Chapter 1
Limits of tangents, Whitney stratifications and a Plücker type formula

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Abstract Let X denote a purely d-dimensional reduced complex analytic space. If it has singularities, it has no tangent bundle, which makes many classical and fundamental constructions impossible directly. However, there is a unique proper map $\nu_X\colon NX \to X$ which has the property that it is an isomorphism over the non-singular part X^0 of X and the tangent bundle T_{X^0} lifted to NX by this isomorphism extends uniquely to a vector bundle on NX. For $x \in X$, the set-theoretical fiber $|\nu_X^{-1}(x)|$ is the set of limit directions of tangent spaces to X^0 at points approaching x. The space NX is reduced and equidimensional, but in general singular. If X is a closed analytic subspace of an open set U of $\mathbf{C}^N$, the space NX is a closed analytic subspace of $X \times \mathbf{G}(d,N)$, where $\mathbf{G}(d,N)$ denotes the Grassmannian of d-dimensional vector subspaces of $\mathbf{C}^N$. The rich geometry of the Grassmannian makes it complicated to study the geometry of the map ν_X using intersection theory. There is an analogous construction where tangent spaces are replaced by tangent hyperplanes, and the map ν_X is replaced by the conormal map $\kappa_X\colon C(X) \to X$, where $C(X)$ denotes the conormal space, which is a subspace of $X \times \check{\mathbf{P}}^{N-1}$, where $\check{\mathbf{P}}^{N-1}$ is the space of hyperplanes of $\mathbf{P}^N$, the dual projective space, so that the intersection theory is simpler. This paper is devoted to these two constructions, their applications to stratification theory in the sense of Whitney and to a Plücker type formula for projective varieties.

Introduction

Let X denote a purely d-dimensional reduced subspace of an affine space $\mathbf{C}^N$ defined in an open subset by algebraic or analytic equations with coefficients in $\mathbf{C}$. The singular locus of X is usually defined as a point where “there is no tangent space” in the sense that the linear equations derived from the original equations of X do not define a unique linear subspace of dimension d. The direction of the tangent space at a non-singular point $x \in X$ is represented by a point in the Grassmannian $\mathbf{G}(d, N)$ of d-dimensional vector subspaces of $\mathbf{C}^N$. Thus, there is a map $\gamma\colon X^0 \to \mathbf{G}(d, N)$, where X^0 denotes the non-singular part of X, which is dense in X since X is reduced. This map is easily seen to be holomorphic, and algebraic if X is. It is called the Gauss map because a similar map was used by Gauss in his study of the curvature of differentiable surfaces, published in 1828.

Around the same time as Gauss, Poncelet, Bobillier, Plücker and others were studying the duality of plane projective curves. Here the motivations did not come from geodesy but rather from the problem of understanding how many tangents can be drawn to a curve C of degree d from a general point in the plane. The plane projective duality which transforms a point in the projective plane $\mathbf{P}^2$ with homogeneous coordinates $(x : y : z)$ into a line in the dual plane simply by exchanging the roles of coefficients and variables in the equation $ax+by+cz = 0$ of lines going through the point $(x : y : z)$ shows that the number of tangents to C from a general point is the degree of the *dual curve* $\check{C} \subset \check{\mathbf{P}}^2$ consisting of the points of $\check{\mathbf{P}}^2$ representing the lines tangent to C. This degree is $d(d-1)$. Thus if $\check{C}$ was non-singular its dual could not be C as the geometry insists it should be, since $d(d-1)((d(d-1)-1) \neq d(d-1)$ unless $d = 2$: this is the Poncelet paradox. Thus $\check{C}$ has singularities and some points of C must represent limits of tangents to $\check{C}$ at non-singular points of $\check{C}$ tending to a singular point. This is perhaps one of the first occurences of limits of tangent spaces.

Singular curves and surfaces were studied throughout the XIX-th century mostly[1] with the goal of generalizing Riemann's work, understanding "conditions of adjunction" and more generally the behavior of differential forms and their integrals. It is perhaps not so surprising that it is only in 1954 that Semple introduced in [45] the space of limit directions of tangent spaces to an algebraic variety, which he called the *first derivate* in [45, §8]. It is the closure NX in $X \times \mathbf{G}(d, N)$ of the graph $NX^0 \subset X^0 \times \mathbf{G}(d, N)$ of the Gauss map. As a subspace of $X \times \mathbf{G}(d, N)$ it is endowed with a projection $\nu\colon NX \to X$ which is proper (since $\mathbf{G}(d, N)$ is compact) and is an isomorphism over X^0. The set-theoretic fiber $|\nu^{-1}(x) \subset \mathbf{G}(d, N)$ above a point $x \in X$ is the set of limit directions at x of tangent spaces at points of X^0 tending to x.
Semple also asked, in the last paragraph of his paper, whether iterating this construction would eventually resolve the singularities of X.

About ten years after Semple, John Nash rediscovered the construction and the question and for a time the construction was called the *Nash blowing-up*, which exp lains the notation NX. Semple's paper is difficult to read and it is only after Monique Lejeune-Jalabert discovered his contribution that the map $\nu_X : NX \to X$ came to be called the *Semple-Nash modification.*

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[1] There are exceptions, for example in work of Cayley, Halphen, M. Noether, Salmon, H.J.S. Smith, often connected with generalizations of the Plücker formulas for curves and the study of linear systems and projective embeddings.

Also about ten years after Semple, and after important preliminary work in the differentiable case by Whitney himself in 1957 and Thom in 1960 (see [58]), in 1965, Hassler Whitney published a study of possible definitions of limits of secants and tangents at a singular point of a complex analytic space, in which he introduced the fundamental notion of regular stratification, nowadays called Whitney stratifications. It is a locally finite partition of a complex analytic space into locally closed non-singular "strata" where each stratum has a "regular" behavior along the strata of its boundary. The definition of "regular" involves both limits of secants and limits of tangents for points tending to the boundary stratum. The definitions extend readily beyond the complex analytic case and in the hands of Thom, Mather, and others it became a most important conceptual and technical tool in the study of singularities of differentiable mappings, in particular when applied to infinite dimensional spaces such as jet spaces and function spaces.

Stratification theory in the large is the subject of David Trotman's contribution (see [58]) to the first volume of this Handbook. In this text we shall concentrate on the complex analytic case for both limits of tangent spaces and stratifications. We consider reduced equidimensional complex spaces and whenever we take the intersection of such a space with a non-singular subspace of some ambiant non-singular space, we endow it with its reduced structure.

Although it can be read independently, this paper is in some ways a continuation of the paper [29] of Lê and Snoussi in Volume II of this Handbook. Also, a version of the content of section 1.1 appears in [51, §3.9] under the name of Nash blowing up (which is more traditional) and a version of the content of section 1.4 appears in [51, §3.3] in Volume I of this Handbook. Some of the topics exposed here can be found exposed in greater detail in [14] from which, with the permission of its authors and of the editors, we have copied some parts of this text.

1.1 Limits of tangent spaces: the Semple-Nash modification

Let X be a reduced and equidimensional closed subspace of an open set $U \subset \mathbf{C}^N$. We denote by X^0 the set of non-singular points of X, which is open and dense in X, by d the dimension of X, and by $\mathbf{G}(d, N)$ the grassmannian of d-dimensional vector subspaces of $\mathbf{C}^N$. The Gauss map

$$\gamma_{X^0} : X^0 \to \mathbf{G}(d, N), \;\; x \mapsto [T_{X^0,x}] \in \mathbf{G}(d, N)$$

associates to every point of X^0 the direction of the tangent space to X at this point. Let us consider the graph $NX^0 \subset X^0 \times \mathbf{G}(d, N)$ of γ. It is a purely d-

dimensional analytic subset of $X^0 \times \mathbf{G}(d,N)$ since it is isomorphic to X^0. The space of limits of (directions of) tangent spaces at points of X^0, the Semple-Nash modification of X, is the closure NX in $X \times \mathbf{G}(d,N)$ of NX^0. So we have to prove that it is a closed analytic subspace of $X \times \mathbf{G}(d,N)$. The singular locus $\mathrm{Sing}X = X \setminus X^0$ is a closed complex subspace of X, of dimension $\leqslant d-1$. However, we cannot apply the Remmert-Stein theorem (see [32, Chap. IV, §6] or [1, Theorem 6]) to prove that NX is analytic because we have to extend $NX^0 \subset U \times \mathbf{G}(d,N)$ through $\mathrm{Sing}X \times \mathbf{G}(d,N)$ which is of dimension $> d$. The proofs in [60, Theorem 16.4] and [40, Theorem 1] build, using jacobian determinants, a system of equations for the closure $NX \subset U \times \mathbf{G}(d,N)$, thus proving its analyticity.

One has then to verify that the map $NX \to X$ is unique up to a unique X-isomorphism, independent of the immersion of X in an open set of an affine space.
Then for any reduced equidimensional complex space X the local Semple-Nash modifications will glue up into a unique proper map, the Semple-Nash modification $\nu_X \colon NX \to X$ (sometimes simplified to ν).
We note that the pull-back by the second projection $\gamma_X \colon NX \to \mathbf{G}(d,N)$ of the tautological bundle on the grassmannian is a vector bundle on NX which extends the tangent bundle of $NX^0 \simeq X^0$.

