On the mathematical work of Professor Heisuke Hironaka

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The authors wish to express their gratitude to Hei Hironaka for his wonderful teaching and his friendship, over a period of many years.

In this succinct and incomplete presentation of Hironaka's published work up to now, it seems convenient to use a covering according to a few main topics: families of spaces and equisingularity, birational and bimeromorphic geometry, finite determinacy and algebraicity problems, flatness and flattening, real analytic and subanalytic geometry. This order follows roughly the order of publication of the first paper in each topic. One common thread is the frequent use of blowing-ups to simplify the algebraic problems or the geometry. For example, in the theory of subanalytic spaces of $\mathbf{R}^n$, Hironaka inaugurated and systematically used this technique, in contrast with the "traditional" method of studying subsets of $\mathbf{R}^n$ by considering their generic linear projections to $\mathbf{R}^{n-1}$.

No attempt has been made to point at generalizations, simplifications, applications, or any sort of mathematical descent of Hironaka's work, since the result of such an attempt must be either totally inadequate or of book length.

• Families of algebraic varieties and analytic spaces, equisingularity.

- Hironaka's first published paper is [**?**], which contains part of his Master's Thesis. The paper deals with the difference between the arithmetic genus and the genus of a projective curve over an arbitrary field. In particular it studies what is today known as the δ invariant of the singularities of curves. Previous work in this direction had been done by Rosenlicht (in his famous 1952 paper where Rosenlicht differentials are introduced), as he points out in his review of Hironaka's paper in Math. Reviews. However, Rosenlicht's treatment is rather "arithmetical", in the style of Chevalley's book on algebraic functions of one variable, while Hironaka's presentation is "geometrical". It allows him to study the behavior of the arithmetic genus under specialization of a curve over a (quasi-excellent) discrete valuation ring and to prove the best possible result in this direction, using Zariski's principle of degeneration.
In a seminar at IHES in the spring of 1968 Hironaka explained his ideas on how to relate the topology of complex analytic singularities and the algebraic properties of the local rings, and stated in particular that the constancy of the Milnor number in an analytic family of plane curve singularities should imply

its equisingularity. In this case the Milnor number is $2\delta - r + 1$ where r is the number of analytic branches. This was later proved by Lê and Ramanujam for analytic families of hypersurfaces of dimension $\neq 2$ with isolated singularities.

- Next comes [**?**], which gives necessary and sufficient conditions for the constancy of the Hilbert characteristic functions of the fibers of a family of projective varieties parametrized by the spectrum of a (quasi-excellent) discrete valuation ring. Again the approach is geometric. This paper contains the well known "Lemma of Hironaka" whose interest was pointed out by Nagata in his review in Math. Reviews. It gives a condition which ensures that the normality of the special fiber of such a family implies the normality of the total space.

- In the late 1960's, Zariski was developing the theory of equisingularity, and one problem was to understand the relationship between Zariski equisingularity, which certainly implies equimultiplicity, and the Whitney conditions. Hironaka used a geometric interpretation of his own ideas about normal flatness to give an analogue, *normal pseudo-flatness*, which makes sense in both the real and the complex case, and to prove that the Whitney conditions along a stratum imply normal pseudo-flatness along that stratum. In the complex analytic case, normal pseudo-flatness implies equimultiplicity, so that the Whitney conditions imply equimultiplicity.

- In the paper [**?**] Hironaka proves the existence of Whitney stratifications for subanalytic sets (see below) and describes an algebraic condition (in terms of blowing-ups) ensuring Thom's a_f condition for pairs of strata in X with respect to a flat map $f\colon X \to S$. Thom's condition concerns the limiting positions of tangent spaces to the fibers of f on the strata. It suffices to ensure a fairly good behavior of the fibers of f, for example the existence of vanishing cycles. Hironaka proves the existence of a stratification of X satisfying Thom's condition whenever S is a non singular curve.

- The paper [**?**] gives a proof of a fundamental result in equisingularity theory: the semicontinuity of Zariski's dimensionality type. Zariski had given an inductive definition of equisingularity of a hypersurface X along a non singular subspace Y which is based on equisingularity along the image of Y of the discriminant of a *generic* projection of X to a non singular space of the same dimension. Here generic does not mean generic among linear projections; one must deal with formal projections given by $\dim X$ formal power series. The dimensionality type of a hypersurface at a point is also defined inductively; roughly speaking, X is of dimensionality type r at a point if it is equisingular at that point along a non singular subspace of codimension r in X. The proof uses subtle constructions and a version of the Weierstrass preparation theorem.

