} (H + \mathcal{P}_\phi) = H$$

and that in addition, if we denote by

$$\Phi = \Phi_0 \to \Phi_1 \to \cdots \to \Phi_i \to \Phi_{i+1} \to \cdots \to \Phi_{h-1} \to (0)$$

the sequence of the groups of the valuations with which ν is composed, for each $\lambda_i\colon \Phi_i \to \Phi_{i+1}$ we have for each $\phi_{i+1} \in \Phi_{i+1+}$

$$\bigcap_{\phi_i\in\lambda_i^{-1}(\phi_{i+1})} (H + \mathcal{P}_{\phi_i}) = H + \mathcal{P}^+_{\phi_{i+1}}.$$

Then the valuation ν gives rise to a loose valuation in R/H whose group is contained in Φ.

Proof Let us first assume that ν is of height one. Given a nonzero element $\overline{x} \in R/H$, I claim that the valuations of all the elements in R whose image in R/H is $\overline{x}$ are bounded. Fix such an element x_0; if the valuations are not bounded, we have $x_0 \in \bigcap_{\phi\in\Phi_+}(H + \mathcal{P}_\phi)$, which is equal to H by assumption, a contradiction. Since Φ is of height one, there is a largest element among these valuations, which we choose as $\overline{\nu}(\overline{x})$.
If $\overline{z} = \overline{x} + \overline{y}$, we may choose representatives x and y with the largest valuations, but this does not guarantee that $x + y$ gives the largest valuation in its class mod H, so that indeed
$\overline{\nu}(\overline{z}) \geq \nu(x + y) \geq \min(\overline{\nu}(\overline{x}), \overline{\nu}(\overline{y}))$.
I leave it as an exercise to check that if $\overline{\nu}(x) \neq \overline{\nu}(y)$ we have equalities. Remark also that the ideal of elements of R/H having valuation $\geq \phi$ is $\mathcal{P}_\phi R/H$.
Similarly, choosing representatives of maximal valuation for $\overline{x}, \overline{y}$ gives

$\overline{\nu}(\overline{xy}) \geq \overline{\nu}(\overline{x}) + \overline{\nu}(\overline{y})$.
Let us now proceed by induction and assume that we have defined a loose valuation $\overline{\nu}_{i+1}$ on R/H. Given an element $\overline{x} \in R/H$, let $\overline{\nu}_{i+1}(\overline{x})$ be its valuation. I claim that the valuations $\nu_i(x)$ of the representatives of $\overline{x}$ in R such that $\nu_{i+1}(x) = \overline{\nu}_{i+1}(\overline{x})$ are bounded. Denote this last valuation by ϕ_{i+1}. Let $x \in R$ be a representative of $\overline{x}$; if the valuations were not bounded in $\lambda_i^{-1}(\phi_{i+1})$, we would have the inclusion

$$x \in \bigcap_{\phi_i \in \lambda_i^{-1}(\phi_{i+1})} (H + \mathcal{P}_{\phi_i}) = H + \mathcal{P}^+_{\phi_{i+1}},$$

where the last equality follows from our assumptions. But this means that $\overline{x}$ has a representative y of valuation $\nu_{i+1}(y) > \phi_{i+1}$, which contradicts our definitions. Since our set of valuations is bounded in $\lambda_i^{-1}(\phi_{i+1})$, by the reference given above, the set of valuations $\nu_i(y) \in \lambda_i^{-1}(\phi_{i+1})$ of representatives y of $\overline{x}$ is finite, and we choose its largest element as $\overline{\nu}_i(\overline{x})$. The proof that it is a loose valuation is almost the same as above. ☐

Remarks 0.12 1) The result still holds, with the same proof, if ν is a loose valuation with values in a group of height one, and more generally if ν is a loose valuation "composed" in the same sense as valuations are, through a sequence of monotone maps $\lambda_i\colon \Phi_i \to \Phi_{i+1}$, with loose valuations.
2) It would be useful to know when $\overline{\nu}$ is a valuation apart from the cases where H is the center of a valuation with which ν is composed and $\overline{\nu}$ the corresponding residual valuation.

Let us consider the case $i = h-1$; denote temporarily by ν the valuation of height one ν_{h-1}, and let H stand for $H_{h-1} = \bigcap_{\phi\in\Phi_+} \mathcal{P}_\phi \hat{R}^m$. The valuation ν extends to a loose valuation $\hat{\nu}$ of $\hat{R}^m/H$ as follows: For an element $x \in (\hat{R}^m/H) \setminus \{0\}$ there is a representative $\tilde{x} \in \hat{R}^m$ and there is a $\phi \in \Phi_+$ such that $\tilde{x} \notin \mathcal{P}_\phi \hat{R}^m$; since Φ is archimedian and R is nœtherian, the elements of $\nu(R\setminus\{0\})$ which are smaller than ϕ (see [Z-S], *loc. cit*) are finite in number, so that there is an element $\eta \in \nu(R\setminus\{0\})$ such that $\tilde{x} \in \mathcal{P}_\eta \hat{R}^m \setminus \mathcal{P}^+_\eta \hat{R}^m$.
Let us check that this η is independant of the choice of the representative of x; if $\tilde{x} - \tilde{y} \in H$, assuming that $\eta(\tilde{x}) < \eta(\tilde{y})$, we must have $\eta(\tilde{x} - \tilde{y}) = \eta(\tilde{x})$ and this contradicts the fact that $\tilde{x} - \tilde{y} \in H$.
We may now set $\hat{\nu}(x) = \eta$; one verifies as in the proof of proposition 0.11 that this defines a loose valuation on $\hat{R}^m/H$. Finally, we have $H \cap R = (0)$ since $\mathcal{P}_\phi \hat{R}^m \cap R = \mathcal{P}_\phi$ and $\bigcap_{\phi\in\Gamma} \mathcal{P}_\phi = (0)$.

Now let us remark that if $\mathbf{p}_\nu$ is the center of ν, we have the inclusions $\mathbf{p}_\nu \mathcal{P}_\phi \subset \mathcal{P}^+_\phi$ and the equalities $\mathcal{P}_\phi \hat{R}^m = \mathcal{P}_\phi \otimes_R \hat{R}^m$ by the flatness of $\hat{R}^m$ over R. We therefore have the following equalities after remarking that $H \subset \mathcal{P}_\phi \hat{R}^m$ for all $\phi \in \Phi_+$:

$$(\mathrm{gr}_{\hat{\nu}} \hat{R}^m/H)_\phi = \frac{\mathcal{P}_\phi \hat{R}^m/H}{\mathcal{P}^+_\phi \hat{R}^m/H} = \frac{\mathcal{P}_\phi \hat{R}^m}{\mathcal{P}^+_\phi \hat{R}^m + \mathbf{p}_\nu \mathcal{P}_\phi \hat{R}^m} = \frac{\mathcal{P}_\phi}{\mathcal{P}^+_\phi} \otimes_R (\hat{R}^m/\mathbf{p}_\nu \hat{R}^m).$$

Note that the graded algebra $\mathrm{gr}_{\hat{\nu}} \hat{R}^m/H$ is not necessarily an integral domain, except in the case where ν has center m studied in [S2]. In this case, we have, setting $\hat{R}_{h-1} = \hat{R}^m/H$, the equalities $\mathcal{P}_\phi/\mathcal{P}^+_\phi = \mathcal{P}_\phi \hat{R}_1/\mathcal{P}^+_\phi \hat{R}_1$, so that for any element $x_1 \in \mathcal{P}_\phi \hat{R}_1$, there is an element $x \in \mathcal{P}_\phi$ such that $x_1 - x \in \mathcal{P}^+_\phi \hat{R}_1$. This suffices, since ν is a valuation, to show that $\hat{\nu}(x_1y_1)=\nu(xy)=\nu(x)+\nu(y)=\hat{\nu}(x_1)+\hat{\nu}(y_1)$, so that $\hat{\nu}$ is a valuation and H_1 is prime.

Remark 0.13 In general, the ideal H is equal to its radical, as one easily verifies, so that $\hat{\nu}_1$ is a loose valuation on a reduced but not necessarily integral nœtherian local ring.

