COMPLEMENTS TO "VALUATIONS, DEFORMATIONS, AND TORIC GEOMETRY"

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PART 1: A QUICK INTRODUCTION TO VALUATION THEORY FROM A TORIC VIEWPOINT[1]

1 Introduction

A valuation of an integral domain R with field of fractions K is a map

$$\nu\colon K^* \to \Phi$$

with values in a totally ordered abelian group Φ and satisfying:
$\nu(xy) = \nu(x) + \nu(y)$,
$\nu(x+y) \geq \min(\nu(x), \nu(y))$,
$\nu(x) \geq 0$ whenever $x \in R$.

Since R is a ring, the image $\Gamma = \nu(R \setminus \{0\}) \subset \Phi_+ \cup \{0\}$ is stable under addition, and so is a sub-semigroup of $\Phi_+ \cup \{0\}$. If R is noetherian it is well-ordered, but not finitely generated in general. According to a result of Zariski, the ordinal type of Γ is $\omega^{h_R(\nu)}$, where $h_R(\nu)$ is an integer, the *length in* R of the valuation ν, which is at most equal to the Krull dimension of R. Determining or even bounding the ordinal type of the minimal set of generators of Γ is an open question as far as I know, although when R is complete it can be shown to be at most $h_R(\nu)\omega$, so one must think of the elements γ_i as indexed by ordinals $\leq \omega^{h_R(\nu)}$.

Since Γ is well ordered it has a *minimal* system of generators

$$\Gamma = \langle \gamma_1, \gamma_2, \ldots, \gamma_i, \ldots \rangle,$$

defined by an eventually transfinite construction: γ_i is the smallest non zero element of Γ not contained in the semigroup generated by the $(\gamma_j)_{j<i}$.

[1] All references in this text are to the paper "Valuations, deformations, and toric geometry", Fields Institute Communications, Vol. 33, 2003, pp. 361-459, also ArXiv, Math.AC/0303200, quoted [VDTG], and to the references of that paper.

• To give a valuation of an integral domain R is to choose an overring $R \subset R_\nu \subset K$ where divisibility is never a problem:
Given $x, y \in R_\nu$, either $x|y$ or $y|x$ in R_ν

Any integral domain with this property is called a *valuation ring.* Then if U_ν denotes the multiplicative group of units of R_ν, we have $U_\nu \subset K^*$ and the preorder $x \leq y \iff x^{-1}y \in R_\nu$ induces a total order of the quotient $\Phi = \frac{K^*}{U_\nu}$. The natural map

$$K^* \to \Phi = \frac{K^*}{U_\nu}$$

is a valuation. Note that the operation in Φ is noted additively although it comes from multiplication in K^*.
Given a valuation ν on K^*, the ring $R_\nu = \{x \in K^*/\nu(x) \geq 0\} \cup \{0\}$ is a valuation ring.

• Valuation rings appear in Geometry as

$$\varinjlim_{X_\alpha, \xi_\alpha} \mathcal{O}_{X_\alpha, \xi_\alpha} = R_\nu$$

where the limit of local rings corresponds to a projective system of points (closed or not) in a projective system of proper birational maps of schemes (you can think of blowing-ups) which is cofinal in the system of all proper birational maps. A *sequence* of points in such a system looks like:

$$\begin{array}{c} \cdots X_i \to X_{i-1} \to \cdots \to X_1 \to X_0 = X \\ \ldots\ \xi_i \to\ \xi_{i-1} \to \cdots \to\ \xi_1 \to\ \xi_0 = \xi \end{array}$$

• If there was a cofinal system in the system of $\mathcal{O}_{X_\alpha, \xi_\alpha}$ consisting of *regular* local rings, then R_ν, which is "regular" but very large, in particular not noetherian in general, could be approximated by regular local rings.

This is the problem of *local uniformization*, solved by Zariski in 1940 for algebraic varieties over a field k of characteristic zero as a step towards resolution of singularities, and which is also a consequence of Hironaka's resolution theorem. For positive characteristic, in dimension ≤ 3, it is a consequence of the results of Abhyankar .

• Given a field extension $k \to K$, the restriction to k of a valuation of K is a valuation of k. The valuations of K which restrict to the trivial valuation with value group 0 on k are known as valuations of K/k. The traditional view is to classify them according to the "size" of their ordered group Φ of values, and the "size" of the extension $k \to k_\nu = R_\nu/m_\nu$. The "size" of k_ν relative to k is measured by the transcendence degree of this field extension. The "size" of Φ is measured in two ways: the *rational rank* $\mathrm{r}(\Phi) = \dim_{\mathbf{Q}} \Phi \otimes_{\mathbf{Z}} \mathbf{Q}$ and the *rank* of Φ, which I prefer to call the height, and which is the length of a maximal chain of convex subgroups of the ordered group Φ. It is also the Krull dimension of the ring R_ν. A classical problem going back to Krull is to see how the valuation ring of a valuation of K/k differs from the ring $k[[t^{\Phi_+}]]$ of formal series with *well ordered* sets of exponents in Φ_+, equipped with the valuation associating to each series the smallest exponent appearing in it.

• We are interested here in pairs (R, ν), where R is a noetherian ring and ν a valuation of its field of fractions, non negative on R. In addition to the measures of the complexity of the valuation just mentionned, we need to consider the values of ν on $R \setminus \{0\}$. These values form a subsemigroup Γ of the semigroup $\Phi_+ \cup \{0\}$. Since R is noetherian, the set Γ is *well ordered*, which is not the case for $\Phi_+ \cup \{0\}$ in general. This implies that the semigroup Γ has a unique minimal set of generators. The group Φ being given, this set of generators contains some information about the singularities of R.

It is therefore interesting to try to build noetherian rings with a valuation having a semigroup of values given in advance. This is the content of the next section.

2 Examples

Let Γ be a sub semigroup of $\mathbf{Q}_+$. We start by building integral domains with a valuation ν having value group $\Phi \subseteq \mathbf{Q}$ and such that the semigroup of the values taken by the valuation on R is exactly Γ.

Example 1: $\Gamma = \langle \gamma_1, \ldots, \gamma_{g+1} \rangle \subset \mathbf{N}$ where $\langle A \rangle$ means the semigroup of all non negative integral linear combinations of the elements of A and the γ_i are collectively coprime integers, ordered according to their indices. We assume that they form a minimal system of generators of Γ, which means that for all i, we have $\gamma_{i+1} \notin \langle \gamma_i, \ldots, \gamma_i \rangle$.

Example 2: Let $(s_i)_{i \geq 1}$ be a sequence of positive integers such that $s_i \geq 2$ for $i \geq 2$, and define inductively rational numbers $(\gamma_i)_{i \geq 1}$ by the relations

$$\gamma_1 = \frac{1}{s_1}\ , \quad \gamma_{i+1} = s_i \gamma_i + \frac{1}{s_1 \ldots s_{i+1}}$$

Then take

$$\Gamma = \langle \gamma_1, \gamma_2, \ldots, \gamma_i, \ldots \rangle \subset \mathbf{Q}_+.$$

If $\forall i\ s_i = i$, the group Φ generated by Γ is $\mathbf{Q}$.

To such semigroups we can associate geometric objects, the spectra of their semigroup algebras over an algebraically closed field k. The Krull dimension of the semigroup algebra is equal to the rational rank $\dim_{\mathbf{Q}} \Phi \otimes_{\mathbf{Z}} \mathbf{Q}$ of the group Φ generated by Γ.

In our case this rank is equal to one, so that our semigroup algebras correspond to curves. In example 1, we have a monomial curve in $\mathbf{A}^{g+1}(k)$:

$$u_i = t^{\gamma_i},\ 1 \leq i \leq g+1, \quad \gamma_i \in \mathbf{N}.$$

Example 2 must also be thought of as a monomial curve $u_i = t^{\gamma_i}$ but now the embedding dimension is infinite.

We are now going to deform these rings, and for that we need equations for them. These equations correspond to relations with integral coefficients between the generators γ_i. The equations defining the monomial curve of example 1, that is, the relations between the γ_i, may be fairly complicated. We shall make the following simplifying assumptions:
a) if e_i is the gcd of $(\gamma_1, \ldots, \gamma_i)$ and if we write $e_i = s_{i+1}e_{i+1}$, then for $1 \leq i \leq g$

$$s_{i+1}\gamma_{i+1} \in \langle \gamma_1, \ldots, \gamma_i \rangle$$

b) $s_i\gamma_i < \gamma_{i+1}$ for $2 \leq i \leq g$.

Then the relations are generated by the following g expressions of condition a):

$$s_{i+1}\gamma_{i+1} = \sum_{k=1}^{i} \ell_k^{(i+1)}\gamma_k, \text{ with } \ell_k^{(i+1)} \in \mathbf{N}$$

These relations are not uniquely determined but in view of condition a) there is a unique way of writing each relation satisfying the condition that $\ell_k^{(i+1)} < s_k$ for $2 \leq k \leq i$. Condition a) implies that in the special case considered the monomial curve is a complete intersection with equations

$$u_{i+1}^{s_{i+1}} - \Pi_{k=1}^{i} u_k^{\ell_k^{(i+1)}} = 0, \ 1 \leq i \leq g.$$

In example 2, it is not difficult to see, using the fact that $(\gamma_1, \ldots, \gamma_i)$ are in the subgroup of $\mathbf{Q}$ consisting of rational numbers which can be written with denominator $s_1 \ldots s_i$, that all relations are generated by the:

$$s_{i+1}\gamma_{i+1} = \sum_{k=1}^{i} \ell_k^{(i+1)}\gamma_k \text{ with } \ell_k^{(i+1)} \in \mathbf{N}, i \geq 1,$$

so that the equations of our monomial curve are

$$u_{i+1}^{s_{i+1}} - \Pi_{k=1}^{i} u_k^{\ell_k^{(i+1)}} = 0, \ 1 \leq i.$$

All these equations are binomial equations defining irreducible varieties in a possibly infinite dimensional affine space. They are (non normal) toric varieties.