There is another approach, based on Grothendieck's Grassmannian of a coherent module (see [17]) which shows directly the canonicity of the Semple-Nash modification.
Let X be a reduced equidimensional complex space and Ω^1_X its coherent module of differentials, which is locally free on X^0. It comes with a morphism of $\mathcal{O}_X$-modules $d_X \colon \mathcal{O}_X \to \Omega^1_X$, the differential, which cannot be confused with the dimension. Since the $\mathcal{O}_X$-module Ω^1_X is coherent, the symmetric algebra $\mathrm{Sym}_{\mathcal{O}_X}\Omega^1_X$ of the $\mathcal{O}_X$-module Ω^1_X is a graded $\mathcal{O}_X$-algebra locally of finite presentation and generated in degree one, and so corresponds to an analytic space $\mathrm{Specan}_X\mathrm{Sym}_{\mathcal{O}_X}\Omega^1_X$ over X.The fibers of the natural map

$$t \colon \mathrm{Specan}_X\mathrm{Sym}_{\mathcal{O}_X}\Omega^1_X \to X$$

are the Zariski tangent spaces $t^{-1}(x) = \mathrm{SpecSym}_{\mathbf{C}}(m_{X,x}/m^2_{X,x})^\vee$, where $\vee$ denotes the dual vector space over $\mathbf{C}$.
Since Ω^1_X is a coherent sheaf of $\mathcal{O}_X$-modules, $\mathrm{Specan}_X\mathrm{Sym}_{\mathcal{O}_X}\Omega^1_X$ is a complex analytic space. The sections $\partial \colon X \to \mathrm{Specan}_X\mathrm{Sym}_{\mathcal{O}_X}\Omega^1_X$ of the projection t correspond to elements of $\mathrm{Hom}_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$, that is, derivations from $\mathcal{O}_X$ to $\mathcal{O}_X$. If X is non-singular $\mathrm{Specan}_X\mathrm{Sym}_{\mathcal{O}_X}\Omega^1_X$ is the tangent bundle to X and the sections ∂ are holomorphic vector fields on X.

Now Grothendieck has shown that for Ω^1_X, as indeed for any coherent $\mathcal{O}_X$-module, just as $t \colon \mathrm{Specan}_X\mathrm{Sym}_{\mathcal{O}_X}\Omega^1_X \to X$ is a relative vector space in the sense that its fibers are vector spaces, there is a relative grassmannian

$$g \colon \mathbf{G}_d(\Omega^1_X) \to X$$

whose fiber at $x \in X$ is the grassmannian of d-dimensional subspaces of the vector space $t^{-1}(x)$.

The defining property of the map g is that for any holomorphic map $h\colon W \to X$ it is equivalent to give, up to isomorphism, a locally free quotient of rank d of the $\mathcal{O}_W$-module $h^*\Omega^1_X$ and to give, up to isomorphism, a factorization of h through g.

Now a rank d locally free quotient of $h^*\Omega^1_X$ corresponds to a vector bundle over W with d-dimensional fibers which is contained in $\mathrm{Specan}_X \mathrm{Sym}_{\mathcal{O}_T} h^*\Omega^1_X$. That is exactly a family of analytically varying d-dimensional subspaces of the Zariski tangent spaces $t^{-1}(h(w))$ for $w \in W$.
In particular, the sheaf $g^*\Omega^1_X$ on $\mathbf{G}_d(\Omega^1_X)$ has a locally free quotient of rank d, which corresponds to the pull back of the tautological bundle on the grassmannian.
If one remembers that in analytic geometry limits can be obtained by moving along analytic arcs (curve selection lemma), we see that since any limit direction T of tangent spaces at a point $x \in X$ is a limit along germs of analytic arcs $h\colon (\mathbf{D}, 0) \to (X, x)$, it is the fiber over 0 of a locally free quotient of $h^*\Omega^1_X$ and so the arc lifts as $\tilde{h}\colon (\mathbf{D}, 0) \to (\mathbf{G}_d(\Omega^1_X), T)$, which (with a little work) defines a map $NX \to \mathbf{G}_d(\Omega^1_X)$ which one shows (with a little more work) to be an X-isomorphism.

The equivalence of this grassmannian construction with the Gauss map construction shows directly that the closure of the graph of the Gauss map is analytic, and that the result of the construction is unique up to a unique isomorphism.
Since the grassmannians embed into projective spaces, the map $NX \to X$ is locally projective and since it is locally bimeromorphic, it is locally on X the blowing-up (see section 1.4 below for the definition) of a sheaf of ideals, a result proved explicitly by Nobile in [40, Theorem 1].

Examples

1. Let $X \subset \mathbf{C}^4$ be the union of two planes meeting at the origin. Then $NX \to X$ maps the disjoint union of two 2-planes to X, each plane mapping isomorphically onto its image. It is a finite bimeromorphic map, and thus a resolution of singularities. If one follows the classical resolution algorithm, one blows up the intersection point. This again separates the two planes, but now the projection restricted to each of the separated planes is the blowing-up of a point, and is not finite.
2. Let $f(z_1, \ldots, z_N) = 0$ be an equation for a germ at the origin of a reduced hypersurface. The Semple-Nash modification is the blowing-up in X of the ideal generated by the partial derivatives of f. More generally, if X is a reduced complete intersection of dimension d in affine space $\mathbf{A}^N(\mathbf{C})$, then the blowing-up in X of the ideal generated by the $(N-d) \times (N-d)$ minors of the jacobian matrix of the equations is isomorphic to NX. For the general case, see [40].

The Semple-Nash modification has been used in the definition of characteristic classes for singular spaces (see [37] and Chapters 5-7 of Volune III of this Handbook), but we shall not go into this here. Much work has been devoted to understanding how the singularities of NX differ from those of X, and in particular to answer the question posed by Semple at the end of his paper and reiterated by Nash a decade later:

Does iterating the Semple-Nash modification resolve the singularities of X in finitely many steps?

In other words, given X as above, is there an integer k_0 such that $N^k X$ is non-singular for $k \geqslant k_0$?
It follows from the definition that if X is non-singular, we have $NX = X$. Nobile proved the converse in [40, Theorem 2]:

Theorem 1.1.1 (Nobile) *The Semple-Nash modification $\nu_X \colon NX \to X$ is an isomorphism if and only if X is non-singular.*

Nobile's original proof of this theorem is somewhat involved and relies on local parametric descriptions of a singular space and results of [60]. A different proof was proposed in [53, §2], based on the second construction of NX.
By definition, if $NX = X$, the module of differentials of X has a locally free quotient. The property of non-singularity being local we may assume after restricting to an open set $U \cap X$ of X that we have a surjection $\Omega^1_{U \cap X} \to \mathcal{O}^d_{U \cap X} \to 0$. Taking germs at $x \in U \cap X$, there is an element $h \in \mathcal{O}_{X,x}$ such that the differential $d_X h \in \Omega^1_{X,x}$ maps to $(1, 0 \ldots, 0) \in \mathcal{O}^d_{X,x}$, and thus a derivation D of $\mathcal{O}_{X,x}$ into itself such that $Dh = 1$. Since D is zero on $\mathbf{C} \subset \mathcal{O}_{X,x}$, we may assume that $h \in m_{X,x}$. Geometrically, the derivation D corresponds to a holomorphic vector field on X not vanishing at x. Its integration (see [53, §2] for details) gives the germ (X, x) a product structure $(X, x) \simeq (X_1 \times \mathbf{C}, x)$, where $X_1 \subset X$ is the reduced equidimensional space defined by the ideal $h\mathcal{O}_{X,x}$ and satisfies $NX_1 = X_1$. The result follows by induction on the dimension.

This theorem has the important consequence that in order to prove Semple's conjecture, it suffices to prove that the sequence of the $N^k X$ eventually becomes stationary.
As already noted by Nobile, it implies immediately that if X is of dimension one $N^k X$ is non-singular for large k. Since a curve has finitely many limit tangent lines at any point, the Semple-Nash modification of a curve is a finite bimeromorphic map, and thus dominated by the normalization. Since the normalization $\overline{\mathcal{O}_{X,x}}$, which is a resolution of singularities, is a finitely generated and thus a noetherian $\mathcal{O}_{X,x}$-module, there cannot be an infinite strictly increasing sequence of subalgebras finite over $\mathcal{O}_{X,x}$.

Apart from some special cases, Semple's conjecture is still open in dimensions $\geqslant 2$. The best result is due to Spivakosky in [50], where he proves that iterating the operation of Semple-Nash modification followed by normalization eventually resolves the singularities of a surface. Spivakovsky's proof sheds light on the change of the dual graph of a minimal resolution when one passes from X to NX.

There are a number of other significant results for surfaces. For example Snoussi in [48] relates the planar components of the tangent cone to a surface to the singularities of its Semple-Nash transform and D. Duarte in [9] shows that interating the Semple-Nash modification for toric surfaces has to stop in some charts.

In dimension $\geqslant 3$ very little is known in general. The resolution problem is open even in the case of toric varieties, where in characteristic zero the Semple-Nash modification is the blowing up of a deceptively simple monomial ideal (see [15, §10]).

Indeed, apart from results of Vaquié in [59] concerning numerical invariants, and precise results for quasi-ordinary singularities (see [4] and [2]), there is no satisfactory description in general of the relation between the geometry of NX and that of X.

However there is another aspect of limits of tangent spaces which is rather well understood: as we shall see below, given $(X, 0) \subset (\mathbf{C}^N, 0)$, a hyperplane in $\mathbf{C}^N$ is said to be tangent to X^0 at a point if it contains the tangent space to X^0 at that point and a hyperplane through 0 is a limit of tangent hyperplanes at points of X^0 if and only if it contains a limit of tangent spaces to X^0.
When X is a hypersurface with isolated singularity it was shown in [52, Chap. II, §1, 1.6] that a hyperplane H through the singular point is *not* a limit of tangent hyperplanes if and only if the Milnor number $\mu(X \cap H)$ is minimal among the Milnor numbers of all intersections $X \cap H'$. Then it was shown in [54, Appendice] that the family of all sections $X \cap H$ where H is not a limit of tangent hyperplanes is equisingular in the sense of Whitney conditions (which we shall see below). These results were generalized, for normal surfaces by Snoussi in [46], for arbitrary reduced equidimensional germs by Gaffney in [12, Theorem 2.1, Corollary 2.4] and in a more topological framework by Tibăr in [57].
The result for isolated hypersurface singularities was used as part of a method to compute limits of tangent spaces in this case. See [39], and [41] for more methods of computation.