Let us now consider the case where ν is of height >1. We proceed by induction on the height of ν and for a while I stop assuming that ν is rational; the valuation ν now stands for one of the valuations with which my original rational valuation is composed, and I assume by induction that I have extended the valuation ν_1 of height $\mathrm{h}(\nu)-1$ with which ν is composed, of center $\mathbf{p}_{\nu_1}$. It means that we have an ideal $H_1\subset\hat{R}^m$ with $H_1\cap R=(0)$ and $H_1\subset\mathbf{p}_{\nu_1}\hat{R}^m$, and such that ν_1 extends to a loose valuation $\hat{\nu}_1$ of $\hat{R}^m/H_1$.
If the centers of ν and ν_1 coincide, there are finitely many ideals $\mathcal{P}_\phi$ between $\mathcal{P}_{\phi_1}$ and $\mathcal{P}^+_{\phi_1}$, so that, given $x\in\hat{R}_1$ with $\hat{\nu}_1(x)=\phi_1$, we have a largest ϕ such that $x\in\mathcal{P}_\phi\hat{R}_1$, and this defines a loose valuation on $\hat{R}_1$ extending ν.

I therefore assume, until the end of this proof, that the centers of ν and ν_1 are distinct, and try to construct an ideal $\tilde{H}$ of $\hat{R}_1$ such that ν extends to a loose valuation of $\hat{R}_1/\tilde{H}$. I denote by Ψ the group of height one which is the kernel of the map from the group Φ of ν to the group Φ_1 of ν_1. The main difficulty is that the ideals $\bigcap_{\lambda(\phi)=\phi_1}\mathcal{P}_\phi\hat{R}_1$ and $\mathcal{P}^+_{\phi_1}\hat{R}_1$ will in general be different. We must arrange that they become equal after passing to $\hat{R}_1/\tilde{H}$.
Let now S be an R-algebra and I an ideal of S. Given an ideal $\mathcal{P}$ of R, I denote as usual by $\mathcal{P}S$ the ideal generated by the image of $\mathcal{P}$ in S.

Lemma 0.14 *a) For each $\phi\in\lambda^{-1}(\phi_1)$ we have an exact sequence of $S/\mathcal{P}^+_{\phi_1}S$-modules*

$$0\to\frac{I\cap\mathcal{P}_\phi S+\mathcal{P}^+_{\phi_1}S}{\mathcal{P}^+_{\phi_1}S}\to\frac{\mathcal{P}_\phi S}{\mathcal{P}^+_{\phi_1}S}\to\frac{\mathcal{P}_\phi S}{\mathcal{P}_\phi S\cap(I+\mathcal{P}^+_{\phi_1}S)}\to 0,$$

and the equalities

$$\frac{\mathcal{P}_\phi S/I}{\mathcal{P}^+_{\phi_1}S/I}=\frac{\mathcal{P}_\phi S}{\mathcal{P}_\phi S\cap(I+\mathcal{P}^+_{\phi_1}S)}=\frac{\mathcal{P}_\phi S}{I\cap\mathcal{P}_\phi S+\mathcal{P}^+_{\phi_1}S}.$$

These $S/\mathcal{P}^+_{\phi_1}$-modules are actually $S/\mathbf{p}_{\nu_1}S$-modules since $\mathbf{p}_{\nu_1}\mathcal{P}_\phi S\subset\mathcal{P}^+_{\phi_1}S$.
b) Similarly, for $\phi\in\Phi_+\cup\{0\}$, we have the equalities

$$\frac{\mathcal{P}_\phi S/I}{\mathcal{P}^+_{\phi}S/I}=\frac{\mathcal{P}_\phi S}{\mathcal{P}_\phi S\cap(I+\mathcal{P}^+_{\phi}S)}=\frac{\mathcal{P}_\phi S}{I\cap\mathcal{P}_\phi S+\mathcal{P}^+_{\phi}S}.$$

Proof This is just a repeated application of the Dedekind-Nœther modular law ([Z-S], vol. II, Chap. 3, p.137). $\square$

Lemma 0.15 *Let $\mathbf{p}$ be the center of the valuation ν in R, and I be an ideal of the m-adic completion $\hat{R}^m$ of R.*
a) The graded algebra

$$\bigoplus_{\phi\in\Phi_+\cup\{0\}}\frac{\mathcal{P}_\phi.(\hat{R}^m/I)}{\mathcal{P}^+_\phi.(\hat{R}^m/I)}$$

is a quotient of the graded algebra

$$\bigoplus_{\phi\in\Phi_+\cup\{0\}}\frac{\mathcal{P}_\phi\hat{R}^m}{\mathcal{P}^+_\phi\hat{R}^m}=\mathrm{gr}_\nu R\otimes_{R/\mathbf{p}}\hat{R}^m/\mathbf{p}\hat{R}^m$$

by the ideal

$$\bigoplus_{\phi\in\Phi_+\cup\{0\}}\frac{I\cap\mathcal{P}_\phi\hat{R}^m+\mathcal{P}^+_\phi\hat{R}^m}{\mathcal{P}^+_\phi\hat{R}^m}.$$

b) If for all $\phi\in\Phi_+$ we have $(I+\mathcal{P}_\phi\hat{R}^m)\cap R=\mathcal{P}_\phi$, the intersection of this ideal with the subalgebra $\mathrm{gr}_\nu R\otimes_{R/\mathbf{p}}R/\mathbf{p}$ is equal to (0).

Proof The proof of a) is the same as for lemma 0.14, and the proof of b) is a direct computation. One may also refer to [B3], Chap. III, §2, No. 8, Theorem 1. □

Note that $\hat{R}_1/\mathbf{p}_{\nu_1}\hat{R}_1=\hat{R}^m/\mathbf{p}_{\nu_1}\hat{R}^m$ is equal to the m-adic completion of $\overline{R_1}=R/\mathbf{p}_{\nu_1}$. In this last ring, we have the valuation $\overline{\nu}$ of height one with group $\Psi=\mathrm{Ker}\lambda$, where $\lambda\colon\Phi\to\Phi_1$ is the map corresponding to the composition of valuations as in section **??**.
By the height one case, there is an ideal $\overline{H}=\bigcap_{\psi\in\Psi_+}\overline{\mathcal{P}}_\psi(\hat{R}/\mathbf{p}_{\nu_1}\hat{R})$, with $\overline{\mathcal{P}}_\psi=\mathcal{P}_\psi/\mathbf{p}_{\nu_1}$ and such that $\overline{\nu}$ extends to a loose valuation of $(\hat{R}/\mathbf{p}_{\nu_1}\hat{R})/\overline{H}$.
Let us choose a system of generators $(\overline{b}_1,\ldots,\overline{b}_s)$ of $\overline{H}$ and lift them to elements $(b_1,\ldots,b_s)$ of $\hat{R}_1$.
Let us denote by $\tilde{H}$ the ideal of $\hat{R}_1$ which they generate.
For simplicity let us write $\mathbf{p}_1$ for $\mathbf{p}_{\nu_1}$

Remark that since $\overline{H}\subset\mathbf{p}\overline{\hat{R}}_1$, where $\overline{\hat{R}}_1=\hat{R}_1/\mathbf{p}_1\hat{R}_1$, we have the inclusion

$$\tilde{H}\subset\mathbf{p}\hat{R}_1.$$

For a given $\phi_1\in\Gamma_1\subset\Phi_{1+}\cap\{0\}$, and $\phi\in\Gamma\cap\lambda^{-1}(\phi_1)$, let us define the ideal

$$\overline{T}_\phi=\{z\in\overline{R_1}/z\frac{\mathcal{P}_{\phi_1}}{\mathcal{P}^+_{\phi_1}}\subset\frac{\mathcal{P}_\phi}{\mathcal{P}^+_{\phi_1}}\}=(\frac{\mathcal{P}_\phi}{\mathcal{P}^+_{\phi_1}}:\frac{\mathcal{P}_{\phi_1}}{\mathcal{P}^+_{\phi_1}}).$$

The last notation is that of [Z-S] and [B3], Chap. 1, §2, No. 10, for the transporter of an $\overline{R_1}$-module, say, into one of its submodules. Let us denote by $\tilde{\phi}_1$ the smallest element of $\Gamma\cap\lambda^{-1}(\phi_1)$, which exists since Γ is well ordered and is equal to $\nu(\mathcal{P}_{\phi_1})$, and, for $\phi\in\lambda^{-1}(\phi_1)$, by $\psi(\phi)$ the smallest element of $\Gamma\cap\Psi$ which is at least equal to $\phi-\tilde{\phi}_1$, so that we have $\mathcal{P}_{\psi(\phi)}=\mathcal{P}_{\phi-\tilde{\phi}_1}$.

