# Chapter 1
# The biLipschitz geometry of complex curves: an algebraic approach

Arturo Giles Flores, Otoniel N. Silva, Bernard Teissier

## Introduction

These are the lecture notes of the course given by Bernard Teissier during the second week of the " International School on Singularities and Lipschitz Geometry" which took place in Cuernavaca, Mexico from june 11 to june 22, 2018. The aim of the course was to explore the concept of "generic plane linear projection" of a complex analytic germ of curve in $\mathbf{C}^N$. The objects of our study will therefore be germs of curves $(X,0)\subset(\mathbf{C}^N,0)$, linear maps germs $\pi\colon(\mathbf{C}^N,0)\to(\mathbf{C}^2,0)$, and the images $(\pi(X),0)\subset(\mathbf{C}^2,0)$.

Intuitively, a projection $\pi$ is generic for $(X,0)$ if a small variation of $\pi$ does not change the "equisingularity type" (or embedded topological type) of the image $(\pi(X),0)$ in $(\mathbf{C}^2,0)$.

The main objective was to provide algebraic criteria for a projection to be generic and to use them to prove two results related to Lipschitz geometry:

(1) That all equisingular (topologically equivalent) germs of reduced plane curves are, up to analytic isomorphism, images of a single space curve $(X,0)\subset(\mathbf{C}^N,0)$ by generic linear projections $\pi\colon\mathbf{C}^N\to\mathbf{C}^2$, and that the restriction $\pi|(X,0)\colon(X,0)\to(\pi(X),0)$ to $(X,0)$ of such a generic projection is a biLipschitz map for the metrics induced by the hermitian metrics of their respective ambient spaces. In particular, all topologically equivalent germs of plane curves are biLipschitz equivalent.

(2) Given a reduced equidimensional germ of a complex space $(X,0)\hookrightarrow(\mathbf{C}^N,0)$, with dimension $d$, we consider a "general" projection $\pi:\mathbf{C}^N\to\mathbf{C}^2$ and the *polar curve on* $X$ associated to the projection $\pi$. It is the closure in $X$ of the critical locus of the restriction of $\pi$ to the smooth part of $X$. If it is not empty, it is a curve usually denoted by $P_{d-1}(X,\pi)$ which plays an important role in the study of the Lipschitz geometry of $X$. We can consider $\pi$ as defining a plane projection of the space curve $(P_{d-1}(X,\pi),0)$ which varies with $\pi$. The result is that if the projection $\pi$ is sufficiently general, then it is a generic plane projection for the curve $(P_{d-1}(X,\pi),0)\subset(\mathbf{C}^N,0)$.

The course assumed a certain familiarity with algebraic or complex analytic geometry, such as the definition of a complex analytic space $X$, the fact that its local algebras of functions are analytic algebras, that is, quotients of rings of convergent power series with complex coefficients, that the singular locus Sing$X$ consisting of points where the local algebra is not isomorphic to a rings of convergent power series, is a closed analytic subspace, etc.

### 1.0.1 What is a germ of complex analytic curve?

A complex analytic curve[1] $X$ may be locally regarded as a family of points in an open subset $U$ of the complex affine space $\mathbf{C}^N$ which is the union of finitely many sets of points depending analytically on one complex parameter. It can also be defined as the zero set of a finite number of holomorphic functions $f_1, \ldots, f_s$ on $U$ satisfying certain algebraic conditions:

$$X = \{z \in U \mid f_1(z) = \cdots = f_s(z) = 0\}.$$

A germ of curve $(X, 0) \subset (\mathbf{C}^N, 0)$ at a point which we take to be the origin is an equivalence class of curves in open neighborhoods of the origin. Two such objects defined respectively in $U$ and $U'$ are equivalent if their restrictions to a third neighborhood of the origin $U'' \subset U \cap U'$ coincide. Of course when we speak of germs we think of representatives in some “sufficiently small" neighborhood of the origin. Because of analyticity, to give a germ is equivalent to giving the convergent power series of $f_1, \ldots, f_s$ around the origin with respect to some coordinate system.

This allows us to associate to the germ $(X, 0) \subset (\mathbf{C}^N, 0)$ the analytic algebra of germs of holomorphic functions on $(X, 0)$:

$$\mathcal{O}_{X,0} := \mathbf{C}\{z_1, \ldots, z_N\}/\langle f_1, \ldots, f_s\rangle$$

where $\mathbf{C}\{z_1, \ldots, z_N\}$ denotes the ring of convergent power series. In these notes we will only be interested in reduced germs, meaning that the ideal $J := \langle f_1, \ldots, f_s\rangle$ is radical and $\mathcal{O}_{X,0}$ is a reduced analytic algebra of pure dimension 1.

In the case of plane curves $(N = 2)$ the ideal $I = \langle f\rangle\mathbf{C}\{x, y\}$ is principal and $f$ is square free, which means that $f$ has a factorization of the form $f = f_1 \cdots f_r$, where each $f_i$ is irreducible in $\mathbf{C}\{x, y\}$ and they are all different. The point is that the $f_i$'s correspond to germs $(X_i, 0) \subset (\mathbf{C}, 0)$ of analytically irreducible curves called the *branches* of the curve.

$$(X, 0) = \bigcup_{i=1}^{r} (X_i, 0)$$

[1] For more details on what follows in this section, we refer the reader to [Tei07].

For arbitrary $N$, the branches $(X_i, 0)$ correspond to the prime ideals appearing in the primary decomposition of the ideal $(0)$ in $\mathcal{O}_{X,0}$

$$(0) = P_1 \cap \ldots \cap P_r, \text{ where each } P_i \text{ is a minimal prime in } \mathcal{O}_{X,0}$$

A germ of curve $(X, 0) \subset (\mathbf{C}^N, 0)$ may also be described parametrically by $r$ sets of power series

$$\varphi_1^i(t_i), \ldots, \varphi_N^i(t_i) \in \mathbf{C}\{t_i\},\ 1 \leqslant i \leqslant r$$

where again $r$ is the number of branches. For each $i$, $z_k = \varphi_k^i(t_i),\ 1 \leqslant k \leqslant N$ defines a germ of map $(\mathbf{D}_i, 0) \longrightarrow (\mathbf{C}^N, 0)$ where $\mathbf{D}_i$ is a disk in $\mathbf{C}$. Together these $r$ $n$-uples of series correspond to a multigerm of map

$$\varphi : \bigsqcup_{i=1}^{r} (\mathbf{D}_i, 0) \longrightarrow (\mathbf{C}^N, 0);\ z_k = \varphi_k^i(t_i) \tag{1.1}$$

The connexion between these two definitions goes back to Newton, who showed that an equation $f(x, y) = 0$, with $f(0, 0) = 0$ has solutions $y(x)$ which are power series in $x$ with *rational exponents with bounded denominators* and coefficients in the algebraic closure of the smallest field containing the coefficients of $f(x, y)$. For Newton $f(x, y)$ is a polynomial with real coefficients, but the method works for series over any field $k$ of characteristic zero. Note that if $\frac{\partial f(x,y)}{\partial y}$ does not vanish at $(0, 0)$ the implicit function Theorem gives us a power series $y(x)$ with integers as exponents. In the general case, such a series $y(x) = \sum_{i \in \mathbf{N}} a_i x^{\frac{i}{n}}$ gives rise to a parametrization

$$x = t^n,\ \ y = \sum_{i \in \mathbf{N}} a_i t^i.$$

of one of the branches of the curve over an algebraic extension of $k$.

### 1.0.2 Structuring a parametrization

Suppose that we have an irreducible and reduced germ of curve in $(\mathbf{C}^N, 0)$, given by $z_k = \phi_k(t) \in \mathbf{C}\{t\},\ k = 1, \ldots, N$. For simplicity we shall write $z_k = \phi_k(t) = \sum_i a_k^{(i)} t^i$. We assume that the group generated by the exponents is $\mathbf{Z}$, which means that they are coprime. Let $n$ be the smallest exponent appearing in all the series $\phi_k(t)$; up to reindexing the variables $z_i$ we may assume that it is the order of $\phi_1(t)$, so that we may write $\phi_1(t) = a_n^{(i)} t^n (1 + \psi(t))$ with $\psi(0) = 0$. By making a homothetic change on the variable $z_1$ we may assume that $a_n^{(i)} = 1$. Since we are in characteristic zero, we may extract an $n$-th root of the unit $1 + \psi(t)$ so that $1 + \psi(t) = u(t)^n$ where $u(t)$ is again invertible in $\mathbf{C}\{t\}$. Now we make the change of parameter $t' = tu(t)$ so that $\phi_1(t') = t'^n$. Now by making a linear change of the form $z_i - a_i z_1$ on the coordinates $z_2, \ldots, z_N$ we may assume that $z_1$ is the only variable where the lowest exponent $n$ appears. Geometrically this means that our curve is tangent to the $z_1$-axis

at the origin: its set-theoretic tangent cone is the $z_1$-axis. Similarly, by making now a non linear change of coordinates of the form $z_i - \sum a_k^{(i)} z_1^k$ we may assume that the first exponent appearing in each $\phi_k(t')$ is not divisible by $n$. This is geometrically more subtle and corresponds to Hironaka's *maximal contact*. Since $t'$ is now our uniformizing parameter, we call it $t$ henceforth.
Let us now compare $z_1 = t^n$ with one of the other coordinates, which we may write (up to a homothetic change of variables) $z_i = \phi_i(t) = t^{b_i} + \cdots$. It may be that the exponents appearing in $\phi_i(t)$ and $n$ are not coprime. As we shall see below it means that the projection of our curve to the $(z_1, z_i)$-plane is not reduced. If that is the case, we may begin by dividing all the exponents by their greatest common divisor. The interesting case is therefore that of two series expansions $t^n, \phi(t)$ with coprime exponents: we are in the case $N = 2$ of a plane branch to which we now turn.

**The case of a plane branch.** As we saw, after a change of coordinates and of uniformizing parameter, we can describe our plane branch by: $z_1 = t^n, z_2 = \phi(t) \in \mathbf{C}\{t\}$ where the smallest exponent of $t$ in $\phi(t)$ is not divisible by $n$. This smallest exponent is traditionally denoted by $\beta_1$. We take the g.c.d. of $n$ and $\beta_1$; set $e_1 = (n, \beta_1) < n$. If $e_1 = 1$, the series $\phi(t)$ is of the form $t^{\beta_1} + \sum_{k\geqslant 1} a_k t^{\beta_1+k}$. If $e_1 > 1$, since the exponents are coprime, there has to be a smallest exponent $\beta_2$ in the series $\phi(t)$ which is not divisible by $e_1$. Then we set $e_2 = (e_1, \beta_2) < e_1$, and we continue in this manner. Since $n > e_1 > e_2 > \cdots$ there exists an integer $g$ such that $e_g = (e_{g-1}, \beta_g) = 1$. Finally we have the following structure for $\phi(t)$: its expansion is decomposed into segments corresponding to the divisibility properties of the exponents.

$$z_2 = t^{\beta_1} + \sum_{k=1}^{s_1} a_{\beta_1+ke_1} t^{\beta_1+ke_1} + a_{\beta_2} t^{\beta_2} + \sum_{k=1}^{s_2} a_{\beta_2+ke_2} t^{\beta_2+ke_2} + \cdots + a_{\beta_j} t^{\beta_j} + \sum_{k=1}^{s_j} a_{\beta_j+ke_j} t^{\beta_j+ke_j} +$$

$$\cdots + a_{\beta_g} t^{\beta_g} + \sum_{k=1}^{\infty} a_{\beta_g+k} t^{\beta_g+k},$$

where all $a_{\beta_i}$ are $\neq 0$ and each sum has to stop before the g.c.d. of the exponents drops and only the last segment is possibly infinite. The set of integers $n, \beta_1, \beta_2, \ldots, \beta_g$, which is often also denoted by $\beta_0, \beta_1, \beta_2, \ldots, \beta_g$, is called the *Puiseux characteristic* of the branch. It determines and is determined by the *embedded topological type* of the branch (see [Za1, §7], [Za3, Theorem 2.1, pg. 983], [Lej73]). This means that if two germs of plane branches $(X_1, 0)$ and $(X_2, 0)$ have the same Puiseux characteristic there exists a homeomorphism $(U_1, 0) \to (U_2, 0)$ of neighborhoods of the origin mapping the representative $X_1 \subset U_1$ to $X_2 \subset U_2$, and conversely. The two germs are also said to be *equisingular*. We shall meet this Puiseux characteristic again after Example 1.4.25 below, where we shall see that it determines not only the topology but also the biLipschitz geometry of the branch.

After what we have seen, the expansion above can be reinterpreted as a Newton expansion in terms of $t = z_1^{\frac{1}{n}}$, but here we have to choose a $n$-th root of $z_1$. The algebraic interpretation is that $\phi(z_1^{\frac{1}{n}}) \in \mathbf{C}\{\{z_1\}\}\{z_1^{\frac{1}{n}}\}$ determines a cyclic extension

of the field $\mathbf{C}\{\{z_1\}\}$ of meromorphic functions in $z_1$ with Galois group equal to the group $\mu_n$ of $n$-th roots of 1. The $n$ series $\phi(\omega z_1^{\frac{1}{n}}),\ \omega\in\mu_n$, are the roots of a unitary polynomial $\prod_{\omega\in\mu_n}(z_2-\phi(\omega z_1^{\frac{1}{n}}))\in\mathbf{C}\{z_1\}[z_2]$ whose vanishing is an equation for our germ of curve in the sense we shall see in the next section.
The structure of the series gives rise to a filtration of the Galois group:

$$\mu_n\supset\mu_{e_1}\supset\mu_{e_2}\supset\cdots\supset\mu_{e_g}=\{1\}$$

with the characteristic property that if we set $n=e_0$ and denote by $\nu_t$ the $t$-adic order of a series, then for $1\leqslant k\leqslant g$, we have that $\omega\in\mu_{e_{k-1}}\setminus\mu_{e_k}\iff\nu_t(\phi(\omega t)-\phi(t))=\beta_k$.

Let us now refine the structure according to [Za2, Chapters III, IV,V]. The parametrization of a branche by $t^n, y(t)$ as above presents its analytic algebra $\mathcal{O}_{X,0}$ as a subalgebra of $\mathbf{C}\{t\}$. The $t$-adic orders of the series in $t$ which are in $\mathcal{O}_{X,0}$ form a numerical semigroup $\Gamma\subset\mathbf{N}$ since one can multiply them and stay in $\mathcal{O}_{X,0}$. Since the exponents are coprime the complement of $\Gamma$ in $\mathbf{N}$ is finite (Dickson's Lemma) and the semigroup $\Gamma$ is finitely generated. The smallest element $c$ of $\mathbf{N}$ such that all integers $\geqslant c$ are in $\Gamma$ is called the *conductor* of the semigroup. It is not difficult to verify (see [Za2, Chapter III, lemma 1.1]) that if the order of a series $\xi(t)\in\mathcal{O}_{X,0}$ is $>\beta_1$, then $\xi(t)\in\langle x,y\rangle^2$, and therefore if the order $s$ of $\xi(t)$ is in $\Gamma$ we can make a change of coordinates $x'=x, y'=y-\xi(t)$ to eliminate a term in $t^s$ from the expansion of $y(t)$. Using this, and the fact that by definition any element of $\Gamma$ is the order of a series in $\mathcal{O}_{X,0}$, Zariski proved in [Za2, Chapter III, proposition 1.2]:

**Proposition 1.0.1** (Zariski) *1) Assume that $n>2$. Let $s_1,\dots,s_q$ be the integers of the set $\{\beta_1+1,\dots,c\}$ which do not belong to $\Gamma$. (One always has $c\geqslant\beta_1+1$ if $n>2$). The branch $(X,0)$ is analytically isomorphic to a branch given parametrically by:*

$$x'(t)=t^n,\quad y'(t)=t^{\beta_1}+\sum_{i=1}^{q}a'_{s_i}t^{s_i}.$$

*2) If $n=2$ then $\beta_1$ is odd since our germ is irreducible and the conductor is $\beta_1$; our curve is isomorphic to $x(t)=t^2, y(t)=t^{\beta_1}$.*

Zariski call this a *short representation*. There are more simplifications of the expansion of $y(t)$ one can make without changing the analytic type. See [Za2, Chapters III, IV,V].

The next thing we need to know is that the semigroup $\Gamma$ determines and is determined by the Puiseux characteristic of the branch: it is a complete invariant of the equisingularity class. See [Za2, Chap. II, §3]. In particular, in the short expansion, the coefficients of the $t^{\beta_i}$ are $\neq 0$.

With this description of branches, we are able to describe the *contact* of two branches, which plays a key role in the characterization of the topological (and biLipschitz) type of a reduced germ of plane curve.

We shall see below how, conversely, the image of a parametrization can be defined by equations.

The modern presentation of the parametrization of a curve goes through the normalization, which is the topic of the next section.

## 1.1 Normalization

The property of being normal has an algebraic aspect which has to do with integral extension of rings.

**Definition 1.1.1** Let $R \subset S$ be rings.

- The inclusion $R \subset S$ is called a finite extension if $S$ is a finitely generated $R-$module.
- An element $s \in S$ is called integral over $R$ if and only if it satisfies an equation

$$s^h + a_1 s^{h-1} + \cdots + a_{h-1} s + a_h = 0$$

  with all $a_i \in R$. The extension is called integral if every element $s \in S$ is integral over $R$. (Just as in field theory, if the extension $R \subset S$ is finite it is integral. See [De-P00, Lemma 1.5.2])
- The ring $R$ is said to be integrally closed in $S$ if every element in $S$ which is integral over $R$ already belongs to R.
- The ring $R$ is called normal if it is reduced and integrally closed in its total quotient ring $Q(R)$.

Suppose that $R$ is a reduced ring. Recall that the set of non-zero divisors of a ring $R$ is a multiplicatively closed set and the corresponding ring of fractions $Q(R)$ is called the total ring of fractions. It has the property that the canonical morphism $R \to Q(R)$ is injective.

The *normalization of* $R$ is defined as the set $\overline{R}$ of all elements of $Q(R)$ which are integral over $R$. It is a reduced ring, integrally closed in $Q(R)$ and whose total ring of fractions coincides with $Q(R)$. In particular, the normalization $\overline{R}$ is a normal ring. Moreover, for the rings appearing in analytic or algebraic geometry, the extension $R \subset \overline{R}$ is finite in the sense that $\overline{R}$ is a finitely generated $R$-module.[2]

So what about if we start with the analytic algebra $O_{X,0}$ of a germ of analytic space $(X,0) \subset (\mathbf{C}^N, 0)$? We will say that the germ $(X,0)$ is normal if $O_{X,0}$ is a normal ring.

- Unique factorization domains are normal ([De-P00, Thm 1.5.5]) so the ring of power series $\mathbf{C}\{z_1, \ldots, z_n\}$ and the corresponding smooth germ $(\mathbf{C}^N, 0)$ are normal.

[2] It is interesting to note that the term "integral" comes from algebraic number theory in the tradition of Dedekind and the definition of the ring of integers of an algebraic number field, while the term "normal" was used by Zariski (see [Za]) in the course of his studies in birational geometry and resolution of singularities to designate an algebraic variety which could *not* be presented as the image of a different one by a finite birational map. This is why the terms "integral closure in the total ring of fractions" and "normalization" are used in algebraic or analytic geometry as names for the algebraic and geometric aspects of the same operation.

- Noetherian normal local rings are integral domains ([De-P00, Thm 1.5.7]), so a normal germ $(X, 0)$ is irreducible.
- Suppose $(X, 0)$ is irreducible. Since $\mathcal{O}_{X,0}$ and its normalization have the same total ring of fractions, which in this case is a field, it follows from what we have just seen that $\overline{\mathcal{O}_{X,0}}$ is a local noetherian domain. Moreover, by [De-P00, Cor. 3.325] it is an analytic algebra and so we can associate to it a normal germ $(\overline{X}, 0)$. In particular we have:
$$\mathcal{O}_{\overline{X},0} = \overline{\mathcal{O}_{X,0}}$$
- Splitting of normalization ([De-P00, Thm. 1.5.20]) tells us that that if we have the irreducible decomposition
$$(X, 0) = (X_1, 0) \cup \ldots \cup (X_s, 0)$$
then the normalization $\overline{\mathcal{O}_{X,0}}$ is equal to a direct sum of analytic algebras which are the normalizations of the analytic algebras $\mathcal{O}_{X_i,0}$ corresponding to the irreducible components $(X_i, 0)$:
$$\overline{\mathcal{O}_{X,0}} = \bigoplus_{i=1}^{s} \overline{\mathcal{O}_{X_i,0}}$$
Note that this implies that $(X, 0)$ and $(\overline{X}, 0)$ have the same dimension.

A *multi-germ* of analytic spaces $(X, x)$ is a finite disjoint union:
$$(X, x) := (X_1, x_1) \sqcup (X_2, x_2) \sqcup \ldots \sqcup (X_r, x_r)$$
of germs of analytic spaces. The ring $\mathcal{O}_{X,x}$ by definition is equal to $\bigoplus_{i=1}^{r} \mathcal{O}_{X_i,x_i}$. The multigerm $(X, x)$ is called normal if $\mathcal{O}_{X,x}$ is a normal ring.

Let $(Y, y) = (Y_1, y_1) \sqcup \ldots \sqcup (Y_s, y_s)$ be another multi-germ. A map $\varphi : (X, x) \to (Y, y)$ of multi-germs is given by a system of maps
$$\varphi_i : (X_i, x_i) \to (Y_{\alpha(i)}, y_{\alpha(i)}), \qquad i \in \{1, \ldots, r\},\ \alpha(i) \in \{1, \ldots, s\}$$
Such a map $\varphi$ induces, and is induced by a $\mathbf{C}-$algebra map $\varphi^* : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$.