In the case where our singular germ $(X, 0)$ is the cone over a projective variety, there is an algebraic approach to the study of the Gauss map in [49]. and a geometric one in [28]. We shall come back to this in the paragraph on projective duality.

Given a flat map $\pi \colon X \to S$ where X is again reduced and equidimensional and say S is non-singular and the open set X^0 of points of X where the map π is smooth is dense in X, with $\dim. X/S = d$, one can define a *relative Semple-Nash modification* as $\mathrm{SpecanSym}_X \Omega^1_{X/S}$ where $\Omega^1_{X/S}$ is the sheaf of relative differentials. In a local presentation of π as the map induced by the first projection in an embedding $X \subset S \times \mathbf{C}^N$ it is the closure of the graph of the relative Gauss map $\gamma_{X^0/S}\colon X^0 \to \mathbf{G}(d,N)$ sending a point $x \in X^0$ to the direction of the tangent space to the fiber of π through x.

This construction is of course useful in the study of families of singularities but the geometry of grassmannians being much more complicated than the geometry of projective spaces, it is time to move to the study of tangent hyperplanes.

1.2 Limits of tangent hyperplanes: The conormal space

Whenever our reduced equidimensional singular space X is not locally a hypersurface in some $\mathbf{C}^N$, the tangent spaces belong to grassmannians instead of projective spaces, and the description of the Semple-Nash modification becomes more complicated, according to the complexity of describing algebraic subvarieties of grassmannians.

It is therefore natural to consider tangent hyperplanes instead of tangent spaces: a tangent hyperplane at a point of $X^0 \subset \mathbf{C}^N$ is a (direction of) hyperplane containing the tangent space to X^0 at that point. This is also the approach which allows the connection with duality of projective varieties, in the case where our singular germ $(X,0)$ is the cone over a projective variety. Most importantly the spaces of limits of tangent hyperplanes to a singular subspace of a non-singular complex variety can be characterized by Lagrangian (or Legendrian) type conditions, a fact which has no direct equivalent for NX [2]. One must emphasize that, in contrast to the Semple-Nash modification, this constructions depends on a local or global embedding of our space X in a non-singular complex analytic variety M.

Let us begin with the case of a local embedding $X \subset \mathbf{C}^N$, where the directions of hyperplanes in $\mathbf{C}^N$ are parametrized by the projective space $\check{\mathbf{P}}^{N-1}$. At a non-singular point $x \in X^0$, by definition a tangent hyperplane is a hyperplane in the tangent space to $\mathbf{C}^N$ at x which contains the tangent space $T_{X^0,x}$. Tangent hyperplanes at a point $x \in X^0$ constitute a $\mathbf{P}^{N-d-1} \subset \mathbf{P}^{N-1}$. Thus we obtain a subspace $C(X^0) \subset X \times \check{\mathbf{P}}^{N-1}$ whose points are pairs (x,H) such that H is a tangent hyperplane at x. The **conormal space** $C(X)$ of $X \subset \mathbf{C}^N$ is the closure of $C(X^0)$ in $X \times \check{\mathbf{P}}^{N-1}$. By definition it is the set of pairs (x,H) such that H is a limit at x of tangent hyperplanes at points of X^0.

The natural map induced by the first projection is denoted by $\kappa_X\colon C(X) \to X$.

Again we have to show that this closure is a closed analytic subspace of $X \times \check{\mathbf{P}}^{N-1}$. Following [14, Section 3.3], we use a diagram relating the conormal space of $(X,0) \subset (\mathbf{C}^N,0)$ and its Semple-Nash modification.

It is convenient here to use the notation of projective duality of linear spaces.

[2] See, however, [16, Theorem 3.14] and [28, Theorem 14].

Given a vector subspace $T \subset \mathbf{C}^N$ we denote by $\mathbf{P}T$ its projectivization, *i.e.*, the image of $T \backslash \{0\}$ by the projection $\mathbf{C}^N \backslash \{0\} \to \mathbf{P}^{N-1}$ and by $\check{T} \subset \check{\mathbf{P}}^{N-1}$ the projective dual of $\mathbf{P}T \subset \mathbf{P}^{N-1}$, which is a $\mathbf{P}^{N-d-1} \subset \check{\mathbf{P}}^{N-1}$, the set of all hyperplanes H of $\mathbf{P}^{N-1}$ containing $\mathbf{P}T$.

We denote by $\check{\Xi} \subset \mathbf{G}(d,N) \times \check{\mathbf{P}}^{N-1}$ the cotautological $\mathbf{P}^{N-d-1}$-bundle over $\mathbf{G}(d,N)$, that is $\check{\Xi} = \{(T,H) \mid T \in \mathbf{G}(d,N),\ H \in \check{T} \subset \check{\mathbf{P}}^{N-1}\}$, and consider the intersection

$$\begin{tikzcd} E := (X \times \check{\Xi}) \cap (NX \times \check{\mathbf{P}}^{N-1}) \arrow[r, hook] \arrow[d, "p_1"'] \arrow[dr, "p_2"] & X \times \mathbf{G}(d,N) \times \check{\mathbf{P}}^{N-1} \arrow[d] \\ NX & X \times \check{\mathbf{P}}^{N-1} \end{tikzcd}$$

and the morphism p_2 induced on E by the projection onto $X \times \check{\mathbf{P}}^{N-1}$. We then have the following:

Proposition 1.2.1 *The set-theoretical image $p_2(E)$ of the morphism p_2 coincides with the conormal space of X in $\mathbf{C}^N$*

$$p_2(E) = C(X) \subset X \times \check{\mathbf{P}}^{N-1}.$$

It is a closed analytic subspace of dimension $N-1$.

Proof If we define $E^0 = \{(x, T_{X,x}, H) \in E \mid x \in X^0, H \in \check{T}_{X,x}\}$, then by construction $E^0 = p_1^{-1}(\nu_X^{-1}(X^0))$, and $p_2(E^0) = C(X^0)$. Since the morphism p_2 is proper it is closed, which finishes the proof since E is a closed analytic subspace of $X \times \mathbf{G}(d,N) \times \check{\mathbf{P}}^{N-1}$ because $\check{\Xi}$ is a closed analytic (in fact algebraic) subspace of $\mathbf{G}(d,N) \times \check{\mathbf{P}}^{N-1}$ and NX is a closed analytic subspace in $X \times \mathbf{G}(d,N)$. The dimension of $C(X)$ is that of its open dense subset $C(X^0)$, which is $N-1$ because it maps to X^0 with fibers $\mathbf{P}^{N-d-1}$. ▫

Corollary 1.2.2 *A hyperplane $H \in \check{\mathbf{P}}^{N-1}$ is a limit of tangent hyperplanes to X at 0, i.e., $H \in \kappa_X^{-1}(0)$, if and only if there exists a d-plane $(0,T) \in \nu_X^{-1}(0)$ such that $T \subset H$.*

Proof Let $(0,T) \in \nu_X^{-1}(0)$ be a limit of tangent spaces to X at 0. By construction of E and Proposition 1.2.1, every hyperplane H containing T is in the fiber $\kappa_X^{-1}(0)$, and so is a limit at 0 of tangent hyperplanes to X^0.
On the other hand, by construction, for any hyperplane $H \in \kappa_X^{-1}(0)$ there is a sequence of points $\{(x_i, H_i)\}_{i \in \mathbf{N}}$ in $\kappa_X^{-1}(X^0)$ converging to $p = (0, H)$. Since the map p_2 is surjective, by definition of E, we have a sequence $(x_i, T_i, H_i) \in E^0$ with $T_i = T_{x_i}X^0 \subset H_i$. By compactness of Grassmannians and projective spaces, this sequence has to converge, up to taking a subsequence, to (x, T, H) with T a limit at x of tangent spaces to X. Since inclusion is a closed condition, we have $T \subset H$. ▫

Corollary 1.2.3 *The morphism $p_1 : E \to NX$ is a locally analytically trivial fiber bundle with fiber $\mathbf{P}^{N-d-1}$.*

Proof By definition of E, the fiber of the projection p_1 over a point $(x, T) \in NX$ is the set of all hyperplanes in $\mathbf{P}^{N-1}$ containing $\mathbf{P}T$. In fact, the tangent bundle T_{X^0}, lifted to NX by the isomorphism $NX^0 \simeq X^0$, extends to a fiber bundle over NX, called the Nash tangent bundle of X. It is the pull-back by γ_X of the tautological bundle of $G(d, N)$, and E is the total space of the $\mathbf{P}^{N-d-1}$-bundle of the projective duals of the projectivized fibers of the Nash bundle. □

Consider the diagram extracted from the diagram we have seen above:

$$\begin{array}{ccc} E & \xrightarrow{p_2} & C(X) \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle \kappa_X} \\ NX & \xrightarrow{\nu_X} & X \end{array}$$

Proposition 1.2.4 *The map $p_2 \colon E \to C(X)$ is isomorphic to the blowing up in $C(X)$ of the lift $\mathcal{J}\mathcal{O}_{C(X)}$ to $C(X)$ by κ_X of an ideal $\mathcal{J}$ of $\mathcal{O}_X$ whose blowing up coincides with the map ν_X.*

Proof By construction, E is a closed subspace of $NX \times_X C(X)$. By definition of E, the map p_2 is an isomorphism over $C(X^0)$ since a tangent hyperplane at a nonsingular point contains only the tangent space at that point. Therefore the map $p_2 \colon E \to C(X)$ is locally bimeromorphic. The lift by $\nu_X \circ p_1$ of the ideal $\mathcal{J}$ is invertible on E. By the universal property of blowing up, any map $W \to C(X)$ such that the lift to W from $C(X)$ of the ideal $\mathcal{J}\mathcal{O}_{C(X)}$ is invertible on W has to factor uniquely through NX and therefore through the fiber product $NX \times_X C(X)$. In particular the blowing-up of $\mathcal{J}\mathcal{O}_{C(X)}$ in $C(X)$ has to factor through a closed subspace of $NX \times_X C(X)$ and has to coincide with E since they coincide over X^0.[3]

In general the fiber of p_2 over a point $(x, H) \in C(X)$ is the set of limit directions at x of tangent spaces to X that are contained in H. If X is a hypersurface, the conormal map coincides with the Semple-Nash modification. In general, the manner in which the geometric structure of the inclusion $\kappa_X^{-1}(x) \subset \check{\mathbf{P}}^{N-1}$ determines the set of limit positions of tangent spaces, *i.e.*, the fiber $\nu_X^{-1}(x)$ of the Semple-Nash modification, is not so simple: by Proposition 1.2.1 and its corollary, the points of $\nu_X^{-1}(x)$ correspond to *some of* the projective subspaces $\mathbf{P}^{N-d-1}$ of $\check{\mathbf{P}}^{N-1}$ contained in $\kappa_X^{-1}(x)$.