Lemma 0.16 *a) The ideal $T_\phi=(\mathcal{P}_\phi:\mathcal{P}_{\phi_1})$ is equal to $\mathcal{P}_{\psi(\phi)}$.*
b) The ideal $\overline{T}_\phi$ is the image of T_ϕ in $\overline{R}_1$; setting $\overline{\mathcal{P}}_\psi=\mathcal{P}_\psi\overline{R_1}$, the ideals $\overline{T}_\phi$ coincide in $\overline{R_1}$ with some of the ideals $\overline{\mathcal{P}}_\psi$; they define the same topology as the ideals $(\overline{\mathcal{P}}_\psi)_{\psi\in\Gamma\cap\Psi}$.

Proof We have $z\mathcal{P}_{\phi_1}\subset\mathcal{P}_\phi$ if and only if $\nu(z)\geq\phi-\tilde{\phi}_1$, so that $T_\phi=\mathcal{P}_{\phi-\tilde{\phi}_1}=\mathcal{P}_{\psi(\phi)}$. Since $\mathcal{P}^+_{\phi_1}\subset\mathcal{P}_\phi$, the image of T_ϕ in $R_1/\mathcal{P}_{\phi_1^+}$ is the transporter of the $R_1/\mathcal{P}_{\phi_1^+}$-module $\frac{\mathcal{P}_{\phi_1}}{\mathcal{P}^+_{\phi_1}}$ into $\frac{\mathcal{P}_\phi}{\mathcal{P}^+_{\phi_1}}$. Since they are both in fact $R_1/\mathbf{p}_1R_1$-modules, the transporter acts through its image $\overline{T}_\phi$ in $R_1/\mathbf{p}_1R_1$. The result on topologies follows from Proposition **??** which implies that for each $\psi \in \Gamma \cap \Psi_+$ there are finitely many ϕ's in $\lambda^{-1}(\phi_1)$ such that $\phi - \tilde{\phi}_1 < \psi$. □

I can now apply Corollary 8, Vol. 2, p. 267, of [Z-S], which tells us that the formation of the transporter of a module into a submodule commutes with m-adic completion in this case, since $\mathcal{P}_{\phi_1}/\mathcal{P}^+_{\phi_1}$ is an $\overline{R_1}$-module of finite type. It gives the following

Lemma 0.17 *a) We have the equality*

$$(\mathcal{P}_\phi \hat{R}^m : \mathcal{P}_{\phi_1}\hat{R}^m) = \mathcal{P}_{\psi(\phi)}\hat{R}^m.$$

b) Given $\phi_1 \in \lambda(\Gamma)$, for $\phi \in \Gamma \cap \lambda^{-1}(\phi_1)$, we have

$$\{z \in \frac{\hat{R}^m}{\mathcal{P}^+_{\phi_1}\hat{R}^m} / z \frac{\mathcal{P}_{\phi_1}\hat{R}^m}{\mathcal{P}^+_{\phi_1}\hat{R}^m} \subset \frac{\mathcal{P}_{\phi}\hat{R}^m}{\mathcal{P}^+_{\phi_1}\hat{R}^m}\} = P_{\psi(\phi)}\hat{R}^m/\mathcal{P}^+_{\phi_1}\hat{R}^m.$$

Proof The first statement follows from the reference just given, and the second from the fact that when we pass from $\hat{R}^m$ to $\hat{R}^m/\mathcal{P}_{\phi_1}\hat{R}^m$ the image of the transporter is equal to the transporter of the images since $\mathcal{P}_{\phi_1}\hat{R}^m \subset \mathcal{P}_\phi\hat{R}^m$. □

Let us remark here, as in Lemma 0.16, that the elements of $\hat{R}^m/\mathcal{P}^+_{\phi_1}\hat{R}^m$ act on $\frac{\mathcal{P}_{\phi_1}\hat{R}^m}{\mathcal{P}^+_{\phi_1}\hat{R}^m}$ via their images in $\hat{R}^m/\mathbf{p}_1\hat{R}^m$. Therefore in statement b) above we may replace $\mathcal{P}_{\psi(\phi)}\hat{R}^m$ by $\tilde{\mathcal{P}}_{\psi(\phi)} + \mathcal{P}^+_{\phi_1}\hat{R}^m$, where $\tilde{\mathcal{P}}_{\psi(\phi)}$ is any other ideal of $\hat{R}^m$ having the same image in $\hat{R}^m/\mathbf{p}_1\hat{R}^m$. From now on $\tilde{\mathcal{P}}_{\psi(\phi)}$ denotes such an ideal, which is *not* a valuation ideal in general.

By [Z-S], Vol. II, Appendix 3, Proposition 1, p. 342 and Proposition 2, p. 345, a primary decomposition for $\phi \in \Gamma$ is of the form

$$\mathcal{P}_\phi = \mathbf{q} \cap \mathbf{q}_{h_1} \cap \cdots \cap \mathbf{q}_{h_q}$$

for $\mathcal{P}_\phi$ in R, where the $\sqrt{\mathbf{q}}_{h_i} = \mathbf{p}_{h_i}$ are among the centers in R of the valuations with which ν is composed and $\mathbf{p}$ is the center of ν.
I assume them ordered as

$$\mathbf{p} \supsetneq \mathbf{p}_{h_1} \supsetneq \cdots \supsetneq \mathbf{p}_{h_q}.$$

By *loc.cit*, the ideal $\mathbf{q}_{h_1} \cap \cdots \cap \mathbf{q}_{h_q}$ is a valuation ideal, equal to $(\mathcal{P}_\phi : \mathbf{q})$ and which has to be equal to $\mathcal{P}_{\phi_1}$, where $\phi_1 = \lambda(\phi)$, since it is contained in the center $\mathbf{p}_1$ of ν_1, which we assumed to be distinct from $\mathbf{p}$, and its ν_1-valuation is ϕ_1. Although the ideal $\mathbf{q}$ is not uniquely defined, we may for each $\phi \in \lambda^{-1}(\phi_1)$ choose one such ideal and write

$$\mathcal{P}_\phi = \mathbf{q}(\phi) \cap \mathcal{P}_{\phi_1}, \quad \forall \phi \in \lambda^{-1}(\phi_1).$$

Since $\hat{R}^m$ is a flat R-algebra, we have

$$\mathcal{P}_\phi\hat{R}^m = \mathbf{q}(\phi)\hat{R}^m \cap \mathcal{P}_{\phi_1}\hat{R}^m.$$

Note that we have also

$$\mathcal{P}^+_{\phi_1}\hat{R}^m \subseteq \mathbf{q}(\phi)\hat{R}^m.$$

Using Lemma 0.17, we obtain for each $\phi \in \lambda^{-1}(\phi_1)$ the inclusion

$$\mathbf{q}(\phi)\hat{R}^m \subseteq \tilde{\mathcal{P}}_{\psi(\phi)}\hat{R}^m + \mathcal{P}^+_{\phi_1}\hat{R}^m,$$

since $\mathbf{q}(\phi)$ is contained in the transporter of $\mathcal{P}_{\phi_1}$ into $\mathcal{P}_\phi$. This gives

(1) $$\mathcal{P}_\phi\hat{R}^m \subseteq (\tilde{\mathcal{P}}_{\psi(\phi)}\hat{R}^m + \mathcal{P}^+_{\phi_1}\hat{R}^m) \cap \mathcal{P}_{\phi_1}\hat{R}^m.$$

The next step is to use Chevalley's theorem to show that the $\tilde{\mathcal{P}}_\psi$ approximate $\tilde{H}$.