Let us now remark that in both examples we have $\gamma_{i+2} > s_{i+1}\gamma_{i+1}$ and let us deform the equations in the following manner: in the first example we consider a variable v and the equations

$$\begin{array}{ll} u_{i+1}^{s_{i+1}} - \Pi_{k=1}^{i} u_k^{\ell_k^{(i+1)}} - vu_{i+2} & = 0, \ 1 \leq i \leq g-1. \\ u_{g+1}^{s_{g+1}} - \Pi_{k=1}^{g} u_k^{\ell_k^{(g+1)}} & = 0 \end{array}$$

In the second example, we introduce a variable v_i for each index $i \geq 2$ and consider for $i \geq 1$ the equations

$$u_{i+1}^{s_{i+1}} - \Pi_{k=1}^{i} u_k^{\ell_k^{(i+1)}} - v_{i+1}u_{i+2} = 0$$

In both cases we have an obvious elimination process in the polynomial ring $k[(v_j^{\pm 1});(u_i)]$. Setting $v_j = 1$, the results are isomorphisms:

• For example 1:

$$\begin{aligned} R &= k[u_1, u_2]/(F) \\ &= k[u_1, \dots, u_{g+1}]/(u_2^{s_2} - u_1^{\ell_1^{(2)}} - u_3, u_3^{s_3} - u_1^{\ell_1^{(3)}} u_2^{\ell_2^{(3)}} - u_4, \dots), \end{aligned}$$

where $F(u_1, u_2)$ is the result of the elimination; for example if $\Gamma = \langle 4, 6, 13 \rangle$, the equations of the monomial curve are

$$\begin{aligned} u_2^2 - u_1^3 &= 0 \\ u_3^2 - u_1^5 u_2 &= 0 \end{aligned}$$

and since the deformation affects only the first equation and is $u_2^2 - u_1^3 - vu_3 = 0$, we find

$$F(u_1, u_2) = (u_2^2 - u_1^3)^2 - u_1^5 u_2 = 0.$$

• For example 2:

$$\begin{aligned} R &= k[u_1, u_2] \\ &= k[u_1, \dots, u_i, \dots]/(u_2^{s_2} - u_1^{\ell_1^{(2)}} - u_3, u_3^{s_3} - u_1^{\ell_1^{(3)}} u_2^{\ell_2^{(3)}} - u_4, \dots) \end{aligned}$$

In both cases, giving to the variable u_i the weight γ_i determines a valuation on the ring and the isomorphism gives a way to compute it: *in the right hand side the value of a polynomial $P(u_1, u_2)$ rewritten replacing systematically each $u_i^{s_i}$ by $\Pi_{k<i} u_k^{\ell_k^{(i)}} + u_{k+1}$ is the minimum of the values (i.e., weights) of its monomials* and this determines a valuation of R with semigroup Γ.

Giving to the variable u_i the weight γ_i determines a monomial order on the polynomial ring $k[(u_i)]$, and therefore a filtration by the minimal order of the monomials in a polynomial. Each of the equations which we have created by deformation has an initial form with respect to this filtration which is precisely the binomial equation which we have deformed.

In fact we have a faithfully flat family parametrized by $k[(v_j)]$ specializing the ring R to the ring of the monomial curve defined by the initial binomial equations. The equations of R play the role of a *standard, or Gröbner, basis* with respect to the monomial order.

Since we have seen that the Krull dimension of that ring is equal to one, the second example contradicts the semicontinuity of fiber dimensions in a family. There is no absurdity since that semicontinuity is proved under a finiteness condition which is not fulfilled here.

The situation in our case can be described by what I call *the abyssal phenomenon*:

Let ut write out the system of equations appearing in the second example:

$$\begin{array}{ll} u_2^{s_2} - u_1^{\ell_1^{(2)}} - v_2 u_3 & = 0 \\ u_3^{s_3} - u_1^{\ell_1^{(3)}} u_2^{\ell_2^{(3)}} - v_3 u_4 & = 0 \\ \vdots & \vdots \\ u_{i+1}^{s_{i+1}} - \Pi_{k=1}^{i} u_k^{\ell_k^{(i+1)}} - v_{i+1} u_{i+2} & = 0 \\ \vdots & \vdots \end{array}$$

When all v_i are nonzero, say equal to one, it amounts actually to an endless sequence of substitutions and therefore it cannot decrease the dimension, while when we specialize to the monomial curve, making all $v_{i+1} = 0$, we obtain equations which also express all u_i, $i \geq 3$ algebraically in terms of (u_1, u_2), but now u_1 and u_2 are algebraically dependent so that the dimension drops to 1.

However, if k is of characteristic zero we can also view these equations as defining a very transcendental curve in $\mathbf{A}^2(k)$ whose Zariski closure, which is all our equations see, is the entire affine plane.

To see this, use the order on the γ_j to order the equations $u_j^{s_j} - \ldots$ as we did above, and for a given $n \in \mathbf{N}$ truncate the system at order n in the following sense: keep all equations of index $< n$, replace the equation of index n by its initial form, which involves only variables of index $\leq n$, and forget all the other equations. We are now reduced to the case of example 1 except that we have to multiply $\gamma_1, \ldots, \gamma_n$ by their common denominator, obtaining a sequence of coprime integers.

If k is of characteristic zero, we can solve the corresponding equation $F_n(u_1, u_2) = 0$ by a Puiseux expansion

$$u_2^{(n)} = \sum_{j=1}^{\infty} a_j^{(n)} u_1^{\frac{j}{s_2 \ldots s_n}}$$

Using the Smith-Zariski formula for the intersection number of parametrized curves one can show that as n increases these Puiseux expansions converge in the ring $k[[t^{\mathbf{Q}_+}]]$ of series with well ordered sets of rational exponents to a series $w(u_1)$ of fractional powers of u_1 *whose exponents have unbounded denominators.* Because of Newton-Puiseux, setting $u_2 = w(u_1)$ cannot cause the vanishing of a polynomial (or even a power series) in (u_1, u_2). It parametrizes a transcendental curve whose Zariski closure is $\mathbf{A}^2(k)$.

We have produced it as a deformation of an algebraic, indeed toric, curve of infinite embedding dimension.

This curve defines a valuation on $k[u_1, u_2]$ by taking, for any polynomial $P(u_1, u_2)$, the order in u_1 of $P(u_1, w(u_1))$, which is finite as we just saw. It is of course the same valuation as that which is obtained as explained above.

Remark that we have many choices to deform our monomial curve of infinite embedding dimension:
Given an integer $t \geq 3$, we could deform by

$$\begin{aligned}
&u_2^{s_2} - u_1^{\ell_1^{(2)}} - v_2 u_t && = 0 \\
&u_3^{s_3} - u_1^{\ell_1^{(3)}} u_2^{\ell_2^{(3)}} - v_3 u_{t+1} && = 0 \\
&\vdots && \vdots \\
&u_t^{s_t} - u_1^{\ell_1^{(t)}} \ldots u_{k-1}^{\ell_{t-1}^{(t)}} - v_t u_{2t-2} && = 0 \\
&\vdots && \vdots \\
&u_{i+1}^{s_{i+1}} - \Pi_{k=1}^{i} u_k^{\ell_k^{(i+1)}} - v_{i+1} u_{i+t-1} && = 0 \\
&\vdots && \vdots
\end{aligned}$$

And then our deformed ring is $k[u_1, \ldots, u_{t-1}]$, with a valuation which still has the same semigroup Γ.

We can also deform our curve into a singular space: fix an integer $t \geq 2$ and for $j = 2, \ldots, t$ a polynomial $P_j(u_1, \ldots, u_t)$ in which every monomial has weight $> s_j\gamma_j$ and which contains no linear term. Fix another integer $t' \geq 2$ and then consider the deformation

$$\begin{array}{ll}
u_2^{s_2} - u_1^{\ell_1^{(2)}} - v_2 P_2(u_1, \ldots, u_t) & = 0 \\
u_3^{s_3} - u_1^{\ell_1^{(3)}} u_2^{\ell_2^{(3)}} - v_3 P_3(u_1, \ldots, u_t) & = 0 \\
\vdots & \vdots \\
u_t^{s_t} - u_1^{\ell_1^{(t)}} \ldots u_{t-1}^{\ell_{t-1}^{(t)}} - v_t P_t(u_1, \ldots, u_t) & = 0 \\
u_{t+1}^{s_{t+1}} - u_1^{\ell_1^{(t+1)}} \ldots u_t^{\ell_t^{(t+1)}} - v_{t+1} u_{t+t'} & = 0 \\
\vdots & \vdots \\
u_{i+1}^{s_{i+1}} - \Pi_{k=1}^{i} u_k^{\ell_k^{(i+1)}} - v_{i+1} u_{i+t'} & = 0 \\
\vdots & \vdots
\end{array}$$

Now taking all $v_s = 1$, we obtain a *singular* ring, with a valuation still having semigroup Γ. If we take infinitely many P_j, the ring is no longer noetherian.

3 Relation with local uniformization

Now what is the use of this?

Suppose that we are interested in building a *regular* local ring R' between R and R_ν, which is essentially of finite type over R; this is the problem of local uniformization. The basic idea is that over an algebraically closed field of any characteristic it is not difficult to resolve the singularities of an irreducible variety defined by binomials, by toric maps. In the case of the first example, that toric resolution will also resolve the singularities of the plane curve obtained by elimination, viewed as embedded in affine $g+1$-space; this is due to the fact that the deformation adds only terms of higher weight than the initial binomial equation.

The second example is not so convincing as far as local uniformization is concerned since the ring is regular, and also because there is no resolution in the usual sense of a space defined by infinitely many binomial equations.

But as we saw above, we can make it more complicated as follows: in some of the equations instead of adding a linear term u_{j+2}, we can add with a factor v_{i+2} a polynomial, or even a series, where every monomial is of weight greater that the weight of the initial binomial and where there is no linear term, and thus manufacture a singular ring, which may be of dimension > 2. When we grow tired, we go back to adding linear terms as above, which we must do anyway if we want the result to be noetherian. The resulting ring still specializes to the ring of the monomial curve wich admits as equations the initial forms, and therefore a toric resolution of the binomial variety corresponding the finitely many binomials which we have deformed in this new manner will provide a local uniformization of the valuation defined on R.

In short, if we have manufactured a complicated singular ring R by adding non linear terms of higher weight to a finite number of the binomial equations of our curve, we may be glad to exchange its noetherianity for the simplicity of dealing with a toric variety, provided we can show that it suffices to resolve **finitely many** of the binomials to uniformize the valuation on R.

Now the claim is that **this is essentially the general situation**, at least when R is a complete equicharacteristic noetherian local ring with an algebraically closed residue field.

