About the Jacobi-Perron algorithm

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Lecture at the Łojasewicz 100 Conference,
Krakow, September 2026.

When trying to understand the geometry of a singular varieties by comparing them with non singular singular ones, a classical approach is to project the singular space X to a non singular one of the same dimension and carefully describe the accidents which occur. Of this, Łojasiewicz was a grand master.

This lecture, however, is about another approach, which is to dominate the algebraic variety X, say over a field k, by non singular spaces by proper algebraic morphisms $\pi\colon X' \to X$ where π is an isomorphism over the non singular locus of X. This implies that the field $k(X')$ of rational functions on X' with coefficients in k is the same as $k(X)$; the morphism π is birational.

This led Zariski to define the Zariski-Riemann manifold $\mathcal{X} = \varprojlim X'$, the projective limit of all proper birational maps over X.

It is the generalisation of the normalisation of curves, which produces Riemann surfaces. In dimensions ≥ 2, it is much more complicated, not even a scheme.

So the question arises to factor the natural map $\Pi\colon \mathcal{X} \to X$ through a birational map $\Pi\colon \mathcal{X} \to X' \to X$, where X' is non singular. In other words, approximate $\mathcal{X}$ by non singular algebraic varieties X' over X.

In characteristic zero this, and much more, was proved by Hironaka. In positive characteristic and dimension > 3 it is not known.

Even though it is not a variety, the space $\mathcal{X}$ has local rings which are valuation rings.
A valuation ring is a commutative ring without zero divisors R_ν such that for any $x, y \in R_\nu$, both $\neq 0$, either x divides y or y divides x. It is equivalent to saying that any non zero finitely generated ideal $(x_1, \ldots, x_n) \subset R_\nu$ is also generated by one of the a_i.

This implies that R_ν is a local ring: the non invertible elements form an ideal m_ν, which is the only maximal ideal.
If R_ν is one of them, corresponding to a point $\nu \in \mathcal{X}$, the case where ν is a *rational* point of $\mathcal{X}$, i.e., $R_\nu/m_\nu = k$ is already interesting. It will be assumed from now on.

A valuation ring is noetherian only if it is a one dimensional regular local ring. The valuation is the order with respect to the maximal ideal.

The quotient $(R_\nu \setminus \{0\})/U_\nu$ of the multiplicative semigroup of R_ν by the group of units of R_ν, is an abelian semigroup $\Phi_{\geq 0}$ totally ordered by the relation $\overline{x} \leq \overline{y}$ if $y/x \in R_\nu$. The natural morphism

$$\nu\colon R_\nu \setminus \{0\} \to (R_\nu \setminus \{0\})/U_\nu = \Phi_{\geq 0},$$

often completed by mapping $0 \in R_\nu$ to an element ∞ larger than any element of $\Phi_{\geq 0}$, is called the valuation associated to R_ν.
It satisfies

$$\nu(xy) = \nu(x) + \nu(y)$$

$$\nu(x+y) \geq \min(\nu(x), \nu(y))$$

$$\nu(x) = \infty \leftrightarrow x = 0.$$

This extends to the field of fractions K of R_ν by $\nu(x/y) = \nu(x) - \nu(y)$.

The totally ordered abelian group Φ of which $\Phi_{\geq 0}$ is the non negative part is called the value group of the valuation and consists of the values taken by ν on K^*, where K is the field of fractions of R_ν.

Thinking of the field of fraction K of R as a field of algebraic functions on an algebraic variety one may think of ν as a generalized order of vanishing, or of pole, with values in Φ, of the rational function x along "something". In dimension one it is the order of vanishing, or of the pole, of a rational function at a point of the curve.