**Definition 1.1.2** Let $(X, x)$ be a germ of analytic space. A normalization of $(X, x)$ is a normal multi-germ $(\overline{X}, \overline{x})$ together with a finite, generically 1-1 map
$$n : (\overline{X}, \overline{x}) \to (X, x)$$

With this definition at hand, for any germ of analytic space $(X, 0)$ with irreducible decomposition
$$(X, 0) = (X_1, 0) \cup \ldots \cup (X_s, 0)$$
we can now obtain a normal multigerm
$$(\overline{X}, \overline{x}) = (\overline{X_1}, x_1) \sqcup \ldots \sqcup (\overline{X_s}, x_s)$$
with associated normal ring

$$\overline{O_{X,0}} = \bigoplus_{i=1}^{s} \overline{O_{X_i,0}} = \bigoplus_{i=1}^{s} O_{\overline{X_i},x_i}$$

And it is not hard to prove that the inclusion map $O_{X,0} \hookrightarrow \overline{O_{X,0}}$ induces a finite and generically 1-1 map, proving thus the existence of normalization ([De-P00, Thm 4.4.8]). Note that, geometrically, the normalization of a germ separates the irreducible components and normalizes each of them separately.

*Example 1.1.3* Let $(X,0) \subset (\mathbf{C}^2,0)$ be the germ of plane curve defined by $f(x,y) = x^2 - y^2$. It has two irreducible components $(X_1,0)$ and $(X_2,0)$ with associated analytic algebras

$$O_{X_1,0} = \mathbf{C}\{x,y\}/\langle x-y\rangle \qquad O_{X_2,0} = \mathbf{C}\{x,y\}/\langle x+y\rangle$$

These two germs are smooth, in particular they are normal and we have:

$$\begin{aligned} O_{X,0} = \frac{\mathbf{C}\{x,y\}}{\langle x^2-y^2\rangle} &\longrightarrow \frac{\mathbf{C}\{x,y\}}{\langle x-y\rangle} \bigoplus \frac{\mathbf{C}\{x,y\}}{\langle x+y\rangle} = \overline{O_{X,0}} \\ f &\longmapsto (f + \langle x-y\rangle, f + \langle x+y\rangle) \end{aligned}$$

Since the germs are smooth and of dimension 1, their analytic algebras are isomorphic to the ring of convergent power series $\mathbf{C}\{t\}$:

$$\mathbf{C}\{x,y\}/\langle x-y\rangle \to \mathbf{C}\{t\} \qquad x \mapsto t,\ y \mapsto t$$

$$\mathbf{C}\{x,y\}/\langle x+y\rangle \to \mathbf{C}\{u\} \qquad x \mapsto u,\ y \mapsto -u$$

This means that the resulting normalization map

$$n : (\mathbf{C},0) \sqcup (\mathbf{C},0) \to (X,0)$$

is the parametrization of each of the branches $t_1 \mapsto (t,t)$ and $t_2 \mapsto (u,-u)$.

It is useful to consider a function-theoretic interpretation of normal spaces. A general result tells us that in a smooth germ $(\mathbf{C}^d,0)$ if you have a meromorphic function which is (locally) bounded then it is actually holomorphic (See for example [Gr-F02, IV.4]). The algebraic version is that a locally bounded meromorphic function $h$ satisfies an integral dependence relation of the form:

$$h^m + c_1 h^{m-1} + \cdots + c_m = 0; \qquad c_j \in O_n := \mathbf{C}\{z_1,\ldots,z_n\}$$

and since $O_n$ is normal then $h \in O_n$.
Now there are many more analytic spaces for which $O_{X,x}$ is normal than just the non singular ones.

**Definition 1.1.4** Proposition For a reduced germ of analytic space $(X,x)$ we call a function $f : X\backslash \operatorname{Sing} X \to \mathbf{C}$ *weakly holomorphic* on $X$ at $x \in X$ if :

- $f$ is holomorphic on $X\backslash \operatorname{Sing} X$ in a neighborhood of $x$.

- $f$ is (locally) bounded near $x$.

A function is weakly holomorphic on $X$ if it is so at every point.

The key point is proving that the germs at $x \in X$ of weakly holomorphic functions on $X$ form a ring which is canonically isomorphic to the normalization of $\mathcal{O}_{X,x}$. That is, $f$ is weakly holomorphic on $X$ if and only if it is meromorphic and satisfies an integral dependence relation. This gives us the following characterization:

**Theorem 1.1.5** [De-P00, Thm 4.4.15]

1) *Let $(X, x)$ be a germ of reduced analytic space. Then a function $f$ is weakly holomorphic on X if and only if $f$ is in the integral closure of $\mathcal{O}_{X,x}$ in its ring of quotients.*
2) *The integral closure of $\mathcal{O}_{X,x}$ in its ring of quotients is a direct sum of analytic algebras.*
3) *The reduced germ $(X, x)$ is normal if and only if every weakly holomorphic function germ can be extended to a holomorphic function.*

*Remark 1.1.6* Since this fact is fundamental for what follows, here is an idea of why boundedness and polynomial equation are related: The roots of a polynomial are bounded in terms of its coefficients, so a solution of a polynomial equation with holomorphic coefficients is bounded because holomorphic functions are. In the other direction, let $h = \frac{f}{g}$, with $f, g \in m_{(X,0)}$ be our meromorphic function, let $(Y, 0) \subset (X, 0)$ be the subset defined by the ideal $\langle f, g\rangle \mathcal{O}_{(X,0)}$, and consider the analytic subspace $X'$ of $X \times \mathbf{P}^1(\mathbf{C})$ which is the closure of the graph of the map $X \backslash Y \to \mathbf{P}^1(\mathbf{C})$ defined by $x \mapsto (f(x) : g(x)) \in \mathbf{P}^1(\mathbf{C})$. It is contained in the hypersurface of $X \times \mathbf{P}^1(\mathbf{C})$ defined by $T_2 f(x) - T_1 g(x) = 0$ where $(T_1 : T_2)$ are projective coordinates on $\mathbf{P}^1$. The first projection induces a holomorphic map $e \colon X' \to X$ (we are blowing-up the ideal $\langle f, g\rangle$). The fiber over 0 is a complex analytic subspace of $\mathbf{P}^1(\mathbf{C})$ and therefore is either $\mathbf{P}^1(\mathbf{C})$ or a finite subset of it. If our meromorphic function is bounded, the point $(1 : 0) \in \mathbf{P}^1(\mathbf{C})$ is not in the fiber, so that by the Weierstrass preparation Theorem (see Theorem 1.1.9 below), for a small enough representative $X$ of the germ $(X, 0)$ the map $X' \to X$ is finite and $X'$ has to be a hypersurface in $X \times \mathbf{C}$: its equation is our integral dependence relation.

*Example 1.1.7* For the germ $(X, 0) \subset (\mathbf{C}^2, 0)$ defined by $xy = 0$ we have

$$\overline{\mathcal{O}_{X,0}} = \mathbf{C}\{x\} \oplus \mathbf{C}\{y\}$$

the function $f = (1, 0)$ meaning it is the constant function 1 on the $x$ axis and the constant function 0 on the $y$ axis is holomorphic on $X \backslash \mathrm{Sing}\, X = X \backslash \{0\}$ and is certainly bounded so it is weakly holomorphic. Note that it can not be continuously extended to $(X, 0)$. As a meromorphic function it can be written as

$$f(x, y) = \frac{x}{x + y}$$

Let us wrap up this discussion on normal spaces and normalization by stating several important properties of which you can find detailed expositions in [Loj91], [G-L-S07] and [Kau83].

1. If $X$ is reduced, the *non normal locus* is the set of points $x \in X$ where the local algebra $\mathcal{O}_{X,x}$ is not normal; it is the complement of the *normal locus* and is a closed analytic subspace contained in the singular locus Sing$X$ of $X$. It is defined by the *conductor sheaf* which is the annihilator of the coherent $\mathcal{O}_X$-module $\overline{\mathcal{O}_X}/\mathcal{O}_X$ and thus a coherent sheaf of ideals.
2. If $T$ is a normal space and $X$ is reduced then any map $T \to X$ which does not map any irreducible component of $T$ to the non-normal locus of $X$ factors uniquely through the normalization $n : \overline{X} \to X$.
3. If $X$ is normal then $\dim \mathrm{Sing}(X) \leqslant \dim X - 2$ (Singular locus of codimension at least 2).
4. If $X$ is normal, the polar locus of a meromorphic function is either of codimension 1 or empty.

Going back to the curve case, a classical result of commutative algebra ([De-P00, Thm 4.4.9]) states that a Noetherian local ring of dimension one is normal if and only if it is regular. This implies that if $(X, 0) = \bigcup_{i=1}^{r}(X_i, 0) \subset (\mathbf{C}^N, 0)$ is a germ of analytic curve with $r$ branches then the normal ring $\overline{\mathcal{O}_{X,0}}$ is isomorphic to a direct sum of $r$ copies of $\mathbf{C}\{t\}$ and the corresponding normalization map is equal to the parametrization of each branch, thus recovering the description in (1.1).
For plane curves, this result can also be seen using algebraic field extensions, but first we need a couple of definitions and the Weierstrass preparation Theorem. A convergent power series $f \in \mathbf{C}\{z_1, \dots, z_N\}$ is called regular of order $b$ in $z_N$ if the power series $f(0, \dots, 0, z_N)$ in the variable $z_N$ has a zero of order $b$. A simple calculation shows that if $f$ is of order $b$ in the sense that $f \in \langle z_1, \dots, z_N\rangle^b \setminus \langle z_1, \dots, z_N\rangle^{b+1}$, then after a general linear change of coordinates, $f$ is regular of order $b$ in $z_N$ (see [De-P00, Lemma 3.2.2]). Geometrically this means that if we consider the germ of hypersurface $(X, 0) \subset (\mathbf{C}^{N-1} \times \mathbf{C}, 0)$ defined by $f$ and the first projection $p : X \to \mathbf{C}^{N-1}$, then for a small enough representative the fiber $p^{-1}(0)$ is the single point 0.

For curve singularities, there is a classical invariant which measures how far the singularity is from being normal, or non singular. It has several geometric interpretations, the classical one being "diminution of genus", and we shall see more about it below.

**Definition 1.1.8** Let $(X, 0)$ be a reduced curve singularity. Its $\delta$ invariant is $\delta = \dim_{\mathbf{C}} \frac{\overline{\mathcal{O}_{X,0}}}{\mathcal{O}_{X,0}}$.

This quotient is a finite dimensional vector space because it is the stalk of a coherent sheaf supported at the origin. For plane, and more generally Gorenstein, branches we have the equality $c = 2\delta$, where $c$ is the conductor defined before proposition 1.0.1. See [Za2, Chap. II, §1].

**Theorem 1.1.9** ***(Weierstrass Preparation Theorem)*** (see [De-P00, Thm 3.2.4])
*Let $f \in \mathbf{C}\{z_1, \ldots, z_N\}$ be regular of order $b$ in $z_N$. Then there exists a unique polynomial monic polynomial $P \in \mathbf{C}\{z_1, \ldots, z_{N-1}\}[z_N]$*

$$P(z_1, \ldots, z_N) = z_N^b + a_1(z_1, \ldots, z_{N-1})z_N^{b-1} + \cdots + a_N(z_1, \ldots, z_{N-1})$$

*with $a_i(0) = 0$, and a unit $u \in \mathbf{C}\{z_1, \ldots, z_N\}$ such that we have the equality of convergent power series*

$$f = uP$$

As a consequence of this result we deduce two important facts: if we choose adequate coordinates such that $f = uP$ then it is equivalent to seek solutions of $f(z_1, \ldots, z_N) = 0$ and of $P(z_1, \ldots, z_N) = 0$. As a geometric consequence of this we get that if we consider the first projection as before and $p^{-1}(0) = \{0\}$, then for any point $q = (q_1, \ldots, q_{N-1}) \in \mathbf{C}^{N-1}$ sufficiently close to the origin the points of the fiber $p^{-1}(q)$ correspond to the roots of the polynomial of degree $b$

$$P(q_1, \ldots, q_{N-1}, z_N) = z_N^b + a_1(q_1, \ldots, q_{N-1})z_N^{b-1} + \cdots + a_N(q_1, \ldots, q_{N-1})$$

and so all nearby fibers are also finite. More generally one uses this result to prove that if a complex analytic map $p\colon X' \to X$ is such that for some point $0 \in X$ we have that $p^{-1}(0)$ is a finite set, then there exists a neighborhood $U$ of 0 in $X$ such that the restricted map $p^{-1}(U) \to U$ is finite. See [De-P00, Thm 3.4.24]

Going back to the plane curve case, that is curves $(X, 0) \subset (\mathbf{C}^2, 0)$ defined by a convergent power series $f \in \mathbf{C}\{x, y\}$, or according to the Weierstrass preparation Theorem and possibly after a linear change of coordinates, by a polynomial $P \in \mathbf{C}\{x\}[y]$. Now from an algebraic point of view, consider the field of fractions $\mathbf{C}\{\{x\}\}$ of the integral domain $\mathbf{C}\{x\}$; the irreducible polynomial $y^n - x \in \mathbf{C}\{\{x\}\}[y]$ defines an algebraic extension of degree $n$ of $\mathbf{C}\{\{x\}\}$, denoted by $\mathbf{C}\{\{x^{\frac{1}{n}}\}\}$, which is a Galois extension with Galois group equal to the group $\mu_n$ of $n$-th roots of unity in $\mathbf{C}$. The action of $\mu_n$ is exactly the change in determination of $x^{\frac{1}{n}}$ determined by $x^{\frac{1}{n}} \mapsto \omega x^{\frac{1}{n}}$ for $\omega \in \mu_n$. A series of the form $y = \sum a_i x^{\frac{i}{n}}$ such that the greatest common divisor of $n$ and all the exponents $i$ which effectively appear is 1 gives $n$ different series as $\omega$ runs through $\mu_n$.

Suppose now that our polynomial $P$ is an irreducible element of $\mathbf{C}\{x\}[y]$ of degree $n$. Then the Newton polygon method (see for example [Tei07], [Che78], or [Br-K86, Section 8.3]) provides a series $y(x^{1/m}) \in \mathbf{C}\{x^{\frac{1}{n}}\}$ such that $P(x, y(x^{\frac{1}{n}})) = 0$ and we have the equality:

$$P(x, y) = \prod_{\omega \in \mu_n} \left(y - y(\omega x^{\frac{1}{n}})\right).$$

In particular we have that

$$\mathbf{C}\{\{x\}\}^* := \bigcup_{n \in \mathbf{N}} \mathbf{C}\{\{x^{\frac{1}{n}}\}\}$$

is an algebraically closed field (See [Wal78, IV.3] or [Che78, Thm 8.2.1]), and so every polynomial $P \in \mathbf{C}\{x\}[y]$ has all its roots in $\mathbf{C}\{\{x\}\}^*$. Finally, the relation with the parametrizations given by the normalization is the following, if

$$y(x^{\frac{1}{n}}) \in \mathbf{C}\{x^{\frac{1}{n}}\} \subset \mathbf{C}\{\{x\}\}^*$$

is a root of $P(x, y)$, then by taking $x = t^n$ we get the parametrization

$$t \mapsto (t^n, y(t)).$$

Let us finish this section by looking at plane projections from an algebraic perspective. For simplicity suppose $(X, 0) \subset (\mathbf{C}^N, 0)$ is a reduced and irreducible germ of complex analytic curve with $N \geqslant 3$. Let us write the associated analytic algebra

$$\mathcal{O}_{X,0} = \frac{\mathbf{C}\{z_1, \ldots, z_N\}}{I}$$

where $I$ is a prime ideal, and so $\mathcal{O}_{X,0}$ is an integral domain. If we choose a sufficiently general coordinate system (or if you prefer after a general linear coordinate change) the Noether normalization Theorem ([De-P00, corollary 3.3.19]) tells us that we have a finite ring extension $\mathbf{C}\{z_1\} \hookrightarrow \mathcal{O}_{X,0}$. This implies that the we have an algebraic field extension

$$\mathbf{C}\{\{z_1\}\} \subset \text{Quot}\,(\mathcal{O}_{X,0})$$

and by the primitive element Theorem there exists an element $f \in \mathcal{O}_{X,0}$ such that Quot $(\mathcal{O}_{X,0}) = \mathbf{C}\{\{z_1\}\}[f]$.

So if we denote by $\mathbf{C}\{z_1, f\}$ the analytic algebra obtained as the quotient of $\mathbf{C}\{x, y\}$ by the kernel of the map $\mathbf{C}\{x, y\} \to \mathcal{O}_{X,0}$ defined by $x \mapsto z_1 + J$, $y \mapsto f$ then we have finite ring extensions with the same field of fractions

$$\mathbf{C}\{z_1, f\} \hookrightarrow \mathcal{O}_{X,0} \hookrightarrow \mathbf{C}\{t\}$$

Now $\mathbf{C}\{z_1, f\}$ is the analytic algebra of a plane curve $(X_1, 0) \subset (\mathbf{C}^2, 0)$ and it has the same normalization as $\mathcal{O}_{X,0}$. We have used the primitive element Theorem as a substitute for the proof of the existence of a projection $\mathbf{C}^N \to \mathbf{C}^2$ sufficiently general for it to induce a " bimeromorphic" map $(X, 0) \to (X_1, 0)$. However the primitive element Theorem does not tell us the nature of the projection. That is the object of the next section.

## 1.2 Fitting Ideals - A good structure for the image of a finite map

In this section, following [Tei77], we will give the definitions of Fitting ideals, which we will use later to give a definition of the image, as a *complex analytic space*, of a finite map between complex analytic spaces.

Let $A$ be a ring, and let $M$ be an $A$-module of finite presentation, that is, there is an exact sequence, called a presentation of $M$:

$$A^q \xrightarrow{\Psi} A^p \longrightarrow M \longrightarrow 0$$

where $p, q \in \mathbf{N}$. For each integer $j$ we associate to $M$ the ideal $F_j(M)$ of $A$ generated by the $(p-j) \times (p-j)$ minors of the matrix (with entries in $A$) representing $\Psi$. Here we need the convention that if there are no $(p-j) \times (p-j)$ minors because $j$ is too large, i.e., $j \geqslant p$, then $F_j(M) = A$ (the empty determinant is equal to 1) and if, at the other extreme, $p - j > q$, set $F_j(M) = 0$ (the ideal generated by the empty set is 0).

A Theorem of Fitting (see [To72, Chap. I, §2], [Eis95, Chap. 20, §2]) asserts that the ideals $F_j(M)$ depend only on the $A$-module $M$ and not on the choice of a presentation. We call $F_j(M)$ the $j$-th Fitting ideal of $M$.

More generally, if $(X, \mathcal{O}_X)$ is a ringed space, and $\mathcal{M}$ a coherent sheaf of $\mathcal{O}_X$-modules, we can define a sheaf of ideals $\mathcal{F}_i(\mathcal{M})$ of $\mathcal{O}_X$, by defining $\mathcal{F}_i(\mathcal{M})$ locally as above, and then by uniqueness the ideals found locally patch up into a sheaf of ideals. Remark also that since $\mathcal{F}_i(\mathcal{M})$ is locally finitely generated, $\mathcal{F}_i(\mathcal{M})$ will be a coherent sheaf of ideals as soon as $\mathcal{O}_X$ is coherent, e.g. for a complex analytic space by Oka's Theorem.

Let now $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a map of complex analytic spaces. We would like to define the image of $f$ as a complex analytic *subspace* of $(Y, \mathcal{O}_Y)$. This is not always possible, and in particular if one hopes to get a closed complex subspace of $Y$ it is better to assume $f$ is proper, and here we will consider only the case where $f$ is *finite* (that is, proper with finite fibres).

The first sheaf of ideals that comes to mind as a candidate to define $f(X)$ is the sheaf of functions $g$ on $Y$ such that $g \circ f = 0$ on $X$, i.e., the annihilator sheaf of the sheaf of $\mathcal{O}_Y$-modules $f_*\mathcal{O}_X$:

$$\mathrm{Ann}_{\mathcal{O}_Y}(f_*(\mathcal{O}_X)) = \text{ sheaf}\{\text{functions } g \text{ on } Y \text{ such that } g \cdot f_*\mathcal{O}_X = 0\ \}$$

This is **not** a good choice because its formation does not commute with base extension, as we will show by an example below (Example 1.2.3).

The second option is the 0th Fitting ideal of $f_*\mathcal{O}_X$, which set theoretically also defines the image of $f$, since as a set the subspace of $Y$ defined by it is $\{y \in Y \mid \dim_{\mathbf{C}}(f_*\mathcal{O}_X) > 0\} = \{y \in Y \mid (f_*\mathcal{O}_X)_y \neq 0\}$.

Since both the formation of direct images and the formation of Fitting ideals commute with base change, this definition of the image will also have this property. So we set:

**Definition 1.2.1** Let $f : X \to Y$ be a finite morphism of complex analytic spaces. The image $\mathrm{im}(f)$ of $f$ is the subspace of $Y$ defined by the coherent sheaf of ideals

$\mathcal{F}_0(f_*\mathcal{O}_X)$. It is sometimes called the *Fitting image* of $f$ to distinguish it from the one defined by the annihilator.

**Proposition 1.2.2** *1. The formation of* $\mathrm{im}(f)$ *commutes with base change: Given a complex analytic map $\phi\colon T \to Y$, consider the map $f_T\colon X \times_Y T \to T$ obtained by base extension, where $X \times_Y T$ is the fiber product. Then* $\mathrm{im}(f_T) = \phi^{-1}(\mathrm{im}(f))$ *as analytic spaces.*
*2. We have the inclusion* $\mathcal{F}_0(f_*\mathcal{O}_X) \subseteq \mathrm{Ann}(f_*\mathcal{O}_X)$ *and the equality* $\sqrt{\mathcal{F}_0(f_*\mathcal{O}_X)} = \sqrt{\mathrm{Ann}(f_*\mathcal{O}_X)}$.