Lemma 0.18 *Let* $\overline{H} \subset \hat{R}^m/\mathbf{p}_1\hat{R}^m$ *be the ideal* $\bigcap_{\psi\in\Psi}\overline{\mathcal{P}}_\psi\hat{R}^m/\mathbf{p}_1\hat{R}^m$ *associated to the valuation* $\overline{\nu}$ *of* $R/\mathbf{p}_1R$ *in the height one case. There is a function* $\psi \mapsto t(\psi) \in \mathbf{N}$ *tending to infinity with* ψ *such that for all* $\psi \in \Psi_+$ *the inclusions*

$$\overline{\mathcal{P}}_\psi\hat{R}^m/\mathbf{p}_1\hat{R}^m \subseteq \overline{H} + m^{t(\psi)}\hat{R}^m/\mathbf{p}_1\hat{R}^m$$

hold.

Proof By the height one case the valuation $\overline{\nu}$ of $R/\mathbf{p}_1$ extends to a valuation of height one $\hat{\overline{\nu}}$ of of $(\hat{R}/\mathbf{p}_1)/\overline{H}$; we have $\bigcap_{\psi\in\Psi_+}\mathcal{P}_\psi(\hat{R}/\mathbf{p}_1)/\overline{H} = (0)$, and the valuation ideals form a simple decreasing sequence of ideals. By Chevalley's theorem ([Z-S], Chap. VIII, §5, Th. 13) there exists a map $t\colon \Psi_+ \to \mathbf{N}$ such that $t(\psi)$ tends to infinity with ψ and such that $\mathcal{P}_\psi(\hat{R}^m/\mathbf{p}_1\hat{R}^m)/\overline{H} \subset m^{t(\psi)}(\hat{R}^m/\mathbf{p}_1\hat{R}^m)/\overline{H}$. □

As a consequence of this lemma, we can lift to $\hat{R}_1$ generators of the ideals $\overline{\mathcal{P}}_\psi\hat{R}^m/\mathbf{p}_1\hat{R}^m$ in such a way that the ideals $\tilde{\mathcal{P}}_\psi\hat{R}_1$ which these lifted elements generate in $\hat{R}_1$ satisfy

$$\tilde{H} \subset \tilde{\mathcal{P}}_\psi\hat{R}_1 \subseteq \tilde{H} + m^{t(\psi)}\hat{R}_1. \tag{2}$$

The second inclusion follows from the lemma, and we may then add $\tilde{H}$ to each $\tilde{\mathcal{P}}_\psi$ in order to satisfy the first.
Now if we go back to our quotient $\hat{R}_1 = \hat{R}^m/H_1$ of $\hat{R}^m$, to which the valuation ν_1 extends as a loose valuation, it follows from the inclusions (1) and (2) above that we have, for $\phi \in \lambda^{-1}(\phi_1)$, the inclusions

$$\mathcal{P}_\phi\hat{R}_1 \subseteq (\tilde{\mathcal{P}}_{\psi(\phi)}\hat{R}_1 + \mathcal{P}^+_{\phi_1}\hat{R}_1) \cap \mathcal{P}_{\phi_1}\hat{R}_1 \subseteq (\tilde{H} + \mathcal{P}^+_{\phi_1}\hat{R}_1 + m^{t(\psi(\phi))}\hat{R}_1) \cap \mathcal{P}_{\phi_1}\hat{R}_1. \tag{3}$$

This implies in turn, taking intersections and using the modular equality , that

$$\bigcap_{\lambda(\phi)=\phi_1}\mathcal{P}_\phi\hat{R}_1 \subseteq \tilde{H} \cap \mathcal{P}_{\phi_1}\hat{R}_1 + \mathcal{P}^+_{\phi_1}\hat{R}_1.$$

Lemma 0.19 *We have the inclusions*

$$\tilde{H}\mathcal{P}_{\phi_1}\hat{R}_1 + \mathcal{P}^+_{\phi_1}\hat{R}_1 \subseteq \bigcap_{\lambda(\phi)=\phi_1}\mathcal{P}_\phi\hat{R}_1 \subseteq \tilde{H} \cap \mathcal{P}_{\phi_1}\hat{R}_1 + \mathcal{P}^+_{\phi_1}\hat{R}_1.$$

Proof Assume by induction that $H_1 \subset \mathbf{p}_1\hat{R}^m$. Then the first inclusion follows from Lemma 0.17 since the image of $\tilde{H}$ in $\hat{R}^m/\mathbf{p}_1\hat{R}^m$ is $\overline{H}$, and we have just proved the second one. □

Let us now check the

Lemma 0.20 *The loose valuation* $\hat{\nu}_1$ *descends to a loose valuation of* $\hat{R}_1/\tilde{H}$*.*

Proof Let us first prove that $\tilde{H}$ is closed in $\hat{R}_1$ for the topology of the $\mathcal{P}_{\phi_1}\hat{R}_1$. This topology is the same as that defined by the loose valuation of height one with which $\hat{\nu}_1$ is composed. But the loose valuation ideals for that loose valuation form a simple decreasing sequence with intersection (0). By Chevalley's Theorem each is contained in an increasing power of the maximal ideal of $\hat{R}_1$; since $\hat{R}_1$ is nœtherian $\tilde{H}$ is closed for the m-adic topology, which shows that it is closed for the $\hat{\nu}_1$-adic topology. If now we go to $\lambda_i\colon \Phi_i \to \Phi_{i+1}$ using the notations of Proposition 0.11, we can apply Chevalley's theorem to $\hat{R}_1/\mathcal{P}_{\phi_{i+1}^+}\hat{R}_1$ and obtain, for $\phi_i \in \lambda_i^{-1}(\phi_{i+1})$,

$$\mathcal{P}_{\phi_i}\hat{R}_1 \subseteq \mathcal{P}^+_{\phi_{i+1}}\hat{R}_1 + m^{s(\phi_i)}\hat{R}_1,$$

from which follows, since $s(\phi_i)$ tends to infinity as ϕ_i grows in $\lambda_i^{-1}(\phi_{i+1})$.

$$\bigcap_{\phi_i\in\lambda_i^{-1}(\phi_{i+1})} (\tilde{H} + \mathcal{P}_{\phi_i}\hat{R}_1) \subseteq \bigcap_{\phi_i\in\lambda_i^{-1}(\phi_{i+1})} (\tilde{H} + \mathcal{P}^+_{\phi_{i+1}}\hat{R}_1 + m^{s(\phi_i)}\hat{R}_1) = \tilde{H} + \mathcal{P}^+_{\phi_{i+1}}\hat{R}_1.$$

The result now follows from Proposition 0.11. ☐

In particular, if we set $S = \hat{R}_1/\tilde{H}$, we have, since $\bigcap_{\lambda(\phi)=\phi_1} \tilde{\mathcal{P}}_{\psi(\phi)}S = (0)$,

$$\bigcap_{\lambda(\phi)=\phi_1} \mathcal{P}_\phi S = \mathcal{P}^+_{\phi_1}S.$$

This implies that the valuation ν extends as a loose valuation $\hat{\nu}$ on S. Let us denote by H the inverse image of $\tilde{H}$ in $\hat{R}^m$. We have by construction that

$$H_1 \subset H \subset \mathbf{p}\hat{R}^m$$

and the

Lemma 0.21 (Assuming that TLU(dim$R-1$) holds) *For each $\phi_1 \in \Phi_1$, we have $(\tilde{H} + \mathcal{P}_{\phi_1}\hat{R}_1) \cap R = \mathcal{P}_{\phi_1}$.*

Proof Fix $x \in R$, and let us first assume that the ideal $\overline{H}$ is generated by a regular sequence in the ring $\hat{R}^m/\mathbf{p}_1\hat{R}^m$ and that there exists a monomial $\overline{\xi} \in R/\mathbf{p}_1$ such that the $(R/\mathbf{p}_1)_{(\overline{\xi})}$-module $\mathcal{P}_\phi/\mathcal{P}^+_\phi \otimes_{R/\mathbf{p}_1} (R/\mathbf{p}_1)_{(\overline{\xi})}$ is free for all $\phi \leq \nu(x)$. Then, if $x \in \tilde{H}$, for some n we have, after choosing a representative $\xi \in R$ of $\overline{\xi}$, $\mathrm{in}_{\nu_1}(\xi^n x) \in \mathrm{gr}_{\nu_1}R \otimes_{R/\mathbf{p}_1} \overline{H}$.