Let $R \subset R_\nu$ be the inclusion of a ring in a valuation ring. The only really important case for local uniformization is when R is an excellent equicharacteristic local ring and R_ν dominates R, i.e., $m_\nu \cap R = m$, and the residue field extension $R/m \subset R_\nu/m_\nu$ is trivial; we say then that ν is a *rational* valuation of R.
We may consider the filtration of R by the ideals

$$\mathcal{P}_\phi(R) = \{x \in R/\nu(x) \geq \phi\}$$
$$\mathcal{P}_\phi^+(R) = \{x \in R/\nu(x) > \phi\}$$

and this gives us an associated graded ring

$$\mathrm{gr}_\nu R = \bigoplus_{\phi \in \Gamma} \frac{\mathcal{P}_\phi(R)}{\mathcal{P}_\phi^+(R)},$$

where $\Gamma \subset \Phi_+ \cup \{0\}$ is the semigroup of the values taken on elements of $R \setminus \{0\}$ by the valuation ν with value group Φ. Remark that if $\phi \notin \Gamma$, then $\mathcal{P}_\phi(R) = \mathcal{P}_\phi^+(R)$ and that if the valuation ν is rational and $\phi \in \Gamma$, then the quotient is a one-dimensional vector space over $k = k_\nu$.

The first basic fact is

Proposition 1: *If ν is a rational valuation of the noetherian ring local R, the associated graded ring can be presented as a quotient of a polynomial ring in countably many variables by a binomial ideal*:

$$\mathrm{gr}_\nu R = k[(U_i)_{i\in I}]/(U^m - \lambda_{mn} U^n)_{m,n\in E}.$$

Now one can associate to (R, ν) a *valuation algebra*

$$\mathcal{A}_\nu(R) = \bigoplus_{\phi \in \Phi} \mathcal{P}_\phi(R) v^{-\phi} \subset R[v^\Phi],$$

where $R[v^\Phi]$ is the group algebra of Φ with coefficients in R.

If R contains a field k such that the valuation takes the value 0 on k^*, we have a natural composed map

$$k[v^{\Phi_+}] \to R[v^{\Phi_+}] \to \mathcal{A}_\nu(R),$$

corresponding to a map of schemes

$$\mathrm{Spec}\mathcal{A}_\nu(R) \to \mathrm{Spec} k[v^{\Phi_+}].$$

And the second basic fact is:
Proposition 2: *If in addition the ring R contains a field of representatives, the $k[v^{\Phi_+}]$-algebra $\mathcal{A}_\nu(R)$ is faithfully flat, the general fiber of the corresponding map of schemes is isomorphic to* Spec$R$ *and its special fiber is* Specgr$_\nu R$.

By a result of Piltant, the Krull dimension of Specgr$_\nu R$ is the rational rank of the group Φ of the valuation, and by Abhyankar's inequality, we have for a rational valuation

$$\mathrm{dimgr}_\nu R \leq \mathrm{dim} R.$$

Strict inequality can occur, as we saw in example 2.

Using the properties of flatness, one can deduce from this a valuative version of Cohen's theorem:

Theorem 3.1. *If the noetherian local ring R is complete and equicharacteristic, given a field of representatives $k \subset R$ and elements $\xi_i \in R$ whose initial forms $\overline{\xi}_i$ generate the k-algebra* gr$_\nu R$*, the surjective map of k-algebras*

$$(*) \qquad\qquad k[(U_i)_{i\in I}] \to \mathrm{gr}_\nu R$$

of Proposition 1, mapping U_i to $\overline{\xi}_i$, extends to a continuous surjective map of k-algebras

$$(**) \qquad\qquad k[\widehat{(u_i)_{i\in I}}] \to R$$

mapping u_i to $\xi_i \in R$, and such that the associated graded map with respect to the natural filtrations coincides with the map (*)

Here the hat means a *scalewise* completion taking into account the structure of the group Φ of the valuation.

Moreover the kernel of the map (**) is generated, up to closure, by equations which are deformations of the binomial equations $U^m - \lambda_{mn} U^n$ generating the kernel of the map (*). They are of the form

$$F_{mn} = u^m - \lambda_{mn} u^n + \sum_p c_p u^p,$$

with $c_p \in k^*$, $w(u^p) > w(u^m) = w(u^n)$, and w is the monomial weight giving to u_i the weight $\gamma_i = \nu(\xi_i) \in \Gamma$.

Now because R is noetherian, its maximal ideal is generated by finitely many of the ξ_i and any variable u_j not in that finite set must appear *linearly* in one of the equations F_{mn}; In fact, modulo an implicit function theorem whose proof is not yet written in full, one can prove that all the equations F_{mn} except finitely many must be of the form

$$F_i = u^{n(i)} - \lambda_i u^{m(i)} + c_i u_i + \sum_p c_p^{(i)} u^p, \quad c_i \in k^*$$

This is beginning to look a lot like our example 2. It suffices to resolve the toric variety defined by the finitely many binomials which do not appear in the

F_i, this will extend to a birational toric map $Z \to \mathrm{Spec}R$ such that Z is non singular at the point picked by the valuation ν, and all the F_i add nothing: it is just a graph.

This reduces us to the case where the graded k-algebra $\mathrm{gr}_\nu R$ is finitely generated: one proves that an irreducible binomial variety over an algebraically closed field of any characteristic has embedded resolutions by toric maps (joint work with P. González Pérez), and then that the same toric map also resolves the space defined by the deformed equations at the point picked by the valuation provided one deforms by adding terms of higher weight. More precisely:

4 Overweight deformations of binomial ideals

Let us go back to the situation of a rational valuation on a local noetherian integral domain R with semigroup $\Gamma = \langle \gamma_1, \gamma_2, \ldots \rangle$. Recall ([VDTG], Proposition 4.15) that the graded algebra $\mathrm{gr}_\nu R_\nu$ of a valuation ring is the union of a nested sequence of polynomial algebras in $N = \mathrm{r}(\nu)$ variables over $k_\nu = R_\nu/m_\nu$ with maps between them sending each variable to a term. These subalgebras are the semigroup algebras of nested free subsemigroups of the non negative part $\Phi_+ \cup \{0\}$ of the value group Φ of the valuation. Let $\mathbf{N}^N \subset \Phi_+ \cup \{0\}$ be such a free subsemigroup of the non negative part of a totally ordered group Φ. Let $(\gamma_i)_{i\in F}$, where $F = \{1, \ldots, f\}$ be the elements of the sequence $(\gamma_i)_{i\geq 1}$ which are in $\mathbf{N}^N$. They form a minimal set of generators of a semigroup $\Gamma_F \subseteq \Gamma \bigcap \mathbf{N}^N$. Given any field k this determines a *monomial order* on $k[(U_i)_{i\in F}]$ corresponding to the order on the weights $w(U_i) = \gamma_i$ given by the order of Φ and a map of semigroup algebras over k

$$k[(U_i)_{i\in F}] \to k[t_1, \ldots, t_N].$$

The kernel of this map is a prime ideal I of $k[(U_i)_{i\in F}]$ generated by binomials. The vectors $m^j - n^j$ associated to these binomials generate the lattice of relations between the elements $(\gamma_j)_{j\in F}$.

Let w be a weight on a polynomial or power series ring over a field k, with values in the positive part Φ_+ of a totally ordered group Φ of finite rational rank.

Let us consider the power series case and the ring $S = k[[u_1, \ldots, u_N]]$. The weight w determines a filtration on S by the weight of series, which is the minimum weight of a monomial appearing in the series. The graded ring associated to this filtration is $k[U_1, \ldots, U_N]$.

Definition 4.1. Given a weight w as above, a (finite dimensional) valuative overweight deformation is the datum of a prime binomial ideal $(u^{m^\ell} - \lambda_\ell u^{n^\ell})_{1\leq \ell \leq s}$ such that the vectors $m^\ell - n^\ell$ generate the lattice of relations between the $\gamma_i = w(u_i)$ and of series

$$(OD) \qquad \begin{array}{l} F_1 = u^{m^1} - \lambda_1 u^{n^1} + \Sigma_{w(p)>w(m^1)} c_p^{(1)} u^p \\ F_2 = u^{m^2} - \lambda_2 u^{n^2} + \Sigma_{w(p)>w(m^2)} c_p^{(2)} u^p \\ \cdots\cdots \\ F_\ell = u^{m^\ell} - \lambda_\ell u^{n^\ell} + \Sigma_{w(p)>w(m^\ell)} c_p^{(\ell)} u^p \\ \cdots\cdots \\ F_s = u^{m^s} - \lambda_s u^{n^s} + \Sigma_{w(p)>w(m^s)} c_p^{(s)} u^p \end{array}$$

in $k[[u_1, \ldots, u_N]]$ such that, with respect to the monomial order determined by w, they form a Gröbner basis for the ideal which they generate.
Here we have written $w(p)$ for $w(u^p)$ and the coefficients $c^{(\ell)}$ are in k.

Let us agree to call X the formal subspace of $\mathbf{A}^N(k)$ defined by te equations F_ℓ.

Proposition 4.2. *a) A valuative deformation determines a valuation ν of the ring $R = S/(F_1, \ldots, F_s)$, and the associated graded ring of R is*

$$\mathrm{gr}_\nu R = k[U_1, \ldots, U_N]/(U^{m^1} - \lambda_1 U^{n^1}, \ldots, U^{m^s} - \lambda_s U^{n^s}).$$

b) There exists a birational toric map $Z \to \mathbf{A}^N(k)$ such that the strict transform of X is regular at the point picked by ν.

Proof. To prove a), let us define the order of a non zero element of R as the maximum weight of a series of S having x as image under the quotient map $S \to R = R/I$ where $I = (F_1, \ldots, F_s)$. This clearly satisfies the inequalities $\nu(x+y) \geq \min(\nu(x), \nu(y))$ and $\nu(xy) \geq \nu(x) + \nu(y)$. Thus, we have defined an order function on R. To prove that it is a valuation is to prove that the second inequality is an equality, and we argue as follows:
The order function ν determines a filtration on R; by construction the associated graded ring of R with respect to this filtration is a quotient of the associated graded ring $k[U_1, \ldots, U_N]$ of S with respect to the weight filtration. Indeed if we denote by $\mathcal{Q}_\phi$ the ideal of elements of weight $\geq \phi$ in S, we see that by definition of ν it maps onto the ideal of elements of R which are of order $\geq \phi$.