• **Birational and bimeromorphic Geometry.**

- The first text is Hironaka's 1960 Harvard Thesis *On the theory of birational blowing-up*. It presents a very complete view of the models of a field K/k of algebraic functions, whether algebraic or "Zariskian" (corresponding to parts of the Zariski-Riemann variety of K/k). The main idea is to classify the models V'

which are projective over a given model V according to the semigroup of those (coherent sheaves of fractional) ideals on V which become locally principal on V'. More precisely, given a projective birational map $V' \to V$ Hironaka introduces an algebraic equivalence relation on the ideals on V which is compatible with the product, shows that equivalent ideals are principalized by the same morphisms and defines $C(V, V')$ to be the commutative semigroup (for the operation induced by the product) of equivalence classes of ideals whose pull-back ideal is locally principal on V'. Then he shows that the smallest group $G(V, V')$ containing $C(V, V')$ is finitely generated and free. Thus $C(V, V')$ generates a convex cone in $G(V, V') \otimes_{\mathbf{Z}} \mathbf{Q}$. This gives naturally rise to a combinatorial complex. The main theorem is that there is a one to one correspondence between the cells of this combinatorial complex and the normal models of K/k which lie between V and V' and are projective over V. The inclusion of cells corresponds to birational domination of models. This is the first occurrence of the characteristic cone of a morphism. The thesis contains many results on the normalized blowing-up of ideals, the behaviour of the Picard group and the additivity of the depths of the base and fiber at a general point of the source of a nice morphism of schemes. (see also [**?**]). It also contains an example of a non projective birational map.

- Hironaka produced in [**?**] (using a very ingenious blowing-up) a one-parameter family of non singular compact complex varieties whose special fiber is non-Kähler and all other fibers are Kähler.

- The famous paper on resolution of singularities in characteristic zero (see [**?**]) introduced a number of new techniques in the subject, indeed in algebraic geometry and commutative algebra, as well as some key ideas which still underlie all the proofs of resolution of singularities in characteristic zero, after more than forty years and a lot of successful work to streamline and simplify Hironaka's proof and to make it effective.

One of the simpler ideas he introduces is to measure the singularity of an algebraic variety X at one of its points x by the Hilbert-Samuel function instead of its multiplicity. It is the function $\mathrm{H}_{X,x}$ from $\mathbf{N}$ to $\mathbf{N}$ defined by: $n \mapsto \dim_{\kappa(x)} \mathcal{O}_{X,x}/m_{X,x}^{n+1}$ which is associated to the local ring $\mathcal{O}_{X,x}$. Asymptotically it behaves like $\mathrm{e}_{X,x}\frac{n^d}{d!}$ where d is the dimension and $\mathrm{e}_{X,x}$ the multiplicity. Multiplicity is blind to lower-dimensional components so that multiplicity one does not imply non singularity in general. If the Hilbert-Samuel function of a local ring is that of a regular ring, the local ring is regular. Moreover, the Hilbert-Samuel function of the local ring of a variety determines its local embedding dimension, while the multiplicity does not. The partition of a variety according to the values taken by the Hilbert-Samuel function of the local rings (Samuel stratification of X) is of an algebraic nature. Next comes the fact that the constancy of the Hilbert-Samuel function of a variety X along a non singular closed subvariety Y (i.e., $H_{X,y}(n)$ is independant of $y \in Y$ for all $n \in \mathbf{N}$) is equivalent to the flatness of the canonical map $C_{X,Y} \to Y$ of the normal cone of Y in X (normal flatness). This makes it possible to prove that under a *permissible* blowing-up $f \colon X' \to X$, the Hilbert-Samuel function cannot increase, in the sense that for all $x' \in X'$ we have $\mathrm{H}_{X',x'}(n) \leq \mathrm{H}_{X,f(x')}(n) \ \ \forall n \in \mathbf{N}$. Here permissible blowing-up means a blowing-up of the ambient non singular space with a non singular center along which our singular space X is normally flat. Moreover we have information on those points $x' \in f^{-1}(x)$ for which the two functions $\mathrm{H}_{X,x}$ and $\mathrm{H}_{X',x'}$ are equal.