To see this, fix a regular sequence of generators of $\overline{H}$ and lift them as generators $(b_1, \ldots, b_s)$ of $\tilde{H}$ in $\hat{R}_1$. Write $x = \sum_1^s \lambda_i b_i$. By induction on the height of the valuation, we may assume that if ν_1 has an immediate succcessor ν_2, the minimum of the $\nu_2(\lambda_i b_i)$ is equal to $\nu_2(x)$. If $\min_{1\leq i\leq s}\nu_1(\lambda_i) = \nu_1(x)$, the initial form is $\sum_{i\in M} \overline{b}_i \mathrm{in}_{\nu_1}\lambda_i$, where M is the set of indices where the minimum of valuations is attained, and we are done. Otherwise, the minimum of the valuations is $< \nu_1(x)$ and we have a relation $\sum_{i\in M} \overline{b}_i \mathrm{in}_{\nu_1}\lambda_i = 0$. Since we assume that the $\overline{\lambda}_i$ form a regular sequence and $\mathcal{P}_\phi/\mathcal{P}^+_\phi \otimes_{R/\mathbf{p}_1} (R/\mathbf{p}_1)_{(\overline{\xi})}$ is free, after multiplication by a suitable power $\overline{\xi}^{n_1}$ of $\overline{\xi}$ we have expressions $\overline{\xi}^{n_1}\mathrm{in}_{\nu_1}\lambda_i = \sum_{k=1}^s \overline{b}_k \mathrm{in}_{\nu_1}\mu_{ik}$ with $\mathrm{in}_{\nu_1}\mu_{ik} \in \mathcal{P}_\phi/\mathcal{P}^+_\phi \otimes_{R/\mathbf{p}_1} (R/\mathbf{p}_1)$ and $\mu_{ki} = -\mu_{ik}$. We may now write

$$\xi^{n_1}x = \sum_{i\in M}(\xi^{n_1}\lambda_i - \sum_k b_k\mu_{ki})b_i + \sum_{i\notin M}\xi^{n_1}\lambda_i b_i,$$

and the minimum of the ν_1 valuations of the coefficients has increased. Since ν_2 is fixed, in a finite number of such steps, we reach the situation where the minimum of the valuation of the coefficients is $\nu_1(x)$, and we are done. Now if we assume $TLU(d)$ for $d <$ dim$R$, since R is excellent by assumption, so is $R/\mathbf{p}_1$ and we can reach the situation taken as a hypothesis above by a toric modification of R as in Lemma **??** which makes $R/\mathbf{p}_1$ regular and $\overline{H}$ generated by part of a system of coordinates of $\hat{R}^m/\mathbf{p}_1\hat{R}^m$, and therefore by a regular sequence.

Without affecting this, we may by a toric modification $R \to R'$ of the same type flatten the image of $\mathcal{P}_{\phi_1}(R)/\mathcal{P}^+_{\phi_1}(R)$ modulo torsion by principalizing a suitable Fitting ideal (see [G-R]) and this, by Lemma **??**, gives us the freeness of $\mathcal{P}_{\phi_1}(R')/\mathcal{P}^+_{\phi_1}(R')$ after localization by some (ξ). In a toric modification such as $R/\mathbf{p}_1 \to R'/\mathbf{p}'_1$, if we denote by $\overline{H}'$ the ideal $\bigcap_{\psi\in\Psi_+}\mathcal{P}_\psi(R')\hat{R}'^{m'}/\mathbf{p}'_1\hat{R}'^{m'}$ of $\hat{R}'^{m'}/\mathbf{p}'_1\hat{R}'^{m'}$, we have $\overline{H}' \cap (\hat{R}^m/\mathbf{p}_1\hat{R}^m) = \overline{H}$. From this follows the inclusion $\tilde{H} \subseteq \tilde{H}' \cap \hat{R}_1$.

Now we observe that for the ϕ_1 which we are considering, we have $(\tilde{H}' + \mathcal{P}_{\phi_1}(R')\hat{R}'_1) \cap R' = \mathcal{P}_{\phi_1}(R')$; if this were not the case, there would be an element x' in the left hand side of valuation $< \phi_1$. Then its initial form would be that of an element of $\tilde{H}'$ so that, by what we have just proved, we would have, remembering that $\nu_1(\xi) = 0$, the inclusion $\mathrm{in}_{\nu_1}(\xi^n x') \in \mathrm{gr}_{\nu_1} R' \otimes_{R'/\mathbf{p}'_1} \overline{H}'$. But since $\xi^n x' \in R'$ and $\overline{H}' \cap (R'/\mathbf{p}'_1) = (0)$, this is impossible, which proves the result. Now since $\mathcal{P}_{\phi_1}(R') \cap R = \mathcal{P}_{\phi_1}$ and $\tilde{H} \subseteq \tilde{H}' \cap \hat{R}_1$, we have the inclusion $(\tilde{H} + \mathcal{P}_{\phi_1}\hat{R}_1) \cap R \subseteq \mathcal{P}_{\phi_1}$, and the result since the reverse inclusion is clear. $\square$

Lemma 0.22 *We have* $H \cap R = (0)$

Proof It suffices to prove that $\tilde{H} \cap R = (0)$ since we know by induction that $H_1 \cap R = (0)$. By Lemma 0.21, we have $\tilde{H} \cap R \subset \bigcap_{\phi_1\in\Phi_{1+}} \mathcal{P}_{\phi_1} = (0)$. $\square$

Finally, we have proved that $H \cap R = (0)$ and ν extends to a loose valuation of $\hat{R}^m/H$ composed with $\hat{\nu}_1$. This also validates our induction hypothesis.

Moreover, by lemmas 0.14 and 0.21, the graded algebra $\mathrm{gr}_{\hat{\nu}_1}\hat{R}^m/H$ is a quotient of

$$\mathrm{gr}_{\nu_1} R \otimes_{R/\mathbf{p}_1} \hat{R}^m/\mathbf{p}_1\hat{R}^m$$

by an homogeneous ideal whose intersection with $\mathrm{gr}_{\nu_1} R \otimes_{R/\mathbf{p}_1} R/\mathbf{p}_1$ is zero.

Going back to our rational valuation ν composed with $\nu_1, \nu_2, \dots, \nu_{h-1}$, we see that we can build a sequence of ideals $H_{h-1} \subset \cdots \subset H_0 = H$ of $\hat{R}^m$ such that $H_i \cap R = (0)$ and, if we set $\nu_0 = \nu$, the valuation ν_i extends as a loose valuation of $\hat{R}^m/H_i$ for $0 \le i \le h-1$. However, since ν is rational, the last graded ring is, in view of Lemma 0.15, a quotient of $\mathrm{gr}_\nu R \otimes_{R/m} k = \mathrm{gr}_\nu R$ which contains $\mathrm{gr}_\nu R$ since $H \cap R = (0)$ and the loose valuation extends ν, and so has to be equal to $\mathrm{gr}_\nu R$. Since that graded ring is an integral domain, this implies that H_0 is a prime ideal of $\hat{R}^m$ and that $\hat{\nu}$ is a valuation of $\hat{R}^m/H_0$, and therefore all the extensions $\hat{\nu}_i$ to $\hat{R}^m/H_0$ have to be valuations too, those with which $\hat{\nu}$ is composed.
This end the proof of Proposition 0.23.