In higher dimension it can be, for example in the case where $K = k(x, y)$, the field of rational functions in two variables, the "order of vanishing" of a rational function $h(x, y)$ along a curve parametrised by $x = t, y = t^\tau$, where τ is a positive irrational number" : the order in t of $h(t, t^\tau)$.
It can also be the order of vanishing along a "curve"

$$x(t) = \sum_{s \in A} a_s t^s, \quad y(t) = \sum_{s \in B} b_s t^s,$$

where A, B are well ordered subsets of $\mathbf{Q}_{\geq 0}$ whose denominators are unbounded, with say $x(0) = y(0) = 0$. This ensures that the series $h(x(t), y(t) \neq 0$ and it has an least exponent, which is the valuation of h.

The possible value groups of valuations of the field of rational functions can be very diverse: Any subgroup of $\mathbf{Q}$, with the induced order, $(\mathbf{Z}^r)_{lex}$, $\mathbf{Z}\tau_1 \oplus \mathbf{Z}\tau_2 \oplus \cdots \oplus \mathbf{Z}\tau_r$, where the $\tau_i \in \mathbf{R}_{>0}$ are rationally independent.

For valuations of a field of algebraic functions in n variables over k, or of an n-dimensional local domain R if we assume that the valuation corresponds to a rational point of $\mathcal{X}$, we have that the rational rank $r(\nu) = \dim_{\mathbf{Q}} \Phi \otimes_{\mathbf{Z}} \mathbf{Q}$ is $\leq n$ and the archimedean rank, the least integer $h(\nu)$ such that Φ is an ordered subgroup of $(\mathbf{R}^h)_{lex}$, is $\leq r(\nu)$.
If $r(\nu) = \dim R$, we have an Abhyankar valuation of R and the value group is $\mathbf{Z}^r$.

A totally ordered abelian group has two main invariants: Its rational rank, $r(\Phi) = \dim_{\mathbf{Q}} \Phi \otimes_{\mathbf{Z}} \mathbf{Q}$, and its archimedean rank which we just saw. Archimedean rank one means that Φ can be embedded in $\mathbf{R}$ as an ordered group.

An aside:

There is a topology on the set of monomial orderings of $\mathbf{Z}^r$, where a basis of open sets is indexed by pairs of distinct points $a, b \in \mathbf{Z}^r$ and elements of the basis are $U_{a,b} = \{\prec / a \prec b\}$. For $r \geq 2$ this topological space is homeomorphic to the Cantor set. This is due to Sikora.

A rational point $\nu \in \mathcal{X}$ has an image in X and an image x' in each birational modification X' of X. At each of these points the variety has a local ring $R_{X',x'}$ so that we have an inductive system of local rings $R_{X',x'} \subset R_\nu$.
One issue is to show that there are regular local rings in this inductive system, and another issue, on which I concentrate today, is to find how to produce nested sequences of noetherian local rings growing from $R_{X,x}$ to R_ν.

A simplified form of a local ring R endowed with a valuation ν is the associated graded ring. Let us define the ideals:

$$\mathcal{P}_\phi(R) = \{x \in R | \nu(x) \geq \phi\}, \quad \mathcal{P}^+_\phi(R) = \{x \in R | \nu(x) > \phi\}, \quad \textit{and}$$

$$\mathrm{gr}_\nu R = \bigoplus_{\phi \in \Phi_{\geq 0}} \mathcal{P}_\phi(R)/\mathcal{P}^+_\phi(R).$$

In the sequel we always assume that R, our $R_{X,x}$, satisfies $R \subset R_\nu$, so that ν takes only positive values on R, and that $m_\nu \cap R$ is the maximal ideal $m = m_{X,x}$ of R, with $R/m = R_\nu/m_\nu = k$ (rational valuation). Thus, $\mathrm{gr}_\nu R$ is a graded k-algebra and so is $\mathrm{gr}_\nu R_\nu$.

For example, if k is algebraically closed, $\mathrm{gr}_\nu R_\nu \simeq k[t^{\Phi_{\geq 0}}]$ the semigroup algebra of the group Φ with coefficients in k.

More generally, if k is algebraically closed, denoting by Γ the semigroup $\nu(R \setminus \{0\}) \subset \Phi_{\geq 0}$, the k-algebra $\mathrm{gr}_\nu R$ is isomorphic to $k[t^\Gamma]$. It is just the subalgebra of $\mathrm{gr}_\nu R_\nu$ consisting of homogeneous components with degree in Γ.