[3] For the reader familiar with bimeromorphic geometry, as for example in [22], [3, Chap. 1, 1.5] and [23, §2], the map p_1 appears as the *strict transform* of the map κ by the blowing-up ν. Since p_1 is a $\mathbf{P}^{N-d-1}$-bundle by Corollary 1.2.3, the map ν is also the flattening map of κ: every blowing-up $t \colon T \to X$ of X such that the strict transform of κ by t is flat must factor uniquely through ν. In this sense κ determines ν.

A linear subspace $\mathbf{P}^{N-d-1} \subset \kappa_X^{-1}(x) \subset \check{\mathbf{P}}^{N-1}$ is dual to a d-dimensional vector subspace $T \subset \mathbf{C}^N$. If T is not a limit at x of tangent spaces, then by corollary 1.2.2 any hyperplane in this $\mathbf{P}^{N-d-1}$ must contain a limit at x of tangent spaces, but this limit cannot be constant. This provides a set-theoretic characterization of those $\mathbf{P}^{N-d-1} \subset \kappa_X^{-1}(x)$ which are dual to a limit at x of tangent spaces, in terms of the diagram we have seen above: they are those which are the image by p_2 of a fiber of p_1. In view of proposition 1.2.4 this gives a geometric characterization, but we would prefer one solely in terms of the geometry of $C(X)$; see [14, Example 3.4].
Note also that given a limit of tangent spaces T at $x \in X$ and a general linear projection $p\colon \mathbf{C}^N \to \mathbf{C}^{d+1}$, the hyperplane $p(T)$ is a limit hyperplane at $p(x)$ for the hypersurface $p(X) \subset \mathbf{C}^{d+1}$. This follows from the fact that given $T \in \nu_X^{-1}(0)$ we can find an analytic arc in NX ending at T and whose image in X is outside of the inverse image by p of the singular locus of $p(X)$.

Definition 1.2.5 The map $\lambda_X\colon C(X) \to \check{\mathbf{P}}^{N-1}$ induced by the second projection $X \times \check{\mathbf{P}}^{N-1} \to \check{\mathbf{P}}^{N-1}$ is called the tangent hyperplane map. It is the analogue of the Gauss map. When there is no ambiguity it will be denoted by λ.

1.3 Some symplectic Geometry

In order to describe this set of tangent hyperplanes, we are going to use the language of symplectic geometry and Lagrangian submanifolds. Let us start with a few definitions. This section is mostly taken from [14, Section 2.1].

Let M be any N-dimensional manifold, and let ω be a de Rham 2-form on M, that is, for each $x \in M$, the map

$$\omega_x : T_{M,x} \times T_{M,x} \to \mathbf{R}$$

is skew-symmetric bilinear on the tangent space to M at x, and ω_x varies smoothly with x. We say that ω is **symplectic** if it is closed and ω_x is non-degenerate for all $x \in M$. Non degeneracy means that the map which to $v \in T_{M,x}$ associates the homomorphism $w \mapsto \omega(v,w) \in \mathbf{R}$ is an isomorphism from $T_{M,x}$ to its dual. A **symplectic manifold** is a pair (M,ω), where M is a manifold and ω is a symplectic form. These definitions extend, replacing $\mathbf{R}$ by $\mathbf{C}$, to the case of a complex analytic manifold *i.e.*, nonsingular space.

For any manifold M, its cotangent bundle T^*M has a canonical symplectic structure as follows. Let

$$\begin{aligned} \pi : T^*M &\longrightarrow M \\ p = (x,\xi) &\longmapsto x, \end{aligned}$$

where $\xi \in T^*_{M,x}$, be the natural projection. The **Liouville 1-form** α on T^*M may be defined pointwise by:

$$\alpha_p(v) = \xi\left(d\pi_p(v)\right), \quad \text{for } v \in T_{T^*M,p}.$$

Note that $d\pi_p$ maps $T_{T^*M,p}$ to $T_{M,x}$, so that α is well defined. The **canonical symplectic 2-form** ω on T^*M is defined as

$$\omega = -d\alpha.$$

And it is not hard to see that if $(U, x_1, \ldots, x_N)$ is a coordinate chart for M with associated cotangent coordinates $(T^*U, x_1, \ldots, x_N, \xi_1, \ldots, \xi_N)$, then locally:

$$\omega = \sum_{i=1}^{N} dx_i \wedge d\xi_i.$$

Definition 1.3.1 Let (M, ω) be a 2n-dimensional symplectic manifold. A submanifold Y of M is a **Lagrangian submanifold** if at each $y \in Y$, $T_{Y,y}$ is a Lagrangian subspace of $T_{M,y}$, *i.e.*, $\omega_y|_{T_{Y,y}} \equiv 0$ and $\dim. T_{Y,y} = \frac{1}{2}\dim. T_{M,y}$. Equivalently, if $i : Y \hookrightarrow M$ is the inclusion map, then Y is **Lagrangian** if and only if $i^*\omega = 0$ and $\dim. Y = \frac{1}{2}\dim. M$.
Let M be a nonsingular complex analytic space of even dimension equipped with a closed non degenerate 2-form ω. If $Y \subset M$ is a complex analytic subspace, which may have singularities, we say that it is a **Lagrangian subspace** of M if it is purely of dimension $\frac{1}{2}\dim. M$ and there is a dense nonsingular open subset of the corresponding reduced subspace which is a Lagrangian submanifold in the sense that ω vanishes on all pairs of vectors in the tangent space.

Example 1.3.2 The **zero section** of T^*M

$$X := \{(x, \xi) \in T^*M | \xi = 0 \text{ in } T^*_{M,x}\}$$

is an n-dimensional Lagrangian submanifold of T^*M.

Exercise 1.3.3 Let $f(z_1, \ldots, z_N)$ be a holomorphic function on an open set $U \subset \mathbf{C}^N$. Consider the differential df as a section $df : U \to T^*U$ of the cotangent bundle. Verify that the image of this section is a Lagrangian submanifold of T^*U. Explain what it means. What is the image in U by the natural projection $T^*U \to U$ of the intersection of this image with the zero section?

1.3.1 The conormal space in general.

Let now M be a complex analytic manifold of dimension N and $X \subset M$ be a possibly singular complex subspace of pure dimension d, and let as before $X^0 = X \backslash \mathrm{Sing} X$ be the nonsingular part of X, which is a submanifold of M.

Definition 1.3.4 Set

$$N^*_{X^0,x} = \{\xi \in T^*_{M,x} | \xi(v) = 0, \ \forall v \in T_{X^0,x}\};$$

this means that the hyperplane $\{\xi = 0\}$ contains the tangent space to X^0 at the point x.
The **conormal bundle** of X^0 is

$$T^*_{X^0}M = \{(x,\xi) \in T^*M | x \in X^0, \ \xi \in N^*_{X^0,x}\}.$$

Definition 1.3.5 A closed subvariety L of the cotangent space T^*M of a manifold M is said to be *conical* if it is left globally invariant by the homotheties on the fibers of the map $T^*M \to M$, described locally by $\rho.(x,\xi) = (x, \rho\xi), \ \rho \in \mathbf{C}$.

Proposition 1.3.6 *Let $i : T^*_{X^0}M \hookrightarrow T^*M$ be the inclusion and let α be the Liouville 1-form in T^*M as before. Then $i^*\alpha = 0$. In particular the conormal bundle $T^*_{X^0}M$ is a conical Lagrangian submanifold of T^*M, and has dimension N.*

Proof
See [8, Proposition 3.6]. □

In the same context we can define the **conormal space of** X **in** M as the closure $T^*_X M$ of $T^*_{X^0}M$ in T^*M, with the **conormal map** $\kappa_X : T^*_X M \to X$, induced by the natural projection $\pi : T^*M \to M$. The conormal space is of dimension N. It may be singular and by Proposition 1.3.6, α vanishes on every tangent vector at a nonsingular point, so it is by construction a Lagrangian subspace of T^*M.

The fiber $\kappa_X^{-1}(x)$ of the conormal map $\kappa_X : T^*_X M \to X$ above a point $x \in X$ consists, if $x \in X^0$, of the vector space $\mathbf{C}^{N-d}$ of all the equations of hyperplanes tangent to X at x, in the sense that they contain the tangent space $T_{X^0,x}$. If x is a singular point, the fiber consists of all equations of limits of hyperplanes tangent at nonsingular points of X tending to x.
Moreover, we can characterize those subvarieties of the cotangent space which are the conormal spaces of their images in M.