By the regularity of $\hat{\overline{R}}^m$, we may choose a representative $\hat{\overline{R}}^m \subset \hat{R}^m$ of the quotient $\hat{\overline{R}}^m = \hat{R}^m/\mathbf{p}_{h-1}\hat{R}^m$. Define the ideal

$$H = \bigcap_{\phi_{h-1}\in\Phi_{h-1+}} \left(\overline{H}\hat{R}^m + \mathcal{P}_{\phi_{h-1}}\hat{R}^m\right) \subset \hat{R}^m.$$

The claim now is that the valuation ν extends to $\hat{R}^m/H$ in the way required by the Proposition. The proof is in several steps:
First, there is a candidate extension of ν_{h-1} to $\hat{R}^m/H$, obtained by defining the candidate valuation ideals

$$\hat{\mathcal{P}}_{\phi_{h-1}} = \left(\overline{H}\hat{R}^m + \mathcal{P}_{\phi_{h-1}}\hat{R}^m\right)/H = P_{\phi_{h-1}}\hat{R}^m/H.$$

Proposition 0.23 **Let R be a local equicharacteristic excellent integral domain with maximal ideal m and an algebraically closed residue field, and let ν a rational valuation on R with value group Φ.*
a) There exists a prime ideal H of the m-adic completion $\hat{R}^m$ of R such that ν extends to a valuation $\hat{\nu}$ of $\hat{R}^m/H$ in such a way that:
b) the associated graded map

$$\mathrm{gr}_\nu R \to \mathrm{gr}_{\hat{\nu}} \hat{R}^m/H$$

is a scalewise birational extension of Φ-graded R/m-algebras, uniquely determined by (R, ν).
*If the valuation ν is of height one, the map $\mathrm{gr}_\nu R \to \mathrm{gr}_{\hat{\nu}} \hat{R}^m/H$ is an isomorphism.**

Remarks 0.24 1) The proof depends on toric local uniformization, an argument for which is only sketched at the end of the paper.
2) The fact that $\hat{\nu}$ extends ν implies that $H \cap R = (0)$.
3) The assumption that the residue field is algebraically closed is due to the use of local uniformization in lower dimensions in the proof. It should eventually disappear when the method is generalized to excellent equicharacteristic local rings.
4) Although the ideal H is unique in the case considered by Zariski and Spivakovsky, it is not uniquely determined in general; it is only the graded map associated to the map $R \to \hat{R}^m/H$ which is uniquely determined. Once H is fixed, the extension of ν is unique. A quotient $\hat{R}^m/H$ as in the Proposition will nevertheless be denoted by $\hat{R}^{(\nu)}$.

4) The fact that the extension of the graded rings is scalewise birational implies that any toric modification we make in $\hat{R}^{(\nu)}$ with respect to representatives $(\eta_j)_{j\in J}$ of the generators of the graded algebra $\mathrm{gr}_{\hat{\nu}}\hat{R}^{(\nu)}$ can be viewed as coming from a toric modification in R with respect to the chosen system of representatives $(\xi_i)_{i\in I}$.

Proof (Sketch) The method is to work by induction on the dimension of R. Let us therefore assume first that R is a one-dimensional nœtherian local domain with a rational valuation ν, which is necessarily of height one. Let us set $H = \bigcap_{\phi\in\Phi_+} \mathcal{P}_\phi(R)\hat{R}^m$. Then we have $H \cap R = \{0\}$ and if we set $\hat{\mathcal{P}}_\phi = \mathcal{P}_\phi \hat{R}^m/H$, we see that $\hat{\mathcal{P}}_\phi \cap R = \mathcal{P}_\phi$ since by the faithful flatness of the extension $R \to \hat{R}^m$ we have $\mathcal{P}_\phi \hat{R}^m \cap R = \mathcal{P}_\phi$, and that the natural inclusion

$$\mathrm{gr}_\nu R \subset \bigoplus_{\phi\in\Phi_+} \frac{\hat{\mathcal{P}}_\phi}{\hat{\mathcal{P}}_\phi^+}$$

is an equality. This shows that $\hat{R}^m/H$ is an integral domain, since its associated graded ring for the filtration by the $\hat{\mathcal{P}}_\phi$ is an integral graded algebra, and that the $\hat{\mathcal{P}}_\phi$ are the valuation ideals of a valuation $\hat{\nu}$ on $\hat{R}^m/H$ which extends ν through the inclusion $R \subset \hat{R}^m/H$. Since $H \cap R = (0)$, the ideal H is a minimal prime ideal of $\hat{R}^m$ uniquely determined by the valuation ν and the quotient $\hat{R}^m/H$ is a one-dimensional local domain; we may call it the formal branch of the curve $\mathrm{Spec}R$ corresponding to ν. The proof in the dimension one case is partly inspired by a proof in [S2], Theorem 4.3.

Assume now that the result is true for excellent local domains of dimension $< d = \dim R$. Let us choose a set of elements $(\xi_i)_{i\in I}$ of R whose images in $\mathrm{gr}_\nu R$ generate it as a k-algebra. It may happen that the $(\xi_i)_{i\in I}$ do not generate the maximal ideal of R. We choose a minimal set of elements $y_1, \ldots, y_a$ such that the $(\xi_i)_{i\in I}, y_1, \ldots, y_a$ generate the maximal ideal. Our toric modifications will always be in terms of the $(\xi_i)_{i\in I}$. Let ν_{h-1} denote the valuation of height one with which ν is composed, and $\mathbf{p}_{h-1}$ denote its center in R. The valuation ν projects as a rational valuation $\overline{\nu}$ of height $\mathrm{h}(\nu)-1$ and value group Ψ_1 of the ring $\overline{R} = R/\mathbf{p}_{h-1}$, and we assume the result to be true for $\overline{\nu}$.

The proof now uses TLU(d-1) and TP(d-1). By using TLU(d-1) we may make a toric birational extension $\overline{R} \to \overline{R}'$ of $\overline{R}$ in the coordinates which are the images of the $\{\xi_i)|\nu(\xi_i) \in \Psi_1\}$ such that the conclusion of TLU is satisfied for $\overline{R}'$. Such a toric modification lifts in a natural way to a birational toric modification $R \to R'$. By applying TP to the relevant Fitting ideals as $\overline{R}$-modules of any finite number of the quotients $\mathcal{P}_{\phi_{h-1}}/\mathcal{P}^+_{\phi_{h-1}}$ we may assume that in addition their transforms modulo torsion, in the sense of Lemma **??**, are free $\overline{R}'$-submodules of $\mathcal{P}_{\phi_{h-1}}(R')/\mathcal{P}^+_{\phi_{h-1}}(R')$ with a cokernel annihilated by some monomials in the images in $\overline{R}'$ of the $\{\xi_i)|\nu(\xi_i) \in \Psi_1\}$.

In order to extend the valuation ν on R to a quotient $\hat{R}^m/H$ in the way required by the proposition it suffices to do the same for R'; the ideal $H \in \hat{R}^m$ will then be the kernel of the map $\hat{R}^m \to \hat{R'}^{m'}/H'$. From now on, I assume therefore that $\overline{R}$ is regular, that $\overline{H} \subset \hat{\overline{R}}^m$ is generated by part of a system of coordinates, and that $\mathcal{P}_{\phi_{h-1}}/\mathcal{P}^+_{\phi_{h-1}}$ contains a free submodule such that the quotient is annihilated by monomials in the $\{\xi_i)|\nu(\xi_i) \in \Psi_1\}$. Moreover, the ideal $\overline{H}$ is generated by elements $\overline{b}_1, \ldots, \overline{b}_s$ of the form $\overline{y}_k - a_k(\overline{\eta})$ where the $\overline{\eta}$ are representatives in $\hat{\overline{R}}^m$ of generators of the graded algebra $\mathrm{gr}_{\hat{\overline{\nu}}}\hat{\overline{R}}^{(\overline{\nu})}$.

Denote by $\hat{\mathbf{p}}$ the kernel of the natural map $\hat{R}^m \to \hat{\overline{R}}^{(\overline{\nu})}$. Let $\overline{b}_1, \ldots, \overline{b}_t$ be the generators of $\overline{H}$ which are not contained in $\mathbf{p}_{h-1}\hat{R}^m_{\hat{\mathbf{p}}} \cap \hat{R}^m/\mathbf{p}_{h-1}\hat{R}^m$. We can lift them to elements $b_1, \ldots, b_t$ of $\hat{R}^m$, of the form $y_k - a_k(\eta)$. Now set

$$H = (b_1, \ldots, b_t)\hat{R}^m + \bigcap_{\phi_{h-1}\in\Phi_{h-1+}} \big(\mathcal{P}_{\phi_{h-1}}\hat{R}^m_{\hat{\mathbf{p}}} \cap \hat{R}^m\big).$$

The idea now is to extend the valuation ν_{h-1} to a valuation $\hat{\nu}_{h-1}$ of the ring $(\hat{R}^m/H)_{\hat{\mathbf{p}}/H}$ which induces a valuation with center $\hat{\mathbf{p}}/H$ in the ring $\hat{R}^m/H$. The valuation $\hat{\nu}$ which we seek will be the composition of $\hat{\overline{\nu}}$ and $\hat{\nu}_{h-1}$.
For each $\phi_{h-1} \in \Phi_{h-1+}$ let us set

$$\hat{\mathcal{P}}_{\phi_{h-1}} = (\mathcal{P}_{\phi_{h-1}}\hat{R}^m_{\hat{\mathbf{p}}} \cap \hat{R}^m + (b_1, \ldots, b_t)\hat{R}^m)/H.$$