Now the key problem is, given a Hilbert-Samuel function H of a point x of X, to prove the existence of a sequence of permissible blowing-ups after which the Hilbert-Samuel function has decreased everywhere above a neighborhood of x. This suffices since one can show that the Hilbert-Samuel function cannot indefinitely decrease, and when it stabilizes with respect to a sequence of permissible blowing-ups, the space is non singular.

The key ideas introduced by Hironaka at this point are those of *maximal contact* and of *idealistic exponent*. They do not appear with these names in the Annals of Mathematics paper, but as parts of *J-stable regular τ-frames* and of *resolution data of type $\mathcal{R}_{II}^{N,n}$*. Their role is explained in [**?**],[**?**],[**?**]. The idea is to find locally at each singular point x of our singular space X a non singular subspace W of the ambient non singular space, which has "maximal contact" with X at x in the sense that it locally contains the Hilbert-Samuel stratum of x in X, and after a permissible blowing-up its strict transform W' contains all the points $x' \in f^{-1}(x)$ of the strict transform X' of X where $\mathrm{H}_{X',x'} = \mathrm{H}_{X,x}$ and has the same property at each of these points as W at x. Given such a W, which exists in characteristic zero and has dimension $\leq \dim_x X$, Hironaka then constructs an *idealistic contact exponent* which in some sense measures the contact of X with W and how difficult it is to separate the strict transforms of X and W through permissible blowing-ups.

An idealistic exponent is an equivalence class of pairs (J, b) of an ideal on W and an integer b. If we call $\nu_x(I)$ the order of an ideal I with respect to the $m_{X,x}$-adic filtration, that is the largest k such that $I_x \subset m_{X,x}^k$, the singular set of an idealistic exponent represented by a pair (J, b) is the set $\{x \in X / \nu_x(J) \geq b\}$. It is independant of the choice of the representative and an idealistic exponent is non singular if its singular locus is empty. There is a notion of permissible blowing-up for idealistic exponents and a rule for their transformation by such blowing-ups. Given X and a W with maximal contact as above, an idealistic contact exponent on W has the following properties:

- Blowing ups $f\colon X' \to X$ that are permissible for X have their centers in W and are permissible for (J, b).
- At all points $x' \in f^{-1}(x)$ of the strict transform X' of X where $\mathrm{H}_{X',x'} = \mathrm{H}_{X,x}$, the transform (J', b') on the strict transform W' of W is again an idealistic contact exponent.
- To make the Hilbert-Samuel function of X drop by a sequence of permissible blowing-ups is equivalent to "resolving the singularities" of the idealistic exponent (J, b) by a sequence of permissible (for (J, b), but it is the same as for X) transformations on W.

In an essential way, the dimension of the problem has dropped to the dimension of W, and Hironaka shows that resolution of singularities of spaces embedded in lower dimension than X implies resolution for idealistic exponents on W, hence the existence of a sequence of permissible blowing-ups which makes the Hilbert-Samuel function of X drop, hence resolution for X.

In the special case where X is defined by one equation of the form

$$Z^n + a_1(x_1, \ldots, x_k)Z^{n-1} + \cdots + a_n(x_1, \ldots, x_k) = 0,$$

the space W is the non singular space $Z + \frac{a_1(x_1,\ldots,x_k)}{n} = 0$ corresponding to the Tschirnhaus transformation: the change of the variable Z to

$$Z' = Z + \frac{a_1(x_1, \ldots, x_k)}{n}$$

makes the term $a_1(x_1, \ldots, x_k)Z^{n-1}$ disappear. Assuming now that $a_1 = 0$, the idealistic contact exponent is $\big((a_i^{b/i})_{2\leq i\leq n}, b = n!\big)$. Note that if $k = 1$ and X is a plane curve, the rational number $\min_{1\leq i\leq n}\nu_x(a_i)/i$ is the slope of the first side of the Newton polygon of the curve. If the coordinates are chosen so that $Z = 0$ has maximal contact, the integral part of this number measures how many blowing-ups (of points) one must make in order to make the multiplicity (or, which is equivalent in this case, the Hilbert-Samuel function) of the strict transform of the curve drop. This is an important part of the philosophy of the general proof, and also of the attempts to prove the positive characteristic case (see [?], [?], [?], [?], [?]).

Note also that one sees immediately from the definition of the Tshirnhaus transformation why maximal contact might fail in positive characteristic, as indeed it does.