***Proof*** 1) Since $\mathcal{O}_X$ is a finitely generated $\mathcal{O}_Y$-module the $\mathcal{O}_T$-module $\mathcal{O}_{X\times_Y T}$ is equal to $\mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{O}_T$ and if $M$ is a finitely presented $A$-module as above and $A \to B$ is a map of algebras, then

$$B^q \xrightarrow{\Psi\otimes_A 1} B^p \longrightarrow M \otimes_A B \longrightarrow 0$$

is a presentation of $M \otimes_A B$ as a $B$-module and the matrix of $\Psi \otimes_A 1$ is the matrix of $\Psi$ so that $F_j(M \otimes_A B) = F_j(M).B$.
2) The inclusion follows directly from Cramer's rule and the equality from the definition of the Fitting ideal as defining the set of points where the cokernel of the second second arrow is not zero. ◻

*Example 1.2.3* Let $f : (\mathbf{C}, 0) \to (\mathbf{C}^2, 0)$ be given by $x = t^{2k}$, $y = t^{3k}$ for some integer $k$. The set-theoretic image of $f$ is the curve $y^2 - x^3 = 0$. However, we wish to obtain an ideal defining a space supported on that curve, but possibly with nilpotent functions. Let us compute $\mathcal{F}_0(f_*(\mathcal{O}_{\mathbf{C}})_0$ as the 0-th Fitting ideal of $\mathbf{C}\{t\}$ considered as $\mathbf{C}\{x, y\}$-module via the map of rings $\mathbf{C}\{x, y\} \to \mathbf{C}\{t\}$ sending $x$ to $t^{2k}$ and $y$ to $t^{3k}$. We must write a presentation of $\mathbf{C}\{t\}$ as $\mathbf{C}\{x, y\}$-module. Let $e_0 = 1$, $e_1 = t, \ldots, e_{2k-1} = t^{2k-1}$. It is easily seen that they form a system of generators of $\mathbf{C}\{t\}$ as $\mathbf{C}\{x, y\}$-module, and that between them we have the following $2k$ relations:

$$\begin{aligned}
xe_k - ye_0 &= 0, & x^2e_0 - ye_k &= 0\\
xe_{k+1} - ye_1 &= 0, & x^2e_1 - ye_{k+1} &= 0\\
&\vdots & &\vdots\\
xe_{2k-1} - ye_{k-1} &= 0, & x^2e_{k-1} - ye_{2k-1} &= 0
\end{aligned}$$

which are independent. Hence we have a sequence of $\mathbf{C}\{x, y\}$-modules:

$$0 \longrightarrow \bigoplus_{i=0}^{2k-1} \mathbf{C}\{x, y\}e_i \xrightarrow{\psi} \bigoplus_{i=0}^{2k-1} \mathbf{C}\{x, y\}e_i \xrightarrow{\varphi} \mathbf{C}\{t\} \longrightarrow 0$$

with $\varphi(e_i) = t^i$, and $\psi$ is given by the $2k \times 2k$ matrix

$$\psi = \begin{bmatrix} -y & 0 & \cdots & 0 & x & 0 & \cdots & 0 \\ 0 & -y & \cdots & 0 & 0 & x & \cdots & 0 \\ \vdots & & \ddots & \vdots & & & \ddots & 0 \\ 0 & 0 & \cdots & -y & 0 & 0 & \cdots & x \\ x^2 & 0 & \cdots & 0 & -y & 0 & \cdots & 0 \\ 0 & x^2 & \cdots & 0 & 0 & -y & \cdots & 0 \\ \vdots & & \ddots & \vdots & & & \ddots & 0 \\ 0 & 0 & \cdots & x^2 & 0 & 0 & \cdots & -y \end{bmatrix}$$

It is not hard to see that the sequence is exact, which means that the independent relations we have found must generate all relations between the $e_i$. Indeed, there is a general reason why $\mathbf{C}\{t\}$ must have a resolution of length 1 as $\mathbf{C}\{x, y\}$-module: the $\mathbf{C}\{x, y\}$-module $\mathbf{C}\{t\}$ is of homological dimension one (see [Mo-P89]) and therefore the module of relations between the $e_i$ is a free submodule of $\bigoplus_{i=0}^{2k-1} \mathbf{C}\{x, y\}$ and thus of rank $\leqslant 2k - 1$.

By permuting rows and columns of $\psi$ one checks that $det(\psi) = (y^2 - x^3)^k$ i.e., we have shown that

$$F_0(f_*\mathcal{O}_\mathbf{C})_0 = (y^2 - x^3)^k \mathbf{C}\{x, y\}$$

Let us now calculate $Ann_{\mathbf{C}\{x,y\}}\mathbf{C}\{t\}$; the annihilator is just the kernel of the map $\mathbf{C}\{x, y\} \to \mathbf{C}\{t\}$, which is the ideal generated by $(y^2 - x^3)$, certainly different from our Fitting ideal if $k > 1$.

Let us now make a base change by restricting our map over the $x$-axis, i.e., by the inclusion $\{y = 0\} \subset (\mathbf{C}^2, 0)$ or algebraically by $\mathbf{C}\{x, y\} \to \mathbf{C}\{x\}$ sending $y$ to 0. Then the annihilator of $\mathbf{C}\{t\} \bigotimes_{\mathbf{C}\{x,y\}} \mathbf{C}\{x\} = \mathbf{C}\{t\}/(t^{3k})$ viewed as $\mathbf{C}\{x\}$-module is $(x^2)\mathbf{C}\{x\}$ while the image in $\mathbf{C}\{x\}$ of $(y^2 - x^3)\mathbf{C}\{x, y\}$ is $(x^3)\mathbf{C}\{x\}$. This shows that the formation of the annihilator does not commute with base change.

### 1.2.1 Equations versus Parametrizations

As we said in subsection 1.0.1, a germ of curve $(X_0, 0)$, abstractly, is a germ of a purely 1-dimensional analytic space, hence it is described by an analytic algebra $\mathcal{O}_{X_0,0}$ of pure dimension 1. Geometrically, $(X_0, 0)$ can be effectively given in two ways:

*By equations:* By giving an ideal $I = \langle f_1, \ldots, f_m \rangle$ in $\mathbf{C}\{x_1, \ldots, x_N\}$ such that $\mathcal{O}_{X_0,0} \simeq \mathbf{C}\{x_1, \ldots, x_N\}/I$. Saying that $\mathcal{O}_{X_0,0}$ is purely one-dimensional means that the ideal $\langle 0 \rangle$ has a primary decomposition $\langle 0 \rangle = Q_1 \cap \ldots \cap Q_r$ where $\sqrt{Q_i} = P_i$ is a minimal prime ideal in $\mathcal{O}_{X_0,0}$, and $\dim(\mathcal{O}_{X_0,0}/I) = 1$.

*By a parametrization:* By giving ourselves a germ of finite map $p : \bigsqcup_{i=1}^{r} (\mathbf{C}, 0) \to (\mathbf{C}^N, 0)$.

Here one has to be very careful: except when $n = 2$, it is not true, even if $r = 1$ and $p$ is generically 1-to-1 that the image (given by the Fitting structure) of this mapping is a reduced curve: it will have "embedded components" concentrated at the singular points, as will be shown in Example 1.2.4. The analysis of this phenomenon is beyond the scope of these notes.

*Example 1.2.4* Consider the curve $(X_0, 0)$ parametrized by $n(t) = (t^4, t^6, t^7)$ which is a complete intersection (with the reduced structure) with ideal

$$\langle y^2 - x^3, z^2 - x^2 y\rangle \mathbf{C}\{x, y, z\}.$$

We have that $\mathbf{C}\{t\}$ is generated as a $\mathbf{C}\{x, y, z\}$-module by $e_0 = 1$, $e_1 = t$, $e_2 = t^2$ and $e_3 = t^3$ and it is not difficult to see that the relations are described by the following matrix

$$\Psi = \begin{bmatrix} y & 0 & -x & 0 \\ 0 & y & 0 & -x \\ -x^2 & 0 & y & 0 \\ 0 & -x^2 & 0 & y \\ z & 0 & 0 & -x \\ -x^2 & z & 0 & 0 \\ 0 & -x^2 & z & 0 \\ 0 & 0 & -x^2 & z \end{bmatrix}$$

that is, $\Psi$ is the matrix of a presentation

$$\mathbf{C}\{x, y, z\}^8 \xrightarrow{\Psi} \mathbf{C}\{x, y, z\}^4 \longrightarrow \mathbf{C}\{t\} \longrightarrow 0.$$

of the $\mathbf{C}\{x, y, z\}$-module $\mathbf{C}\{t\}$. Computing the $4 \times 4$ minors of $\Psi$ we find that:

$$F_0(\mathbf{C}\{t\}) =$$
$$\langle y^2 - x^3, z^2 - x^2 y\rangle \cap \langle z^2, xy^3, y^4, xy^2 z - x^4 z, y^3 z, x^4 y, x^3 y^2, x^3 yz, x^6, x^5 z\rangle \mathbf{C}\{x, y, z\}$$

where $\sqrt{\langle z^2, xy^3, y^4, xy^2 z - x^4 z, y^3 z, x^4 y, x^3 y^2, x^3 yz, x^6, x^5 z\rangle} = \langle x, y, z\rangle \mathbf{C}\{x, y, z\}$.

In general, given a morphism $\mathbf{C}\{x_1, \ldots, x_n\} \to \mathbf{C}\{t\}$ corresponding to a parametric (and primitive) representation of a curve, we can be certain that $F_0(\mathbf{C}\{t\})$ will define a curve, in the sense that $\mathbf{C}\{x_1, \ldots, x_n\}/F_0(\mathbf{C}\{t\})$ is purely 1-dimensional, only if $n = 2$ or if our germ of curve is non-singular.

Now, we will consider deformations of a curve. We will follow the presentation given in [Bu-G80]. The following results in this section are due to B. Teissier (see [Tei77]).

Let $(X_0, 0) \subset (\mathbf{C}^N, 0)$ be a germ of a reduced curve and $X_0 \subset \mathbf{B}_0$ a representative, where $\mathbf{B}_0 \subset \mathbf{C}^N$ is a small open ball with center 0. Let

$$\varphi_0 : \overline{X}_0 = \bigsqcup_{j=1}^{r} \mathbf{D}_j \to X_0 \subset \mathbf{B}_0,$$
$$\varphi_0(s) = (\varphi_{0,1}(s), \ldots, \varphi_{0,n}(s)),$$

be a representative of the normalization of $X_0$, where $\bigsqcup_{j=1}^{r} \mathbf{D}_j$ is the disjoint union of $r$ open discs centered at the origin in $\mathbf{C}$, such that for each $j$ the restriction $\phi_0|\mathbf{D}_j$ is a homeomorphism $(\mathbf{D}_j, 0) \to (X_0^j, 0)$, where $(X_0^j, 0)$ is the $j$-th branch of $(X_0, 0)$.

**Definition 1.2.5** Let $\mathbf{D} \subset \mathbf{C}^q$ be a small disc with center 0. A deformation of the normalization of $X_0$ is a holomorphic mapping

$$\varphi : \overline{X}_0 \times \mathbf{D} = \bigsqcup_{j=1}^{r} (\mathbf{D}_j \times \mathbf{D}) \to \mathbf{B}_0,$$

such that $\varphi(s, v) = \varphi_0(s) + v\psi(s, v)$, $s \in \overline{X}_0$, $v \in \mathbf{D}$.

Then for sufficiently small $\mathbf{D}_j$ and $\mathbf{D}$ we have that $\phi = (\varphi, v) : \overline{X}_0 \times \mathbf{D} \to \mathbf{B}_0 \times \mathbf{D}$ is a finite mapping and therefore

$$Y = \phi(\overline{X}_0 \times \mathbf{D}) \subset \mathbf{B}_0 \times \mathbf{D}$$

is a two dimensional analytic subset. Let

$$f : Y \to \mathbf{D}$$

be the projection on the second factor and set $Y_v = f^{-1}(v)$.

Now, let $I = \langle f_1, \ldots, f_k \rangle \subset O_N$ be the defining ideal of $X_0$, that is, $X_0$ is the set of points in $\mathbf{C}^N$ such that $f_i(z_1, \ldots, z_N) = 0$, for $i = 1, \ldots, k$. We can also consider a deformation[3] of the equations $f_1, \ldots, f_k$, given by $F_i(z_1, \ldots, z_N, v) = f_i(z_1, \ldots, z_N) + vg_i(z_1, \ldots, z_N, v)$, where $g_i \in \mathbf{C}\{z_1, \ldots, z_N, v\}$. Since $X_0$ is reduced, the space $Y^{'} = V(F_1, \ldots, F_k) \subset \mathbf{B}_0 \times \mathbf{D}$ is a reduced surface. Again, let

$$f^{'} : Y^{'} \to \mathbf{D}$$

be the projection on the second factor and set $Y_v^{'} = f^{'-1}(v)$.

Note that $Y$ is given as the image of $\phi$, so it is not necessarily reduced when we consider the Fitting ideal structure. Consider now the reduction map $\tau : Y_{red} \to Y$, where $Y_{red}$ denotes the space $Y$ with the reduced structure. Set $X_v = (f \circ \tau)^{-1}(v)$. It is a reduced curve in $\mathbf{B}_0$ which by the Weierstrass preparation Theorem has finitely many singularities which all tend to 0 when $v$ tends to 0, provided $\mathbf{D}$ and $\mathbf{B}_0$ are sufficiently small. Each of those singularities has a $\delta$ invariant (see definition 1.1.8) and we denote by $\delta(X_v)$ the sum of these invariants. So, a natural question is:

**Question:** Is deforming the normalization of $X_0$ the same as deforming the equations of $X_0$? More precisely, given deformations of the series parametrizing the branches of $X_0$ is there a corresponding deformation of the equations of $X_0$ such that $Y_{red} = Y^{'}$? and vice-versa?

[3] In general, a deformation of the equations is called an unfolding and does not produce a flat deformation of the special fiber, unless it is a complete intersection. See [G-L-S07, Chap. II, 1.2]. Here we tacitly assume that the deformation of the equations produces a flat family.

**Proposition 1.2.6** (Teissier, [Tei77, §3]) *(a) The projection $f : Y_{red} \to \mathbf{D}$ is flat, $X_0 \backslash \{0\}$ is reduced and $X_v$ is reduced for each $v \neq 0$.*

*(b) Given a deformation of the equations of $X_0$ there is a deformation of the normalization of $X_0$ such that $Y_{red} = Y'$ if and only if $\delta(X_v)$ is constant for all $v \in \mathbf{D}$.*

***Proof*** See [Tei77, Lemma 7.1.1 and Prop. 7.1.3] and [G-L-S07, Theorem 2.54]. □

*Example 1.2.7* Consider the of curve $X_0$ in $\mathbf{C}^3$ given by the equations $x = 0,\ z^2 - y^3 = 0$. The normalization of $X_0$ is given by

$$\varphi(t) = (0, t^2, t^3).$$

Consider the deformation $\Phi(v, t) = (vt, t^2, t^3, v)$. So, the reduced image $Y_{red}$ of $\Phi$ is given by the following equations:

$$x^2 - v^2 y = 0, \quad xy - vz = 0, \quad xz - vy^2 = 0, \quad z^2 - y^3 = 0.$$

Now, when we consider the projection $f : Y_{red} \to \mathbf{D}$, the fiber $f^{-1}(0) = X_0$ is given by

$$x^2 = 0, \quad xy = 0, \quad xz = 0, \quad z^2 - y^3 = 0.$$

Note that it is not reduced at the origin, hence there is no deformation of $x = z^2 - y^3 = 0$ such that $Y_{red} = Y'$. One can understand this as follows: while the special fiber of our family of curves has embedding dimension two, the general fiber has embedding dimension three. In an analytic family the embedding dimension of the fibers can only increase by specialization so that in our analytic family $f : Y_{red} \to$ d the ideal defining the special fiber has in its primary decomposition an infinitesimal embedded component with ideal $\langle x^2, y, z \rangle$ sticking out of the $x = 0$ plane, which makes the embedding dimension of $f^{-1}(0)$ equal to three as it must be. This fact was stressed also in [Tei77, §3, section 3.5].

*Remark 1.2.8* We note that one can use Mond-Pellikaan's algorithm in [Mo-P89] to find a presentation matrix of a finite analytic map germ $g : (X, 0) \to (\mathbf{C}^{d+1}, 0)$, where $(X, x)$ is a germ of Cohen-Macaulay analytic space of dimension $d$. For the computations one can use also the software Singular [D-G-P-S] and the implementation of Mond-Pellikaan's algorithm given by Hernandes, Miranda, and Peñafort-Sanchis in [H-M-P18]. At the web page of Miranda [Mir19] one can find a Singular library to compute presentation matrices based on the results of [H-M-P18].

## 1.3 General projections

For a reduced and equidimensional germ of complex analytic variety $(X, 0) \subset (\mathbf{C}^N, 0)$ Whitney gave 6 possible definitions of tangent vectors ([Whi65b]), the sets of which constitute tangent cones:

$$C_1(X,0) \subset C_2(X,0) \subset C_3(X,0) \subset C_4(X,0) \subset C_5(X,0) \subset C_6(X,0),$$

and when the germ $(X, 0)$ is smooth they all coincide with the tangent space $T_0X$.

What is usually known as the tangent cone $C_{X,0}$ is what Whitney defined as the cone $C_3(X, 0)$ and is constructed by taking limits of secants through the origin. This means that if we take a representative $(X, 0) \subset (\mathbf{C}^N, 0)$ then a vector $\mathbf{v} \in \mathbf{C}^N$ is in $C_3(X, 0)$ if there exists a sequence of points $\{p_i\} \subset X \backslash \{0\}$ tending to 0 and a sequence of complex numbers $\{\lambda_i\} \subset \mathbf{C}^*$ such that

$$\lambda_i p_i \to \mathbf{v}$$

Algebraically it is constructed by blowing up the point

$$e_0 : \mathrm{Bl}_0 X \to X$$

and the fiber over the origin is the projectivized tangent cone $e_0^{-1}(0) = \mathbf{P}C_3(X, 0)$. In particular it is a pure d-dimensional algebraic cone where $d$ is the dimension of $(X, 0)$.

If $(X, 0)$ is a curve then the cone $C_3(X, 0)$ is a finite number of lines, one for each branch of $X$. Of course different branches may have the same tangent.

**Definition 1.3.1** A linear projection $\pi : (\mathbf{C}^N, 0) \to (\mathbf{C}^M, 0)$ with kernel $D$ is called $C_3$-general (with respect to X) if it is transversal to the tangent cone. That is

$$D \cap C_3(X, 0) = \{0\}$$

Note that the condition implies $M \geqslant d$ and that restriction of a $C_3$-general projection to $X$

$$\pi|X : (X, 0) \to (\mathbf{C}^M, 0)$$

satisfies $\pi^{-1}(0) = \{0\}$ since otherwise the tangent cone to $\pi^{-1}(0)$, which is contained in $C_3(X, 0)$, would be contained in $D$. This finiteness is equivalent to it being finite (proper with finite fibers) by [De-P00, Thm 3.4.24]. Since $C_3(X, 0)$ is of dimension $d$ then almost all (an open dense set of) linear projections $\pi : (\mathbf{C}^N, 0) \to (\mathbf{C}^{d+1}, 0)$ are $C_3$-general. This tells us that $\pi(X) \subset \mathbf{C}^{d+1}$ is a hypersurface and by [Chi89, Cor. 8.2] we have that $C_3(\pi(X), 0) = \pi\,(C_3(X, 0))$. We leave it as an exercise for the reader to verify that this last equality is an equality of Fitting images. Hint: use the specialization spaces $\mathcal{X}$ and $\mathcal{Y}$ to the tangent cones for $X$ and $\mathbf{C}^{d+1}$ respectively (see [Gi-T18, §2, 2.4]) and the fact that the natural map $\mathcal{X} \to \mathcal{Y}$ is finite by Weierstrass

preparation because the genericity assumption is equivalent to the finiteness of the map $C_3(X,0) \to T_{\mathbf{C}^{d+1},0}$, and apply [Mo-P89, Prop. 1.6].
Moreover, a projection $\pi : (\mathbf{C}^N, 0) \to (\mathbf{C}^d, 0)$ is $C_3$-general for $(X,0)$ if and only if the map $C_3(X,0) \to C_3(\mathbf{C}^d, 0) = T_{\mathbf{C}^d,0}$ which it induces is finite, which is equivalent to $D \cap C_3(X,0) = \{0\}$. These $C_3$-general maps are all ramified analytic covers of $\mathbf{C}^d$ of degree equal to the multiplicity of $(X,0)$.
In the curve case (d=1) this guarantees the existence of linear projections with image a plane curve.

The cone $C_4(X,0)$ is constructed by taking limits of tangent vectors at smooth points. One can prove that it is equivalent to taking limits at 0 in the appropriate Grasmannian of tangent spaces at non singular points of $X$ and so it is determined by the fiber of the Semple-Nash modification of $X$. Of course there is an analogous definition of a $C_4$-general linear projection and they do have interesting equisingularity properties. However, since in the curve case the cones $C_3$ and $C_4$ coincide we will skip this part and ask the interested reader to look at [Chi89], [Stu72a] and [Stu72b].