The ideal defining the quotient is the ideal of $k[U_1, \ldots, U_r]$ generated by the initial forms of the elements of I with respect to the weight filtration of S. Since by hypothesis this initial ideal is generated by the $w(m^\ell) = w(n^\ell)$, the graded algebra $\mathrm{gr}_\nu R$ is equal to $k[U_1, \ldots, U_N]/I_0$ where I_0 is the ideal generated by the binomials. It is therefore an integral domain, which shows that the order function ν is actually a valuation.

To prove b), consider the system of hyperplanes $H_\ell = H_{m^\ell - n^\ell}$ of $\check{\mathbf{R}}^N$ dual to the vectors $m^\ell - n^\ell$ and remember from [VDTG] (or see below) that if Σ is a regular fan subdividing the first quadrant of $\check{\mathbf{R}}^N$ and compatible with all the the H_ℓ, then the toric modification $\pi(\Sigma)\colon Z \to \mathbf{A}^N$ of $\mathbf{A}^N$ determined by Σ is an embedded resolution of singularities of the toric variety X_0 corresponding to the binomial ideal. Remember also that the charts of Z where the strict transform of X_0 meets the exceptional divisor are those which correspond to cones $\sigma = \langle a^1, \ldots, a^N \rangle$ which meet the *weight cone* which is the intersection with the first quadrant of the d dimensional vector space $\bigcap_{\ell=1}^t H_\ell$.

More precisely:

To the regular fan Σ with support $\mathbf{R}^N_{\geq 0}$ corresponds a proper and birational toric map of non singular toric varieties $\pi(\Sigma)\colon Z(\Sigma) \to \mathbf{A}^N(k)$. To each cone of maximal dimension $\sigma = \langle a^1, \ldots, a^N \rangle$ corresponds a chart $Z(\sigma)$ of $Z(\Sigma)$ which is isomorphic to $\mathbf{A}^N(k)$. If we choose adapted coordinates $y_1, \ldots, y_N$ in that chart, the restriction

$$\pi(\sigma)\colon Z(\sigma) \to \mathbf{A}^N(k)$$

is described by monomials:

$$(*) \qquad \begin{array}{ll} u_1 = & y_1^{a_1^1} \ldots y_N^{a_1^N} \\ u_2 = & y_1^{a_2^1} \ldots y_N^{a_2^N} \\ \ldots & \\ u_N = & y_1^{a_N^1} \ldots y_N^{a_N^N} \\ , & \end{array}$$

and so for each monomial u^m we have

$$u^m \mapsto y_1^{\langle a^1, m \rangle} \cdots Y_N^{\langle a^N, m \rangle},$$

where $\langle a^i, m \rangle = \sum_{j=1}^N a^i_j m_j$.
Let us compute the transform by this map of one of our binomial generators, denoted by $u^m - \lambda u^n$. We may assume that $a^1, \ldots, a^t$ are those among the a^j which lie on the hyperplane H_{m-n} dual to $m - n$, i.e., such that $\langle a^j, m - n \rangle = 0$, $1 \leq j \leq t$. *Because our fan is compatible with H_{m-n}, all the other $\langle a^j, m-n \rangle$ are of the same sign*, say $\langle a^j, m - n \rangle > 0$. We have then

$$u^m - \lambda u^n \mapsto y_1^{\langle a^1, n \rangle} \cdots y_N^{\langle a^N, n \rangle} (y_{t+1}^{\langle a^{t+1}, m-n \rangle} \cdots y_N^{\langle a^N, m-n \rangle} - \lambda).$$

The strict transform $y_{t+1}^{\langle a^{t+1}, m-n \rangle} \cdots y_N^{\langle a^N, m-n \rangle} - \lambda = 0$ of our binomial hypersurface is non singular, and transversal to the exceptional divisor defined by $\Pi_{j \in J} y_j = 0$, where J is the set of those $j, 1 \leq j \leq N$ such that a^j is not a basis vector.
It is shown in [VDTG] that the strict transform by the map $\pi(\Sigma)$ of the toric variety $X \subset \mathbf{A}^N(k)$ defined by our prime binomial ideal is non singular and transversal to the exceptional divisor.
Furthermore, if the intersection of σ with W is d-dimensional, then the equations of the strict transform depend on exactly d variables, say $y_{t+1}, \ldots, y_N$ and in view of their binomial nature their only solutions in the charts $Z(\sigma$ are given by $y_{t+j} = c_{t+j} \in k^*$.

We are going to show that there exist regular fans refining such a fan Σ such that the corresponding toric modification resolves the strict transform of X at the point picked by the valuation ν.

Let us examine how valuative overweight deformations behave with respect to toric resolutions of singularities. Let w be a weight on the variables $u_1, \ldots, u_N$ with values in a well ordered subsemigroup of the positive semigroup Φ_+ of a totally ordered group Φ of finite height. Let us say that a convex rational cone $\sigma \subset \check{\mathbf{R}}^N$ is *w-centering* if the monomial valuation on $k(u_1, \ldots, u_N)$ determined by w has a center in $k[\check{\sigma} \cap \mathbf{Z}^N]$. If we assume that σ is simplicial and write as above $\sigma = \langle a^1, \ldots, a^N \rangle$, this means that in the map determined by σ, we have that the $w(y_i)$ are ≥ 0.

We remark that since the matrix of the a_1^j is invertible, the weights of the y_j are uniquely determined by the $w(u_i)$.
If the weight w is of rank one and we identify its value group Φ with a subgroup of $\mathbf{R}$, we can consider the vector $\mathbf{w} = (w(u_1), \ldots, w(u_N)) \in \mathbf{R}^N_{\geq 0}$. Then the positivity of the $w(y_j)$ is equivalent to the fact that $\mathbf{w}$ is in σ; a simplicial convex cone σ is w-centering if and only if it contains the vector $\mathbf{w}$.

If the rank r of Φ is greater than one, we consider the sequence of convex subgroups, with the convention that $\Psi_0 = \Phi$:

$$(0) = \Psi_h \subset \Psi_{h-1} \subset \ldots \Psi_1 \subset \Phi,$$

and we notice that since Φ has no torsion and we are interested only in inequalities and equalities we can work in the divisible hull $\Phi \otimes_{\mathbf{Z}} \mathbf{Q}$ of Φ with the natural extension of the ordering on Φ and therefore assume that Φ is the lexicographic product of groups of rank one

$$\Phi = \Xi_1 \times \ldots \times \Xi_h$$

with $\Psi_j = \{0\} \times \Xi_{j+1} \times \ldots \times \Xi_h$.

Now let us choose an ordered embedding of $\Phi \otimes_{\mathbf{Z}} \mathbf{Q}$ in $(\mathbf{R}^h)_{\text{lex}}$. For each j, $1 \leq j \leq h$, we can define a vector $\mathbf{w}(j) \in \check{\mathbf{R}}^N$; it is the vector whose coordinates are the projections in Ξ_j of the $w(u_i)$.

Lemma 4.3. *Given N elements $w_1, \ldots, w_N$ of $\mathbf{R}$, the rational rank of the subgroup of $\mathbf{R}$ generated by the w_i is equal to the dimension of the smallest vector subspace $\langle w \rangle_{\mathbf{Q}}$ of $\mathbf{R}^N$ defined over $\mathbf{Q}$ containing the vector $\mathbf{w} = (w_1, \ldots, w_N)$.*

Proof. Let $\mathcal{L}$ be the kernel of the $\mathbf{Z}$-linear map $b\colon \mathbf{Z}^N \to \mathbf{R}$ sending the i-th basis vector to w_i. The image of the map b is the subgroup generated by the w_i and the rank of $\mathcal{L}$ is the codimension of $\langle w \rangle_{\mathbf{Q}}$ in $\mathbf{R}^N$. □

Let us denote by r_j the rational rank of the group Ξ_j.

Lemma 4.4. *The vectors $\mathbf{w}(j)$, $1 \leq j \leq h$, of $\check{\mathbf{R}}^N$ are linearly independant over $\mathbf{Q}$.*

Proof. The subgroup Φ of $\mathbf{R}^h$ is the image of the map $\mathbf{Z}^N \to \mathbf{R}^h$ sending the i-th basis vector to $w(u_i)$. The kernel of this map is a lattice $\mathcal{L}$ of rank $N - r$ where r is the rational rank of Φ and its image is spanned by the coordinates of the vectors $\mathbf{w}(j)$.

Consider the $\mathbf{Z}$-linear map $B\colon (\mathbf{Z}^N)^h \to \mathbf{R}^h$ defined by sending the basis vector $e_i^{(k)}$ to $\mathbf{w}(k)_i$ for $1 \leq i \leq N,\ 1 \leq k \leq h$. Since each $\Xi_k \subset \mathbf{R}$ is generated by the $\mathbf{w}(k)_i, 1 \leq i \leq N$ the group $\Phi = \Xi_1 \times \cdots \times \Xi_h$ is the image of this map.

The kernel is the lattice $\mathcal{M}$ in $(\mathbf{Z}^N)^h$ determined by the kernels of the maps $\mathbf{Z}^N \to \Xi_k$ sending $e_i^{(k)}$ to $\mathbf{w}(k)_i$ for each $k,\ 1 \leq k \leq h$. The rank of this lattice is $N - r_1 + \cdots + N - r_h = hN - (r_1 + \cdots + r_h)$.

If there was a linear relation with integral coefficients $\sum_{k=1}^h d_k \mathbf{w}(k) = 0$, it would give N elements $\sum_{k=1}^h d_k e_i^{(k)},\ 1 \leq i \leq N$ of the kernel of B, which must be rationally independant of $\mathcal{M}$ since they do not imply any relation among the generators of the groups Ξ_k. The existence of such a relation contradicts the fact that the rational rank of Φ is $r_1 + \cdots + r_h$. □

For each $j,\ 1 \leq j \leq h$, let us denote by S_j the smallest vector subspace of $\check{\mathbf{R}}^N$ defined over $\mathbf{Q}$ and containing the $\mathbf{w}(k),\ 1 \leq k \leq j$.
Notice that all the vector spaces S_i meet the first quadrant $\check{\mathbf{R}}^N_{\geq 0}$ outside of the origin since $\mathbf{w}(1)$ is in it. By the properties of the lexicographic order, the vector space S_h meets the interior of $\check{\mathbf{R}}^N_{\geq 0}$, so that we have $\dim(S_h \bigcap \mathbf{R}^N_{>0}) = \dim S_h$.