In the general case, the construction of local systems of coordinates and equations which allow one to describe locally the Samuel stratum of the point and of the points obtained after permissible blowing-ups and to follow the behavior of idealistic exponents under permissible blowing ups at all the points where the Hilbert-Samuel function does not decrease requires a lot of work. In particular, if X is defined locally at x in a non singular space T, in order for local equations to contain enough information to describe locally the Hilbert-Samuel stratum, their initial forms (or leading forms) for the $m_{T,x}$-adic filtration of $\mathcal{O}_{T,x}$ must generate the ideal of the tangent cone $C_{X,x}$ (and similarly for the normal cone $C_{X,Y}$ along a permissible center): this is the foundation of the notion of *standard basis*, closely related to what is called nowadays Gröbner basis or Macaulay basis.

The problem of resolution of singularities for complex analytic spaces, about which Hironaka explained his first ideas in a seminar in Ruuponsaari (Finland) in the summer of 1968, and which was studied in [?], [?], [?], [?], [?] (a survey of the method), and [?], presents entirely new difficulties.

- Starting from an ideal generated by holomorphic functions, in order to find useful generators for the ideal I of the idealistic exponent, one must first apply a generalized form of the Weierstrass preparation theorem, invented by Hironaka for this purpose, and which will appear again in the paragraph devoted to flattening. It is in fact rather a division theorem, dividing an element g in a convergent power series ring $\mathbf{C}\{z_1, \ldots, z_N\}$ by the generators $f_1, \ldots, f_\ell$ of an ideal I, that is, writing $g = h_1 f_1 + \cdots + h_\ell f_\ell + r$ in such a way that no monomial z^A of r is divisible by the initial monomials of the series f_i. It is closely related to a theorem of Grauert (1972).

- The centers of blowing-up which one can construct locally for the analytic topology do not necessarily extend as global centers of blowing-up as the centers constructed locally for the Zariski topology do in algebraic geometry. To overcome this difficulty, Hironaka invented the *Gardening of infinitely near singularities* explained in [?] and in the Madrid notes [?]. It is a sort of sheafification of local sequences of permissible blowing-ups which can be used to prove by induction on the dimension the existence of global centers of permissible blowing-ups.

More recently (see [?], [?]) Hironaka has taken up again the proof of resolution of singularities of excellent schemes in arbitrary characteristic. He has introduced a graded algebra which is finitely generated in any characteristic and is associated to sequences of permissible blowing-ups. In principle the finite-generatedness of this algebra and its very special properties with respect to differential operators and some restrictions of ambient space shall make up for the absence of maximal contact in positive characteristic. This is work in progress.

- Hironaka has also made a substantial contribution to the problem of resolution of singularities by Nash modification. The Nash modification $\nu\colon N(X) \to X$ of an algebraic variety or a reduced equidimensional complex analytic space is the minimal proper birational (or bimeromorphic) map such that $\nu^*\Omega^1_X$ has a locally free quotient of rank $\dim X$. The fibers $\nu^{-1}(x)$ are set-theoretically the limit directions at x of tangent spaces to X at non singular points tending to x. The problem of proving that after finitely many Nash modifications one obtains a non singular space was first stated by Semple in 1954, and essentially no progress was made until Hironaka proved in [?] by a valuative argument that if one considers the same problem for surfaces and with *normalized* Nash modification instead of Nash modification, one could reduce to the case where X has only rational singularities.

- Another important contribution of Hironaka to bimeromorphic geometry is the definition in [?] and [?] of the complex-analytic analogue of the Zariski-Riemann manifold, the *Voûte étoilée*. Instead of taking the projective limit of birational blowing-ups, one considers compositions of local blowing-ups. A local blowing-up $X' \to X$ of X is the blowing-up in an open subset U of X of a closed analytic subset of U, composed with the inclusion $U \subset X$. It is not proper, and compositions of local blowing-ups do not form a projective system. An element of the Voûte étoilée $\mathcal{E}_X$ is a maximal projective subsystem of the system of all compositions of local blowing-ups above X, satisfying in addition some technical condition. There is a canonical proper morphism $\mathcal{E}_X \to X$. This object renders the same services as the Zariski-Riemann manifold, and in particular is crucial in the proof of the local flattening theorem below.