The cone $C_5(X,0)$ is constructed by taking limits of secants. This means that if we take a representative $(X,0) \subset (\mathbf{C}^N, 0)$ then a vector $\mathbf{v} \in \mathbf{C}^N$ is in $C_5(X,0)$ if there exist sequences of pairs of distinct points $\{p_i\}, \{q_i\} \subset X \backslash \{0\}$ tending to 0 as $i \to \infty$ and a sequence of complex numbers $\{\lambda_i\} \subset \mathbf{C}^*$ such that

$$\lambda_i (p_i - q_i) \to \mathbf{v}$$

To prove that $C_5(X,0)$ is an algebraic cone and have a bound for its dimension, take a small representative $X \subset \mathbf{C}^n$ consider the (closed) diagonal embedding $\delta : X \hookrightarrow X \times X$ and blow up its image $\Delta$:

$$e_\Delta : \mathrm{Bl}_\Delta(X \times X) \to X \times X$$

If we choose coordinates $(z_1, \ldots, z_N, w_1, \ldots, w_N)$ of the ambient space $\mathbf{C}^{2N}$, then we can obtain the space $\mathrm{Bl}_\Delta(X \times X)$ as the closure of the graph of the secant map defined away from the diagonal $\Delta$ by:

$$\begin{aligned} X \times X \backslash \Delta &\longrightarrow \mathbf{P}^{N-1} \\ (z,w) &\longmapsto [z_1 - w_1 : \cdots : z_N - w_N] \end{aligned}$$

So we have $\mathrm{Bl}_\Delta(X \times X)$ as a closed subspace of the product $X \times X \times \mathbf{P}^{N-1}$, the map $e_\Delta$ is induced by the projection to $X \times X$, and the exceptional fiber is the divisor $D := e_\Delta^{-1}(\Delta) \subset \Delta \times \mathbf{P}^{N-1}$ which comes with a map $D \to \Delta$ such that for every point $(q,q) \in \Delta$ the fiber is the projective subvariety corresponding to the projectivization of the $C_5$-cone of $X$ at $q$, that is $\mathbf{P}C_5(X,q)$. This is roughly the way Whitney proved that the $C_5$-cone is an algebraic variety in [Whi65b, Th. 5.1]. Now $C_5(X)$ is the analytic space obtained by deprojectivization of the (fibers of) the divisor $D$ and $\psi$ corresponds to the pullback of $e_\Delta$ by $\delta$:

$$
\begin{tikzcd}
X \times \mathbf{C}^N \supset & C_5(X) \arrow[r, dotted] \arrow[d, "\psi"'] & \mathrm{Bl}_\Delta(X \times X) \arrow[d, "e_\Delta"] \\
& X \arrow[r, "\delta"'] & X \times X
\end{tikzcd}
$$

where the upper arrow is defined only outside of $X \times \{0\}$. Note that the dimension of $C_5(X)$ is $2d$, and the dimension of $\psi^{-1}(p) = C_5(X, p)$ for a smooth point $p \in X$ is equal to $d$ since in this case we have $C_5(X, p) = T_p X$. By the semicontinuity of the dimensions of the fibers of an analytic morphism, this implies that:

$$d \leqslant \dim C_5(X, 0) \leqslant 2d$$

**Definition 1.3.2** A linear projection $\pi : (\mathbf{C}^N, 0) \to (\mathbf{C}^M, 0)$ with kernel $D$ is called *generic* (or $C_5$-general) with respect to X if it is transversal to the cone $C_5(X, 0)$. That is

$$D \cap C_5(X, 0) = \{0\}$$

In other words, no limit at 0 of secants to $X$ is contained in $D$.
Note that a generic projection is in particular $C_3$-general and $C_4$-general.

**Proposition 1.3.3** *Let* $(X, 0) \subset (\mathbf{C}^N, 0)$ *be a reduced equidimensional germ of complex analytic variety of dimension* $d$ *and* $\pi : (\mathbf{C}^N, 0) \to (\mathbf{C}^M, 0)$ *a linear projection.*

*a) If* $\pi$ *is generic then the restriction to* $X$ *induces a homeomorphism with its image.*
*b)* $(X, 0)$ *is smooth if and only if dim* $C_5(X, 0) = d$

***Proof*** First of all note that the transversality to the cone $C_5(X, 0)$ implies that the restriction $\pi|X$ is injective for a small enough representative of $X$. But then the induced map $\pi|X \colon X \to \mathbf{C}^M$ is injective, continous and the map $X \to \pi(X)$ is open since $\pi$ is and so $\pi|X$ induces an homeomorphism of $X$ with its image $\pi(X)$. Now for $b$) sufficiency is clear since $(X, 0)$ smooth implies $C_5(X, 0) = T_0 X$ and so it is of dimension d. For necessity note that the dimension of $C_5(X, 0)$ equal to $d$ implies the existence of a generic linear projection to $\mathbf{C}^d$

$$\pi|X \colon (X, 0) \to (\mathbf{C}^d, 0)$$

By $a$) this gives us a homeomorphism between $(X, 0)$ and $(\mathbf{C}^d, 0)$. Note that $\pi$ is also $C_3$-general so it induces a ramified covering of degree equal to the multiplicity of $(X, 0)$, but the injectivity gives us multiplicity 1 and so $(X, 0)$ is smooth. □

(For more on this and more general results see [Stu72a],[Stu77] and [Chi89, Section 9.4])

An important thing to notice is that in the reducible case the cone $C_5(X, 0)$ contains but IS NOT EQUAL to the union of the $C_5$-cones of its irreducible components. For instance if $(X, 0)$ is a curve consisting of two smooth branches $X_1$ and $X_2$ then both cones $C_5(X_i, 0)$ are one-dimensional but since $(X, 0)$ is singular then by the previous result $C_5(X, 0)$ can not have dimension 1.

So now we have that if $(X, 0)$ is singular then $d + 1 \leqslant \dim C_5(X, 0) \leqslant 2d$, and in the curve case this just gives dim $C_5(X, 0) = 2$. This guarantees the existence of generic projections to $\mathbf{C}^2$.

**Corollary 1.3.4** *Let $(X, 0) \subset (\mathbf{C}^N, 0)$ be a germ of reduced analytic curve. Then almost all (an open dense set of) linear projections $\pi : (\mathbf{C}^N, 0) \to (\mathbf{C}^2, 0)$ are generic and its image $\pi(X) \subset \mathbf{C}^2$ is a plane curve homeomorphic to $X$.*

### 1.3.1 The case of dimension 1.

In the case of curves we have the following important results:

**Proposition 1.3.5** (see [B-G-G80, Prop IV.1])
*Let $(X, 0) \subset (\mathbf{C}^N, 0)$ be a germ of reduced analytic curve. If $(X, 0)$ is singular then the cone $C_5(X, 0)$ is a finite union of 2-planes each one of them containing at least one tangent line to $(X, 0)$.*

***Proof*** We will only give an idea of the proof.
By Proposition 1.3.3 the cone $C_5(X, 0)$ is two dimensional and by the blowup construction it has a finite number of irreducible components. So what one has to prove is that all the irreducible components are 2-planes. Again, by this blowup construction, any (direction of) line contained in $C_5(X, 0)$ can be picked off by lifting an arc

$$(\psi_1, \psi_2) : (\mathbf{C}, 0) \to (X \times X, (0, 0))$$

to $\mathrm{Bl}_\Delta(X \times X)$ like $(\psi_1(t), \psi_2(t), [\psi_1(t) - \psi_2(t)])$. Now each $\psi_i(t)$ is an arc $(\mathbf{C}, 0) \to (X, 0)$ and can be obtained using the parametrization of one of the branches of $(X, 0)$. Once you see this, what you have to do is consider the different cases and work out the calculations.

The first case is when $(X, 0) \subset (\mathbf{C}^N, 0)$ is irreducible of multiplicity $n$ so we have a parametrization of the form:

$$\varphi(t) = \left(t^n, \sum_{i>n} a_{2,i} t^i, \dots, \sum_{i>n} a_{n,i} t^i\right)$$

with the tangent line being the $z_1$-axis $[1 : 0 : \cdots : 0]$. For every $n$-th root of unity $\omega \neq 1$ the lifted arc

$$t \mapsto (\varphi(t), \varphi(\omega t), [\varphi(t) - \varphi(\omega t)]) \in X \times X \times \mathbf{P}^{N-1}$$

will define a limit line $\ell_\omega \in \mathbf{P}^{N-1}$ as $t \to 0$ and if you define $H_\omega$ as the 2-plane generated by the $z_1$-axis and the line in $\mathbf{C}^N$ corresponding to $\ell_\omega$ then you can prove that

$$C_5(X,0) = H_{\omega_1} \cup \ldots \cup H_{\omega_{n-1}}$$

by verifying that any line obtained by lifting an arc is contained in one of these 2-planes.

For the reducible case it is enough to consider two branches $(X,0) = (X_1,0) \cup (X_2,0)$. In this case you have that the $C_5$-cone of each irreducible component $(X_i,0)$ will be contained in $C_5(X,0)$ but you will have additional components that come from the configuration of these two branches. For instance if they have different tangent lines $\ell_1$ and $\ell_2$ then all you have to add is the plane $H_{12}$ generated by these two lines.i.e:

$$C_5(X,0) = C_5(X_1,0) \cup C_5(X_2,0) \cup H_{12}$$

When the two branches are tangent (have the same tangent line) then you have to play a game very similar to the irreducible case by reparametrizing your branches in such a way as to travel through them at the same "speed" and using roots of unity to find lines $\ell_\omega$ in the $C_5(X,0)$ that are different from the tangent line and these will give you the additional 2-planes. i.e.,:

$$C_5(X,0) = C_5(X_1,0) \cup C_5(X_2,0) \cup H_{\omega_1} \cup \ldots \cup H_{\omega_k}$$

**Proposition 1.3.6** (see [B-G-G80, Prop IV.2])
*Let $(X,0) \subset (\mathbf{C}^N,0)$ be a germ of reduced analytic curve, and let $\Omega \subset G(N-2,N)$ be the non-empty Zariski open set of the Grassmannian of $(N-2)$-planes of $\mathbf{C}^N$ which are transversal to $C_5(X,0)$. Then:*

*a) For $H \in \Omega$ the plane curve $(\pi_H(X),0)$ is reduced and of constant topological (equisingularity) type with Milnor number $\mu_0$.*
*b) If $H \notin \Omega$ then one of the following statements is verified:*

- *0 is not an isolated point of $H \cap X$.*
- *0 is an isolated point of $H \cap X$ but the curve $(\pi_H(X),0)$ is not reduced.*
- *0 is an isolated point of $H \cap X$, the curve $(\pi_H(X),0)$ is reduced but its Milnor number is greater than $\mu_0$.*

***Proof*** Let $W' \subset G(N-2,N)$ be the open subset of the the Grassmannian of $(N-2)$-planes of $\mathbf{C}^N$ defined by the condition that $H \in W'$ if and only if $0 \in \mathbf{C}^N$ is an isolated point of $H \cap X$. Let $W \subset \mathbf{C}^{2N}$ with coordinate system $(a_1,\ldots,a_N,b_1,\ldots,b_N)$ be the associated open subset, where $d = (a,b) \in W$ if and only if the linear forms

$$a_1 z_1 + \cdots + a_N z_N \quad \text{and} \quad b_1 z_1 + \cdots + b_N z_N$$

are linearly independent and the $N-2$ plane $H_d \subset \mathbf{C}^N$ they define is in $W'$. Let $\pi_d$ be the linear projection

$$\begin{aligned} \pi_d : \mathbf{C}^N &\longrightarrow \mathbf{C}^2 \\ (z_1,\ldots,z_N) &\mapsto (a_1 z_1 + \cdots + a_N z_N, b_1 z_1 + \cdots + b_N z_N) \end{aligned}$$

Note that for $d \in W$ the germ $\pi_d : (X, 0) \to (\mathbf{C}^2, 0)$ is finite, and if we denote by $(\pi_d(X), 0) \subset (\mathbf{C}^2, 0)$ the image germ with the Fitting structure then by [Mo-P89, Lemma 2.1] it is a (not necessarily reduced) plane curve.

We put all these projections in an analytic family by considering the map

$$\Pi : \mathbf{C}^N \times W \longrightarrow \mathbf{C}^2 \times W$$
$$(z_1, \dots, z_N, a, b) \mapsto (\pi_d(z_1, \dots, z_N), a, b)$$

Note that for every $d \in W$ the map germ

$$\Pi : (X \times W, (0, d)) \to \left(\mathbf{C}^2 \times W, (0, d)\right)$$

is finite. And since the analytic algebra $O_{X \times W,(0,d)}$ is Cohen-Macaulay again by [Mo-P89, Lemma 2.1] we have a germ of hypersurface $(\Pi(X\times, W), (0, d)) \subset \left(\mathbf{C}^2 \times W, (0, d)\right)$. By projecting to $W \subset \mathbf{C}^{2N}$ we obtain (by [G-L-S07, Thm B.8.11]) a flat map:

$$G : (\Pi(X\times, W), (0, d)) \to (W, d)$$

Since the Fitting structure commutes with base change we have that the germ $\left(G^{-1}(d), (0, d)\right)$ is isomorphic to $(\pi_d(X), 0)$, and so we have a flat deformation of $(\pi_d(X), 0)$ where all the fibers are plane curves.

Note that if $\varphi : (\mathbf{C}, 0) \to (\mathbf{C}^N, 0)$, $t \mapsto (\varphi_1(t), \dots, \varphi_N(t))$ is the normalization of a branch of $(X, 0)$ then the plane curve $(\pi_d(X), 0)$ is parametrized by:

$$t \mapsto (a_1\varphi_1(t) + \cdots + a_N\varphi_N(t), b_1\varphi_1(t) + \cdots + b_N\varphi_N(t))$$

and by varying $d$ we get that the deformation space of $G$ admits a parametrization in family.

**Proof of a):** When $H_d$ is transversal to $C_5(X, 0)$ then for every $d'$ in a small neighborhood of $d$ the $(n-2)$-plane $H_{d'}$ is also transversal to $C_5(X, 0)$ and all the corresponding projections $\pi_{d'}$ are therefore generic. By corollary 1.3.4 this tells us that $\pi_{d'} : X\backslash\{0\} \to G^{-1}(d')\backslash\{0\}$ is an analytic isomorphism for every $d'$ sufficiently close to $d$. This implies:

- All the curves in the family $G^{-1}(d')$ have the same number of branches as $X$.
- The parametrization in family is actually a normalization in family and by [Tei77, §3], see also [G-L-S07, II, Thm 2.56] the family is $\delta$ constant.

By the Milnor formula $\mu = 2\delta - r + 1$ the family $G : (\Pi(X\times, W), (0, d)) \to (W, d)$ is $\mu$-constant and so equisingular by [B-G-G80, Thm II.4].

**Proof of b):** For $H_d \in W\backslash\Omega$ we have that the map

$$\pi_d : (X, 0) \to (\mathbf{C}^2, 0)$$

is finite but if it is generically $k$ to 1 then by [Mo-P89, Prop. 3.1] the Fitting structure of $(\pi_d(X), 0)$ is not reduced.

When $\pi_d$ is generically $1-1$ then $(\pi_d(X), 0)$ is reduced but by assumption there is a line $\ell \subset H_d \cap C_5(X,0)$. Take a sequence of secants $\ell_k$ going through the points $x_k, y_k \in X\backslash\{0\}$ such that $\ell_k$ converges (in direction) to $\ell$, since $\Omega$ is Zariski open we can find a sequence $d_k$ tending to $d$ such that $H_{d_k} \in \Omega$ and it contains (the direction of) $\ell_k$. Note that $\pi_{d_k}(\ell_k) = q_k \neq 0$ and so the plane curve $(G^{-1}(d_k), q_k)$ is singular which implies that $\mu\left((\pi_d(X),0)\right) > \mu\left((\pi_{d_k}(X),0)\right)$. □

*Example 1.3.7* Let $(X,0) \subset (\mathbf{C}^3,0)$ the germ of irreducible curve parametrized by

$$t \mapsto (t^4, t^5, t^7)$$

then the tangent cone $C_3(X,0)$ is the $z_1$-axis.
By taking other arcs $t \mapsto (t^4, \omega t^5, \omega^3 t^7)$ were $\omega \in \mu_4\backslash\{1\}$ and taking the limit at $t \to 0$ of the difference $(0 : (1-\omega)t^5 : (1-\omega^3)t^7)$ we get the $z_2-$ axis as a limit of secants and we can deduce that the cone $C_5(X,0)$ is the $z_1z_2$-plane.
For $d = (1,0,0,0,1,0)$ the corresponding projection

$$\pi_{d_0}(z_1,z_2,z_3) = (z_1,z_2)$$

is $C_5$-general and its image $\pi_d(X,0) \subset (\mathbf{C}^2,0)$ is the reduced plane curve $y^4 - x^5 = 0$ with Milnor number $\mu = 12$.
On the other hand For $d_0 = (1,0,0,0,0,1)$ the corresponding projection

$$\pi_{d_0}(z_1,z_2,z_3) = (z_1,z_3)$$

is not $C_5$-general and its image $\pi_{d_0}(X,0) \subset (\mathbf{C}^2,0)$ is the reduced plane curve $y^4 - x^7 = 0$ with Milnor number $\mu = 18$.
By taking $d_\alpha = (1,0,0,0,-\alpha^2,1)$ we get a sequence of $C_5$-general projections $\pi_{d_\alpha}$ converging to $\pi_{d_0}$

$$\pi_{d_\alpha}(z_1,z_2,z_3) = (z_1, z_3 - \alpha^2 z_2)$$

Note that the plane curve $X_\alpha := \pi_{d_\alpha}(X)$ has a singular point in $(\alpha^4,0)$ coming from the image of the secant going through the points $(\alpha^4,\alpha^5,\alpha^7)$ and $(\alpha^4,-\alpha^5,-\alpha^7)$ in $X$. Moreover as $\alpha$ tends to 0 these secants $d_\alpha = [0:1:\alpha^2]$ converge to the $z_2$-axis $[0:1:0]$ in $\mathbf{P}^2$ which is precisely the intersection $H_{d_0} \cap C_5(X,0)$.

## 1.4 Main result

We have just seen that all ($C_5$-)generic plane projections of a reduced analytic curve are equisingular. Now our objective is to prove that all equisingular germs of reduced plane curves are generic projections of a single space curve. As we shall see, given a reduced plane curve $(X, 0) \subset (\mathbf{C}^2, 0)$ this space curve corresponds to the one dimensional analytic algebra which is the Lipschitz saturation $\mathcal{O}^s_{X,0}$ of $\mathcal{O}_{X,0}$ in the sense of [P-T69]. In doing so we will also give another reason why a) of Proposition 1.3.6 is true, since we shall see that a projection $\pi$ is generic for a space curve $(X, 0) \subset (\mathbf{C}^N, 0)$ if and only if it induces an isomorphism of the saturated algebras $\mathcal{O}^s_{X,0}$ and $\mathcal{O}^s_{\pi(X),0}$. In particular, two germs of reduced plane curves are equisingular (topologically equivalent) if and only if their saturations are *analytically* isomorphic. In order to define these saturations we need the theory of integral closure of ideals.

### 1.4.1 Integral closure of ideals

Our main references for this subsection are, [Lej-T08], [Lip82], [Tei74] and [Hu-S06].

**Definition 1.4.1** Let $I$ be an ideal in a ring $R$. An element $r \in R$ is said to be integral over $I$ if there exists an integer $h$ and elements $a_j \in I^j$, $j = 1, \ldots, h$, such that

$$r^h + a_1 r^{h-1} + a_2 r^{h-2} + \cdots + a_{h-1} r + a_h = 0$$

The set of all elements of $R$ that are integral over $I$ is an ideal called the **integral closure of** $I$ and denoted by $\overline{I}$. We say that $I$ is integrally closed if $I = \overline{I}$. If $I \subset J$ are ideals we say that $J$ is integral over $I$ if $J \subset \overline{I}$.

*Remark 1.4.2* The following properties are easily verified:

1. $I \subset \overline{I}$. For each $r \in I$ choose $n = 1$ and $a_1 = -r$.
2. If $I \subset J$ are ideals then $\overline{I} \subset \overline{J}$ since an integral dependence equation for $r$ over $I$ is also an an integral dependence equation for $r$ over $J$.
3. $\overline{I} \subset \sqrt{I}$ since the integral dependence equation implies $r^n \in \langle a_1, \ldots, a_n \rangle \subset I$.
4. Radical ideals are integrally closed.
5. If $\varphi : R \to S$ is a ring morphism and $I \subset S$ is an integrally closed ideal of S then $\varphi^{-1}(I)$ is an integrally closed ideal of $R$.

A related concept is that of *reduction*: For ideals $J \subset I \subset R$ we say that $J$ is a **reduction of** $I$ if there exists a non-negative integer $n$ such that $I^{n+1} = JI^n$. This implies that $\overline{I} = \overline{J}$. We can express integral dependence using equalities of ideals and modules.

**Proposition 1.4.3** (see [Lej-T08, Chapter 1], [Hu-S06, Prop 1.1.7, Cor. 1.1.8 & Cor. 1.2.2]) *For any element $r \in R$ and ideal $I \subset R$. The following are equivalent:*

*a)* $r \in \overline{I}$.
*b)* *There exists an integer* $k$ *such that* $(I + r)^k = I\,(I + r)^{k-1}$.
*c)* $I$ *is a reduction of* $I + < r >$.
*d)* *There exists a finitely generated* $R-$*module* $M$ *such that* $rM \subset IM$ *and if there exists* $a \in R$ *such that* $aM = 0$*, then there exists an integer* $\ell$ *such that* $ar^\ell = 0$. .

A very important corollary of this Proposition is that $\overline{I} \subset R$ is an integrally closed ideal of $R$ and you can find a complete proof of this fact in [Hu-S06, Cor. 1.3.1].