If $R \subset \mathbf{C}\{t\}$ is the ring of a complex analytic branch, with the t-adic valuation ν_t, then

$$\mathrm{gr}_\nu R \simeq k[t^{\gamma_1}, \ldots, t^{\gamma_g}],$$

where the $\gamma_i \in \mathbf{N}$ minimally generate the numerical semigroup of values $\nu_t(R \setminus \{0\})$.

The k-algebra $\mathrm{gr}_\nu R$ is not in general finitely generated, but its Krull dimension is the rational rank r. For rational valuations, Abhyankar's inequality is

$$\dim \mathrm{gr}_\nu R \leq \dim R.$$

The philosophy is that for rational valuations, with k algebraically closed, the semigroup of values contains all the information necessary to produce local uniformization.

The Jacobi-Perron algorithm is a generalisation to higher dimensions of the continued fraction expansion of a real number viewed geometrically. It has a role in the approximation of valuation rings by local rings of non singular algebraic varieties and I will try to explain the connection.

If

$$\tau = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ldots a_k + \cfrac{1}{\ldots}}}}$$

shortened into $\tau = [a_0, a_1, \ldots, a_k, \ldots]$ is the continued fraction expansion of a positive real number τ, and the $\frac{p_k}{q_k} = [a_0, a_1, \ldots, a_k]$ are the *reduced fractions*.

The construction of the integers a_k is as follows: $a_0 = \lfloor \tau \rfloor$, the integral part of τ, $a_1 = \lfloor \frac{1}{\tau - a_0} \rfloor$.
Set $\tau^{(1)} = \frac{1}{\tau - a_0}$ and $a_1 = \lfloor \tau^{(1)} \rfloor$, then $\tau^{(2)} = \frac{1}{\tau^{(1)} - a_1}$, and so on.

A little experimentation gives: $p_0 = a_0, q_0 = 1;\ p_1 = a_0 a_1 + 1, q_1 = a_1,\ p_2 = a_0 + a_2(a_0 a_1 + 1), q_2 = 1 + a_2 a_1$ so that we are led to *conjecture* the general rule of formation for the numerators and denominators of the reduced fractions

$$p_{k+1} = p_{k-1} + a_{k+1} p_k, \quad q_{k+1} = q_{k-1} + a_{k+1} q_k,$$

with the initial conditions $p_0 = a_0,\ q_0 = 1$.
It follows by induction that

$$p_k q_{k+1} - p_{k+1} q_k = (-1)^k.$$

It is time for a geometric interpretation, which goes back to H.J. Stephen Smith

We see that the vectors $A_k = (p_k, q_k)$ and $A_{k+1} = (p_{k+1}, q_{k+1})$ generate in the first quadrant a *regular* cone σ_k, which means that they for a basis of the integral lattice, which contains the vector $\mathbf{w} = (1, \tau)$.

Since the a_k are all ≥ 1, for every k and $i \geq 1$, the vectors A_{k+1+i} are strictly inside the cone σ_k.

Let now $\tau_1, \ldots, \tau_m$ be positive real numbers. Let us first assume that they are rationally independent. We are going to approximate the vector $\mathbf{w} = (\tau_1, \ldots, \tau_m) \in \mathbf{R}^m_{\geq 0}$ by cones generated by integral vectors. forming a basis of the integral lattice.

Define $\tau_1^{(1)}, \dots, \tau_m^{(1)}$ as follows:

$$\tau_1 = \tau_m^{(1)}, \tau_2 = \tau_1^{(1)} + a_2^{(0)} \tau_m^{(1)}, \dots, \tau_m = \tau_{m-1}^{(1)} + a_m^{(0)} \tau_m^{(1)},$$

with $a_j^{(0)} = \lfloor \frac{\tau_1}{\tau_j} \rfloor$.
Now repeat starting from from $\tau_1^{(1)}, \dots, \tau_m^{(1)}$ and iterate.
After h steps, one has written

$$\tau_i = A_i^{(h)} \tau_1^{(h)} + \dots + A_m^{(h)} \tau_m^{(h)},$$

with $\tau_j^{(h)} \geq 0$.