- In [?] it is shown that on a non singular projective variety X of dimension n every algebraic cycle of dimension $d \leq \min(3, \frac{n-1}{2})$ is rationally equivalent to a linear combination of non singular subvarieties of X. The method is to desingularize, move into general position in the blown-up space, and push down. The first paragraph contains the theorem that a proper flat morphism of noetherian schemes with smooth fibers which has a fiber which is a projective space is a projective bundle.

- Under the pseudonym of Hej Iss'sa (Kobayashi Issa is a famous Japanese poet) Hironaka settled in 1966 (see [?]) a long standing problem in the theory of Riemann surfaces. Here we consider, for non compact connected Riemann surfaces, the ring $A(X)$ of holomorphic functions on X, and its field of fractions $F(X)$. A famous theorem of Chevalley-Kakutani and Bers showed that X and Y are conformally equivalent if and only if $A(X)$ and $A(Y)$ are isomorphic as $\mathbf{C}$-algebras. Iss'sa proved that X and Y are conformally equivalent if and only if the fields $F(X)$ and $F(Y)$ are $\mathbf{C}$-isomorphic. The argument is in fact valid in a much more general setting. The key fact is that a valuation of the field $M(X)$ of meromorphic functions on a complex variety X whose value group is "not too large" in a precise sense, and in particular not divisible, is necessarily ≥ 0 on the ring of holomorphic functions on X. The crucial case is the case $X = \mathbf{C}$.

• Finite determinacy, algebraicity.

The general result stated (but proved only in special cases) in [?] is that given a finite type flat map $\pi\colon X \to Y$ of schemes, which, over a neighborhood of a point $y_0 \in Y$, has reduced equidimensional fibers, all of the same dimension and is endowed with a section $\epsilon\colon Y \to X$, and given a closed subscheme Z of X containing $\epsilon(Y)$ and such that $\pi | X \setminus Z\colon X \setminus Z \to Y$ is formally smooth, then there exists an H-adic (t, r)-index for $(Y, y_0, \pi, X, \epsilon)$, where H is the ideal defining Z in X. The completion of X along Z is defined as the completion of the sheaf of algebras of X with respect to the H-adic topology. It is the inductive limit of the infinitesimal neighborhoods of Z in X, which are defined by the powers of H. Roughly speaking, the existence of a (t, r)-index means that, if the dimensions of the fibers are fixed, the completion of X along Z is, in a neighborhood of $\epsilon(y_0)$ and up to Y-isomorphism, determined by a finite infinitesimal neighborhood of Z in X. There is a subtlety in that, if we assume that we have two data $(Y, y_0, \pi, X, \epsilon, Z)$ and $(Y, y_0, \pi', X', \epsilon', Z')$ with the same fiber dimension, it is not an isomorphism of the t-th infinitesimal neighborhood Z_t of Z in X with an infinitesimal neighborhood Z'_t of Z' in X', which extends to the completions, but its restriction to some Z_{t-r}.

Hironaka writes at the end of [?] that one of his sources of inspiration was a result of Grauert stating that the formal completion along the exceptional divisor of a point-blowing-up is determined by a finite infinitesimal neighborhood. Generalizing this to exceptional divisors with normal crossings (see [?]) and applying resolution of singularities, one can hope to prove for example that any isolated singularity is analytically or formally determined by a finite infinitesimal neighborhood of the singular point, and in particular is algebraic, but there are difficulties. In any case, for complete intersections with isolated singularities, this result is proved in [**?**], and the determinacy of a small complex neighborhood by a finite infinitesimal neighborhood for an exceptional divisor A in a complex space X such that $X \setminus A$ is non singular is proved in [**?**].

The papers [**?**], [**?**], [**?**] all deal with the problem of comparing formal or analytic structures (embeddings, line bundles, etc.) with algebraic or rational ones.

• Flatness and Flattening.