We have that $I \subset \overline{I} \subset \sqrt{I}$, but in fact the integral closure is much “closer" to $I$ than to the radical and a very good family of examples in which it is easy to calculate and compare is that of monomial ideals in $\mathbf{C}\{z_1, \ldots, z_d\}$, which are the ideals generated by monomials. We begin with an example:

*Example 1.4.4* For the ideal $I = \langle x^4, xy^2, y^3 \rangle \mathbf{C}\{x, y\}$ we have that

$$\overline{I} = \langle x^4, x^3y, xy^2, y^3 \rangle$$

and

$$\sqrt{I} = \langle x, y \rangle$$

The exponent set of $I$ consists of all integer lattice points in the yellow region below:

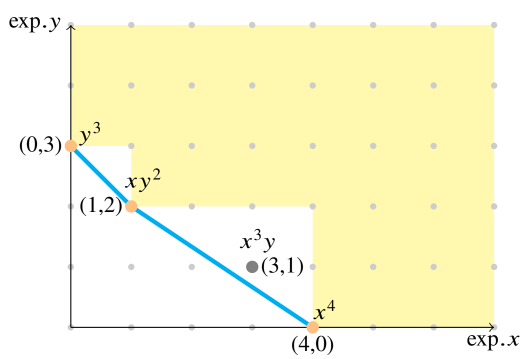


**Fig. 1.1** The point (3, 1) representing the monomial $x^3y$ is in the convex hull of the yellow region, whose integral points represent monomials in $I$. The integral dependence relation is $(x^3y)^2 - x^5.xy^2 = 0$.

Similarly, in $\mathbf{C}\{z_1, \ldots, z_d\}$ we have $\overline{\langle z_1^n, \ldots, z_d^n \rangle} = \langle z_1, \ldots, z_d \rangle^n$.

The *exponent vector* of a monomial $m = z_1^{m_1} \cdots z_d^{m_d}$ is $(m_1, \ldots, m_d) \in \mathbf{N}^d$. For any monomial ideal $I$, the set of all exponent vectors of all the monomials is $I$ is called the **exponent set of** $I$. Since a monomial $m$ is in $I$ if and only if it is a multiple in $\mathbf{C}\{z_1, \ldots, z_d\}$ of one of the monomial generators of $I$, the exponent set of $I$ consists of all those points of $\mathbf{N}^d$ which are componentwise greater or equal than the exponent vector of one of the monomial generators of $I$. Moreover one can prove that $\overline{I}$ is monomial and its exponent set is equal to all the integer lattice points in the convex hull of the set of exponents of elements of $I$. (See [Tei04, §3, §4], [Tei82, Chap.1, §2], [Hu-S06, Props 1.4.2 & 1.4.6]).

To understand how this theory can be used in the setting of complex analytic geometry the following result is fundamental.

**Theorem 1.4.5** ([Lej-T08, Thm 2.1, p. 799]) *Let $X$ be a reduced complex analytic space. Let $Y \subset X$ be a closed, nowhere dense, analytic subspace of $X$, and $x$ a point in $Y$. Let $\mathcal{I} \subset \mathcal{O}_X$ be the coherent ideal defining $Y$, and let $\mathcal{J} \subset \mathcal{O}_X$ be another coherent ideal. Let $I$ (resp. $J$) be the stalk of $\mathcal{I}$ (resp. $\mathcal{J}$) at $x$. Then the following statements are equivalent:*

1. *$J \subset \overline{I}$*
2. *For every germ of morphism $\phi : (\mathbf{C}, 0) \to (X, x)$*

$$\phi^* J \cdot \mathcal{O}_{\mathbf{C},0} \subseteq \phi^* I \cdot \mathcal{O}_{\mathbf{C},0}$$

3. *For every morphism $\pi : X' \to X$ such that $X'$ is a normal analytic space, $\pi$ is proper and surjective, and $\mathcal{I} \cdot \mathcal{O}_{X'}$ is locally invertible, there exists an open subset $U \subset X$ containing $x$, such that:*

$$\mathcal{J} \cdot \mathcal{O}_{X'}|\pi^{-1}(U) \subseteq \mathcal{I} \cdot \mathcal{O}_{X'}|\pi^{-1}(U)$$

3*. *If $\Pi : \overline{Bl_{\mathcal{I}} X} \to X$ denotes the normalized blowup of $X$ along $\mathcal{I}$, then there exists an open subset $U \subset X$ containing $x$, such that:*

$$\mathcal{J} \cdot \mathcal{O}_{\overline{Bl_{\mathcal{I}} X}}|\Pi^{-1}(U) \subseteq \mathcal{I} \cdot \mathcal{O}_{\overline{Bl_{\mathcal{I}} X}}|\Pi^{-1}(U)$$

4. *Let $V \subset X$ be a neighborhood of $x$, where both $\mathcal{J}$ and $\mathcal{I}$ are generated by their global sections. Then for every system of generators $g_1, \ldots, g_m$ of $\Gamma(V, \mathcal{I})$ and every $f \in \Gamma(V, \mathcal{J})$, there exist a neighborhood $V'$ of $x$ in $V$ and a constant $C$ such that:*

$$|f(y)| \leqslant C \sup_{i=1,\ldots,m} |g_i(y)|$$

*for every $y \in V'$.*

Let us take a closer look at statement 2: For any arc $\varphi : (\mathbf{C}, 0) \to (X, 0) \subset (\mathbf{C}^N, 0)$ we have a corresponding morphism of analytic algebras

$$\varphi^* : \mathcal{O}_{X,0} = \mathbf{C}\{z_1, \ldots, z_N\}/\mathfrak{a} \longrightarrow \mathbf{C}\{t\}$$
$$z_i + \mathfrak{a} \mapsto \varphi_i(t) = t^{m_i} u_i(t)$$

where $m_i \geqslant 1$ and $u_i(t)$ is a unit in $\mathbf{C}\{t\}$. So if $I \subset \mathcal{O}_{X,0}$ is an ideal then $\varphi^*(I)\mathcal{O}_{\mathbf{C},0} = \langle t^k \rangle \mathbf{C}\{t\}$ for some integer $k$ and an element $g \in \mathcal{O}_{X,0}$ is in $\overline{I}$ if and only if for any

such arc $\varphi(t)$ the order of the series $g\,(\varphi_1(t),\ldots,\varphi_N(t))$ is greater or equal than this $k$.
The fact that the normalized blowing-up map is proper implies that the condition of statement 2 needs to be verified only for finitely many arcs. Since the general statement is somewhat cumbersome, let us illustrate how this works in the case where the ideal $I$ a complete intersection defining the origin in $(X,0)$. Let $I=\langle h_1,\ldots,h_d\rangle\subset O_{X,0}$. The blowing up $Bl_IX$ of $I$ in $X$ is the subspace of $X\times\mathbf{P}^{d-1}$ defined by the $d-1$ equations $\frac{h_1}{T_1}=\frac{h_2}{T_2}=\cdots=\frac{h_d}{T_d}$, again a complete intersection. The fiber of the natural projection $Bl_IX\to\mathbf{P}^{d-1}$ over a point $t\in\mathbf{P}^{d-1}$ with coordinates $(t_1:t_2:\cdots:t_d)$ is a curve in $Bl_IX$ which is isomorphic to its image in $X$ defined by the equations $h_it_j-h_jt_i=0$. So we can view $Bl_IX$ as a family of curves $C_t$ on $X$ parametrized by $\mathbf{P}^{d-1}$, which is the exceptional divisor of the map $Bl_IX\to X$. When we pass to the normalization $n\colon\overline{Bl_IX}\to Bl_IX$, by general Theorems on normalization (see Proposition 1.2.6 and use the fact that there is a dense open $U\subset\mathbf{P}^{d-1}$ where $\delta$ is constant), there exists a Zariski dense open subset $U\subset\mathbf{P}^{d-1}$ such that $n^{-1}(U)$ is a non singular divisor in a non singular space $n^{-1}((X\times U)\cap Bl_IX)$, and for each point $t\in U$ the map $n$ induces a normalisation of the curve $C_t$. This normalization is then a union of disks, one for each irreducible component of $C_t$, and each disk transversal to $n^{-1}(\mathbf{P}^{d-1})$ in $n^{-1}((X\times U)\cap Bl_IX)$. Because a meromorphic function on a normal space is holomorphic if it has no poles in codimension one, to verify that an element $g\in O_{X,0}$ is in $\overline{I}$, it suffices to verify that for some $t\in U$, the order of vanishing of $g$ along each arc parametrizing a branch of $C_t$ is larger than the order of vanishing of the ideal $I$. Because of what we have just seen, the order of vanishing along these arcs will, after lifting to $\overline{Bl_IX}$, translate as the order of vanishing along some irreducible component of the exceptional divisor in $\overline{Bl_IX}$. Since the ideal $I$ is locally principal on $\overline{Bl_IX}$, to prove that $g\in\overline{I}$ it suffices to prove that after lifting to $\overline{Bl_IX}$ the function $g$ becomes a multiple of the local equations of the exceptional divisor. But the polar set of the quotient of $g$ by that equation is contained in that exceptional divisor and the inequalities of orders imply that there are no poles at a general point of each irreducible component. Because $\overline{Bl_IX}$ is normal, there are no poles anywhere and on $\overline{Bl_IX}$ the pull back of the function $g$ is indeed in the pull back of the ideal $I$ so that $g$ is in $\overline{I}$.

We shall use this below to describe the saturation.

With this at hand we can now characterize $C_3$-general projections in terms of integral closure of ideals. Let $(X,0)\subset(\mathbf{C}^N,0)$ be a reduced germ of analytic space of pure dimension $d$. Let us choose coordinates $z_1,\ldots,z_N$ on $\mathbf{C}^N$, denote by $L$ the linear subspace of $\mathbf{C}^N$ defined by $z_1=\cdots=z_d=0$ and let $\mathfrak{a}$ be the ideal of $O_{X,0}$ generated by the images of $z_1,\ldots,z_d$.

**Proposition 1.4.6** *The linear projection*

$$\pi:(X,0)\to(\mathbf{C}^d,0)\qquad(z_1,\ldots,z_N)\mapsto(z_1,\ldots,z_d)$$

*with kernel L is $C_3$-general if and only if $\overline{\mathfrak{a}}=\mathfrak{m}$ where $\mathfrak{m}=\langle z_1,\ldots,z_N\rangle O_{X,0}$ is the maximal ideal of the analytic algebra $O_{X,0}$.*

***Proof*** Recall that $\pi$ is $C_3$-general if and only if $C_3(X,0) \cap L = \{0\}$. Let $\ell = [a_1 : \cdots : a_N] \in \mathbf{P}^{N-1}$ be a line in the (projectivized) tangent cone $C_3(X,0)$, then $\ell \not\subset L$ if and only if $a_i \neq 0$ for some $i \in \{1,\dots,d\}$. Note that any arc $\varphi : (\mathbf{C},0) \to (X,0)$ determines a line in $C_3(X,0)$, the limit as $t \to 0$ of

$$t \longrightarrow [\varphi_1(t) : \cdots : \varphi_N(t)] \in \mathbf{P}^{N-1}$$

and conversely any line in the tangent cone can be obtained through an arc since it corresponds to a point in the fiber over 0 of the blowing-up $Bl_0X \to X$. On the other hand, for every arc $\varphi : (\mathbf{C},0) \to (X,0)$ we have that

$$\varphi^*(\mathfrak{a})\mathcal{O}_{\mathbf{C},0} = \langle \varphi_1(t),\dots,\varphi_d(t)\rangle \mathbf{C}\{t\} = \langle t^k \rangle \mathbf{C}\{t\}$$

where $k = \min\{\mathrm{ord}_0\varphi_i(t) \,|\, i = 1,\dots,d\}$. Finally $a_i \neq 0$ for some $i \in \{1,\dots,d\}$ if and only if for all $j \in \{d+1,\dots,N\}$

$$\mathrm{ord}_0\varphi_j(t) \geqslant k = \min\{\mathrm{ord}_0\varphi_i(t) \,|\, i = 1,\dots,d\}$$

if and only if $\varphi^*(z_j) \in \varphi^*(I)\mathcal{O}_{\mathbf{C},0}$ if and only if $z_j \in \overline{\mathfrak{a}}$ for all $j \in \{d+1,\dots,N\}$, that is, $\overline{\mathfrak{a}} = \mathfrak{m}$. □

By a linear change of coordinates in $\mathbf{C}^N$ we can always place ourselves in the setting of the previous result. But the theory of integral closure also gives us an algebraic way to prove that for a given germ $(X,0)$ of pure dimension $d$ almost all linear projections $\pi : (\mathbf{C}^N,0) \to (\mathbf{C}^d,0)$ are $C_3$-general as stated in the following result (For a proof see [Mat89, Thm 14.14])

**Theorem 1.4.7** *(Rees-Samuel) Let $\mathcal{O}_{X,0}$ be a $d$-dimensional analytic algebra with maximal ideal $\mathfrak{m} = \langle z_1,\dots,z_N\rangle$. Then if $y_i = \sum_{j=1}^N \lambda_{ij}z_j$ for $1 \leqslant i \leqslant d$ are $d$ "sufficiently general" $\mathbf{C}$-linear combinations of $z_1,\dots,z_n$ the ideal $\mathfrak{a} = \langle y_1,\dots,y_d\rangle$ satisfies $\overline{\mathfrak{a}} = \mathfrak{m}$.*

We can take this one step further by considering another important aspect of this theory, namely its relation with multiplicity. For a local Noetherian ring $(R,\mathfrak{m})$ and an $\mathfrak{m}-$primary ideal $\mathfrak{a} \subset R$ we can define a Hilbert Samuel function

$$k \in \mathbf{N} \mapsto \dim_{R/\mathfrak{m}} R/\mathfrak{a}^k$$

The result is that for large enough $k$ the Hilbert-Samuel function behaves like a polynomial of degree equal to the dimension of $R$ and its leading coefficient is of the form $e(\mathfrak{a})k^d/d!$, where $e(\mathfrak{a})$ is a positive integer called the multiplicity of the ideal $\mathfrak{a}$. In the case $R$ is the analytic algebra $\mathcal{O}_{X,0}$ of a germ $(X,0)$ and $\mathfrak{a} = \mathfrak{m}$ it IS the multiplicity of the germ. (See [De-P00, Section 4.2])

**Theorem 1.4.8** *(Rees)*(see [Ree61, Thm 3.2],[Hu-S06, Thm 11.3.1])
*Let $(\mathcal{O}_{X,0},\mathfrak{m})$ be a reduced and equidimensional analytic algebra and $\mathfrak{a} \subset \mathfrak{b}$ two $\mathfrak{m}-$primary ideals. Then $\overline{\mathfrak{a}} = \overline{\mathfrak{b}}$ if and only if $e(\mathfrak{a}) = e(\mathfrak{b})$.*

A geometric interpretation of this result is described by Lipman in [Lip82]. Let $(X,0)$ be a germ of reduced and equidimensional singularity of dimension $d$ with associated analytic algebra $(\mathcal{O}_{X,0},\mathfrak{m})$. Every $\mathfrak{m}-$primary ideal is generated by at least $d$ elements, and every $d-$tuple $(f_1,\ldots,f_d)$ of elements of $\mathfrak{m}$ defines a map-germ $F:(X,0)\to(\mathbf{C}^d,0)$.

Now, the ideal $\mathfrak{a}=\langle f_1,\ldots,f_d\rangle$ is $\mathfrak{m}-$primary if and only if $F$ is finite. As we have mentioned before you can prove that such an $F:(X,0)\to(\mathbf{C}^d,0)$ is then a ramified analytic cover of degree equal to $e(\mathfrak{a})$ and by Rees' TTheoremheorem this degree will be the multiplicity of $(X,0)$ $(=e(\mathfrak{m}))$ if and only if $\overline{\mathfrak{a}}=\mathfrak{m}$.

Moreover using Nakayama's Lemma one checks that $\mathfrak{a}$ is a reduction of $\mathfrak{m}$ (equivalently $\overline{\mathfrak{a}}=\mathfrak{m}$) if and only if in the graded $\mathbf{C}-$algebra

$$\mathrm{gr}_{\mathfrak{m}}\mathcal{O}=\bigoplus_{k\geqslant 0}\mathfrak{m}^k/\mathfrak{m}^{k+1},\ \ \text{with } \mathfrak{m}^0=\mathcal{O}$$

(which "corresponds" to the homogeneous coordinate ring of the projectivized tangent cone $\mathbf{P}C_3(X,0)$ see [Gi-T18, Section 2.4]) the images $\overline{f_i}$ of the $f_i$ in $\mathfrak{m}/\mathfrak{m}^2$ generate an irrelevant ideal (that is, an ideal containing all elements of $G$ of sufficiently large degree so that its zero locus in projective space is empty).

What this last condition means is that first of all the $\overline{f_i}$ are linearly independent over $\mathbf{C}$, so that there is an embedding of the germ $(X,0)$ into $(\mathbf{C}^N,0)$ for some $N$ and a linear projection $\pi:(\mathbf{C}^N,0)\to(\mathbf{C}^d,0)$ such that its restriction to $(X,0)$ is germwise the $F$ associated above to $(f_1,\ldots,f_d)$ and secondly, since $\overline{\mathfrak{a}}=\mathfrak{m}$ by Proposition 1.4.6 the projection $\pi$ is $C_3$-general.

We end this section by establishing a result analogous to Proposition 1.4.6 but with respect to generic projections of curves.

**Definition 1.4.9** Let $\varphi_1:R\to A_1$ and $\varphi_2:R\to A_2$ morphisms of $\mathbf{C}$-analytic algebras. There is a unique $\mathbf{C}$-analytic algebra, denoted $A_1\widehat{\otimes}_R A_2$, together with morphisms $\theta_i:A_i\to A_1\widehat{\otimes}_R A_2$, $i=1,2$, such that $\theta_1\circ\varphi_1=\theta_2\circ\varphi_2$ and for every pair of morphisms of $\mathbf{C}$-analytic algebras $\psi_1:A_1\to B$, $\psi_2:A_2\to B$ satisfying $\psi_1\circ\varphi_1=\psi_2\circ\varphi_2$ there is a unique morphism of $\mathbf{C}$-analytic algebras $\psi:A_1\widehat{\otimes}_R A_2\to B$ making the whole diagram commute. The algebra $A_1\widehat{\otimes}_R A_2$ is called the analytic tensor product of $A_1$ and $A_2$ over $R$.

$$\begin{tikzcd}
 & A_1 \arrow[dr, "\theta_1"] \arrow[drr, bend left, "\psi_1"] & & \\
R \arrow[ur, "\varphi_1"] \arrow[dr, "\varphi_2"'] & & A_1\widehat{\otimes}_R A_2 \arrow[r, dotted, "\psi"] & B \\
 & A_2 \arrow[ur, "\theta_2"'] \arrow[urr, bend right, "\psi_2"'] & &
\end{tikzcd}$$

Geometrically this analytic tensor product is the operation on the analytic algebras that corresponds to the fibre product of analytic spaces. Given holomorphic maps $\phi_1 : (X_1, p_1) \to (Y, q)$ and $\phi_2 : (X_2, p_2) \to (Y, q)$ we have the fibre product:

$$\begin{tikzcd}
X_1 \times_Y X_2 \arrow[r, "\Pi_1"] \arrow[d, "\Pi_2"] & X_1 \arrow[d, "\phi_1"] \\
X_2 \arrow[r, "\phi_2"'] & Y
\end{tikzcd}$$

which induces the corresponding diagram of analytic algebras

$$\begin{tikzcd}
\mathcal{O}_{Y,q} \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"] & \mathcal{O}_{X_1,p_1} \arrow[d] \\
\mathcal{O}_{X_2,p_2} \arrow[r] & \mathcal{O}_{X_1\times_Y X_2,(p_1,p_2)}
\end{tikzcd}$$

that is, the analytic algebra $\mathcal{O}_{X_1\times_Y X_2,(p_1,p_2)}$ is isomorphic to $\mathcal{O}_{X_1,p_1}\widehat{\otimes}_{\mathcal{O}_{Y,q}}\mathcal{O}_{X_2,p_2}$.

*Remark 1.4.10* See [Gr-P07, Def 1.28, Example 1.46.1 & Lemma 1.89] and [Ada12].