Lemma 4.5. *a) For each $j,\ 1 \leq j \leq h$ the dimension of the vector space S_j is $\sum_{k=1}^j r_j$.*
b) Let Σ be a regular fan with support $\check{\mathbf{R}}^N_{\geq 0}$ which is compatible with the vector spaces S_j. There exist N-dimensional cones σ of Σ such that $\mathbf{w}(1) \in \sigma$ and for all j the face $\sigma \bigcap S_j$ of σ is a cone of maximal dimension in S_j.
c) For such cones σ, for each j the number of support hyperplanes of σ which contain $S_{j-1} \bigcap \sigma$ but do not contain $\mathbf{w}(j)$ is equal to r_j.

Proof. Assertion a) follows directly from Lemmas 4.3 and 4.4.
To prove b) it suffices to remark that since Σ is compatible with the S_j, for each j the intersection $S_j \bigcap \Sigma$ is a fan of $S_j \bigcap \check{\mathbf{R}}^N_{\geq 0}$, which is of the same dimension as S_j as we saw above. Since the support of Σ is $\check{\mathbf{R}}^N_{\geq 0}$ this intersection must contain cones of the maximal dimension, which are faces of cones of Σ. These cones have the required property.
Assertion c) is a reformulation of b). □

Assume that σ is a regular convex cone of dimension N belonging to a regular fan with support $\mathbf{R}^N_{\geq 0}$ which is compatible with the rational vector spaces $S_j,\ 1 \leq j \leq h$. Assume that $\mathbf{w}(1) \in \sigma$. Let us denote by $(L_s)_{1 \leq s \leq N}$ the hyperplanes bounding σ. For each j there is a largest subset $I(j) \subset \{1, \ldots, N\}$ such that $S_j \subseteq \bigcap_{s \in I(j)} L_s$. By convention we set $I(0) = \{1, \ldots, N\}$.

Let us denote by $L_s^{\geq 0}$ the closed half space of $\mathbf{R}^N$ determined by L_s which contains σ.

Lemma 4.6. *In this situation, the N-dimensional regular convex cone σ is w-centering if and only if the following holds:*
For each $j,\ 0 \leq j \leq h-1$, we have $\mathbf{w}(j+1) \in \bigcap_{s \in I(j)} L_s^{\geq 0}$.

Proof. Since σ is regular, the determinant of its generating vectors is ± 1. According to the description $(*)$ of the monomial map associated to σ, the weights of the u_i uniquely determine the weights of the y_i in Φ since the determinant is $\neq 0$. Now writing that $w(y_i)$ is ≥ 0 in the lexicographic product $\Xi_1 \times \ldots \times \Xi_h$ reduces exactly to the expression given in the lemma. We observe that the projections in Ξ_k of the valuations of the $(y_i)_{1 \leq i \leq N}$ are the barycentric coordinates of the vector $\mathbf{w}(k)$ with respect to the generators of σ. If all the barycentric coordinates of $\mathbf{w}(1)$ are positive, then all the $w(y_i)$ are also positive and σ is w-centering. If some of these barycentic coordinates are zero, it means that $\mathbf{w}(1)$ is in a face of σ whose linear span is the intersection of the L_s for $s \in I(1)$, by the definition of $I(1)$. Then, in order for the corresponding $w(y_j)$ to be nonnegative in $\mathbf{R}^h$, it is necessary that the corresponding barycentric coordinates of $\mathbf{w}(2)$ are ≥ 0, which is equivalent to the inclusion $\mathbf{w}(2) \in \bigcap_{s \in I(1)} L_s^{\geq 0}$, and so on. The proof of the converse statement is obtained in the same way. $\square$

Remarks 4.7. : 1) If the group Φ is of rank one, the condition is simply that the vector $\mathbf{w}(1)$ is in σ, as we have noted above.
2) The argument uses only the fact that σ is simplicial and N-dimensional. In fact, given the monomial map associated to the simplicial cone σ, for each j the images in the group Ξ_j of the orders $w(y_i)$ are the barycentric coordinates of the vector $\mathbf{w}(j)$ with respect to the vertices of σ.

Lemma 4.8. *Let $\mathbf{w}(1), \mathbf{w}(2), \ldots, \mathbf{w}(h)$ be rationally independant vectors in a d-dimensional rational vector subspace $W \subset \mathbf{R}^N$, all lying in $W \bigcap \mathbf{R}^N_{\geq 0}$. Let $\sigma \subset \mathbf{R}^N_{\geq 0}$ be a d-dimensional regular cone of a fan Σ supported in $\mathbf{R}^N_{\geq 0}$ which is compatible with W and the vector spaces S_j defined above and containing $\mathbf{w}(1)$. Assume that $\sum_{i=1}^h r_i = d$ and is such that $\sigma \bigcap W$ is of dimension d. Then there exists a regular cone $\sigma' \subset \mathbf{R}^N_{\geq 0}$ satisfying the conditions of Lemma 4.6 and whose intersection with W is of dimension d.*

Proof. let $(L_s)_{s \in I}, I = \{1, \ldots, d\}$ be the collection the supporting hyperplanes σ. By construction there is a largest subset $I(1) \subset I$ such that $S_1 = \bigcap_{s \in I(1)} L_s$. In view of Lemma 4.5, c) and the compatibility of Σ with S_1, it is of cardinality $N - r_1$. By Lemma 4.4 we know that $\mathbf{w}(2)$ is not in $\bigcap_{s \in I(1)} L_s$. Let us denote by $I(2) \subset I(1)$ the set $\{s \in I(1) | \mathbf{w}(2) \in L_s\}$. For each $s \in I(1) \setminus I(2)$, which is of cardinality r_2 by Lemma 4.5, c), we denote by $L_s^{\geq 0}$ the closed half space determined by L_s which contains $\mathbf{w}(2)$. Again by Lemma 4.4 we know that $\mathbf{w}(3) \notin \bigcap_{s \in I(2)} L_s$ so we can define a subset $I(3) \subset I(2)$ by the condition that $\mathbf{w}(2) \in \bigcap_{s \in I(3)} L_s$ and a closed half space $L_s^{\geq 0}$ for each $s \in I(2) \setminus I(3)$, and so on.

In the end we have built a sequence of subsets

$$\{1, \ldots, N\} \supset I(1) \supset I(2) \supset \cdots \supset I(h)$$

such that $S_t = \bigcap_{s \in I(t)} L_s$ and we have determined half spaces $L_s^{\geq 0}$ corresponding to all the hyperplanes L_s for $s \in \{1, \ldots, N\} \setminus I(h)$ in such a way that each

$\mathbf{w}(k)$ is always in the half space $L_s^{\geq 0}$ if L_s vanishes on $\mathbf{w}(k-1)$. According to Lemma 4.5, c), at step i we define r_i half-spaces. Since $\sum_{i=1}^{h} r_i = d$, the set $I(h)$ is the set of those hyperplanes L_s which contain W. Thus $\bigcap_{s\in\{1,\ldots,N\}\setminus I(h)} L_s^{\geq 0}$ is a rational cone which is the intersection of d half spaces in $\mathbf{R}^N$. If now we define for $s \in I(h)$ the half-space $L_s^{\geq 0}$ as the one containing σ, we see that $\sigma' = \bigcap_{s\in\{1,\ldots,N\}} L_s^{\geq 0}$ is a rational regular cone which satisfies the conditions of Lemma 4.6. Moreover, its intersection with W is a regular d-dimensional cone of the fan $\Sigma \bigcap W$. □

Remark 4.9. If we transform σ into the first quadrant, contained in a larger cone which is the transform of the first quadrant, then σ' becomes another quadrant.

Proposition 4.10. *Keeping the same notations, let Σ be a regular fan with support $\check{\mathbf{R}}^N_{\geq 0}$. Assume that it is compatible with the S_j and the H_ℓ. Then there is a cone σ of dimension N of Σ which is w-centering and whose intersection with W is of dimension d.*

Proof. Let σ be a cone of Σ containing $\mathbf{w}(1)$. Piltant's theorem tells us that the dimension d of the toric variety defined by the initial binomial ideal, which is also the dimension of the vector space W, is equal to the rational rank of the group Φ:

$$r_1 + \cdots + r_h = \text{rat.rk}\Phi = d.$$

It suffices now to apply Lemma 4.8.

□

Remarks 4.11. 1) The key point in the proof of Proposition 4.10 is the linear independance over $\mathbf{Q}$ of the $\mathbf{w}(j)$.
2) We could have proved Proposition 4.10 by translating the weights into valuations on the ring $k[u_1,\ldots,u_N]$ and its quotient by the binomial ideal, and then invoking the valuative criterion of properness. However some work is required to obtain the result in this form, and more importantly this combinatorial proof prepares the extension to the overweight deformations.

Given an overweight deformation as in (OD) above, let us define for each ℓ indexing the binomial $u^{m^\ell} - \lambda_\ell u^{n^\ell}$ the following cones in $\mathbf{R}^N$:

$$\begin{aligned} E_\ell^{(1)}(j) = &\quad \langle \{p - n^\ell / |w(u^{p-n^\ell}) \in \Psi_j \setminus \Psi_{j+1}| c_p^{(\ell)} \neq 0, \ \}, m^\ell - n^\ell \rangle \\ E_\ell^{(2)}(j) = &\quad \langle \{p - m^\ell / |w(u^{p-m^\ell}) \in \Psi_j \setminus \Psi_{j+1}| c_p^{(\ell)} \neq 0, \ \}, n^\ell - m^\ell \rangle \end{aligned}$$

Lemma 4.12. *For all ℓ and $0 \leq j \leq h-1$ the cones $E_\ell^{(1)}(j)$ and $E_\ell^{(2)}(j)$ are contained in strictly convex polyhedral rational cones whose elements satisfy $\langle \mathbf{w}(k), q\rangle = 0$ for $1 \leq k \leq j-1$ and $\langle \mathbf{w}(j), q\rangle \geq 0$, with $\langle \mathbf{w}(j), q\rangle = 0$ if and only if q is on the half-line generated by $m^\ell - n^\ell$ (respectively $n^\ell - m^\ell$).*

Proof. Since the ring $k[[u_1,\ldots,u_N]]$ is noetherian, for each ℓ the ideal generated by the monomials u^p appearing in the ℓ-th series is generated by finitely many of them, say $u^{p_1},\ldots u^{p_{s_\ell}}$. In view of the convexity of the subgroups Ψ_j the cones

$E_\ell^{(1)}(j)$ and $E_\ell^{(2)}(j)$ are contained respectively in the convex cones generated by the $p_k - n^\ell + \mathbf{R}^N_{\geq 0},\ m^\ell - n^\ell$ and by the $p_k - m^\ell + \mathbf{R}^N_{\geq 0},\ n^\ell - m^\ell$ (for $1 \leq k \leq s_\ell$) for which $w(p - n^\ell)$ (respectively $w(p - m^\ell)$) is in $\Psi_j \setminus \Psi_{j+1}$. These cones are rational since the p_k are finite in number and they are strictly convex since they can be defined using strict inequalities and thus cannot contain a vector subspace. The second part of the statement follows from the definition of the vectors $\mathbf{w}(k)$. $\square$

Since what we want in the end is to find regular convex cones contained in the convex duals $\check{E}_\ell^{(i)}(j)$ of the cones $E_\ell^{(i)}(j)$, we may in view of this lemma assume that the cones $E_\ell^{(i)}(j)$ themselves are rational strictly convex cones, which we shall do henceforth.