Let us first prove that the graded algebra $\bigoplus_{\phi_{h-1}\in\Phi_{h-1}} \hat{\mathcal{P}}_{\phi_{h-1}}/\hat{\mathcal{P}}_{\phi_{h-1}+}$ is a $\hat{\overline{R}}^{(\overline{\nu})}$-algebra, which is a birational extension of $\mathrm{gr}_{\nu_{h-1}}R \otimes_{\overline{R}} \hat{\overline{R}}^{(\overline{\nu})}$ and is an integral domain. We see that $\hat{\mathbf{p}} = (b_1, \ldots, b_t)\hat{R}^m + \mathbf{p}_{h-1}\hat{R}^m_{\hat{\mathbf{p}}} \cap \hat{R}^m$, and it follows from this that in $\hat{R}^m/H$ we have $\hat{\mathbf{p}}\hat{\mathcal{P}}_{\phi_{h-1}} \subset \hat{\mathcal{P}}_{\phi_{h-1}+}$, so that our algebra is indeed a $\hat{\overline{R}}^{(\overline{\nu})}$-algebra. By construction, it is contained in $\mathrm{gr}_{\nu_{h-1}}R \otimes_{\overline{R}} \kappa(\hat{\mathbf{p}}_{h-1})$, where $\kappa(\hat{\mathbf{p}}_{h-1})$ is the field of fractions of $\hat{\overline{R}}^{(\overline{\nu})}$, so that it is a birational extension of $\mathrm{gr}_{\nu_{h-1}}R \otimes_{\overline{R}} \hat{\overline{R}}^{(\overline{\nu})}$.

Now we have a candidate graded algebra of $\hat{R}^m/H$ with respect to the $\hat{\nu}_{h-1}$ valuation. We are going to prove that it is an integral domain by building a filtration on it such that the associated graded ring is a domain. This filtration will turn out to be the filtration associated to a valuation $\hat{\nu}$ extending ν to $\hat{R}^m/H$, and its associated graded ring will be a scalewise birational extension of $\mathrm{gr}_\nu R$.
First, one must make a distinction concerning birational extensions of rings such as $\hat{R}^m/H$; this distinction is essentially due to the fact that if $R \to R' \subset R_\nu$ is a birational extension, the extension $R/\mathbf{p}_{h-1} \to R'/\mathbf{p}'_{h-1}$ is not in general birational: Given a birational toric extension $\hat{\overline{R}}^{(\overline{\nu})} \to (\hat{\overline{R}}^{(\overline{\nu})})'$ of $\hat{\overline{R}}^{(\overline{\nu})}$, we can lift it to a birational toric extension of $\hat{R}^m/H$ with respect to representatives $(\eta_j)_{j\in J}$ in $\hat{R}^m/H$ of the $(\overline{\eta}_j)_{j\in J}$. Let us call such an extension $\hat{R}^m/H \to (\hat{R}^m/H)'$ a basic birational extension of $\hat{R}^m/H$. There is a natural map $(\hat{R}^m/H)' \to (\hat{\overline{R}}^{(\overline{\nu})})'$. We use the fact that after a toric modification $\overline{R} \to \overline{R}'$, for all further toric extensions $\overline{R}' \to \overline{R}''$ the ideal $\overline{H}''$ is the strict transform of $\overline{H}'$ under the map $\hat{\overline{R}}' \to \hat{\overline{R}}''$. This is part of the embedded resolution "package" in lower dimension, see Proposition **??**.
There is another type of birational extension:
Given ϕ_{h-1}, let us choose elements $(e_1, \ldots, e_r)$ of $\mathcal{P}_{\phi_{h-1}}(R)$ such that $\nu(e_1) < \nu(e_2) < \ldots < \nu(e_r)$ and their images modulo $\mathcal{P}^+_{\phi_{h-1}}$ form a system of generators $(\overline{e}_1, \ldots, \overline{e}_r)$ of the $R/\mathbf{p}_{h-1}$-module $\mathcal{P}_{\phi_{h-1}}/\mathcal{P}^+_{\phi_{h-1}}$.
Let $\kappa\colon R \to R'$ be the morphism of local rings obtained by blowing up the ideal $\mathcal{P}_{\phi_{h-1}}(R)$ in $\mathrm{Spec} R$ and localizing at the point picked by the valuation ν. We have a map (not necessarily injective)

$$\hat{R}^m \to \hat{R'}^{m'}.$$

The idea now is that to an element $x \in \hat{\mathcal{P}}_{\phi_{h-1}}$ may be attributed a $\hat{\nu}$-valuation as follows:
There is an element $s \notin \hat{\mathbf{p}}$ such that we may write $sx = \sum_{i=1}^{s} \overline{c}_i e_i$ with $\overline{c}_i \in \hat{R}^m/H$; in the ring $\hat{R'}^{m'}$, we may write $e_i = d_i e_1 \quad (2 \le i \le s)$, and therefore $sx = (c_1 + \sum_{i=2}^{s} c_i d_i) e_1$.

By our induction hypothesis, the valuation $\overline{\nu}'$ on $R'/\mathbf{p}'_{h-1}$ induced by the valuation ν on R' extends to a valuation $\hat{\overline{\nu}}'$ with value group Ψ_1 on $\widehat{R'/\mathbf{p}'_{h-1}}^{(\overline{\nu}')}$, which is a quotient of $\hat{R'}^{m'}$.

Now we may view $c = c_1 + \sum_{i=2}^{s} \lambda_i c_i$ as an element of $\hat{R'}^{m'}$; its image in $\widehat{R'/\mathbf{p}'_{h-1}}^{(\overline{\nu}')}$ is not zero by the construction of the ideals $\overline{H}'$ and we can define the $\hat{\nu}$-valuation of sx as the sum of $\nu(e_1)$ and the $\hat{\overline{\nu}}'$-valuation of the image of c in $\widehat{R'/\mathbf{p}'_{h-1}}^{(\overline{\nu}')}$. This is well defined since $H'\hat{R'}^{m'} \cap \hat{R}^m$ and H are equal modulo $\mathbf{p}_{h-1}\hat{R}^m$. Then define the $\hat{\nu}$ valuation of x as $\hat{\nu}(sx) - \hat{\nu}(s)$. If we replace R' by an R'' such that $\mathbf{p}_{h-1}R'' = e_1 R''$, the $\hat{\overline{\nu}}'$ value of c and its $\hat{\overline{\nu}}$" value are equal.
For $\phi \in \Phi_+ \cup \{0\}$ having image ϕ_{h-1} in Φ_{h-1}, define the ideal $\hat{\mathcal{P}}_\phi \subset \hat{\mathcal{P}}_{\phi_{h-1}} \subset \hat{R}^m/H$ to be the set of those elements $x \in \hat{\mathcal{P}}_{\phi_{h-1}}$ such that in the local ring of some (not necessarily basic) birational extension $(\hat{R}^m/H)'$ of $\hat{R}^m/H$ we have $sx = ce_1$, with $s \notin \hat{\mathbf{p}}'/H'$ and $\phi \le \hat{\overline{\nu}}(\overline{c}) + \nu(e_1) - \hat{\overline{\nu}}(\overline{s})$, where $\overline{c}$ is the image of c in $(\hat{\overline{R}}^{(\overline{\nu})})'$. By