1. When $R = \mathbf{C}$ in the definition, the analytic tensor product $\mathcal{O}_{X,x}\widehat{\otimes}_{\mathbf{C}}\mathcal{O}_{Y,y}$ is the analytic algebra corresponding to the product germ $(X \times Y, (x, y))$. Moreover if $\mathcal{O}_{X,x} = \mathbf{C}\{\mathbf{z}\}/I$ and $\mathcal{O}_{Y,y} = \mathbf{C}\{\mathbf{w}\}/J$ with $\mathbf{z} = (z_1, \ldots, z_N)$ and $\mathbf{w} = (w_1, \ldots, w_M)$ then

$$\mathcal{O}_{X,x}\widehat{\otimes}_{\mathbf{C}}\mathcal{O}_{Y,y} = \frac{\mathbf{C}\{\mathbf{z},\mathbf{w}\}}{\langle I\mathbf{C}\{\mathbf{z},\mathbf{w}\} + J\mathbf{C}\{\mathbf{z},\mathbf{w}\}\rangle}$$

2. In general if $(X_1, p_1) \subset (\mathbf{C}^N, 0)$, $(X_2, p_2) \subset (\mathbf{C}^M, 0)$ and $(Y, q) \subset (\mathbf{C}^k, 0)$ then

$$\begin{aligned}
\mathcal{O}_{X_1,p_1}\widehat{\otimes}_{\mathcal{O}_{Y,q}}\mathcal{O}_{X_2,p_2} &= \frac{\mathbf{C}\{\mathbf{z}\}}{I}\widehat{\otimes}_{\mathcal{O}_{Y,q}}\frac{\mathbf{C}\{\mathbf{w}\}}{J} \\
&\cong \frac{\mathbf{C}\{\mathbf{z},\mathbf{w}\}}{\langle I\mathbf{C}\{\mathbf{z},\mathbf{w}\} + J\mathbf{C}\{\mathbf{z},\mathbf{w}\}\rangle + \langle \phi_1^1(\mathbf{z}) - \phi_2^1(\mathbf{w}), \ldots, \phi_1^k(\mathbf{z}) - \phi_2^k(\mathbf{w})\rangle\mathbf{C}\{\mathbf{z},\mathbf{w}\}}
\end{aligned}$$

Let $(X,0) \subset (\mathbf{C}^N,0)$ be a germ of reduced singular curve. By Proposition 1.3.5 the $C_5$ cone is a finite union of 2-planes of $\mathbf{C}^n$

$$C_5(X,0) = H_1 \cup \ldots \cup H_r$$

If we let $\pi : (\mathbf{C}^N,0) \to (\mathbf{C}^2,0)$ denote the linear projection to the first two coordinates $(z_1,\ldots,z_N) \mapsto (z_1,z_2)$ then $\pi$ is generic if and only if $\pi(H_i) = \mathbf{C}^2$ for $i = 1,\ldots,r$. Recall that the construction of $C_5(X,0)$ goes through the blowup of the diagonal of $X \times X$, so let $I_\Delta \subset O_{X\times X,(0,0)}$ be the ideal defining this diagonal

$$I_\Delta = \langle z_1 - w_1, \ldots, z_N - w_N \rangle O_{X\times X,(0,0)}$$

**Proposition 1.4.11** *Let $I_{\Delta_2} \subset O_{X\times X,(0,0)}$ be the ideal coming from the projection $\pi$, that is, $I_{\Delta_2} = \langle z_1 - w_1, z_2 - w_2\rangle O_{X\times X,(0,0)}$. Then $\pi$ is generic if and only if $\overline{I_{\Delta_2}} = \overline{I_\Delta}$.*

***Proof*** The proof is now very similar to the $C_3$-general case, and since $I_{\Delta_2} \subset I_\Delta$ all we have to prove is that genericity is equivalent to the inclusion $I_\Delta \subset \overline{I_{\Delta_2}}$.

Let $L = V(z_1,z_2)$ be the kernel of $\pi$. Then $\pi$ is generic if and only if $C_5(X,0) \cap L = \{0\}$. Let $\ell = [a_1 : \cdots : a_N] \in \mathbf{P}^{N-1}$ be a line in the (projectivized) cone $C_5(X,0)$, then $\ell \subsetneq L$ if and only if $a_i \neq 0$ for some $i \in \{1,2\}$. This time the lines in $C_5(X,0)$ are determined by taking the limit as $t \to 0$ of the secants associated to pairs of arcs $(\varphi,\psi) : (\mathbf{C},0) \to (X\times X,(0,0))$

$$t \longrightarrow [\varphi_1(t) - \psi_1(t) : \cdots : \varphi_N(t) - \psi_N(t)] \in \mathbf{P}^{N-1}$$

Again for every such pair of arcs $(\varphi,\psi) : (\mathbf{C},0) \to (X\times X,(0,0))$ we have that

$$(\varphi,\psi)^*(I_{\Delta_2})O_{\mathbf{C},0} = \langle \varphi_1(t) - \psi_1(t), \varphi_2(t) - \psi_2(t)\rangle \mathbf{C}\{t\} = \langle t^k \rangle \mathbf{C}\{t\}$$

where $k = \min\{\mathrm{ord}_0(\varphi_1(t) - \psi_1(t)), \mathrm{ord}_0(\varphi_2(t) - \psi_2(t))\}$. Finally $a_i \neq 0$ for some $i \in \{1,2\}$ if and only if for all $j \in \{3,\ldots,N\}$

$$\mathrm{ord}_0(\varphi_j(t) - \psi_j(t)) \geqslant k = \min\{\mathrm{ord}_0(\varphi_1(t) - \psi_1(t)), \mathrm{ord}_0(\varphi_2(t) - \psi_2(t))\}$$

if and only if $(\varphi,\psi)^*(z_j - w_j) \in (\varphi,\psi)^*(I_{\Delta_2})O_{\mathbf{C},0}$ if and only if $z_j - w_j \in \overline{I_{\Delta_2}}$ for all $j \in \{3,\ldots,N\}$, that is, $I_\Delta \subset \overline{I_{\Delta_2}}$. $\square$

### 1.4.2 Lipschitz Saturation

Let $n^* : O_{X,0} \hookrightarrow \overline{O_{X,0}}$ be the integral closure of a reduced complex analytic algebra which is a quotient of $\mathbf{C}\{z_1,\ldots,z_N\}$. Recall that $\overline{O_{X,0}}$ is a direct sum of normal analytic algebras (in particular integral domains), one for each irreducible component of the germ $(X,0)$. By definition 1.4.9 the following commutative diagram determines a unique morphism $\Psi$ of direct sums of analytic algebras:

$$
\begin{tikzcd}
 & & \overline{\mathcal{O}_{X,0}} \arrow[dr, "\theta_1"] \arrow[drr, bend left, "\psi_1"] & & \\
\mathbf{C} \arrow[r] \arrow[urr, bend left] \arrow[drr, bend right] & \mathcal{O}_{X,0} \arrow[ur] \arrow[dr] & & \overline{\mathcal{O}_{X,0}}\widehat{\otimes}_{\mathbf{C}}\overline{\mathcal{O}_{X,0}} \arrow[r, dotted, "\Psi"] & \overline{\mathcal{O}_{X,0}}\widehat{\otimes}_{\mathcal{O}_{X,0}}\overline{\mathcal{O}_{X,0}} \\
 & & \overline{\mathcal{O}_{X,0}} \arrow[ur, "\theta_2"] \arrow[urr, bend right, "\psi_2"'] & &
\end{tikzcd}
$$

where $\theta_1(f) = f\widehat{\otimes}_{\mathbf{C}}1$ and $\theta_2(f) = 1\widehat{\otimes}_{\mathbf{C}}f$. Note that the map $\Psi : \overline{\mathcal{O}_{X,0}}\widehat{\otimes}_{\mathbf{C}}\overline{\mathcal{O}_{X,0}} \to \overline{\mathcal{O}_{X,0}}\widehat{\otimes}_{\mathcal{O}_{X,0}}\overline{\mathcal{O}_{X,0}}$ is the morphism of sums of analytic algebras corresponding to the inclusion $\left(\overline{X} \times_X \overline{X}, (0,0)\right) \hookrightarrow \left(\overline{X} \times \overline{X}, (0,0)\right)$. By remark 1.4.10 if we denote by $n : (\overline{X}, 0) \to (X, 0)$ the normalization map and $\overline{\mathcal{O}_{X,0}} = \bigoplus_{i=1}^{r} \mathbf{C}\{t_1, \ldots, t_{m_i}\}/J_i(\underline{t})$ with $\underline{t}_j = (t_1, \ldots, t_{m_i})$, then

$$
\overline{\mathcal{O}_{X,0}}\widehat{\otimes}_{\mathbf{C}}\overline{\mathcal{O}_{X,0}} = \bigoplus_{i,j} \frac{\mathbf{C}\{t_1, \ldots, t_{m_i}, u_1, \ldots, u_{m_j}\}}{\langle J_i(\underline{t}_j), J_j(\underline{u}_j)\rangle}
$$

and $\Psi$ is a surjection with kernel

$$
I_\Delta = \langle z_1\widehat{\otimes}_{\mathbf{C}}1 - 1\widehat{\otimes}_{\mathbf{C}}z_1, \ldots, z_N\widehat{\otimes}_{\mathbf{C}}1 - 1\widehat{\otimes}_{\mathbf{C}}z_N\rangle \overline{\mathcal{O}_{X,0}}\widehat{\otimes}_{\mathbf{C}}\overline{\mathcal{O}_{X,0}}.
$$

**Definition 1.4.12** The $I_\Delta$ be the kernel of the morphism $\Psi$ above. We define the Lipschitz saturation $\mathcal{O}^s_{X,0}$ of $\mathcal{O}_{X,0}$ as the algebra

$$
\mathcal{O}^s_{X,0} := \left\{ f \in \overline{\mathcal{O}_{X,0}} \mid \theta_1(f) - \theta_2(f) \in \overline{I_\Delta} \right\} = \left\{ f \in \overline{\mathcal{O}_{X,0}} \mid f\widehat{\otimes}_{\mathbf{C}}1 - 1\widehat{\otimes}_{\mathbf{C}}f \in \overline{I_\Delta} \right\}.
$$

*Example 1.4.13* Let $(X, 0) \subset (\mathbf{C}^3, 0)$ be the irreducible curve with normalization map:

$$
\begin{aligned}
\eta : (\mathbf{C}, 0) &\longrightarrow (X, 0) \\
t &\mapsto (t^4, t^6, t^7)
\end{aligned}
$$

In this setting the map $\Psi$ above is

$$
\begin{aligned}
\Psi : \overline{\mathcal{O}_{X,0}}\widehat{\otimes}_{\mathbf{C}}\overline{\mathcal{O}_{X,0}} &\to \overline{\mathcal{O}_{X,0}}\widehat{\otimes}_{\mathcal{O}_{X,0}}\overline{\mathcal{O}_{X,0}} \\
\Psi : \mathbf{C}\{t, u\} &\longrightarrow \frac{\mathbf{C}\{t, u\}}{\langle t^4 - u^4, t^6 - u^6, t^7 - u^7\rangle}.
\end{aligned}
$$

The maps $\theta_i$ are just inclusions, $\mathbf{C}\{t\} \hookrightarrow \mathbf{C}\{t,u\}$, $\mathbf{C}\{u\} \hookrightarrow \mathbf{C}\{t,u\}$ and the ideal $I_\Delta = \langle t^4 - u^4, t^6 - u^6, t^7 - u^7\rangle$. By definition $O^s_{X,0} := \left\{ f \in \mathbf{C}\{t\} \mid f(t) - f(u) \in \overline{I_\Delta} \right\}$ and note that $O_{X,0} \subset O^s_{X,0}$. For example $t^5 \in \mathbf{C}\{t\}$ is in $O^s_{X,0}$ if and only if $t^5 - u^5$ is in $\overline{I_\Delta}$. By taking the arc $\phi(\tau) = (\tau, -\tau)$ we have that $\phi^* I_\Delta O_{\mathbf{C},0} = \langle \tau^7 \rangle$ and $\phi^*(t^5 - u^5) = 2\tau^5 \notin \phi^* I_\Delta O_{\mathbf{C},0}$, so by Theorem 1.4.5-2 the element $t^5 \in \mathbf{C}\{t\} = \overline{O_{X,0}}$ is not in the Lipschitz saturation $O^s_{X,0}$. For this particular arc we have $\phi^*(t^9 - u^9) = 2\tau^9 \in \phi^* I_\Delta O_{\mathbf{C},0}$, and one can actually prove that $t^9 \in O^s_{X,0}$. As we shall see later on, in fact $O^s_{X,0} = \mathbf{C}\{t^4, t^6, t^7, t^9\}$.

We are going to show that the Lipschitz saturation $O^s_{X,0}$ is always an analytic algebra, even if the germ $(X,0)$ is not irreducible. To begin to understand why this is true, let's look at the irreducible case. Define the map

$$\begin{aligned} \alpha : \overline{O_{X,0}} &\to \overline{O_{X,0}} \widehat{\otimes}_{\mathbf{C}} \overline{O_{X,0}} \\ f &\mapsto \theta_1(f) - \theta_2(f) = f(z) - f(w) \end{aligned}$$

It is not a ring map, however if $n^* : O_{X,0} \hookrightarrow \overline{O_{X,0}}$ denotes the inclusion coming from the normalization map $n : \overline{X} \to X$ then $\alpha(n^*(O_{X,0})) = \alpha(n^*(\mathfrak{m}_{X,0}))$ and $I = \operatorname{Ker} \Psi = \langle \alpha(n^*(\mathfrak{m}_{X,0})) \rangle$.

By definition 1.4.9 $\overline{O_{X,0}} \widehat{\otimes}_{O_{X,0}} \overline{O_{X,0}}$ is an $O_{X,0}$-algebra, in particular an $O_{X,0}$-module. However, an interesting point is that since $n : \overline{X} \to X$ is a finite map, by [Gr-P07, Lemma 1.89] this algebra is isomorphic to the algebraic tensor product $\overline{O_{X,0}} \otimes_{O_{X,0}} \overline{O_{X,0}}$, so for instance

$$\frac{\mathbf{C}\{t,u\}}{\langle t^2 - u^2, t^3 - u^3 \rangle} \cong \mathbf{C}\{t\} \otimes_{\mathbf{C}\{t^2,t^3\}} \mathbf{C}\{u\}$$

**Lemma 1.4.14** *The map*

$$\Psi \circ \alpha : \overline{O_{X,0}} \longrightarrow \overline{O_{X,0}} \widehat{\otimes}_{O_{X,0}} \overline{O_{X,0}}$$

*is a morphism of $O_{X,0}$-modules.*

***Proof*** Indeed for $r \in O_{X,0}$ and $f \in \overline{O_{X,0}}$:

$$rf \overset{\alpha}{\mapsto} r(z)f(z) - r(w)f(w) \overset{\Psi}{\mapsto} r(z)f(z) - r(w)f(w) + I$$

but $r(z) = r(w) \bmod (I)$ so $r(z)f(z) - r(w)f(w) = (r(z) + I)(f(z) - f(w) + I) = r(\Psi \circ \alpha)(f)$.

By definition $\overline{I_\Delta}$ is an ideal of $\overline{O_{X,0}} \widehat{\otimes}_{\mathbf{C}} \overline{O_{X,0}}$ and since $\Psi$ is a surjective ring homomorphism we have that $\Psi(\overline{I_\Delta}) \subset \overline{O_{X,0}} \widehat{\otimes}_{O_{X,0}} \overline{O_{X,0}}$ is an ideal, in particular it is an $O_{X,0}$-module. But this implies that

$$(\Psi \circ \alpha)^{-1}\left(\Psi(\overline{I_\Delta})\right) = \alpha^{-1}(\overline{I_\Delta}) = \mathcal{O}^s_{X,0} \subset \overline{\mathcal{O}_{X,0}}$$

is an $\mathcal{O}_{X,0}$-module.

**Lemma 1.4.15** *The Lipschitz saturation $\mathcal{O}^s_{X,0}$ is an $\mathcal{O}_{X,0}$-algebra and a direct sum of analytic algebras.*

***Proof*** Since $\mathcal{O}^s_{X,0}$ is a submodule of the Noetherian module $\overline{\mathcal{O}_{X,0}}$, it is a finitely generated $\mathcal{O}_{X,0}$-module. Even more, you can easily check that $\mathcal{O}^s_{X,0}$ is closed under multiplication, so it is an $\mathcal{O}_{X,0}$-algebra and by [De-P00, Cor. 3.3.25 & 3.3.26] this implies that $\mathcal{O}^s_{X,0}$ is a direct sum of analytic algebras.

Indeed, take $f_1, f_2 \in \mathcal{O}^s_{X,0}$ then $(\Psi \circ \alpha)(f_1) = f_1(z) - f_1(w) + I_\Delta \in \Psi(\overline{I_\Delta})$, but it is an ideal so $(f_2(z) + I_\Delta)(f_1(z) - f_1(w) + I_\Delta) \in \Psi(\overline{I_\Delta})$. Analogously $(f_1(w) + I_\Delta)(f_2(z) - f_2(w) + I_\Delta) \in \Psi(\overline{I_\Delta})$ by taking their sum we get that $(\Psi \circ \alpha)(f_1 f_2) = f_1(z) f_2(z) - f_1(w) f_2(w) + I_\Delta \in \Psi(\overline{I_\Delta})$ which implies that $f_1 f_2 \in \mathcal{O}^s_{X,0}$ as claimed. □

Before proving that $\mathcal{O}^s_{X,0}$ is actually an analytic algebra we would like to give an idea of how things work in the non-irreducible case so suppose there are two irreducible components $(X,0) = (X_1,0) \cup (X_2,0)$. As we said before $\overline{X}$ is then a multigerm $(\overline{X_1}, p) \sqcup (\overline{X_2}, q)$ and $\overline{\mathcal{O}_{X,0}} = \overline{\mathcal{O}_{X_1,0}} \bigoplus \overline{\mathcal{O}_{X_2,0}} = \mathcal{O}_{\overline{X_1},p} \bigoplus \mathcal{O}_{\overline{X_2},q}$. Since the analytic tensor product should be the algebraic counterpart of the fibre product then we should consider/define

$$\overline{\mathcal{O}_{X,0}} \widehat{\otimes}_{\mathcal{O}_{X,0}} \overline{\mathcal{O}_{X,0}} =$$

$$\mathcal{O}_{\overline{X_1},p} \widehat{\otimes}_{\mathcal{O}_{X,0}} \mathcal{O}_{\overline{X_1},p} \oplus \mathcal{O}_{\overline{X_1},p} \widehat{\otimes}_{\mathcal{O}_{X,0}} \mathcal{O}_{\overline{X_2},q} \oplus \mathcal{O}_{\overline{X_2},q} \widehat{\otimes}_{\mathcal{O}_{X,0}} \mathcal{O}_{\overline{X_1},p} \oplus \mathcal{O}_{\overline{X_2},q} \widehat{\otimes}_{\mathcal{O}_{X,0}} \mathcal{O}_{\overline{X_2},q}$$

and analogously for $\overline{\mathcal{O}_{X,0}} \widehat{\otimes}_{\mathbf{C}} \overline{\mathcal{O}_{X,0}}$. By componentwise taking the ring maps $\Psi_{ij}$ coming from the universal property of the irreducible case, for example:

$$\Psi_{12} : \mathcal{O}_{\overline{X_1},p} \widehat{\otimes}_{\mathbf{C}} \mathcal{O}_{\overline{X_2},q} \to \mathcal{O}_{\overline{X_1},p} \widehat{\otimes}_{\mathcal{O}_{X,0}} \mathcal{O}_{\overline{X_2},q}$$

we get the ring map $\Psi$ as before with kernel $I_\Delta = I_{11} \oplus I_{12} \oplus I_{21} \oplus I_{22}$. The map $\alpha$ should now be defined as

$$\begin{aligned} \alpha : \overline{\mathcal{O}_{X,0}} &\longrightarrow \overline{\mathcal{O}_{X,0}} \widehat{\otimes}_{\mathbf{C}} \overline{\mathcal{O}_{X,0}} \\ (f_1, f_2) &\mapsto (f_1(z) - f_1(w), f_1(z) - f_2(w), f_2(z) - f_1(w), f_2(z) - f_2(w)) \end{aligned}$$

and we get the same definition for the Lipschitz saturation

$$\mathcal{O}^s_{X,0} := \left\{ f = (f_1, f_2) \in \overline{\mathcal{O}_{X,0}} \mid \alpha(f) \in \overline{I_\Delta} \right\}$$

More importantly both Lemmas remain valid. Note that in this context of two irreducible components we have $\alpha(f) \in \overline{I_\Delta}$ if and only if $f_1(z) - f_1(w) \in \overline{I_{11}}$, $f_1(z) - f_2(w) \in \overline{I_{12}}$, $f_2(z) - f_1(w) \in \overline{I_{21}}$ and $f_2(z) - f_2(w) \in \overline{I_{22}}$.

**Proposition 1.4.16** (See [P-T69, Theorem 1.2], [Tei82, Prop. 6.1.1]) *The algebra $\mathcal{O}^s_{X,0}$ is the ring of germs of meromorphic functions on $(X, 0)$ which are locally Lipschitz with respect to the ambient metric.*

***Proof*** Recall that $\overline{\mathcal{O}_{X,0}}$ is the ring of meromorphic functions on $(X, 0)$ that are locally bounded and a Lipschitz meromorphic function is locally bounded. Now if $h \in \mathcal{O}^s_{X,0}$ we need to prove that there exists a real positive constant $C > 0$ such that for every couple $(x_1, x_2) \in X \setminus \mathrm{Sing}\, X \times X \setminus \mathrm{Sing}\, X$ (in a small enough representative) we have

$$|h(x_1) - h(x_2)| \leqslant C||x_1 - x_2||$$

Let $n : \overline{X} \to X \subset \mathbf{C}^n$ be the normalization map. In the irreducible case where $\mathcal{O}_{X,0} = \mathbf{C}\{z_1, \dots, z_N\}/\langle f_1, \dots, f_s\rangle$ and $\overline{\mathcal{O}_{X,0}} = \mathbf{C}\{t_1, \dots, t_m\}/J(\underline{t})$, the map $n$ induces a morphism of analytic algebras which may be described by

$$\begin{aligned} n^* : \mathcal{O}_{X,0} &\longrightarrow \overline{\mathcal{O}_{X,0}} \\ z_i &\mapsto z_i(t_1, \dots, t_m) = z_i(\underline{t}) \end{aligned}$$

and refering to the maps $\alpha$ and $\Psi$ as above we have that

$$I_\Delta = \mathrm{Ker}\, \Psi = \langle z_1(\underline{t}) - z_1(\underline{u}), \dots, z_N(\underline{t}) - z_N(\underline{u}).\rangle$$

By definition $h \in \mathcal{O}^s_{X,0}$ if $\alpha(h) = h(\underline{t}) - h(\underline{u}) \in \overline{I_\Delta}$ and by Theorem 1.4.5-4 there exists a constant $C$ such that

$$|h(\underline{t}) - h(\underline{u})| \leqslant C \sup |z_i(\underline{t}) - z_i(\underline{u})| = C||z(\underline{t}) - z(\underline{u})||$$

with $\big(z(\underline{t}), z(\underline{u})\big) \in X \times X$ and so $h$ is Lipschitz. Reading the proof in the opposite sense gives that a meromorphic, locally Lipschitz function $h$ is necessarily in $\mathcal{O}^s_{X,0}$.