- In his study of algebraicity conditions and the (t,r)-index ([**?**]) Hironaka used in the 1960's blowing-ups to transform an arbitrary subscheme X of a non singular ambient scheme, defined by an ideal I, into a local complete intersection, which amounts essentially to making the (strict transform of the) normal space corresponding to the coherent sheaf of $\mathcal{O}_X$-modules I/I^2 flat. This can be systematized in at least two ways in analytic geometry. The main idea is that given any analytic map $f\colon X \to S$ it can be made flat by base change in the following sense:
- If the map f is proper and S is reduced and countable at infinity, given any coherent sheaf on X, there exists a proper bimeromorphic map $\pi\colon S' \to S$ such that in the diagram

$$\begin{array}{ccc} X \times_S S' & \xrightarrow{\pi'} & X \\ f' \downarrow & & f \downarrow \\ S' & \xrightarrow{\pi} & S \end{array}$$

the sheaf $\pi'^* F$ modulo its f'-torsion is f'-flat.
- We no longer assume f to be proper and consider a point s of S and a compact subset $L \subset f^{-1}(s)$. Then there exists a finite family of maps $\pi_k\colon S'_k \to S$, each of which is a finite composition of local blowing-ups, such that the strict transform of f by each π_k is flat at every point mapped to a point of L. Here the strict transform is the closure of the part of the fiber product whose image by the projection to S'_k is not contained in the exceptional divisor of π_k. Moreover, the union of the images of the maps π_k is a neighborhood of s in S.

In both cases, the first difficulty is to define a *flattener* of the sheaf (or the map). The proof of flattening in the proper case is given in [**?**] (see also [**?**]). In the local case, the flattener is the largest closed subspace B of S *containing the point* s such that the restriction of f above B (in the sense of fiber product with B over S) is flat.

The first proof of local flattening in [**?**] used generalized Newton polygon techniques. There is in [**?**] a proof of the existence of the flattener using a parametrized version (with respect to S) of Hironaka's division theorem for the local equations of X in $S \times \mathbf{C}^N$.

In particular, if $A \to B = A\{t_1, \ldots, t_n\}/I$ is a flat morphism of complex-analytic algebras, Hironaka's division theorem provides, given a system of generators of I and after a suitable linear change of the coordinates $t_1, \ldots, t_n$, an explicit presentation of the A-module B as a finite direct sum of free A-modules of the form $A\{t_1, \ldots, t_j\}$. If B is not flat over A, the division theorem makes the obstructions to this presentation appear as series in $A\{t_1, \ldots, t_n\}$, whose coefficients in A provide equations for the flattener of the corresponding map.

It should be pointed out that flattening has important consequences in birational and bimeromorphic geometry. For example if $f\colon X \to Y$ is a proper bimeromorphic map of complex spaces, after base change by a blowing-up $Y' \to Y$ the induced map (strict transform) $f'\colon X' \to Y'$ is bimeromorphic and flat, so it is an isomorphism. Inverting it we see that f is dominated by a blowing-up, a form of Chow's lemma.

• **Real analytic and subanalytic Geometry; subanalytic sets.**

- In the 1970's Hironaka developed his theory of subanalytic sets. The existence and usefulness of such a theory had been foreseen by Thom and Łojasiewicz in the 1960's, and in 1968 Gabrielov proved the theorem of the complement: if we call a subset of $\mathbf{R}^n$ subanalytic when it is locally at every point of $\mathbf{R}^n$ a finite union of differences of images of semi-analytic sets by proper analytic maps, the complement of a subanalytic set is subanalytic. From there the theory could be developed, and indeed was later, from the viewpoint of projections of subsets of $\mathbf{R}^n$ to affine spaces of lower dimensions introduced by Łojasiewicz in the study of semi-analytic sets.

In [**?**], [**?**] and [**?**](which contains a complete exposition), Hironaka built the whole theory from a different viewpoint, as explained in the beginning. The main theorem is a resolution of singularities of subanalytic sets: Every subanalytic set is locally the finite union of images of spheres by real-analytic maps. This is a consequence of the *Rectilinearization theorem* which states that a subanalytic set can locally be transformed by finitely many finite sequences of local blowing-ups into a union of quadrants in affine space. One important step in the proof is the use of the local flattening theorem in complex analytic geometry to the complexifications of suitable real analytic maps defining the subanalytic set to prove that after suitable blowing-ups of the ambient space a subanalytic set becomes semi-analytic. Then one can apply resolution of singularities to the definining analytic functions.

In [**?**] Hironaka gives a proof of the triangulability of semi-algebraic sets (a known theorem, reputedly difficult) which is so streamlined and clear that it allows him, almost without change, to prove the triangulability of subanalytic sets, a new theorem. Finally by 1976 Hironaka had provided a complete analytic description of subanalytic sets and their finiteness properties, including Whitney stratifications, triangulability and the Łojasiewicz inequalities.

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