Lemma 4.13. *Still denoting by H_ℓ the hyperplane of $\check{\mathbf{R}}^N$ dual to $m^\ell - n^\ell$ and by W the intersection of the H_ℓ, for each j and each ℓ we have :*

- *The cones $\check{E}_\ell^{(1)}(j)$ and $\check{E}_\ell^{(2)}(j)$ are N-dimensional, and their intersection $\check{E}_\ell^{(1)}(j) \bigcap \check{E}_\ell^{(2)}(j)$ is equal to $\check{E}_\ell^{(1)}(j) \bigcap H_\ell = \check{E}_\ell^{(2)}(j) \bigcap H_\ell$.*
- *For $i = 1, 2$ the dimension of $\check{E}_\ell^{(i)}(j) \bigcap H_\ell$ is $N - 1$.*
- *The interior in H_ℓ of $\check{E}_\ell^{(1)}(j) \bigcap \check{E}_\ell^{(2)}(j)$ is contained in the interior of $\check{E}_\ell^{(1)}(j) \bigcup \check{E}_\ell^{(2)}(j)$.*
- *For each k the cone $\mathbf{R}\mathbf{w}(1) + \cdots + \mathbf{R}\mathbf{w}(k-1) + \mathbf{R}_{\geq 0}\mathbf{w}(k)$ is contained in $\check{E}_\ell^{(1)}(k) \bigcap \check{E}_\ell^{(2)}(k)$ and meets its relative interior in H_ℓ.*
- *The same statements are true if one replaces each $\check{E}_\ell^{(i)}(k)$ by $\bigcap_\ell \check{E}_\ell^{(i)}(k)$ and H_ℓ by W.*

Proof. The dimensionality statement is nothing but the fact that the $E_\ell^{(i)}(j)$ are strictly convex. An element $a \in \check{\mathbf{R}}^N$ which is in $\check{E}_\ell^{(1)}(j) \bigcap \check{E}_\ell^{(2)}(j)$ has to be both ≥ 0 and ≤ 0 on $m^\ell - n^\ell$, so it is in H_ℓ. Since $p - n^\ell = p - m^\ell + m^\ell - n^\ell$ an element of H_ℓ which is ≥ 0 on $E_\ell^{(1)}(j)$ is ≥ 0 on $E_\ell^{(2)}(j)$ and conversely.
The second statement follows by convex duality from Lemma 4.12 which implies that $\mathbf{R}\langle m^\ell - n^\ell \rangle$ is the largest vector space contained in $E_\ell^{(i)}(j) + \mathbf{R}\langle m^\ell - n^\ell \rangle$.
The third statement is true because we join two convex cones along a common face of codimension 1; the boundary of the union does not meet the interior of the face.
The fourth statement also follows by convex duality from Lemma 4.12 if one observes that convex duality, which reverses inclusions, transforms intersection into Minkowski sum. Furthermore, if we add to a vector $a\mathbf{w}(k)$ with $a > 0$ a vector $\epsilon\mathbf{b}$ with $b \in H_\ell$, for $|\epsilon|$ small enough the vector $a\mathbf{w}(k) + \epsilon\mathbf{b}$ will still take positive values on the $p_k - n^\ell$, where the p_k are defined in the proof of Lemma 4.12, and therefore will belong to $\check{E}_\ell^{(1)}(k) \bigcap \check{E}_\ell^{(2)}(k)$.
Finally, the same arguments apply to W.

$\square$

Proposition 4.14. *Given an overweight deformation as in (OD), there exist regular fans* Σ *with support* $\check{\mathbf{R}}^N_{\geq 0}$ *compatible with the hyperplanes* H_ℓ*, the vector spaces* S_j *and all the cones* $\check{E}^{(i)}_\ell(k), i = 1, 2,\ k = 1, \ldots, h$.
Such fans contain w*-centering regular cones* $\sigma = \langle a^1, \ldots, a^N \rangle$ *contained for all* ℓ *and* k *in one of the two cones* $\check{E}^{(i)}_\ell(k) \quad i = 1, 2$*. For such a cone we have* $\langle a^i, p - m^\ell \rangle \geq 0$ *if* $\langle a^i, n^\ell - m^\ell \rangle \geq 0$ *(resp.* $\langle a^i, p - n^\ell \rangle \geq 0$ *if* $\langle a^i, m^\ell - n^\ell \rangle \geq 0$*) for all monomials* p *with* $w(p) > w(m^\ell) = w(n^\ell)$ *appearing in the overweight deformation.*

Proof. By the resolution theorem for toric varieties we know that there exist regular fans Σ with support $\check{\mathbf{R}}^N_{\geq 0}$ compatible with the H_ℓ, the S_j and the $\check{E}^{(i)}_\ell(k)$, which all determine rational cones in $\check{\mathbf{R}}^N_{\geq 0}$. According to Lemma 4.10 such fans contain w-centering cones. Let us show that such a cone is contained in every $\check{E}^{(1)}_\ell(k) \bigcup \check{E}^{(2)}_\ell(k)$. In view of the compatibility, it suffices to show that σ meets the interior of $\check{E}^{(1)}_\ell(k) \bigcup \check{E}^{(2)}_\ell(k)$. To prove that, in view of Lemma 4.13 it is enough to check that σ meets the interior of $\check{E}^{(1)}_\ell(k) \bigcap H_\ell$ in H_ℓ.

We see that by construction the cone σ contains points of the cone $\mathbf{Rw}(1) + \cdots + \mathbf{Rw}(k-1) + \mathbf{R}_{\geq 0}\mathbf{w}(k)$ which are in the interior of $\check{E}^{(1)}_\ell(k) \bigcap W$ in W. Thus the cone σ meets the interior of each $\check{E}^{(1)}_\ell(k) \bigcap \check{E}^{(1)}_\ell(k)$ and by compatibility of T with the $\check{E}^{(i)}_\ell(k)$ it is contained in the union $\check{E}^{(1)}_\ell(k) \bigcup \check{E}^{(2)}_\ell(k)$. By compatibility with the H_ℓ the cone σ has to be entirely on one side of H_ℓ, which means that it must be in $\check{E}^{(1)}_\ell(k)$ or $\check{E}^{(2)}_\ell(k)$. But this is decided for each ℓ by the fact that one generating vector of σ is on one side of H_ℓ.

If we write $\sigma = \langle a^1, \ldots, a^N \rangle$ we see that it has the property that whenever for a given ℓ we have that $\langle a^i, m^\ell - n^\ell \rangle$ is > 0 for some i then for all i we have $\langle a^i, m^\ell - n^\ell \rangle \geq 0$, and $\langle a^i, p - n^\ell \rangle \geq 0$ for all p appearing in the ℓ-th equation, and if $\langle a^i, n^\ell - m^\ell \rangle$ is > 0 for some i, then for all i we have $\langle a^i, n^\ell - m^\ell \rangle \geq 0$ and $\langle a^i, p - m^\ell \rangle \leq 0$ for those p. Given ℓ there has to be an index i for which $\langle a^i, m^\ell - n^\ell \rangle \neq 0$. □

Let us now finish the proof of Proposition 4.2. Take a regular fan T with support $\check{\mathbf{R}}^N_{\geq 0}$ and compatible with the H_ℓ, the S_j and the $\check{E}^{(i)}_\ell \quad i = 1, 2$ (and so depending on the deformation), and a w-centering cone $\sigma = \langle a^1, \ldots, a^N \rangle$ of that fan as above. Let us write the transforms of the equations $F_1, \ldots, F_s$, with the convention that $y^{\langle a,m \rangle} = y_1^{\langle a^1,m \rangle} \ldots y_N^{\langle a^N,m \rangle}$.

$$\begin{aligned}
&\tilde{F}_1 = y^{\langle a,m^1 \rangle} - \lambda_1 y^{\langle a,n^1 \rangle} + \Sigma_{w(p)>w(m^1)} c^{(1)}_p y^{\langle a,p \rangle} \\
&\tilde{F}_2 = y^{\langle a,m^2 \rangle} - \lambda_2 y^{\langle a,n^2 \rangle} + \Sigma_{w(p)>w(m^2)} c^{(2)}_p y^{\langle a,p \rangle} \\
&\ldots.. \\
&\tilde{F}_\ell = y^{\langle a,m^\ell \rangle} - \lambda_\ell y^{\langle a,n^\ell \rangle} + \Sigma_{w(p)>w(m^\ell)} c^{(\ell)}_p y^{\langle a,p \rangle} \\
&\ldots.. \\
&\tilde{F}_s = y^{\langle a,m^s \rangle} - \lambda_s y^{\langle a,n^s \rangle} + \Sigma_{w(p)>w(m^s)} c^{(s)}_p y^{\langle a,p \rangle}
\end{aligned}$$