Proposition 1.3.7 (see [42, Chap. II, §10]) *Let M be a nonsingular analytic variety of dimension N and let L be a closed conical irreducible analytic subvariety of T^*M, also of dimension N. The following conditions are equivalent:*

1) The variety L is the conormal space of its image in M.
2) The Liouville 1-form α vanishes on all tangent vectors to L at every nonsingular point of L.
3) The symplectic 2-form $\omega = -\mathrm{d}\alpha$ vanishes on every pair of tangent vectors to L at every nonsingular point of L.

Since conormal varieties are conical we may as well projectivize with respect to vertical homotheties of T^*M and work in $\mathbf{P}T^*M$. This means that we consider hyperplanes and identify all linear equations defining the same hyperplane. In $\mathbf{P}T^*M$ it still makes sense to be Lagrangian since α is homogeneous by definition[4].

Going back to our original problem we have $X \subset U$ where U is open in $\mathbf{C}^N$, so $T^*U = U \times \check{\mathbf{C}}^N$ and $\mathbf{P}T^*U = U \times \check{\mathbf{P}}^{N-1}$. So we have the **(projective) conormal space** $\kappa_X : C(X) \to X$ with $C(X) \subset X \times \check{\mathbf{P}}^{N-1}$, where $C(X)$ denotes the projectivization of the conormal space $T^*_X M$. Note that we have not changed the name of the map κ_X after projectivizing since there is no ambiguity, and that the dimension of $C(X)$ is $N-1$, which shows immediately that it depends on the embedding of X in an affine space.
When there is no ambiguity we shall often omit the subscript in κ_X. We have the following result showing that this projectivized conormal is the same as that of section 1.2 :

Proposition 1.3.8 *Given a reduced closed complex analytic subspace X of an open set $U \subset \mathbf{C}^N$ The (projective) conormal space $C(X)$ is a closed, reduced, complex analytic subspace of $X \times \check{\mathbf{P}}^{N-1}$ of dimension $N-1$. For any $x \in X$ the fiber $|\kappa_X^{-1}(x)|$ is the set of limit positions at x of tangent hyperplanes at points of X^0. Its dimension is at most $N-2$.*

Proof
These are classical facts. See [8, Chap. III] or [55, Chap. II, §4, Proposition 4.1, p. 379]. □

1.3.2 Conormal spaces and projective duality

Let us assume for a moment that $V \subset \mathbf{P}^{N-1}$ is a projective algebraic variety. In the spirit of last section, let us take $M = \mathbf{P}^{N-1}$ with homogeneous coordinates $(z_1 : \ldots : z_N)$, and consider the dual projective space $\check{\mathbf{P}}^{N-1}$ with coordinates $(\xi_1 : \ldots : \xi_N)$; its points are the hyperplanes of $\mathbf{P}^{N-1}$ with equations $\sum_{i=1}^{N} z_i \xi_i = 0$.

Definition 1.3.9 Define the **incidence variety** $I \subset \mathbf{P}^{N-1} \times \check{\mathbf{P}}^{N-1}$ as the set of points satisfying:

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[4] In symplectic geometry it is called **Legendrian** with respect to the natural contact structure on $\mathbf{P}T^*M$.

$$\sum_{i=1}^{N} z_i \xi_i = 0,$$

where $(z_1 : \ldots : z_N ; \xi_1 : \ldots : \xi_N) \in \mathbf{P}^{N-1} \times \check{\mathbf{P}}^{N-1}$

Lemma 1.3.10 (Kleiman; see [25, §4]) *The projectivized cotangent bundle of $\mathbf{P}^{N-1}$ is naturally isomorphic to I.*

Proof
Let us first take a look at the cotangent bundle of $\mathbf{P}^{N-1}$:

$$\pi : T^*\mathbf{P}^{N-1} \longrightarrow \mathbf{P}^{N-1}.$$

Remember that the fiber $\pi^{-1}(x)$ over a point x in $\mathbf{P}^{N-1}$ is by definition isomorphic to $\check{\mathbf{C}}^{N-1}$, the vector space of linear forms on $\mathbf{C}^{N-1}$. Recall that projectivizing the cotangent bundle means projectivizing the fibers, and so we get a map:

$$\Pi : \mathbf{P}T^*\mathbf{P}^{N-1} \longrightarrow \mathbf{P}^{N-1}$$

where the fiber is isomorphic to $\check{\mathbf{P}}^{N-2}$. So we can see a point of $\mathbf{P}T^*\mathbf{P}^{N-1}$ as a pair $(z, \xi) \in \mathbf{P}^{N-1} \times \check{\mathbf{P}}^{N-2}$. On the other hand, if we fix a point $z \in \mathbf{P}^{N-1}$, the equation defining the incidence variety I tells us that the set of points $(z, \xi) \in I$ is the set of hyperplanes of $\mathbf{P}^{N-1}$ that go through the point z, which we know is isomorphic to $\check{\mathbf{P}}^{N-2}$.

Now to explicitly define the map, take a chart $\mathbf{C}^{N-1} \times \left\{\check{\mathbf{C}}^{N-1} \backslash \{0\}\right\}$ of the manifold $T^*\mathbf{P}^{N-1} \backslash \{\text{zero section}\}$, where the $\mathbf{C}^{N-1}$ corresponds to a usual chart of $\mathbf{P}^{N-1}$ and $\check{\mathbf{C}}^{N-1}$ to its associated cotangent chart. Define the map:

$$\phi_i : \mathbf{C}^{N-1} \times \left\{\check{\mathbf{C}}^{N-1} \backslash \{0\}\right\} \longrightarrow \mathbf{P}^{N-2} \times \check{\mathbf{P}}^{N-2}$$

$$(z_1, \ldots, z_{N-1}; \xi_1, \ldots, \xi_{N-1}) \longmapsto \left(\varphi_i(z), (\xi_1 : \ldots : \xi_{i-1} : -\sum_{j=1}^{N-1 *_i} z_j \xi_j : \xi_{i+1} : \ldots : \xi_{N-1})\right)$$

where $\varphi_i(z) = (z_1 : \ldots : z_{i-1} : 1 : z_{i+1} : \ldots : z_{N-1})$ and the star means that the index i is excluded from the sum.

An easy calculation shows that ϕ_i is injective, has its image in the incidence variety I and is well defined on the projectivization $\mathbf{C}^{N-1} \times \check{\mathbf{P}}^{N-2}$. It is also clear, that varying i from 1 to $N-1$ we can reach any point in I. Thus, all we need to check now is that the ϕ_j's paste together to define a map. For this, the important thing is to remember that if φ_i and φ_j are charts of a manifold, and $h := \varphi_j^{-1}\varphi_i = (h_1, \ldots, h_{N-1})$ then the change of coordinates in the associated cotangent charts $\tilde{\varphi}_i$ and $\tilde{\varphi}_j$ is given by:

$$\begin{tikzcd} & T^*M \arrow[dr, "\tilde{\varphi_j}^{-1}"] & \\ \mathbf{C}^{N-1}\times\check{\mathbf{C}}^{N-1} \arrow[ur, "\tilde{\varphi_i}"] \arrow[rr, "h"'] & & \mathbf{C}^{N-1}\times\check{\mathbf{C}}^{N-1} \end{tikzcd}$$

$$(z_1,\dots,z_{N-1};\xi_1,\dots,\xi_{N-1}) \longmapsto (h(z);(Dh^{-1}|_z)^T(\xi))$$

By Lemma 1.3.10 the incidence variety I inherits the Liouville 1-form α which is $\sum \xi_i dz_i$ in local coordinates) from its isomorphism with $\mathbf{P}T^*\mathbf{P}^{N-1}$. Exchanging $\mathbf{P}^{N-1}$ and $\check{\mathbf{P}}^{N-1}$, I is also isomorphic to $\mathbf{P}T^*\check{\mathbf{P}}^{N-1}$ so it also inherits the 1-form $\check{\alpha}(:= \sum z_i d\xi_i$ locally).

Lemma 1.3.11 (Kleiman; see [26, §4]) *Let I be the incidence variety as above. Then $\alpha + \check{\alpha} = 0$ on I.*

Proof
Note that if the polynomial $\sum_{i=1}^N z_i\xi_i$ defined a function on $\mathbf{P}^{N-1}\times\check{\mathbf{P}}^{N-1}$, we would obtain the result by differentiating it. The idea of the proof is basically the same, it involves identifying the polynomial $\sum_{i=1}^N z_i\xi_i$ with a section of the line bundle $p^*O_{\mathbf{P}^{N-1}}(1)\otimes\check{p}^*O_{\check{\mathbf{P}}^{N-1}}(1)$ over I, where p and $\check{p}$ are the natural projections of I to $\mathbf{P}^{N-1}$ and $\check{\mathbf{P}}^{N-1}$ respectively and $O_{\mathbf{P}^{N-1}}(1)$ denotes the canonical line bundle, introducing the appropriate flat connection on this bundle, and differentiating. □

In particular, this lemma tells us that if at some point $z \in I$ we have that $\alpha = 0$, then $\check{\alpha} = 0$ too. Thus, a closed conical irreducible analytic subvariety of $T^*\mathbf{P}^{N-1}$ as in Proposition 1.3.7 is the conormal space of its image in $\mathbf{P}^{N-1}$ if and only if it is the conormal space of its image in $\check{\mathbf{P}}^{N-1}$. So we have $\mathbf{P}T^*_V\mathbf{P}^{N-1} \subset I \subset \mathbf{P}^{N-1}\times\check{\mathbf{P}}^{N-1}$ and the restriction of the two canonical projections:

$$\begin{tikzcd} & \mathbf{P}T^*_V\mathbf{P}^{N-1}\subset I \arrow[dl, "p"'] \arrow[dr, "\check{p}"] & \\ V\subset\mathbf{P}^{N-1} & & \check{\mathbf{P}}^{N-1}\supset\check{V} \end{tikzcd}$$

Definition 1.3.12 The **dual variety** $\check{V}$ of $V \subset \mathbf{P}^{N-1}$ is the image by the map $\check{p}$ of $\mathbf{P}T^*_V\mathbf{P}^{N-1} \subset I$ in $\check{\mathbf{P}}^{N-1}$. So by construction $\check{V}$ is the closure in $\check{\mathbf{P}}^{N-1}$ of the set of hyperplanes tangent to V^0.