In other words,

$$\mathbf{w} = \tau_1^{(h)} A^{(h)} + \tau_2^{(h)} A^{(h+1)} + \cdots + \tau_m^{(h)} A^{(h+m-1)},$$

The coefficients $A_i^{(j)}$ are non negative integers, and the matrix of the vectors

$$A^{(h)}, A^{(h+1)}, \ldots, A^{(h+m-1)}$$

has determinant $(-1)^{h(m-1)}$.

The $\tau_i^{(h)}$ are the coordinate of $\mathbf{w}$ in the coordinate system $A^{(h)}, A^{(h+1)}, \ldots, A^{(h+m-1)}$.

If we apply this to $(1,\tau) \in \mathbf{R}^2_{\geq 0}$. i.e, $m = 2$ we see that it produces the (p_k, q_k) as vectors $A^{(h)}$.

Moreover, as h grows the directions in $\mathbf{P}^{m-1}(\mathbf{R})$ of the vectors $A^{(h)}$ tend to the direction of w. So we have a sequence of vectors $A^{(h)}$ with positive integral coordinates whose directions in $\mathbf{P}^{m-1}(\mathbf{R})$ spiral to the direction of w and such that any consecutive m of them as above form a basis of the integral lattice such that $\mathbf{w}$ is contained in the convex cone $\sigma^{(h)} = \langle A^{(h)}, A^{(h+1)}, \ldots, A^{(h+m-1)} \rangle$ which they generate.

The convex dual of a cone $\sigma \subset \mathbf{R}^m$ is the set of $(a_1, \ldots, a_m) \in \check{\mathbf{R}}^m$ such that

$$\sum_1^m a_i x_i \geq 0 \ \ \forall x \in \sigma$$

It is a cone, generated by the linear forms which vanish on the faces of σ.

Note that by construction we have for each $h \geq 1$ the equality

$$A^{(h)} - A^{(h+m)} + a_2^{(h)} A^{(h+1)} + \cdots + a_m^{(h)} A^{(h+m-1)} = 0,$$

which shows, since the $a_j^{(h)} = [\tau_j^{(h)} / \tau_1^{(h)}]$ are non-negative, that we have

$$A^{(h+m)} \in \sigma^{(h)} = \langle A^{(h)}, \ldots, A^{(h+m-1)} \rangle$$

and therefore $\sigma^{(h+1)} \subset \sigma^{(h)}$, that is $\check{\sigma}^{(h)} \subset \check{\sigma}^{(h+1)}$.

Geometrically, starting from the first quadrant, or indeed any regular rational cone $\sigma^{(0)}$ in the first quadrant containing the vector $\mathbf{w}$, the Jacobi-Perron algorithm builds a nested sequence

$$\sigma^{(0)} \supset \sigma^{(1)} \supset \sigma^{(2)} \supset \cdots \supset \sigma^{(h)} \supset \cdots \ni \mathbf{w}$$

of regular rational cones

$$\sigma^{(h)} = \langle A^{(h)}, A^{(h+1)}, \ldots A^{(h+m-1)} \rangle,$$

converging to $\mathbf{w}$.

Let us now consider the subgroup $\Phi \subset \mathbf{R}$ generated by $\tau_1, \ldots, \tau_m$, with the induced order.

We shall be interested in the totally ordered semigroup $\Phi_{\geq 0}$ of non negative elements.
It coincides with the m-uples of integers $a_1, \ldots, a_m$ such that $\sum_{i=1}^{m} a_i \tau_i \geq 0$.