If $(X, 0)$ has $r$ irreducible components then $\overline{X}$ is a multigerm and then we have $r$ maps $n_k : (\overline{X_k}, x_k) \to (X_k, 0) \subset (X, 0)$ with coordinate functions $z_1(\underline{t}_k), \dots, z_N(\underline{t}_k)$. Then for $h = (h_1, \dots, h_r) \in \overline{\mathcal{O}_{X,0}}$ we have that $\alpha(h) = \Big(h_i(\underline{t}_i)\widehat{\otimes}1 - 1\widehat{\otimes}h_j(\underline{u}_j)\Big)_{i,j} \in \bigoplus_{i,j=1}^{i,j=r} \overline{\mathcal{O}_{X_i,0}}\widehat{\otimes}_{\mathbf{C}}\overline{\mathcal{O}_{X_j,0}}$, and

$$I_\Delta = \bigoplus_{i,j=1}^{i,j=r} I_{ij} \text{ with } I_{ij} = \langle z_1(\underline{t}_i)\widehat{\otimes}1 - 1\widehat{\otimes}z_1(\underline{u}_j), \dots, z_N(\underline{t}_i)\widehat{\otimes}1 - 1\widehat{\otimes}z_N(\underline{u}_j)\rangle \overline{\mathcal{O}_{X_i,0}}\widehat{\otimes}_{\mathbf{C}}\overline{\mathcal{O}_{X_j,0}}$$

and $\alpha(h) \in \overline{I_\Delta}$ if and only if $h_i(\underline{t}_i)\widehat{\otimes}1 - 1\widehat{\otimes}h_j(\underline{u}_j) \in \overline{I_{ij}}$ for all $(i, j)$.
So in the spirit of example 1.1.7 the "coordinate" $h_i$ of $h$ indicates you how to evaluate $h$ in points of the corresponding irreducible component $(X_i, 0)$ of $(X, 0)$ and for $i \neq j$ the condition $h_i(\underline{t}_i)\widehat{\otimes}1 - 1\widehat{\otimes}h_j(\underline{u}_j) \in \overline{I_{ij}}$ tells you that the Lipschitz condition must also be satisfied when you take points in different irreducible components. □

**Corollary 1.4.17** (See [P-T69, Corollary 1.3]) *Let $(X, 0) \subset (\mathbf{C}^N, 0)$ be a reduced germ of complex analytic singularity. The ring $\mathcal{O}^s_{X,0}$ is an analytic algebra.*

***Proof*** We already proved in Lemma 1.4.15 that $\mathcal{O}^s_{X,0}$ is a direct sum of analytic algebras, but if there were more than one, the function $(1, 0, \ldots, 0) \in \mathcal{O}^s_{X,0}$ would not be Lipschitz, contradicting Proposition 1.4.16. □

From Lemma 1.4.15 we have injective ring morphisms

$$\mathcal{O}_{X,0} \hookrightarrow \mathcal{O}^s_{X,0} \hookrightarrow \overline{\mathcal{O}_{X,0}}$$

Since $\overline{\mathcal{O}_{X,0}}$ is contained in the total ring of fractions $Q(\mathcal{O}_{X,0})$, the total ring of fractions of the Lipschitz saturation $\mathcal{O}^s_{X,0}$ coincides with $Q(\mathcal{O}_{X,0})$ and by transitivity of integral dependence the normalizations also coincides i.e., $\overline{\mathcal{O}^s_{X,0}} = \overline{\mathcal{O}_{X,0}}$. In terms of holomorphic maps we have:

$$\overline{X} \xrightarrow{n_s} X^s \xrightarrow{\zeta} X$$

where $X^s$ is the germ of complex analytic singularity corresponding to the analytic algebra $\mathcal{O}^s_{X,0}$, the map $n_s : \overline{X} \to X^s$ is the normalization map of $X^s$, $\zeta : (X^s, 0) \to (X, 0)$ is finite and induces an isomorphism outside the non-normal locus of $X$, and $n = \zeta \circ n_s : \overline{X} \to X$ is the normalization map of $X$.

**Definition 1.4.18** The germ $(X^s, 0)$ together with the finite map $\zeta : (X^s, 0) \to (X, 0)$ is called the Lipschitz saturation of $(X, 0)$.

**Lemma 1.4.19** *Let* $(X, 0) \subset (\mathbf{C}^n, 0)$ *be a reduced germ of complex analytic singularity, then* $\left(\mathcal{O}^s_{X,0}\right)^s = \mathcal{O}^s_{X,0}$.

***Proof*** Following the notation of Lemma 1.4.14 we have the maps:

$$\mathcal{O}_{X,0} \hookrightarrow \mathcal{O}^s_{X,o} \hookrightarrow \overline{\mathcal{O}_{X,0}} \xrightarrow{\alpha} \overline{\mathcal{O}_{X,0}} \widehat{\otimes}_{\mathbf{C}} \overline{\mathcal{O}_{X,0}}$$

and this induces a map

$$\overline{\mathcal{O}_{X,0}} \widehat{\otimes}_{\mathcal{O}_{X,0}} \overline{\mathcal{O}_{X,0}} \cong \frac{\overline{\mathcal{O}_{X,0}} \widehat{\otimes}_{\mathbf{C}} \overline{\mathcal{O}_{X,0}}}{\langle \alpha(\mathcal{O}_{X,0}) \rangle} \longrightarrow \frac{\overline{\mathcal{O}_{X,0}} \widehat{\otimes}_{\mathbf{C}} \overline{\mathcal{O}_{X,0}}}{\langle \alpha(\mathcal{O}^s_{X,0}) \rangle} \cong \overline{\mathcal{O}_{X,0}} \widehat{\otimes}_{\mathcal{O}^s_{X,0}} \overline{\mathcal{O}_{X,0}}$$

that makes the following diagram commute

$$\begin{tikzcd}
\overline{\mathcal{O}_{X,0}} \widehat{\otimes}_{\mathbf{C}} \overline{\mathcal{O}_{X,0}} \arrow[d, "\Psi"'] \arrow[dr, "\Psi_s"] & \\
\overline{\mathcal{O}_{X,0}} \widehat{\otimes}_{\mathcal{O}_{X,0}} \overline{\mathcal{O}_{X,0}} \arrow[r] & \overline{\mathcal{O}_{X,0}} \widehat{\otimes}_{\mathcal{O}^s_{X,0}} \overline{\mathcal{O}_{X,0}}
\end{tikzcd}$$

If we denote $I_\Delta = \operatorname{Ker} \Psi$ and $I_{\Delta^s} = \operatorname{Ker} \Psi_s$ we have $I_\Delta \subset I_{\Delta^s}$. Now by definition we have $\mathcal{O}^s_{X,0} = \{h \in \overline{\mathcal{O}_{X,0}} \mid \alpha(h) \in \overline{I_\Delta}\}$ so $\alpha(\mathcal{O}^s_{X,0}) \subset \overline{I_\Delta}$ which implies $\overline{I_\Delta} = \overline{I_{\Delta^s}}$ and so $\left(\mathcal{O}^s_{X,0}\right)^s = \{h \in \overline{\mathcal{O}_{X,0}} \mid \alpha(h) \in \overline{I_{\Delta^s}} = \overline{I_\Delta}\} = \mathcal{O}^s_{X,0}$. □

### 1.4.3 The case of dimension 1

Let $(X,0) \subset (\mathbf{C}^2,0)$ be a germ of reduced plane curve, and $\zeta : (X^s,0) \to (X,0) \subset (\mathbf{C}^2,0)$ the finite map given by the Lipschitz saturation of $(X,0)$. What we want to emphasize is that this map can always be realized as a linear projection on suitable representatives. Indeed, any representative $(X^s,0) \subset (\mathbf{C}^m,0)$ can be re-embedded as the graph of $\zeta$ in $\mathbf{C}^{m+2}$, namely by the map $X^s \to \mathbf{C}^2 \times \mathbf{C}^m : \ p \mapsto (\zeta_1(p), \zeta_2(p), p)$. The map $\zeta$ is now the projection of $(X^s,0)$ to $(X,0)$ by the first two coordinates: $(z_1,\dots,z_{m+2}) \mapsto (z_1,z_2)$.

**Proposition 1.4.20** (See [Tei82, Proposition 6.2.1]) *For a germ of reduced plane curve $(X,0) \subset (\mathbf{C}^2,0)$ the Lipschitz saturation map $\zeta : (X^s,0) \to (X,0)$ is a generic projection.*

***Proof*** Suppose first that $(X,0)$ is irreducible, in this case we have the holomorphic maps

$$(\mathbf{C},0) \xrightarrow{\eta_s} (X^s,0) \subset (\mathbf{C}^{m+2},0) \xrightarrow{\zeta} (X,0) \subset (\mathbf{C}^2,0)$$

$$t \mapsto (z_1(t), z_2(t), z_3(t), \dots, z_{m+2}(t)) \mapsto (z_1(t), z_2(t))$$

By Proposition 1.4.11 we have to prove that the ideals $I_{\Delta^s} = \langle z_1 - w_1, \dots, z_{m+2} - w_{m+2}\rangle$ and $I_{\Delta_2^s} = \langle z_1 - w_1, z_2 - w_2\rangle$ have the same integral closure in $\mathcal{O}_{X^s \times X^s,(0,0)}$. In this coordinate system we have the normalization map $\eta_s^* : \mathcal{O}_{X^s,0} \hookrightarrow \overline{\mathcal{O}_{X,0}}$ given by

$$\frac{\mathbf{C}\{z_1,\dots,z_{m+2}\}}{I} \hookrightarrow \mathbf{C}\{t\}$$

$$z_i + I \mapsto z_i(t) \quad i = 1,2\,; \qquad z_{j+2} + I \mapsto z_{j+2}(t) \quad j = 1,\dots m$$

Which induces the morphism

$$\theta : \mathcal{O}_{X^s \times X^s,(0,0)} = \frac{\mathbf{C}\{z_1,\dots,z_{m+2}, w_1,\dots,w_{m+2}\}}{I(z)+I(w)} \hookrightarrow \mathbf{C}\{t,u\} = \overline{\mathcal{O}_{X,0}}\widehat{\otimes}_{\mathbf{C}}\overline{\mathcal{O}_{X,0}}$$

$$z_i + I \mapsto z_i(t) \quad i = 1,2\,; \qquad z_{j+2} + I \mapsto z_{j+2}(t) \quad j = 1,\dots m$$

$$w_i + I \mapsto z_i(u) \quad i = 1,2\,; \qquad w_{j+2} + I \mapsto z_{j+2}(u) \quad j = 1,\dots m$$

But from the proof of Lemma 1.4.19 we have that the ideals $I_{\Delta_2^s} = \langle z_1(t) - z_1(u), z_2(t) - z_2(u)\rangle$ and
$I_{\Delta^s} = \langle z_1(t) - z_1(u), z_2(t) - z_2(u), z_3(t) - z_3(u), \dots, z_{m+2}(t) - z_{m+2}(u)\rangle$ have the same integral closure in $\mathbf{C}\{t,u\}$ and so by remark 1.4.2-5 the ideals $\theta^{-1}(I_{\Delta^s})$ and $\theta^{-1}(I_{\Delta_2^s})$ have the same integral closure in $\mathcal{O}_{X^s \times X^s,(0,0)}$, which is what we wanted.

In the reducible case the proof works exactly the same way, it is just a lot messier to write down. The only thing you have to keep track off is the following. Suppose $(X,0)$ has two irreducible components $(X_1,0) \cup (X_2,0)$ then $(X^s,0)$ also has two irreducible components and $\overline{\mathcal{O}_{X,0}} \cong \mathbf{C}\{t_1\} \oplus \mathbf{C}\{t_2\}$. This implies that the normalization map $\eta_s^* : \mathcal{O}_{X^s,0} \hookrightarrow \overline{\mathcal{O}_{X,0}}$ is given by

$$\frac{\mathbf{C}\{z_1,\ldots,z_{m+2}\}}{I} \hookrightarrow \mathbf{C}\{t_1\}\oplus\mathbf{C}\{t_2\}$$

$$z_i \mapsto (z_i(t_1), z_i(t_2)) \quad i=1,2\,; \qquad z_{j+2} \mapsto (z_{j+2}(t_1), z_{j+2}(t_2)) \quad j=1,\ldots m$$

In this case $\overline{O_{X,0}}\widehat{\otimes}_{\mathbf{C}}\overline{O_{X,0}} \cong \mathbf{C}\{t_1,u_1\}\oplus\mathbf{C}\{t_1,u_2\}\oplus\mathbf{C}\{t_2,u_1\}\oplus\mathbf{C}\{t_2,u_2\}$ and the induced morphism $\theta$ looks like:

$$\theta : O_{X^s\times X^s,(0,0)} = \frac{\mathbf{C}\{z_1,\ldots,z_{m+2},w_1,\ldots,w_{m+2}\}}{I(z)+I(w)} \hookrightarrow \overline{O_{X,0}}\widehat{\otimes}_{\mathbf{C}}\overline{O_{X,0}}$$

$$z_i \mapsto (z_i(t_1), z_i(t_1), z_i(t_2), z_i(t_2)) \quad i=1,2$$

$$z_{j+2} \mapsto (z_{j+2}(t_1), z_{j+2}(t_1), z_{j+2}(t_2), z_{j+2}(t_2)) \quad j=1,\ldots m$$

$$w_i \mapsto (z_i(u_1), z_i(u_2), z_i(u_1), z_i(u_2)) \quad i=1,2$$

$$w_{j+2} \mapsto z_{j+2}(u_1), z_{j+2}(u_2), z_{j+2}(u_1), z_{j+2}(u_2)) \quad j=1,\ldots m$$

then you have the map $\alpha$ as in the proof of Proposition 1.4.16 and the rest follows through. ◻

*Remark 1.4.21* 1. Since the Lipschitz saturation map $\zeta : (X^s,0)\to(X,0)$ is a generic projection the multiplicity of $(X^s,0)$ is equal to the multiplicity of $(X,0)$.
2. Except if the plane branch $(X,0)$ is non singular, the map $(\overline{X},0)\to(X,0)$ is never obtained as a generic projection since the multiplicity changes. However, among all germs $(X',0)$ which dominate $(X,0)$ and are dominated by $(\overline{X},0)$, and in addition are such that the map $(X',0)\to(X,0)$ can be represented by a *generic* linear projection, there is a unique one, up to isomorphism, which dominates all the others: it is the saturation.

**Corollary 1.4.22** *Let* $(X,0)\subset(\mathbf{C}^2,0)$ *be a reduced plane curve. The Lipschitz saturation map* $\zeta : (X^s,0)\to(X,0)$ *is a biLipschitz homeomorphism.*

***Proof*** We already know that a generic projection induces a homeomorphism with its image (Prop. 1.3.3), so by Proposition 1.4.20 the map $\zeta$ is a homeorphism and since it is the restriction to $X^s$ of the linear projection $(z_1,\ldots,z_{m+2})\mapsto(z_1,z_2)$ it is Lipschitz. The inverse of $\zeta$ can be described on each irreducible component $X_k$ by

$$(z_1(t_k), z_2(t_k)) \mapsto (z_1(t_k), z_2(t_k), z_3(t_k),\ldots,z_{m+2}(t_k)),$$

and since for all $j\in\{1,\ldots,m\}$, $z_{j+2}(\underline{t})\in O^s_{X,0}$ Proposition 1.4.16 tells us that it is also Lipschitz. ◻

Our main result now follows from the following result.

**Theorem 1.4.23** (See [P-T69, §4], [B-G-G80, Prop. VI.3.2]) *Let* $O_{X,0}$ *be the analytic algebra of a germ of reduced plane curve* $(X,0)\subset(\mathbf{C}^2,0)$. *The Lipschitz saturation* $O^s_{X,0}$ *determines and is determined by the characteristic exponents of its branches (irreducible components) and the intersection multiplicities* $m_{ij}=(X_i,X_j)$ *of each pair of branches. In particular the saturated curve* $(X^s,0)$ *is an invariant (up to isomorphism) of the equisingularity class of* $(X,0)$.

This implies that every member of the equisingularity class of a germ of reduced plane curve $(X, 0) \subset (\mathbf{C}^2, 0)$ can be obtained by a generic projection of the Lipschitz saturation $(X^s, 0)$ of any one of them. The proof of the Proposition involves a lot of calculations and can be found in the references. For this reason we would rather describe how to calculate the saturated curve $(X^s, 0)$. Let us start with the irreducible case:

**Definition 1.4.24** Let $h \in \mathbf{C}\{t\}$ be a power series with coprime exponents. If

$$h = \sum_{j=0}^{\infty} a_j t^j$$

we define the set of exponents of $f$ as $Ex(f) = \{j \in \mathbf{N} \mid a_j \neq 0\}$. And for a germ of analytically irreducible plane curve $(X, 0) \subset (\mathbf{C}^2, 0)$ we define the set of exponents of $O_{X,0}$ as

$$E(O_{X,0}) = \bigcup_{h \in \mathfrak{m}} Ex(h)$$

Note that the semigroup $\Gamma(X)$ of the plane branch $(X, 0)$ is contained in $E(O_{X,0})$.(See [Tei07, Section 8]).

If $(X, 0) \subset (\mathbf{C}^2, 0)$ is irreducible then:

1. For every $j \in E(O_{X,0})$ we have that $t^j \in O^s_{X,0}$.
2. The analytic algebra $O^s_{X,0}$ is monomial, in particular:

$$O^s_{X,0} = \mathbf{C}\{t^j \mid j \in E(O^s_{X,0})\}$$

For a numerical semigroup (i.e., a subsemigroup of $(\mathbf{N}, +)$ with finite complement) there is the concept of saturated semigroup (see [Ro-G09, Chapter 3, Section 2]) which can be described in the following way.

For $A \subset \mathbf{N}$ and $a \in A \backslash \{0\}$ define

$$d_A(a) = \gcd\{x \in A \mid x \leqslant a\}$$

Then a non-empty subset $A \subset \mathbf{N}$ such that $0 \in A$ and $\gcd(A) = 1$ is a saturated numerical semigroup if and only if $a + kd_A(a) \in A$ for all $a \in A\backslash\{0\}$ and $k \in \mathbf{N}$.

*Example 1.4.25* Let $(X, 0) \subset (\mathbf{C}^2, 0)$ be the cusp singularity defined by $y^2 - x^3 = 0$. Its normalization map is $t \mapsto (t^2, t^3)$ and so its semigroup is generated by $\langle 2, 3\rangle$. Since $\Gamma(X) = \mathbf{N}\backslash\{1\}$ then $E(O_{X,0}) = \Gamma(X)$ is a saturated numerical semigroup.

This characterization tells us how to obtain a saturated semigroup from any $A \subset \mathbf{N}$ with $\gcd(A) = 1$, for example the set of exponents $E(O_{X,0})$.
Let $e_0 = \beta_0 = \min\{x \in A\}$ and define

$$\widetilde{A_0} := A \cup \beta_0 \cdot \mathbf{N}$$

In the case of $E(O_{X,0})$ we have that $e_0 = \beta_0$ is the multiplicity of $(X,0)$. Lemma Let $\beta_1 := \min\{x \in A \mid e_0 \,\text{does not divide}\, x\}$ and $e_1 = \gcd\{\beta_0, \beta_1\}$; note that $e_1 = d_A(\beta_1)$. Again define

$$\widetilde{A_1} := \widetilde{A_0} \cup \{\beta_1 + ke_1 \mid k \in \mathbf{N}\}$$

Continuing this way we obtain two sequences of natural numbers $e_0 > e_1 > \cdots > e_g = 1 = \gcd(A)$ and $\beta_0 < \beta_1 < \cdots < \beta_g$ and an associated sequence of subsets of $\widetilde{A_0} \subset \widetilde{A_1} \subset \cdots \subset \widetilde{A_g} \subset \mathbf{N}$ where $\beta_{i+1} := \min\{x \in A \mid e_i \,\text{does not divide}\, x\}$, $e_i := \gcd\{\beta_0, \ldots, \beta_i\} = d_A(\beta_i)$ and

$$\widetilde{A_{i+1}} := \widetilde{A_i} \cup \{\beta_{i+1} + ke_{i+1} \mid k \in \mathbf{N}\}$$

Note that $\widetilde{A} := \widetilde{A_g}$ is a saturated semigroup which is completely determined by its *characteristic sequence* $\{\beta_0, \ldots, \beta_g\}$. Moreover if $t \mapsto \left(t^n, \sum_{i \geqslant n} a_i t^i\right)$ is a Puiseux parametrization of the plane branch $(X,0) \subset (\mathbf{C}^2, 0)$, the characteristic sequence of $E(O_{X,0})$ is the set of characteristic exponents of $(X,0)$ and so it determines its equisingularity class.

**Proposition 1.4.26** (Pham-Teissier), see [P-T69, §4], [B-G-G80, Thm VI.1.6] *For a germ of irreducible plane curve singularity* $(X,0) \subset (\mathbf{C}^2, 0)$ *the Lipschitz saturation* $O^s_{X,0}$ *is given by*

$$O^s_{X,0} = \mathbf{C}\{t^p \mid p \in \widetilde{E(O_{X,0})}\}$$

*In particular* $E\left(O^s_{X,0}\right) = \widetilde{E(O_{X,0})}$.