We immediately get by symmetry that $\check{\check{V}} = V$. What is more, we see that establishing a projective duality is equivalent to finding a Lagrangian subvariety in I; its images in $\mathbf{P}^{N-1}$ and $\check{\mathbf{P}}^{N-1}$ are necessarily dual.

Lemma 1.3.13 *Let us assume that* $(X, 0) \subset (\mathbf{C}^N, 0)$ *is the cone over a projective algebraic variety* $V \subset \mathbf{P}^{N-1}$*. Let* $x \in X^0$ *be a nonsingular point of* X*. Then the tangent space* $T_{X^0,x}$*, contains the line* $\ell = \overline{0x}$ *joining* x *to the origin. Moreover, the tangent map at* x *to the projection* $\pi\colon X\backslash\{0\} \to V$ *induces an isomorphism* $T_{X^0,x}/\ell \simeq T_{V,\pi(x)}$.

Proof
This is due to Euler's identity for a homogeneous polynomial of degree m:

$$m.f = \sum_{i=1}^{N} z_i \frac{\partial f}{\partial z_i}$$

and the fact that if $\{f_1, \ldots, f_r\}$ is a set of homogeneous polynomials defining X, then $T_{X^0,x}$ is the kernel of the matrix:

$$\begin{pmatrix} df_1 \\ \cdot \\ \cdot \\ df_r \end{pmatrix}$$

representing the differentials df_i in the basis $dz_1, \ldots, dz_N$. □

It is also important to note that the tangent space to X^0 is constant along all non-singular points x of X in the same generating line since the partial derivatives are homogeneous as well, and contains the generating line. By Lemma 1.3.13, the quotient of this tangent space by the generating line is the tangent space to V at the point corresponding to the generating line.

So, $\mathbf{P}T^*_X\mathbf{C}^N$ has an image in $\check{\mathbf{P}}^{N-1}$ which is the projective dual of V.

$$\begin{tikzcd}
 & \mathbf{P}T^*_V\mathbf{P}^{N-1} \arrow[dl] \arrow[dr] & & \mathbf{P}T^*_X\mathbf{C}^N \subset \check{\mathbf{P}}^{N-1} \times \mathbf{C}^N \arrow[dll] \arrow[dr] & \\
V \subset \mathbf{P}^{N-1} & & \check{\mathbf{P}}^{N-1} \supset \check{V} & & X \subset \mathbf{C}^N
\end{tikzcd}$$

The fiber over 0 of $\mathbf{P}T^*_X\mathbf{C}^N \to X$ is equal to $\check{V}$ as subvariety of $\check{\mathbf{P}}^{N-1}$: it is the set of limit positions at 0 of hyperplanes tangent to X^0.
For more information on projective duality, in addition to Kleiman's papers one can consult [56].

A *relative* version of the conormal space and of projective duality will play an important role in these notes. Useful references are [19], [26], [55, Chap. IV]. The relative conormal space is used in particular to define the *relative polar varieties.*

1.3.3 Polar varieties and the control of the dimension of the fibers of $\kappa_X : C(X) \to X$.

The simplest measure of the complexity of the space of limits of tangent hyperplanes at a point $x \in X$ is the dimension of the fiber $\kappa_X^{-1}(x) \subset \check{\mathbf{P}}^{N-1}$. This dimension is the difference between $N-1$ and the maximum codimension of a linear subspace of $\check{\mathbf{P}}^{N-1}$ whose intersection with $\kappa^{-1}(x)$ is not empty. We are thus led to consider the subspaces $C(X) \cap (X \times L^{d-k})$ of $C(X)$, where $0 \leqslant k \leqslant d = \dim. X$ and L^{d-k} is a linear subspace of $\check{\mathbf{P}}^{N-1}$ of dimension $d-k$, dual to a vector subspace $D_{d-k+1} \subset \mathbf{C}^N$ of codimension $d-k+1$ in the sense that it is the space of directions of hyperplanes containing it. We remark that, with the notations introduced above, we have $C(X) \cap (X \times L^{d-k}) = \lambda^{-1}(L^{d-k})$.

The next proposition provides the relation between the geometry of $\kappa_X^{-1}(x) \subset \check{\mathbf{P}}^{N-1}$ as read by linear subspaces and geometrically defined subspaces of X, the *local polar varieties* of $X \subset \mathbf{C}^N$ which are defined as the closures in X of sets of critical points on X^0 of projections $X \to \mathbf{C}^{d-k+1}$ induced by general linear maps $\mathbf{C}^N \to \mathbf{C}^{d-k+1}$. They were originally defined in [33]. Recall the definition of the map λ in Definition 1.2.5.

Proposition 1.3.14 *For a sufficiently general D_{d-k+1}, the image $\kappa(\lambda^{-1}(L^{d-k}))$ is the closure in X of the set of points of X^0 which are critical for the projection $\pi|_{X^0} : X^0 \to \mathbf{C}^{d-k+1}$ induced by the projection $\mathbf{C}^N \to \mathbf{C}^{d-k+1}$ with kernel $D_{d-k+1} = (L^{d-k})\check{}$.*

Proof
Note that $x \in X^0$ is critical for π if and only if the tangent map $d_x\pi : T_{X^0,x} \longrightarrow \mathbf{C}^{d-k+1}$ is not onto, which means $\dim. \ker d_x\pi \geqslant k$ since $\dim T_{X^0,x} = d$, and $\ker d_x\pi = D_{d-k+1} \cap T_{X^0,x}$.
Note that the conormal space $C(X^0)$ of the nonsingular part of X is equal to $\kappa^{-1}(X^0)$ so by definition:

$$\lambda^{-1}(L^{d-k}) \cap C(X^0) = \{(x,H) \in C(X) | x \in X^0,\ H \in L^{d-k},\ T_{X^0,x} \subset H\}$$

equivalently:

$$\lambda^{-1}(L^{d-k}) \cap C(X^0) = \{(x,H), \in C(X) | x \in X^0,\ H \in (D_{d-k+1})\check{},\ H \in (T_{X^0,x})\check{}\ \}$$

thus $H \in (D_{d-k+1})\check{} \cap (T_{X^0,x})\check{}$, and from the equality $(D_{d-k+1})\check{} \cap (T_{X^0,x})\check{} = (D_{d-k+1} + T_{X^0,x})\check{}$ we deduce that the intersection is not empty if and only if $D_{d-k+1} + T_{X^0,x} \neq \mathbf{C}^N$, which implies that $\dim D_{d-k+1} \cap T_{X^0,x} \geqslant k$, and consequently $\kappa(H) = x$ is a critical point.

According to [55, Chap. IV, 1.3], there exists an open dense set U_k in the Grassmannian of $(N-d+k-1)$-planes of $\mathbf{C}^N$ such that if $D_{d-k+1} \in U_k$, the intersection $\lambda^{-1}(L^{d-k}) \cap C(X^0)$ is dense in $\lambda^{-1}(L^{d-k})$. So, for any $D_{d-k+1} \in U_k$, since κ is a proper map and thus closed, we have that $\kappa(\lambda^{-1}(L^{d-k})) =$

$\kappa\left(\overline{\lambda^{-1}(L^{d-k}) \cap C(X^0)}\right) = \overline{\kappa(\lambda^{-1}(L^{d-k}))}$, which finishes the proof. See [55, Chap. IV, 4.1.1] for a complete proof of a more general statement. □

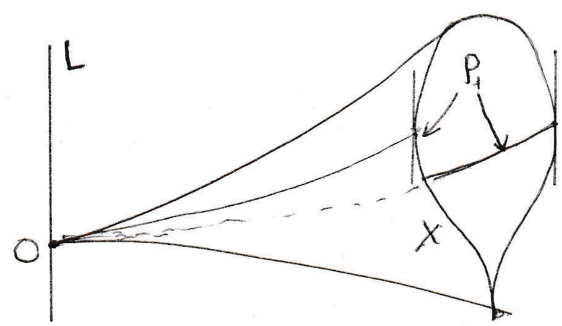
Remark 1.3.15 It is important to have in mind the following easily verifiable facts:

a) As we have seen before, the fiber $\kappa^{-1}(x)$ over a regular point $x \in X^0$ in the (projectivized) conormal space $C(X)$ is a $\mathbf{P}^{N-d-1}$, so by semicontinuity of fiber dimension we have that $\dim \kappa^{-1}(0) \geqslant N - d - 1$.
b) For a general L^{d-k}, the intersection $C(X) \cap (X \times L^{d-k})$ is of pure dimension $N - 1 - N + d - k + 1 = d - k$ if it is not empty.
The proof of this is not immediate because we are working over an open neighborhood of a point $x \in X$, so we cannot assume that $C(X)$ is compact. However (see [55, Chap. IV]) we can take a Whitney stratification of $C(X)$ (these stratifications are explained below) such that the closed algebraic subset $\kappa^{-1}(0) \subset \check{\mathbf{P}}^{N-1}$, which is compact, is a union of strata. By general transversality theorems in algebraic geometry (see [25]) a sufficiently general L^{d-k} will be transversal to all the strata of $\kappa^{-1}(0)$ in $\check{\mathbf{P}}^{N-1}$ and then because of the Whitney conditions (see [58, section 4.9]) $\mathbf{C}^N \times L^{d-k}$ will be transversal in a neighborhood of $\kappa^{-1}(0)$ to all the strata of $C(X)$, which will imply in particular the statement on the dimension. Since κ is proper, the neighborhood of $\kappa^{-1}(0)$ can be taken to be the inverse image by κ of a neighborhood of 0 in X. The meaning of "general" in Proposition 1.3.14 is that of Kleiman's transversality theorem. Moreover, since $C(X)$ is a reduced equidimensional analytic space, for a general L^{d-k}, the intersection of $C(X)$ and $\mathbf{C}^N \times L^{d-k}$ in $\mathbf{C}^N \times \check{\mathbf{P}}^{N-1}$ is generically reduced and since according to our general rule we remove embedded components when intersecting with linear spaces, $\lambda^{-1}(L^{d-k})$ is a reduced equidimensional complex analytic space.
Note that the existence of Whitney stratifications does not depend on the existence of polar varieties. In [55, Chap. III, Proposition 2.2.2] it is deduced from the idealistic Bertini theorem.
c) The fact that $\lambda^{-1}(L^{d-k}) \cap C(X^0)$ is dense in $\lambda^{-1}(L^{d-k})$ means that if a limit of tangent hyperplanes at points of X^0 contains D_{d-k+1}, it is a limit of tangent hyperplanes which also contain D_{d-k+1}. This equality holds because transversal intersections preserve the frontier condition; see [58, Theorem 4.2.15] or [7, Lemme 2.2.2], [55, Remarque 4.2.3].
d) Note that for a fixed L^{d-k}, the germ $(P_k(X; L^{d-k}), 0)$ is empty if and only if the intersection $\kappa^{-1}(0) \cap \lambda^{-1}(L^{d-k})$ is empty. From a) we know that $\dim \kappa^{-1}(0) = N - d - 1 + r$ with $r \geqslant 0$. Thus, by the same argument as in b), this implies that the polar variety $(P_k(X; L^{d-k}), 0)$ is not empty if and only if $\dim(\kappa^{-1}(0) \cap \lambda^{-1}(L^{d-k})) \geqslant 0$ and if and only if $r \geqslant k$.

Definition 1.3.16 With the notation and hypotheses of Proposition 1.3.14, define for $0 \leqslant k \leqslant d-1$ the **local polar variety** is defined as:

$$P_k(X; L^{d-k}) = \kappa(\lambda^{-1}(L^{d-k})).$$

A priori, we have just defined local polar varieties set-theoretically, but since $\lambda^{-1}(L^{d-k})$ is empty or reduced and κ is a projective fibration over the smooth part of X we have the following result, for which a proof can be found in [55, Chap. IV, 1.3.2].



Proposition 1.3.17 *For a general linear subspace $L^{d-k} \subset \check{\mathbf{P}}^{N-1}$ and $0 \leqslant k \leqslant d$ the local polar variety $P_k(X; L^{d-k}) \subset X$ is a reduced closed analytic subspace of X, either of pure codimension k in X or empty.*

We have thus far defined a local polar variety that depends on both the choice of the embedding $(X,0) \subset (\mathbf{C}^N, 0)$ and the choice of the general linear space D_{d-k+1}. However, an important information we will extract from these polar varieties is their multiplicities at 0, and these numbers are analytic invariants provided the linear spaces used to define them are general enough.

Proposition 1.3.18 (Teissier, see [55, Chap. IV, §3])*Let $(X,0) \subset (\mathbf{C}^N, 0)$ be as before, then for every $0 \leqslant k \leqslant d-1$ and a sufficiently general linear space $D_{d-k+1} \subset \mathbf{C}^N$ the multiplicity of the polar variety $P_k(X; L^{d-k})$ at 0 depends only on the analytic type of $(X,0)$.*

Exercise 1.3.19 Let $0 \in Y \subset X \subset \mathbf{C}^N$ where Y is one dimensional and non-singular and X is d-dimensional. Show that the following conditions are equivalent:

1. The germ of polar curve $(P_{d-1}(X; L^{d-k}), 0)$ is empty;
2. dim. $\kappa^{-1}(0) < N-2$.

and imply:

A Zariski open and dense subset of the $\check{\mathbf{P}}^{N-2} \subset \check{\mathbf{P}}^{N-1}$ consisting of hyperplanes containing $T_{Y,0}$ is not contained in $\kappa^{-1}(0)$: a general hyperplane containing $T_{Y,0}$ is not a limit of tangent hyperplanes to X^0. Compare with example 1.5.4 below.

1.4 Limits of secants: the blowing-up

In this section we present the blowing up of a coherent sheaf of ideals in a way which is adapted to the construction of the normal/conormal diagram which is used in the study of Whitney conditions.

Let $\mathcal{I}$ be a coherent sheaf of ideals on X defining a closed analytic subspace $Y \subset X$. Let $U \subset X$ be an open set on which we have a presentation

$$\mathcal{O}_U^q \to \mathcal{O}_U^p \to \mathcal{I}|U \to 0.$$

We have thus a set of global generators $f_1, \ldots, f_p$ for $\mathcal{I}|U$. Consider the map $U \backslash Y \to \mathbf{P}^{p-1}$ defined by $x \mapsto (f_1(x) : \ldots : f_p(x)$, and its graph $E_Y(U \backslash Y) \subset (U \backslash Y) \times \mathbf{P}^{p-1}$. The closure $E_Y U$ of this graph in $U \times \mathbf{P}^{p-1}$ is a closed analytic subspace which, up to a unique isomorphism, depends only on $\mathcal{I}|U$.
To see this, consider the graded $\mathcal{O}_X$ algebra

$$P(\mathcal{I}) = \bigoplus_{n \in \mathbf{N}} \mathcal{I}^n,$$

which is locally finitely generated in degree one.
Because $\mathcal{I}$ is locally finitely presented, this algebra has also locally a finite presentation by an exact sequence of finitely generated graded $\mathcal{O}_U$ algebras and modules (see [3, Chap. 1, 1.3]).

$$0 \to \mathcal{K}_U \to \mathcal{O}_U[T_1, \ldots, T_p] \to P(\mathcal{I})|U \to 0,$$

where each T_j is mapped to $f_j \in \mathcal{I}|U$. The ideal $\mathcal{K}_U$ is generated by finitely many homogeneous polynomials in $T_1, \ldots, T_p$ which by definition generate all the algebraic homogeneous relations beween $f_1, \ldots f_p$. The vanishing of these polynomials defines a closed subspace of $U \times \mathbf{P}^{p-1}$ which, by construction, is the closure $E_Y U$ of the graph we have just seen. One verifies that this subspace is independent of the choice of the generators $f_1, \ldots, f_p$ and so by uniqueness the local constructions glue up into a space E_Y over X, say

$$e_Y : E_Y X \to X$$

which is called the *blowing-up* of $\mathcal{I}$ (or Y) in X.

The construction we have just described is, when we give the subspace of $U \times \mathbf{P}^{p-1}$ its natural structure as a complex analytic space, the Projan$P(\mathcal{I})$ of the locally finitely presented graded $\mathcal{O}_X$-algebra $P(\mathcal{I})$.
The inverse image $e_Y^{-1}(Y)$ is the projan of the graded $\mathcal{O}_Y$-algebra

$$P(\mathcal{I}) \otimes_{\mathcal{O}_X} \mathcal{O}_X/\mathcal{I} = \bigoplus_{n \in \mathbf{N}} \mathcal{I}^n/\mathcal{I}^{n+1} = \mathcal{O}_Y \oplus \mathcal{I}/\mathcal{I}^2 \oplus \mathcal{I}^2/\mathcal{I}^3 \oplus \cdots$$

Besides the fact that the blowing-up is locally the closure of a graph, its essential feature is that $e_Y^{-1}(Y) \subset E_Y X$ is locally on $E_Y X$ defined by one

equation which is not a zero divisor and is one of the generators of the pull-back $\mathcal{I}\mathcal{O}_{E_Y X}$ of the ideal $\mathcal{I}$. It is the *exceptional divisor* of the blowing-up. Indeed, in each affine chart V_j defined by $T_j \neq 0$ of $\mathbf{P}^{p-1}$ the T_i/T_j are coordinates, which implies that on the intersection of $E_Y X$ with $X \times V_j$ the functions f_i/f_j are regular and thus the ideal $(f_1 \circ e_Y, \dots, f_p \circ e_Y)$, which is the restriction of $\mathcal{I}\mathcal{O}_{E_Y X}$ to the intersection of $E_Y X$ with $X \times V_j$, is principal and generated by $f_j \circ e_Y$.
The following universal property of blowing-up, which we state here in the complex analytic framework, is due to Hironaka:

Theorem 1.4.1 *A complex-analytic map $\pi\colon T \to X$ such that $\pi^{-1}(Y)$ is locally on T defined by a single equation which is not a zero divisor in the local rings of T factors uniquely through e_Y. This property characterizes the map e_Y.*