The convex dual $\check{\sigma}^{(h)}$ of $\sigma^{(h)}$ is contained in the half space $\check{\mathbf{w}} = \sum_{i=1}^{m} a_i \tau_i \geq 0$, the integral points of which form the semigroup $\Phi_{\geq 0}$. Since $\sigma^{(h)}$ is a regular cone, so is its convex dual.

So we have a nested sequence of regular rational cones

$$\check{\sigma}^{(0)} \subset \check{\sigma}^{(1)} \subset \cdots \subset \check{\sigma}^{(h)} \subset \cdots \subset \check{\mathbf{w}}$$

and a nested sequence of free semigroups isomorphic to $\mathbf{N}^m$

$$\check{\sigma}^{(0)} \cap \mathbf{Z}^m \subset \check{\sigma}^{(1)} \cap \mathbf{Z}^m \subset \cdots \subset \check{\sigma}^{(h)} \cap \mathbf{Z}^m \subset \cdots \subset \check{\mathbf{w}} \cap \mathbf{Z}^m = \Phi_{\geq 0}$$

with $\mathbf{Z}$-linear injections.

The algebra of the free semigroup $\check{\sigma}^{(h)} \cap \mathbf{Z}^m$ is a polynomial algebra $k[x_1^{(h)}, \ldots, x_m^{(h)}]$ contained in $k[t^{\Phi_{\geq 0}}]$, and since by assumption there are no integral points on the hyperplane $\sum_{i=1}^m a_i \tau_i = 0$ except the origin, the semigroup $\Phi_{\geq 0}$ is the union of the $\check{\sigma}^{(h)} \cap \mathbf{Z}^m$ as $h \to \infty$.

This proves that $k[t^{\Phi_{\geq 0}}]$ is the union, or direct limit, of nested polynomial subalgebras.

The morphisms between these polynomial algebras correspond by duality to the expressions of the $A^{(h+m)}$ as linear combinations with non negative integral coefficients of $(A^{(h)}, \ldots, A^{(h+m-1)})$ and are therefore monomial as announced.

If we now consider a group with one more generator $\tau_{m+1} > 0$ which is rationally dependent on $\tau_1, \ldots, \tau_m$, the new weight vector $\mathbf{w} = (\tau_1, \ldots, \tau_m, \tau_{m+1}) \in \mathbf{R}^{m+1}$ is contained in a rational simplicial cone $\sigma \subset \mathbf{R}^{m+1}$ generated by m integral vectors $v_1, \ldots, v_m$ of the first quadrant forming part of a basis of the integral lattice.

In the end, we can prove again that for any totally ordered abelian group of rank one and rational rank r, one has the same approximation of $\Phi_{\geq 0}$ by free subsemigroups of rank r. For example:

$$\mathbf{N} = \mathbf{N}_{(1)} \to \mathbf{N}_{(2)} \cdots \to \mathbf{N}_{(h)} \to \mathbf{N}_{(h+1)} \to \cdots \to \mathbf{Q}_{\geq 0}.$$

By induction on the archimedean rank, one can extend this result to any totally ordered group of finite rational rank, in an algorithmic form. The idea is based on the approximation of $\mathbf{Z}^2_{lex,\geq 0}$ by free subsemigroups $\mathbf{N}^2_{(h)} = \{a(0,1) + b(1,-h) \text{ with } (a,b) \in \mathbf{N}^2\}$.

From now on, we change the notation m coming from Jacobi-Perron to the notation r of rational rank.

Let R_ν be the valuation ring of a valuation with value group Φ, of finite rational rank r. Choose r homogeneous elements $x_1^{(0)},\ldots,x_r^{(0)}\in \mathrm{gr}_\nu R_\nu$ whose valuations are rationally independent. The graded algebra $\mathrm{gr}_\nu R_\nu$ is the union of a nested family of polynomial algebras in r variables over $k_\nu = R_\nu/m_\nu$ with maps between them sending each variable to a term:

$$k_\nu[x_1^{(0)},\ldots,x_r^{(0)}]\subset\ldots\subset k_\nu[x_1^{(h)},\ldots,x_r^{(h)}]\subset k_\nu[x_1^{(h+1)},\ldots,x_r^{(h+1)}]\subset\ldots\subset \mathrm{gr}_\nu R_\nu.$$

This can be viewed as implying a graded version of local uniformization: a finitely generated graded k_ν-subalgebra of $\mathrm{gr}_\nu R_\nu$ is contained in a polynomial (i.e., graded regular) subalgebra.