what we have just seen, the $\hat{\mathcal{P}}_\phi/\hat{\mathcal{P}}_{\phi_{h-1}}$ form an exhaustive filtration of the quotient $\hat{\mathcal{P}}_{\phi_{h-1}}/\hat{\mathcal{P}}^+_{\phi_{h-1}}$. The quotient $\hat{\mathcal{P}}_\phi/\hat{\mathcal{P}}^+_\phi$ is either (0) or k. Given $x \in \hat{\mathcal{P}}_{\phi_{h-1}}$ and $y \in \hat{\mathcal{P}}_{\psi_{h-1}}$, let f_1, g_1 be generators of minimal valuation of $\mathcal{P}_{\psi_{h-1}}$ and $\mathcal{P}_{\phi_{h-1}+\psi_{h-1}}$ respectively; after blowing up the product of ideals $\mathcal{P}_{\phi_{h-1}}\mathcal{P}_{\psi_{h-1}}\mathcal{P}_{\phi_{h-1}+\psi_{h-1}}$ and localizing at the point picked by ν, we may write $e_1 f_1 = q g_1$, together with $x = ce_1, y = df_1$. Using the definitions above, we see that $\hat{\nu}(xy) = \hat{\overline{\nu}}(\overline{cdq}) + \nu(g_1)$, which means $\hat{\nu}(xy) = \hat{\nu}(x) + \hat{\nu}(y)$. This shows that the graded algebra $\bigoplus_{\phi\in\Phi_+\cup\{0\}} \hat{\mathcal{P}}_\phi/\hat{\mathcal{P}}^+_\phi$ is a graded domain; it is therefore the graded algebra $\mathrm{gr}_{\hat{\nu}}\hat{R}^m/H$ with respect to the valuation $\hat{\nu}$. In particular, the ideal $\mathrm{gr}_{\nu_{h-1}}R\otimes_{\overline{R}}\overline{H}$ of $\mathrm{gr}_{\nu_{h-1}}R\otimes_{\overline{R}}\hat{\overline{R}}^m$ is prime. □

Now we have to show that the valuation $\hat{\nu}$ induces ν on R. Let us first prove that for all $\phi_{h-1} \in \Phi_{h-1}$ we have that the inverse image in R of $\hat{\mathcal{P}}_{\phi_{h-1}}2$ is equal to $\mathcal{P}_{\phi_{h-1}}$.
The inverse image clearly contains $\mathcal{P}_{\phi_{h-1}}$. If we did not have the opposite inclusion, there would be an element $x \in R$ such that $\nu_{h-1}(x) = \mu_{h-1} < \phi_{h-1}$ and an element $s \in \hat{R}^m \setminus \hat{\mathbf{p}}$ such that $sx \in (b_1, \ldots, b_t)\hat{R}^m + \mathcal{P}^+_{\phi_{h-1}}\hat{R}^m$. The image of sx in

$$\mathcal{P}_{\mu_{h-1}}\hat{R}^m/\mathcal{P}^+_{\mu_{h-1}}\hat{R}^m = \mathcal{P}_{\mu_{h-1}}/\mathcal{P}^+_{\mu_{h-1}} \otimes_{\overline{R}} \hat{R}^m/\mathbf{p}_{h-1}\hat{R}^m$$

has to come from an element y of $(b_1, \ldots, b_t)\hat{R}^m$. Let us write $y = \sum_{i=1}^t b_i c_i$ with $c_i \in \hat{R}^m$. If the least $\mathcal{P}_\phi\hat{R}^m$ to which all c_i belong is μ_{h-1}, the image of y in $\mathcal{P}_{\mu_{h-1}}/\mathcal{P}^+_{\mu_{h-1}} \otimes_{\overline{R}} \hat{R}^m/\mathbf{p}_{h-1}\hat{R}^m$ is in $\mathcal{P}_{\mu_{h-1}}/\mathcal{P}^+_{\mu_{h-1}} \otimes_{\overline{R}} \overline{H}$. Now $\overline{s} \notin \overline{H}$ since $s \notin \hat{\mathbf{p}}$ and so we must have that the initial form of x is in $\mathcal{P}_{\mu_{h-1}}/\mathcal{P}^+_{\mu_{h-1}} \otimes_{\overline{R}} \overline{H}$. Since $\overline{H} \cap (R/\mathbf{p}_{h-1}) = (0)$, and the image of y is also that of $x \in R$, this is impossible, and so an element such as x cannot exist.
If the smallest ϕ such that $c_i \in \mathcal{P}_\phi$ for all i is $\kappa < \mu_{h-1}$, there must be a relation

$$\sum_{j\in J\subset\{1,\ldots,t\}} \overline{b}_j\overline{c}_j = 0,$$

where $\overline{c}_j$ is the image of c_j modulo $\mathcal{P}^+_\kappa\hat{R}^m$. After possibly making a new toric modification we may assume, since ν_{h-1} is of height one, that all $\mathcal{P}_\kappa/\mathcal{P}^+_\kappa$ for $\kappa \leq \phi_{h-1}$ contain a free submodule M_κ such that the quotient by this submodule is annihilated by monomials in the $\{\xi_i|\nu(\xi_i) \in \Psi_1\}$. Multiplying x by a suitable monomial ξ^e, and since $\xi^e x \in R$, we may assume that the $\overline{c}_j$ are in M_κ, in view of the fact that the $\hat{R}^m/\mathbf{p}_{h-1}\hat{R}^m$-module $\mathcal{P}_\kappa\hat{R}^m/\mathcal{P}^+_\kappa\hat{R}^m$ is torsion-free. Since $\overline{H}$ is generated by a regular sequence, the relation between the $\overline{c}_j$ implies that for all $j \in J$ we can write $\overline{c}_j = \sum_k \overline{e}_{jk}\overline{b}_k$ with $\overline{e}_{jk} = -\overline{e}_{kj}$. Taking representatives $e_{jk} \in \hat{R}^m$ such that $e_{jk} = -e_{kj}$, we can finally write

$$\xi^e x = \sum_{i\notin J} \xi^e c_i b_i + \sum_{j\in J}(\xi^e c_j - \sum_k e_{jk}b_k)b_j,$$

and the minimum of the ν_{h-1}-valuations has increased; in a finite number of such steps, we reach the situation where the minimum valuation of the coefficients is μ_{h-1} and a contradiction as above.
This also shows that $H \cap R = \bigcap_{\phi_{h-1}\in\Phi_{h-1+}} \mathcal{P}_{\phi_{h-1}} = (0)$. Finally, the valuation $\hat{\nu}$ on $\hat{R}^m/H$ is indeed the composition of the extension $\hat{\nu}_{h-1}$ of ν_{h-1} to $\hat{R}^m/H$ and $\hat{\overline{\nu}}$ on $\hat{\overline{R}}^{(\overline{\nu})}$.

Since we have by the induction hypothesis that the graded algebra $\mathrm{gr}_{\hat{\overline{\nu}}'}(\hat{\overline{R}'}^{(\overline{\nu})})'$ is a scalewise birational extension of $\mathrm{gr}_{\overline{\nu}}(\overline{R}')$, the group of values of $\hat{\overline{\nu}}'$ is Ψ_1, and therefore the group of values of $\hat{\nu}$ is Φ. The inclusion $\mathrm{gr}_\nu R \subset \bigoplus_{\phi\in\Phi_+\cup\{0\}} \hat{\mathcal{P}}_\phi/\hat{P}^+_\phi$ is scalewise graded. From this and Lemma **??** follows that the inclusion

$$\mathrm{gr}_\nu R \subset \mathrm{gr}_{\hat{\nu}} \hat{R}^m/H$$

is scalewise birational.

From the fact that the graded algebra $\mathrm{gr}_{\hat{\nu}_{h-1}} \hat{R}^m/H$ is independent from the choice of represntatives and that inside each of its homogeneous components the definition of $\hat{\mathcal{P}}_\phi/\hat{\mathcal{P}}_{\phi_{h-1}}$ is itself by construction independent of the choice of the representatives ξ_i, we see that the inclusion $\mathrm{gr}_\nu R \subset \bigoplus_{\phi\in\Phi_+\cup\{0\}} \hat{\mathcal{P}}_\phi/\hat{P}_{\phi_{h-1}}$ is independent of the choice of the representatives

If the valuation ν is of height one, we set $H = \bigcap_{\phi\in\Phi_+} \mathcal{P}_\phi \hat{R}^m$ and $\hat{\mathcal{P}}_\phi = \mathcal{P}_\phi \hat{R}^m/H$. Since for all ϕ the ideal $\hat{\mathcal{P}}_\phi$ contains H and $m\mathcal{P}_\phi \subset \mathcal{P}^+_\phi$, one sees that $\hat{\mathcal{P}}_\phi/\hat{\mathcal{P}}^+_\phi = \mathcal{P}_\phi/\mathcal{P}^+_\phi$, and the fact that in this case the graded algebra does not change.