Let us give a sketch of the proof: we start from a structured parametrization $(t^n, y(t))$ of our branch $X$ as in subsection 1.0.2 and we have to study integral dependence over the ideal $I_\Delta = (t-u)\mathcal{N} := \langle t^n - u^n, y(t) - y(u)\rangle \subset \mathbf{C}\{t,u\}$. Here $\mathcal{N}$ is the primary ideal $\langle \frac{t^n-u^n}{t-u}, \frac{y(t)-y(u)}{t-u}\rangle \mathbf{C}\{t,u\}$[4]. According to what we saw after Theorem 1.4.5, to verify that $g(t) - g(u)$ is integral over $I$, which is the same as $\frac{g(t)-g(u)}{t-u}$ being integral over $\mathcal{N}$, it suffices to verify that its order along any of the branches of a plane curve $C_T \subset \mathbf{C}^2$ defined by $T_1 \frac{t^n-u^n}{t-u} - T_2 \frac{y(t)-y(u)}{t-u} = 0$ is larger than that of the ideal $I$ for $T = (T_1 : T_2)$ in the open set $U \subset \mathbf{P}^1$. Now we claim that the open set $U$ is $T_1 \neq 0$. Indeed, since the order of $y(t)$ is $> n$ all the plane curves $C_T$ with $T_1 \neq 0$ have a tangent cone consisting of $n-1$ lines in general position. It is not difficult then to show (see [Tei74, Chap. II, Lemma 2.6, Proposition 2.7]) that they are equisingular with their tangent cone, and therefore are all equisingular, with

[4] It is shown in [Tei80, 5.2] that the multiplicity, in the sense we saw after Theorem 1.4.7, of the primary ideal $\mathcal{N}$ is equal to twice the invariant $\delta$ which appears in Propositions 1.2.6 and 1.3.6. It is also shown there that $\delta$ is the maximum number of different singular points (then necessarily ordinary double points) which can appear when deforming the parametrization of the plane branch. Both results extends to reducible curves. One can define an analogous ideal $\mathcal{N}$ for a non-plane branch but then, in view of Theorem 1.4.8 and Proposition 1.4.11, its multiplicity is twice the $\delta$ invariant of a generic plane projection and no longer the classical $\dim_{\mathbf{C}} \frac{\overline{O_{X,0}}}{O_{X,0}}$ in this case.

simultaneous normalization. So the curve $\frac{t^n-u^n}{t-u}=0$ is in $U$, and its branches are the lines $u=\omega t,\ \omega\in\mu_n\backslash\{1\}$, which means that a function $g(t)\in\mathbf{C}\{t\}$ is in the saturation if and only if we have

$$\mathrm{ord}_t(g(t)-g(\omega t))\geqslant \mathrm{ord}_t(y(t)-y(\omega t)) \text{ for all } \omega\in\mu_n\backslash\{1\}.$$

The result now follows easily from what we saw at the end of subsection 1.0.2 about the orders of the $y(t)-y(\omega t)$.
It may be interesting to remark here that this construction gives an intrinsic (coordinate free) definition of the Puiseux characteristic as the set of valuations (orders of vanishing) of the ideal $\mathcal{N}$ along the irreducible components of the exceptional divisor of the normalized blowing up of $\mathcal{N}$ in $\overline{X}\times\overline{X}$. For more details, see [P-T69, §4] and [B-G-G80, Thm VI.1.6].

*Example 1.4.27* Let $(X,0)\subset(\mathbf{C}^2,0)$ be the plane branch with normalization map:

$$\begin{aligned}\eta:(\mathbf{C},0)&\longrightarrow(X,0)\\ t&\mapsto(t^4,t^6+t^7)\end{aligned}$$

Then the exponent set $E(\mathcal{O}_{X,0})$ contains the semigroup $\Gamma(X)=\langle 4,6,13\rangle\mathbf{N}$ but by definition it also contains 7. Now $\beta_1=6$ and $e_1=2$ so

$$\widetilde{E_1}=E(\mathcal{O}_{X,0})\cup\{6+2k\mid k\in\mathbf{N}\}$$

In the next step $\beta_2=7$ and $e_2=1$ so $g=2$ and we get the saturated semigroup

$$\widetilde{E_2}=\widetilde{E_1}\cup\{7+k\mid k\in\mathbf{N}\}$$

Note that $\widetilde{E(\mathcal{O}_{X,0})}=\langle 4,6,7,9\rangle\mathbf{N}$ and so we have the normalization map for the Lipschitz saturation $(X^s,0)\subset(\mathbf{C}^4,0)$ given by:

$$\begin{aligned}\eta^s:(\mathbf{C},0)&\longrightarrow(X^s,0)\\ t&\mapsto(t^4,t^6,t^7,t^9)\end{aligned}$$

By making the change of coordinates in $(\mathbf{C}^4,0)$, $(x,y,z,w)\mapsto(x,y+z,z,w)$ we can view the Lipschitz saturation map

$$\zeta:(X^s,0)\to(X,0)$$

as the projection on the first two coordinates as before.

*Remark 1.4.28* (see [Tei82, Chap. I, theorem 6.3.1], [B-G-G80, Appendice]) A more concrete way of seeing that all plane branches with the same Puiseux characteristic are generic plane projections of a single space curve is to go back to the notations of subsection 1.0.2 to write down explicitly the saturation of a plane branch $(X,0)$ with given characteristic $(n,\beta_1,\ldots,\beta_g)$: it is isomorphic to the monomial curve with analytic algebra

$$\mathbf{C}\{t^n,t^{2n},\ldots,t^{\beta_1},t^{\beta_1+e_1},\ldots,t^{\beta_2},t^{\beta_2+e_2},\ldots,t^{\beta_3},\ldots,t^{\beta_g},t^{\beta_g+1},\ldots\},$$

where $n = e_0 = \beta_0$ as above. The semigroup generated by these exponents, which are those of $\widetilde{E(O_{X,0})}$, is finitely generated by Dickson's Lemma and because the Puiseux exponents are coprime its complement in $\mathbf{N}$ is finite. For more details on the saturation of semigroups we refer to [Ro-G09, Chapter 3, Section 2].
As we saw in subsection 1.0.2, up to isomorphism, the image of this monomial curve by a generic linear projection can be parametrized by $x = t^n, y = \sum_{p \in \widetilde{E(O_{X,0})} \setminus \{n\}} a_p t^p$. Now we see that the generic projections are precisely those which are such that the coefficient of $t^n$ is $\neq 0$ and for $p = \beta_1, \ldots, \beta_g$ we have $a_p \neq 0$, which means that the projection has characteristic $(n, \beta_1, \ldots, \beta_g)$.

Remark that, except if $n = 2$, the semigroup of integers generated by the exponents of the monomials belonging to the saturation $O^s_{X,0}$ is different from the semigroup $\Gamma$ we saw in subsection 1.0.2.

When $(X, 0)$ is not irreducible it is a bit more complicated, nevertheless the Lipschitz saturation $O^s_{X,0}$ can be described in the following way:

**Theorem 1.4.29** (see [P-T69, §4] and [B-G-G80, Thm VI.2.2]) *Let $O_{X,0}$ be the analytic algebra of a reduced plane curve $(X, 0) = (X_1, 0) \cup \ldots \cup (X^r, 0)$ with normalization $\overline{O_{X,0}} = \mathbf{C}\{t_1\} \oplus \cdots \oplus \mathbf{C}\{t_r\}$. We may assume that the image of $x$ in $\overline{O_{X,0}}$ is $\left(t_1^{n_1}, \ldots, t_r^{n_r}\right)$ where $n_i$ is the multiplicity of the branch $(X_i, 0)$. Let $\mu$ be the least common multiple of $\{n_1, \ldots, n_r\}$. Then the element $\underline{h} = (h_1, \ldots, h_r) \in \overline{O_{X,0}}$ is in the Lipschitz saturation $O^s_{X,0}$ if and only if the following two conditions are satisfied:*

1. *For every $j \in \{1, \ldots, r\}$ we have that $h_j \in O^s_{X_j,0}$.*
2. *For every $\mu$-th root of unity $\epsilon$ and every couple $i \neq j$ we have the inequality*

$$m_{i,j,\epsilon}(h) \geqslant m_{i,j,\epsilon} := \inf_{\underline{g} \in \overline{O_{X,0}}} \left\{ \nu_\tau \left( g_i(\tau^{\mu/n_i}) - g_j([\epsilon\tau]^{\mu/n_j}) \right) \right\}$$

*where $m_{i,j,\epsilon}(\underline{h}) = \nu_\tau \left( h_i(\tau^{\mu/n_i}) - h_j([\epsilon\tau]^{\mu/n_j}) \right)$ and $\nu_\tau$ is the valuation of $\mathbf{C}\{\tau\}$ given by the order of the series. The number $m_{i,j,\epsilon}$ depends only on the characteristic exponents and the intersection multiplicity of the branches $X_i$ and $X_j$.*

*Example 1.4.30* Let $(X, 0) = (X_1, 0) \cup (X_2, 0)$ be the plane curve with normalization map:

$$\begin{aligned} \eta : (\mathbf{C}, 0) \sqcup (\mathbf{C}, 0) &\longrightarrow (X, 0) \\ t_1 &\mapsto (t_1^4, t_1^6 + t_1^7) \\ t_2 &\mapsto (t_2^3, t_2^5) \end{aligned}$$

In the previous example we already calculated the Lipschitz saturation $O^s_{X_1,0} = \mathbf{C}\{t_1^4, t_1^6, t_1^7, t_1^9\}$ and following the algorithm we get the Lipschitz saturation $O^s_{X_2,0} = \mathbf{C}\{t_2^3, t_2^5, t_2^7\}$. Since the branches are tangent, their intersection multiplicity is greater than the product of their multiplicities and it is equal to order of the series in $t_1$

obtained by substituting the parametrization of $(X_1,0)$ in the equation $y^3 - x^5 = 0$ defining $(X_2,0)$. In this case it is equal to 18.

By definition $\mu = \operatorname{lcm}\{3,4\} = 12$ and it is not hard to prove that for any 12−th root of unity $\epsilon$

$$\begin{aligned} m_{1,2,\epsilon} &= \nu_\tau\left(y_1(\tau^3) - y_2([\epsilon\tau]^4)\right) \\ &= \nu_\tau\left(\tau^{18} + \tau^{21} - \epsilon^8\tau^{20}\right) = 18 \end{aligned}$$

So from the Theorem 1.4.29 we have that $h = (h_1(t_1), h_2(t_2))$ is in $O^s_{X,0}$ if and only if $h_1(t_1) \in O^s_{X_1,0}$, $h_2(t_2) \in O^s_{X_2,0}$ and $m_{1,2,\epsilon}(h) \geqslant 18$. For example if $h = (t_1^4, t_2^5)$ then

$$\begin{aligned} m_{1,2,\epsilon}(h) &= \nu_\tau\left((\tau^3)^4 - ([\epsilon\tau]^4)^5\right) \\ &= \nu_\tau(\tau^{12} - \epsilon^8\tau^{20}) = 12 \Rightarrow h \notin O^s_{X,0} \end{aligned}$$

On the other hand if $h = (t_1^6 + t_1^7, t_2^5)$ then

$$\begin{aligned} m_{1,2,\epsilon}(h) &= \nu_\tau\left((\tau^3)^6 + (\tau^3)^7 - ([\epsilon\tau]^4)^5\right) \\ &= \nu_\tau(\tau^{18} + \tau^{21} - \epsilon^8\tau^{20}) = 18 \Rightarrow h \in O^s_{X,0} \end{aligned}$$

We will finish this section with the following consequence of the Theorem:

**Corollary 1.4.31** (see [B-G-G80, VI.3.7]) *Let $(X,0) = (X_1,0) \cup \ldots \cup (X_r,0)$ be a reduced plane curve with normalization $\overline{O_{X,0}} = \mathbf{C}\{t_1\} \oplus \cdots \oplus \mathbf{C}\{t_r\}$. If $\Pi_j : \overline{O_{X,0}} \to \mathbf{C}\{t_j\}$ denotes the canonical projection to the $j$−th factor then*

$$\Pi_j(O^s_{X,0}) = O^s_{X_j,0}.$$

### 1.4.4 Application to local polar curves

Let $(X,0) \subset (\mathbf{C}^N,0)$ be a reduced equidimensional germ of complex analytic space. Consider linear projections $\pi\colon \mathbf{C}^N \to \mathbf{C}^2$ and the critical locus of $\pi$ restricted to the smooth part $X^0$ of $X$. It is proved in [L-T81], where the theory of (absolute)[5] local polar varieties was initiated, that for a Zariski dense open set $U$ in the space $G(N, N-2)$ of linear projection, this critical locus is either empty or a reduced curve. The *closure* of this curve in $X$ is an (absolute) *polar curve* of $X$ and is denoted by $P_{d-1}(X,\pi)$ where $d$ is the dimension of $X$. It is also denoted by $P_{d-1}(X,D)$, where $D = \ker\pi$. These curves play an important role in the local study of singularities, and

[5] This precision refers to a distinction between absolute and relative polar varieties, which is not relevant here but should be mentioned to avoid confusions. See [Tei82, Chap. IV, p. 417]

especially in the study of the Lipschitz geometry of surfaces. See [L-T81], [Tei82, Chap. IV] for more details.

Of course, if it is not empty, $P_{d-1}(X,\pi)$ varies with the projection $\pi \in U$ and *a priori* it could be that $\pi$ remains constantly a non generic projection for $P_{d-1}(X,\pi)$. That seems unlikely but still we need a proof for the following:

**Theorem 1.4.32** (See [Tei82, Chap. V, Lemme 1.2.2]) *Given $(X,0) \subset (\mathbf{C}^N,0)$ as above and assuming that $P_{d-1}(X,\pi)$ is a reduced curve for $\pi \in U \subset G(N,N-2)$, there exists a non empty Zariski open set $V \subset U$ such that for $\pi \in V$, the projection $\pi\colon \mathbf{C}^N \to \mathbf{C}^2$ is a generic projection for the curve $P_{d-1}(X,\pi) \subset \mathbf{C}^N$.*

The proof, which we only sketch, gives an example of the notion of Lipschitz equisaturation , which is found in [P-T69, §6]. Fixing coordinates $z_1,\dots,z_N$ on $\mathbf{C}^N$ and $x,y$ on $\mathbf{C}^2$, we can parametrize by $\mathbf{C}^{2(N-2)}$ a dense open set of the space of linear projections $\mathbf{C}^N \to \mathbf{C}^2$ as follows:

$$x = z_1 + \sum_3^N a_i z_i,\quad y = z_2 + \sum_2^N b_i z_i\ ,\ (\underline{a},\underline{b}) \in \mathbf{C}^{2(N-2)}.$$

To simplify notations while keeping the ideas, we assume that $X$ is a hypersurface defined by $f(z_1,\dots,z_N) = 0$. One can also consult [B-H80, Lemme 3.7] which gives the proof for isolated singularities of surfaces in $\mathbf{C}^3$.
For any series $h(z_1,\dots,z_N) \in \mathbf{C}\{z_1,\dots,z_N\}$ let us denote by $h_{\underline{a},\underline{b}}$ the series

$$h_{\underline{a},\underline{b}}(\underline{z},\underline{a},\underline{b}) = h(x - \sum_3^N a_i z_i, y - \sum_3^N b_i z_i, z_3,\dots,z_N).$$

The equation $f_{\underline{a},\underline{b}} = 0$ defines a germ of hypersurface $\mathcal{Z}$ in $\mathbf{C}^N \times \mathbf{C}^{2(N-2)}$ and if we consider the projection $\pi\colon \mathbf{C}^N \times \mathbf{C}^{2(N-2)} \to \mathbf{C}^2 \times \mathbf{C}^{2(N-2)}$ defined by

$$x = z_1 + \sum_3^N a_i z_i,\quad y = z_2 + \sum_2^N b_i z_i,\ \underline{a} = \underline{a},\ \underline{b} = \underline{b},$$

and the closure of its critical locus on the non singular part of $\mathcal{Z}$, we obtain a subspace which, over a Zariski open subset of $\mathbf{C}^{2(N-2)}$, contains the family of polar curves associated to the family of projections $\pi_{\underline{a},\underline{b}}$ defined by $x = z_1 + \sum_3^N a_i z_i,\ \ y = z_2 + \sum_2^N b_i z_i$. Over a possibly smaller Zariski open subset $V$ of $\mathbf{C}^{2(N-2)}$ this family of curves is equisingular in the sense that it has a simultaneous parametrization. The number $r$ of irreducible components of $\mathcal{Z}$ at points of $\{0\} \times V \subset \mathbf{C}^N \times V$ is constant and after choosing as origin of $\mathbf{C}^{2(N-2)}$ a point of $V$ we can parametrize each irreducible component in a neighborhood of $\{0\} \times \{0\}$ by:

$$z_1 = t_\ell^{n_\ell}, z_2 = \upsilon(t_\ell,\underline{a},\underline{b}), z_i = \zeta_i(t_\ell,\underline{a},\underline{b}),$$

with $\upsilon(t_\ell,\underline{a},\underline{b}), \zeta_i(t_\ell,\underline{a},\underline{b}) \in \mathbf{C}\{t_\ell,\underline{a},\underline{b}\}$ for $i = 3,\dots,N$. The normalization of $O_{\mathcal{Z},0}$ being $\overline{O_{\mathcal{Z},0}} = \prod_{i=1}^r \mathbf{C}\{t_\ell,\underline{a},\underline{b}\}$.

By definition of $\mathcal{Z}$ we have for each $\ell = 1, \ldots, r$ the identity in $\mathbf{C}\{t_\ell, \underline{a}, \underline{b}\}$

$$f(t^{n_\ell} - \sum_3^N a_i \zeta_i(t_\ell, \underline{a}, \underline{b}), \upsilon(t_\ell, \underline{a}, \underline{b}) - \sum_3^N a_i \zeta_i(t_\ell, \underline{a}, \underline{b}), \zeta_3(t_\ell, \underline{a}, \underline{b}), \ldots, \zeta_N(t_\ell, \underline{a}, \underline{b})) \equiv 0.$$

Differentiating $f_{\underline{a},\underline{b}} = 0$ with respec to $z_i$ gives the following equations on $\mathcal{Z}$:

$$-a_i \frac{\partial f_{\underline{a},\underline{b}}}{\partial z_1} - b_i \frac{\partial f_{\underline{a},\underline{b}}}{\partial z_2} + \frac{\partial f_{\underline{a},\underline{b}}}{\partial z_i} \equiv 0,$$

for $i = 3, \ldots, N$. which by definition are satisfied on the polar curve.

Differentiating the first identity with respect to $b_k$ and taking into account the second set of identities, we obtain that the equation

$$\left(\zeta_k(t_\ell, \underline{a}, \underline{b}) - \frac{\partial \upsilon(t_\ell, \underline{a}, \underline{b})}{\partial b_k}\right) \frac{\partial f_{\underline{a},\underline{b}}}{\partial z_2} = 0$$

must be satisfied in each $\mathbf{C}\{t_\ell, \underline{a}, \underline{b}\}$. By general transversality results found in [L-T81, Cor. 4.1.6] and [Tei82, Chap. IV, 5.1], $\frac{\partial f_{\underline{a},\underline{b}}}{\partial z_2}$ does not vanish because it would entail a lack of $C_3$ transversality (see Definition 1.3.1) of the polar curve with the kernel of the projection which defines it. So we must have on $\overline{\mathcal{Z}}$ the identity $z_k = \frac{\partial \upsilon}{\partial b_k}$.
By [Tei82, Proposition 6.4.2], after perhaps shrinking $V$ to a smaller Zariski open dense subset $V_1$ of $\mathbf{C}^{2(N-2)}$ we have that over $V_1$ the family $\mathcal{Z}_1$ of plane curves given parametrically by the parametrizations $x = t^{n_\ell}, y = \upsilon(t_\ell, \underline{a}, \underline{b})$, which consists of the plane projections of our polar curves, is equisaturated. This implies that the derivations $\frac{\partial}{\partial b_k}$ of $\mathbf{C}\{\underline{a}, \underline{b}\}$ extend to derivations $D_k$ of $\overline{O_{\mathcal{Z}_1,(0,v)}} = \overline{O_{\mathcal{Z},(0,v)}} = \prod_{i=1}^r \mathbf{C}\{t_\ell, \underline{a}, \underline{b}\}$ into itself which preserve the relative saturated ring (see [P-T69]). Since of course the functions $\upsilon(t_\ell, \underline{a}, \underline{b})$ belong to the relative saturation of $\overline{O_{\mathcal{Z}_1,(0,v)}}$, so do the $\zeta_k(t_\ell, \underline{a}, \underline{b}))$ which are their images by $D_k$. But $\zeta_k$ belonging to this relative saturation means precisely that for $v \in V_1$, the saturations of the rings $\overline{O_{\mathcal{Z}_1(v)}}$ and $\overline{O_{\mathcal{Z}(v)}}$ of the fibers over $v$ of $\mathcal{Z}$ and $\mathcal{Z}_1$ are equal for $v \in V_1$, which is the condition for $C_5$ genericity according to Proposition 1.4.11.

The fact that the plane projection of a generic polar curve by the map which defines it is generic plays an important role in the following three domains: the comparison of Zariski equisingularity and Whitney equisingularity for surfaces (see [B-H80], [N-P1]), the comparison of Zariski equisingularity and Lipschitz equisingularity for surfaces (see [N-P1], [PaP]), the numerical characterization of Whitney equisingularity (see [Tei82, Chap. V]) and the valuative study of the metric geometry of surface singularities in view of their biLipschitz classification (see [B-F-P]).

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Arturo Giles Flores, Departamento de Matemáticas y Física, Universidad Autónoma de Aguascalientes.
Avenida Universidad 940, Ciudad Universitaria C.P. 20131,
Aguascalientes, Aguascalientes, México.
arturo.giles@cimat.mx

Otoniel Nogueira da Silva, Instituto de Matemáticas UNAM, Unidad Cuernavaca,
Av. Universidad s/n. Col. Lomas de Chamilpa C.P. 62210,

Cuernavaca, Morelos, México.
otoniel@im.unam.mx

Bernard Teissier, Institut mathématique de Jussieu-Paris Rive Gauche,
UP7D - Campus des Grands Moulins, Boite Courrier 7012.
Bât. Sophie Germain, 8 Place Aurélie Nemours, 75205 PARIS Cedex 13, France.
bernard.teissier@imj-prg.fr