The main result

Let $R \subset R_\nu$ be a rational Abhyankar valuation of an excellent equicharacteristic local domain R of dimension r with algebraically closed residue field k. Assume that R contains a field of representatives of k. Choose r homogeneous elements $x_1^{(0)}, \ldots, x_r^{(0)} \in \mathrm{gr}_\nu R$ whose valuations are rationally independent. The ring R_ν is the union of a nested family indexed by $\mathbf{N}$ of local domains $R^{(h)}$ essentially of finite type over R, which are regular for $h \geq 1$:

$$R = R^{(0)} \subset R^{(1)} \subset \ldots \subset R^{(h)} \subset R^{(h+1)} \subset \ldots \subset R_\nu,$$

where each inclusion $R^{(h)} \subset R^{(h+1)}$, including the first one $R^{(0)} \subset R^{(1)}$, is obtained by localizing at the point picked by the valuation a birational map of finite type, which for $h \geq 1$ is monomial with respect to suitable minimal sets of generators of the maximal ideals of $R^{(h)}$ and $R^{(h+1)}$.

The graded version

The proof of the main result relies on local uniformization of Abhyankar valuations to prove the existence of a regular $R^{(1)}$ with coordinates having rationally independent values, and then follows from what we have seen:
The associated graded rings $\mathrm{gr}_\nu R^{(h)}$ for $h \geq 1$ form a nested system of polynomial rings in r variables over k containing $k[x_1^{(0)}, \ldots, x_r^{(0)}]$, where the inclusion maps

$$k[x_1^{(0)}, \ldots, x_r^{(0)}] \subset \ldots \subset k[x_1^{(h)}, \ldots, x_r^{(h)}] \subset k[x_1^{(h+1)}, \ldots, x_r^{(h+1)}] \subset \ldots \subset \mathrm{gr}_\nu R_\nu$$

send each variable to a term, and the inclusions are birational, with the possible exception of the first one, which is composed:
$k[x_1^{(0)}, \ldots, x_r^{(0)}] \subset \mathrm{gr}_\nu R \subset k[x_1^{(1)}, \ldots, x_r^{(1)}]$.

The proof of he main result is essentially to lift the monomial relations of the nested sequence of polynomial rings to create new monomials to add to the rings $R^{(h)}$ at each step. For details see the proof of Theorem 7.27 of reference [T2] below.

If you want to know more, 1

There are more details in

- §4.3 of:

[T1] B. Teissier, *Valuations, deformations, and toric geometry*, in *Valuation Theory and its applications, Vol. II*, Fields Inst. Commun. 33, AMS., Providence, RI., 2003, 361-459. Available at http://webusers.imj-prg.fr/ bernard.teissier/

See in particular page 395.

-[T2] B. Teissier, *Overweight deformations of affine toric varieties and local uniformization*, in *Valuation theory in interaction*, Proceedings of the second international conference on valuation theory, Segovia–El Escorial, 2011. Edited by A. Campillo, F-V. Kuhlmann and B. Teissier. European Math. Soc. Publishing House, Congress Reports Series, Sept. 2014, 474-565.

There is certainly a geometric relation with Zariski's Perron transform, but it is not clear to me at this point; my viewpoint is different. See

-[EH] Samar ElHitti, *Perron transforms*, Communications in Algebra, 42 (2014) 2003-2045.

If you want to know more, 2

A good reference for the Jacobi-Perron algorithm and its use for local uniformisation is:

-[H-P] W.V.D. Hodge and D. Pedoe, *Methods of Algebraic Geometry, Volume 3*, Cambridge Mathematical Library, Cambridge University Press, 1954.