Thanks to the properties of our cone σ we may factor out of each $\tilde{F}_\ell$ either $y^{\langle a,m^\ell\rangle}$ or $y^{\langle a,n^\ell\rangle}$. This leaves us with a deformation of the strict transform of the toric variety, which is regular in the chart corresponding to σ. More precisely, if for convenience of notation we rearrange the binomials in such a way that all $\langle a^i, m^\ell - n^\ell\rangle$ are ≥ 0, by writing $n^\ell - \lambda_\ell^{-1}m^\ell$ if $\langle a^k, m^\ell - n^\ell\rangle < 0$ we can write

$$\begin{array}{l}
\tilde{F}_1 = y^{\langle a,n^1\rangle}\big(y^{\langle a,m^1-n^1\rangle} - \lambda_1 + \Sigma_{w(p)>w(m^1)}c_p^{(1)}y^{\langle a,p-n^1\rangle}\big)\\
\tilde{F}_2 == y^{\langle a,n^2\rangle}\big(y^{\langle a,m^2-n^2\rangle} - \lambda_2 + \Sigma_{w(p)>w(m^2)}c_p^{(2)}y^{\langle a,p-n^2\rangle}\big)\\
.....\\
\tilde{F}_\ell = y^{\langle a,n^\ell\rangle}\big(y^{\langle a,m^\ell-n^\ell\rangle} - \lambda_\ell + \Sigma_{w(p)>w(m^\ell)}c_p^{(\ell)}y^{\langle a,p-n^\ell\rangle}\big)\\
.....\\
\tilde{F}_s = y^{\langle a,n^s\rangle}\big(y^{\langle a,m^s-n^s\rangle} - \lambda_s + \Sigma_{w(p)>w(m^s)}c_p^{(s)}y^{\langle a,p-n^s\rangle}\big)
\end{array}$$

This end the proof of the Proposition. □

5 The complete case

Now let us go back to the situation and notations of section 3 and especially of Theorem 3.1. In this section we assume that R is a complete equicharacteristic local domain. Assume that we are in the situation of Theorem 3.1 and that there exists a finite set $(u_i)_{i\in F}$ of coordinates such that for every $j \in I\setminus F$ there exists among the F_{mn} one, which we denote by F_j, in which u_j appears linearly. Let us write for simplicity

$$F_j = u^{n^j} - \lambda_j u^{m^j} + c_j u_j + \Sigma_p c_p^{(j)} u^p$$

with $c_j \in k^*$, and note that since the weight of u_j is greater that the weight of the initial binomial, it cannot appear there.

By the implicit function theorem (see the Appendix), we know that u_j can be expressed, modulo the closure I in $k[\widehat{(u_i)_{i\in I}}]$ of the ideal generated by the F_{mn}, as a series in the $(u_i)_{i\in F}$. We know also that we can invert the equation F_j as

$$u_j = c_j^{(-1)}\big(u^{n^j} - \lambda_j u^{m^j} + \Sigma d_p u^p\big),$$

an expression where we only know that u_j does not appear on the right-hand side and the terms $d_p u^p$ are of weight greater than the weight of u^{n^j}.

Lemma 5.1. *The images of the F_{mn} other than F_j in the ring $k[\widehat{(u_i)_{i\in I}}]/\big(u_j - c_j^{(-1)}\big(u^{n^j} - \lambda_j u^{m^j} + \Sigma d_p u^p\big)\big) \approx k[\widehat{(u_i)_{i\in I\setminus\{j\}}}]$ form a Gröbner basis of the ideal which they generate, with respect to the filtration determined by $w(u_i) = \gamma_i$ for $i \neq j$.*

Proof. We know that the F_{mn} form a Gröbner basis of the ideal they generate in $k[\widehat{(u_i)_{i\in I}}]$. So the initial form of a sum $\Sigma_{mn}A_{mn}F_{mn}$ is in the ideal of $k[(u_i)_{i\in I}]$

generated by the binomials $(u^m - \lambda_{mn}u^n)_{(m,n)\in E}$, where E is the our set of generators of the binomial relations defining $\mathrm{gr}_\nu R$. Now we have to see how the initial terms in $(u_i)_{i\neq j}$ appear when we substitute u_j by $u^{m^j} - \lambda_j u^{n^j} + \Sigma_{w(p)>w(u^{m_j})} d_p u^p$ in a series $\Sigma A_{mn}F_{mn}$. Let us denote by u' the set of variables $(u_i)_{i\neq j}$. In view of the definition of $k[\widehat{(u_i)_{i\in I}}]$ we can write our sum

$$\Sigma_{mn}A_{mn}F_{mn} = \Sigma_{k=0}^{\infty} D_k(u')u_j^k.$$

Given an element $S \in k[\widehat{(u_i)_{i\in I}}]$, let us denote by $\tilde{S} \in k[\widehat{(u_i)_{i\in I\setminus\{j\}}}]$ the image via the isomorphism $k[\widehat{(u_i)_{i\in I}}]/\overline{(F_j)} \cong k[\widehat{(u_i)_{i\in I\setminus\{j\}}}]$ of the class of S modulo the closure $\overline{(F_j)}$ of the ideal generated by F_j. Consider the filtration of $k[\widehat{(u_i)_{i\in I\setminus\{j\}}}]$ determined by giving u_i the weight γ_i. We denote by $\mathrm{In}_w\tilde{S} \in k[(U_i)_{i\in I\setminus\{j\}}]$ the initial form of $\tilde{S}$.

Lemma 5.2. *a) If the sum* $S = \Sigma A_{mn}F_{mn}$ *is not contained in* $\overline{(F_j)}$, $\mathrm{In}_w\tilde{S}$ *is a nonzero element of* $k[(U_i)_{i\in I\setminus\{j\}}]$ *of the form* $\Sigma_k M_k(u^{n^j} - \lambda_j u^{m^j})^k$.
b) If we denote by $\mathrm{In}_w\tilde{I}$ *the initial ideal in* $k[(U_i)_{i\in I\setminus\{j\}}]$ *of the ideal generated by the* $\tilde{F_{mn}}$, $(m,n) \neq (m^j, n^j)$ *with respect to the filtration by weight, then there is a nonzero homogeneous* $E \in k[(U_i)_{i\in I\setminus\{j\}}]$ *such that* $E(U^{m^j} - \lambda_j U^{n^j}) \in \mathrm{In}_w\tilde{I}$.

Proof. To prove b), remark that by definition of F, the value group of the valuation is rationally generated by the $(\gamma_j)_{j\in F}$. Since $j \notin F$, among all the relations between the γ_i, there must be at least one involving γ_j. Therefore among our binomials there is one of the form $u_j^{m(j)}u'^{m'} - \lambda' u'^{n'}$ where monomials denoted with u' do not contain u_j. The corresponding equation is

$$u_j^{m(j)}u'^{m'} - \lambda' u'^{n'} + \Sigma c_{k,q}u'^k u_j^q.$$

It cannot be in the ideal $\overline{(F_j)}$. Let us estimate its weight in $k[\widehat{(u_i)_{i\in I\setminus\{j\}}}]$ after the substitution $u_j \mapsto c_j^{(-1)}\big(u^{n^j} - \lambda_j u^{m^j} + \Sigma d_p u^p\big)$. If the lowest weight comes from the initial binomial, it is the weight of $(u^{m^j} - \lambda_j u^{n^j})^{m(j)}u'^{m'}$. The terms $c_{k,0}u'^k$ cannot contribute to the initial form since their weight is greater than the weight in $k[\widehat{(u_i)_{i\in I}}]$ of the initial binomial and is not affected by the substitution, while the weight of the initial binomial is strictly lowered. Therefore all the terms in the initial form are divisible by $U^{m^j} - \lambda_j U^{n^j}$, which proves b). □

□

6 The excellent case

Then there is the difficulty of reducing the excellent equicharacteristic case to the complete case. This is not entirely settled yet, although there is a precise program to deal with it.

• page 362, table of contents, section 4, read: ...valuation rings and valued nœtherian rings.
• page 366, line -15, read: ... by Abhyankar's inequality we have $\mathrm{r}(\nu) = \dim\mathrm{gr}_\nu R \leq \dim R$.
• page 369, footnote, the last letters in both words of the first Greek quotation are sigmas.
• page 370 before 2.2 add the following: **Remark** Given a ring R and a group Φ, the datum of a graded R-subalgebra $\mathcal{G}$ of the group algebra $R[v^\Phi]$ is equivalent to the datum of a family of ideals $(I_\phi)_{\phi\in\Phi}$ of R such that $I_\phi.I_\psi \subseteq I_{\phi+\psi}$, the correspondence being described by the equality $\mathcal{G} = \bigoplus_{\phi\in\Phi} I_\phi v^{-\phi}$. If in addition Φ is a totally ordered group, and $I_\phi = R$ for $\phi \leq 0$, then $\mathcal{G}$ is a graded $R[v^{\Phi_+}]$-algebra if and only if we have $I_\phi \supseteq I_\psi$ whenever $\psi \geq \phi$.
• page 371, lines 11-13: it is implicitely assumed that each $\overline{x}_\phi$ is $\neq 0$.
• page 373 line -3, read: ...are respectively isomorphic to $\mathrm{Spec}(R\otimes_k(v^{\Phi_+})^{-1}k[v^{\Phi_+}])$ and to $\mathrm{Spec}R$.
• page 374, line 3, the correct equality is

$$\mathcal{A}_\nu(R)/(\bigoplus_{\phi\in\Phi_+} \mathcal{P}_\phi(R)v^{-\phi} + m\mathcal{A}_\nu(R)) \simeq k[v^{\Phi_+}]$$

• page 383, at the end of subsection 3.3, add:
Indeed, except for $\phi = 0$, all these ideals are equal to the maximal ideal of R, which is also the ideal $m_{\nu_1} \cap R$. The image in $\overline{R}_1[v^\Phi] = k[v^\Phi]$ of $\mathbf{A}_\Psi(R)$ is equal to k. This is as it should be since the valuation $\overline{\nu}$ is trivial.
• page 384, line -20, read: "...which can also be deduced from subsection 3.3"
• page 389, title of section 4, read: ...valuation rings and valued nœtherian rings.
• page 390, line 14, read: ...with $c_{m,m'} \in k, \quad \lambda_{m,m'} \in k^*$.
• page 392, line -14, read: "...version of (non embedded) local uniformization;"
• page 393, line 14, read: ([Z1], B.I, page 861; see also [H-P], Chap. XVIII, §5).
• page 397, line -3, read:..irrational and > 1,
• page 398, lines 20,21, read: where $U_1 = V_2$ is of degree 1 and $U_2 = V_1$ of degree τ.
Addendum: In the notations of the Perron algorithm, the degree of V_i is $\tau_2^{(i)}$ and the degree of V_{i+1} is $\tau_1^{(i)}$.
• page 401, line 18, read: $P(u_2, u_3)$.
• page 401, footnote: the third letter of the Greek word is an upsilon and the last letter is a sigma.
• page 415, lines 4 and 5 of the proof of Lemma 5.25, replace $(X_i)_{i\in I}$ by $(U_i)_{i\in I}$, cX^a by cU^a, X^a by U^a.
• page 415, in Example 5.27, read:

$$x - \widetilde{\mathrm{in}}_\nu x = (a - \tilde{a})f^\ell \mod. f^{\ell+1}$$

• page 415, last line of the proof of lemma 5.25: replace "the sequence $x^{(j)}$" by " the sequence $x - x^{(j)}$ of elements of $k[(\xi_i)_{i\in I}]$".
• page 416, line -6, add a $(-1)^n$ before the last sum sign in the formula.
• page 417 From "given in advance", and until "we may now repeat", replace the text by:

Let s be the least integer such that for infinitely many values of the integer j we have $\nu_s(x - x^{(j+1)}) > \nu_s(x - x^{(j)})$. If $s = 1$, since the valuation ν_1 is of height one, for any $\phi_1 \in \Phi_{1+}$, after finitely many steps we have $\nu_1(x-x^{(j)}) > \phi_1$, hence for any $\phi \in \Phi_+$, after finitely many steps we have $\nu(x - x^{(j)}) > \phi$, so we may take $r_j = x - x^{(j)}$ and get $\nu(x - \sum_{r=0}^{j} P_r(\xi_i)) = \nu(x - r_j) > \phi$ and this shows that $x \in \hat{R}_h$. Assume now $s > 1$, and let $x^{(1)}$ be the representative in $\hat{R}_{s-1}[(\xi_i)_{i\in I_s}]$ of the initial form $\mathrm{in}_{\nu_{s-1}}(x) \in \mathrm{gr}_{\nu_{s-1}} R$. This makes sense because we may assume by induction on the dimension of R that the subring $\hat{R}_{s-1} \subset R$ is a representative of $R/(m_{\nu_{s-1}} \cap R)$. We define inductively the $x^{(j)}$ as representatives in $\hat{R}_{s-1}[(\xi_i)_{i\in I_s}]$ of $\mathrm{in}_{\nu_{s-1}}(x - x^{(j-1)})$. By definition of s, there exists a j_0 such that $\nu_{s-1}(x - x^{(j+1)}) = \nu_{s-1}(x - x^{(j)})$ for $j \geq j_0$. By substracting from x a polynomial in the ξ_i, we may assume that $j_0 = 1$.
Note that the center of ν_{s-1} is necessarily distinct from the center of ν_s, since if the two centers are equal, there are only finitely many distinct ν_s-ideals between two consecutive ν_{s-1}-ideals of R, according to [Z-S], Vol. II, Appendix 3, Corollary p.345.
We consider the initial forms $\mathrm{in}_{\nu_{s-1}}(x-x^{(j)}) \in \mathrm{gr}_{\nu_{s-1}} R$; for $j \geq j_0$; they all have the same degree, say ϕ_{s-1} and they are in $\mathcal{P}_{\phi_{s(j)}}/\mathcal{P}^+_{\phi_{s-1}}$ with $\phi_s(j)$ increasing strictly with j. The $R/\mathbf{p}_{s-1}$-submodules $\mathcal{P}_{\phi_s}/\mathcal{P}^+_{\phi_{s-1}} \subset \mathcal{P}_{\phi_{s-1}}/\mathcal{P}^+_{\phi_{s-1}}$ form a simple infinite sequence in view of Proposition 3.17. This sequence of submodules has intersection (0) and by the module-theoretic version of Chevalley's theorem ([B3], Chap. IV, §2, No.5, Cor.4) there is a sequence of integers $t(\phi_s(j))$ tending to infinity with the image of $\phi_s(j)$ in the height one group Ψ_{s-1}/Ψ_s and such that $\mathcal{P}_{\phi_s(j)}/\mathcal{P}^+_{\phi_{s-1}} \subset m^{t(\phi_s(j))}(\mathcal{P}_{\phi_{s-1}}/\mathcal{P}^+_{\phi_{s-1}})$. Since each homogeneous component $\mathcal{P}_{\phi_{s-1}}/\mathcal{P}^+_{\phi_{s-1}}$ of $\mathrm{gr}_{\nu_{s-1}} R$ is a $R/(m_{\nu_{s-1}} \cap R)$-module of finite type, it is complete for the $m/(m_{\nu_{s-1}} \cap R)$-topology. It is also complete for the $\overline{\nu}_s$ topology by (the proof of) Proposition 5.10. The sequence of the initial forms $\mathrm{in}_{\nu_{s-1}}(x-x^{(j)})$ then converges in $(\mathrm{gr}_{\nu_{s-1}} R)_{\phi_{s-1}}$ for the m-adic topology, and therefore also for the topology defined by the $\mathcal{P}_{\phi_s}/\mathcal{P}^+_{\phi_{s-1}}$, to a unique limit $\overline{x_1^{(1)}}$. By the definition of $\hat{R}_s$, we can lift $\overline{x_1^{(1)}}$ to an element $x_1^{(1)} \in \hat{R}_s$. This element has the property that for all $j \geq 1$, we have $\nu_{s-1}(x - x_1^{(1)}) > \nu_{s-1}(x - x^{(j)}) = \nu_{s-1}(x)$. Replacing now x by $x - x_1^{(1)}$, and repeating this construction, we build a sequence of elements $x_1^{(q)} \in \hat{R}_h$ such that $\nu_{s-1}(x - x_1^{(q)})$ increases at each step.
• page 418, from the beginning of 5.3 to: "This is a ν-adic..", replace the text by:
Given a system $(\overline{\xi}_i)_{i\in I}$ of generators of the k-algebra $\mathrm{gr}_\nu R$, giving rise to a surjective map $k[(U_i)_{i\in I}] \to \mathrm{gr}_\nu R$, a field of representatives $k \subset \hat{R}^{(\nu)}$ and representatives $(\xi_i)_{i\in I}$ in R of the $\overline{\xi}_i$, the main result of this subsection is the extension

of the map $k[(u_i)_{i\in I}]\to \hat{R}^{(\nu)}$ mapping u_i to ξ_i to a continuous surjective map

$$k[\widehat{(w_j)_{j\in J}}]\to \hat{R}^{(\nu)}$$

from the scalewise completion of the polynomial ring to the scalewise ν-adic completion of R. When $k[\widehat{(w_j)_{j\in J}}]$ is endowed with the term order obtained by giving to w_j the valuation of its image $\eta_j\in \hat{R}^{(\nu)}$, the associated graded map is the map

$$k[(W_j)_{j\in J}]\to \mathrm{gr}_{\hat{\nu}}\hat{R}^{(\nu)}$$

mapping W_j to $\overline{\eta}_j$, where each $\overline{\eta}_j$ is a Laurent monomial in the $\overline{\xi}_i$.

• page 424, replace line 5 by: Using the fact that $\widehat{\mathcal{A}_\nu(R)}^{(\nu_{\mathbf{A}})}$ is a faithfully flat $k[v^{\Phi_+}]$-algebra, one can show:
• page 425, replace line 6 of 'Corollary 5.51 by:

$$\hat{c}\colon k[\widehat{(w_j)_{j\in J}}]\to \hat{R}^{(\nu)},\quad w_j\mapsto c(-\hat{\nu}(\eta_j))\eta_j$$

• page 426, first line of the proof of Cor. 5.53: delete "of the".
• page 426, change the statement of Cor. 5.54 to:
Let R be a complete noetherian local ring endowed with a rational valuation ν of height $\mathrm{h}(\nu)$. *In the minimal system of generators of the semigroup* Γ *of* (R,ν) *there are at most* $\mathrm{h}_R(\nu)-1$ *elements without predecessor.*
• page 426, in the proof of Corollary 5.54, from "All the monomials" to the end, replace the text by:
Finitely many of these monomials $\xi''^{\alpha''}$ generate the $R/\mathbf{p}$-module $\mathcal{P}_{\phi_1}/\mathcal{P}^+_{\phi_1}$; denote them by $\xi''^{\alpha''_\ell}$. If a linear combination $\Sigma_\ell a_\ell\xi''^{\alpha''_\ell}$ of ν_1 valuation ϕ_1 has ν valuation greater than the minimum of the ν-valuations of its terms, it must contain binomials $\xi'^{\alpha'}\xi''^{\alpha''_i}-\lambda_{ij}\xi'^{\alpha'_j}\xi''^{\alpha''_j}$. But such a binomial can be written as a series beginning with a monomial $\xi'^{\alpha'_k}\xi''^{\alpha''_k}$. After making all such substitutions we obtain an expression for our sum in which the ν valuation of each term has increased. After finitely many steps we reach the situation where the ν-valuation of the sum is the minimum of the valuations of its terms, and this shows that the ν-valuation of every element of $\mathcal{P}_{\phi_1}\setminus\mathcal{P}^+_{\phi_1}$ is a linear combination with coefficients in $\mathbf{N}\cup\{0\}$ of elements of the semigroup $\overline{\Gamma}$ of the residual valuation $\overline{\nu}$ and the valuations of the monomials $\xi''^{\alpha''_\ell}$.
• page 426, first line of the proof of Cor. 5.54, read:..height $\mathrm{h}_R(\nu)$ of ν in R.
• page 426, at the end of the proof of Cor. 5.54, read ..height of ν in R minus one.
• page 426, line -2, replace "fact" by "condition"
• page 427 line -4, replace $\partial_{u_{h+1}}$ by ∂_{w_h}.
• page 430
• page 436, lines -12, -13: remove the sentence "Conversely,...$\{1,\dots,L\}$".
• page 440 line 1: ...of the canonical basis vectors...
• page 440 in the firs paragraph, after "..first quadrant of $\mathbf{R}^N$.", add:

Conversely if $\tilde{f}(e_i) \geq 0$ and $\tilde{f} \in W$ then $\tilde{f}$ is in the image of σ. This shows that σ is exactly the intersection of W with the first quadrant of $\mathbf{R}^N$.

• page 444 line -15, read : Spec$k[v][[u_1, \ldots, u_N]]$

• page 449 add at the bottom of the page: One difference between example 5.7 and example 4.20 is that the first one is weakly Abhyankar while the second one is not since the completion of R in this second case is $k[[u_1, u_2]]$.

• page 451, first line of the statement of Corollary 7.7, read : $\mathrm{gr}_{\hat{\nu}}\hat{R}^{(\nu)}$ and not $\mathrm{gr}_{\nu}\hat{R}^{(\nu)}$.

• page 457, after [H2], add

[H-P] W.V.D. Hodge and D. Pedoe, *Methods of Algebraic Geometry, Vol. 3*, Cambridge mathematical library, Cambridge University Press.

• page 457, reference [Kr], the year is 1932.

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