In what follows we shall consider the case where $Y \subset X \subset \mathbf{C}^N$, where $\mathbf{C}^N$ is endowed with coordinates $z_1, \dots, z_N$ and Y is non-singular of dimension t. We may assume that the coordinates are adapted to Y in the sense that it is defined by the vanishing of coordinates $z_{t+1}, \dots, z_N$ on $\mathbf{C}^N$. The map $X \setminus Y \to \mathbf{P}^{N-t-1}$ defined by $(z_1, \dots, z_N) \mapsto (z_{t+1} : \dots : z_N) \in \mathbf{P}^{N-t-1}$ can be deemed to associate to a point of $X \setminus Y$ the direction of the secant line joining this point to the point in Y with coordinates $z_1, \dots, z_t$. The closure in $X \times \mathbf{P}^{N-t-1}$ of the graph of this map is the blowing up in X of the subspace Y. Although the secant lines clearly depend on the choice of coordinates, the blowing up does not.
A point of $E_Y X \subset X \times \mathbf{P}^{N-t-1}$ is therefore a pair $(x, [\ell])$ where if $x \in X \setminus Y$, $[\ell]$ is the direction of the secant line joining x to its linear projection on Y according to the coordinate system, and if $x \in Y$, the direction $[\ell]$ is a limit direction of such secant lines along a sequence of points of $X \setminus Y$ tending to x. Denoting by $\mathcal{I}_Y$ the coherent sheaf of ideals defining $Y \subset X$, and by $\mathrm{gr}_{\mathcal{I}_Y}\mathcal{O}_X$ the graded $\mathcal{O}_Y$-algebra

$$\mathrm{gr}_{\mathcal{I}_Y}\mathcal{O}_X = \bigoplus_{n \in \mathbf{N}} \mathcal{I}_Y^n / \mathcal{I}_Y^{n+1},$$

the space $\mathrm{Specan}(\mathrm{gr}_{\mathcal{I}_Y}\mathcal{O}_X)$ with its natural mapping $\mathrm{Specan}(\mathrm{gr}_{\mathcal{I}_Y}\mathcal{O}_X) \to Y$ corresponding to the inclusion $\mathcal{O}_Y \subset \mathrm{gr}_{\mathcal{I}_Y}\mathcal{O}_X$ is called the *normal cone* of Y in X and usually denoted by $C_{X,Y} \to Y$. In the case where Y is a point, say $x \in X$, it is for historical reasons the *tangent cone* of X at x. If X is non-singular these notions coincide with the normal bundle of Y in X and the tangent space of X at y.

In the case where Y is a point $x \in X$, $\mathcal{I}_{\{x\}}$ corresponds to the maximal ideal $m_x \subset \mathcal{O}_{X,x}$ which is generated by the local coordinates $z_1, \dots, z_N$. The degree of the tangent cone is the *multiplicity* of X at the point x. It is also the degree of the projective variety $e_x^{-1}(x) \subset \mathbf{P}^{N-1}$ associated to the tangent cone.

Remark 1.4.2 One may ask for the interpretation of the fiber of $e_Y : E_Y \to X$ at a point $y \in Y$. It is called the *analytic spread* of the ideal $\mathcal{I}$ at this point and plays an important role in detecting equimultiplicity of X along Y.

1.5 The normal/conormal diagram

In this section we construct a space which, given a non-singular subspace $Y \subset X \subset \mathbf{C}^N$ and a local retraction $r : \mathbf{C}^N \to Y$ does for limit positions of pairs (ℓ, T) at a point $x \in X^0 \backslash Y$ of the direction of secant line $\overline{xr(x)}$ and a direction of tangent hyperplane $H \supset T_{X^0, x}$ what the conormal space and the blowing up of Y in X do separately.

With the help of the normal/conormal diagram and the polar varieties we will be able to obtain information on the limits of tangent spaces to X at 0, assuming that $(X, 0)$ is reduced and purely d-dimensional. This method is based on Whitney's lemma and the two results which follow it:

Lemma 1.5.1 ***(Whitney's lemma for** X^0***)***

Let $(X, 0)$ *be a pure-dimensional germ of analytic subspace of* $\mathbf{C}^N$, *choose a representative* X *and let* $\{x_n\} \subset X^0$ *be a sequence of points tending to* 0, *such that*

$$\lim_{n\to\infty} [0x_n] = l \ \ \textit{and} \ \ \lim_{n\to\infty} T_{x_n} X = T.$$

Then $l \subset T$.

A stronger form of this lemma originally appeared in [60, Theorem 22.1], and you can also find a proof due to Hironaka in [31] and yet another below in assertion a) of Theorem 1.5.2.

Given $X \subset \mathbf{C}^N$ as above, consider the normal/conormal diagram

$$\begin{tikzcd}
X \times \mathbf{P}^{N-1} \times \check{\mathbf{P}}^{N-1} \arrow[dd, "\kappa"] & \supset & E_0C(X) \arrow[r, "\hat{e}_0"] \arrow[dd, "\kappa'"] \arrow[ddr, "\xi"] & C(X) \arrow[r, hook] \arrow[dd, "\kappa"] \arrow[dr, "\lambda"] & X \times \check{\mathbf{P}}^{N-1} \arrow[d, "pr_2"] \\
 & & & & \check{\mathbf{P}}^{N-1} \\
X \times \mathbf{P}^{N-1} & \supset & E_0X \arrow[r, "e_0"'] & X &
\end{tikzcd},$$

where e_0 is the blowing up of the point $0 \in X$, $\hat{e}_0$ is the blowing up of the subspace $\kappa^{-1}(0)$ and κ' is the map coming from the universal property of blowing ups applied to the map $\xi = \kappa \circ \hat{e}_0$.

Theorem 1.5.2 (Lê-Teissier, see [35, §2])

In the normal/conormal diagram, consider the irreducible components D_j of the exceptional divisor $D = |\xi^{-1}(0)|$. Then we have:
I) *The following hold*

1. *Each $D_j \subset \mathbf{P}^{N-1} \times \check{\mathbf{P}}^{N-1}$ is contained in the incidence variety $I \subset \mathbf{P}^{N-1} \times \check{\mathbf{P}}^{N-1}$.*

2. *Each D_j is Lagrangian in I and therefore establishes a projective duality of its images:*

$$\begin{tikzcd} D_j \arrow[r] \arrow[d] & W_j \subset \check{\mathbf{P}}^{N-1} \\ V_j \subset \mathbf{P}^{N-1} & \end{tikzcd}$$

Note that, from commutativity of the diagram we obtain $\kappa^{-1}(0) = \bigcup_j W_j$, and $e_0^{-1}(0) = \bigcup_\alpha V_j$. It is important to notice that these expressions are not necessarily the irreducible decompositions of $\kappa^{-1}(0)$ and $e_0^{-1}(0)$ respectively, since there may be repetitions; it is the case for the surface of Example 1.5.4 below, where the dual of the tangent cone, a point in $\check{\mathbf{P}}^2$, is contained in the projective line dual to the exceptional tangent. However, it is true that they contain the respective irreducible decompositions.

In particular, note that if $\dim V_{j_0} = d - 1$, then the cone $O(V_{j_0}) \subset \mathbf{C}^N$ is an irreducible component of the tangent cone $C_{X,0}$ and its projective dual $W_{j_0} = \check{V}_{j_0}$ is contained in $\kappa^{-1}(0)$. That is, any tangent hyperplane to the tangent cone is a limit of tangent hyperplanes to X at 0. The converse is very far from true and we shall see more about this below.

II) *For any integer k, $0 \leqslant k \leqslant d-1$, and sufficiently general $L^{d-k} \subset \check{\mathbf{P}}^{N-1}$ the tangent cone $C_{P_k(X,L),0}$ of a non empty polar variety $P_k(X,L)$ at the origin consists of:*

- *The union of the cones $O(V_j)$ which are of dimension $d-k$ $(= \dim P_k(X,L))$.*
- *The polar varieties $P_\ell(O(V_j), L)$ of dimension $d-k$, for the projection p associated to L, of the cones $O(V_j)$, for j such that $\dim O(V_j) = d-k+\ell$ for some $1 \leqslant \ell \leqslant k$.*

Note that $P_k(X,L)$ is not unique, since it varies with L, but we are saying that its tangent cone may have parts which do not vary with L. The V_α's are fixed, so the first part is the fixed part of $C_{P_k(X,L),0}$ because it is independent of L, the second part is the mobile part, since we are talking of polar varieties of certain cones, which by definition move with L.

Proof
The proof of **I)**, which can be found in [35, §2], is essentially a strengthening of Whitney's lemma (Lemma 1.5.1) using the normal/conormal diagram and the fact that the vanishing of a differential form (the symplectic form in our case) is a closed condition.
The proof of **II)**, a special case of [35, Proposition 2.2.1], is somewhat easier to explain geometrically:
Using our normal/conormal diagram, remember that we can obtain the blowing up $E_0(P_k(X,L))$ of the polar variety $P_k(X,L)$ by taking its strict transform under the morphism e_0, and as such we will get the projectivized tangent cone $\mathbf{P}C_{P_k(X,L),0}$ as the fiber over the origin.

The first step is to prove that set-theoretically the projectivized tangent cone can also be expressed as

$$|\mathbf{P}C_{P_k(X,L),0}| = \bigcup_j \kappa'(\hat{e}_0^{-1}(\lambda^{-1}(L)\cap W_j)) = \bigcup_j \kappa'(D_j \cap (\mathbf{P}^{N-1}\times L))$$

Now recall that the intersection $P_k(X,L)\cap X^0$ is dense in $P_k(X,L)$, so for any point $(0,[l]) \in \mathbf{P}C_{P_k(X,L),0}$ there exists a sequence of points $\{x_n\}\subset X^0$ such that the directions of the secants $\overline{0x_n}$ converge to it. So, by definition of a polar variety, if $D_{d-k+1} = \check{L}$ and $T_n = T_{x_n}X^0$ then by Proposition 1.3.14 we know that $\dim T_n \cap D_{d-k+1} \geqslant k$ which is a closed condition. In particular if T is a limit of tangent spaces obtained from the sequence $\{T_n\}$, then $T\cap D_{d-k+1}\geqslant k$ also. But if this is the case, since the dimension of T is d, there exists a limit of tangent hyperplanes $H\in\kappa^{-1}(0)$ such that $T+D_{d-k+1}\subset H$ which is equivalent to $H\in\kappa^{-1}(0)\cap\lambda^{-1}(L)\neq\emptyset$. Therefore the point $(0,[l],H)$ is in $\bigcup_j \hat{e}_0^{-1}(\lambda^{-1}(L)\cap W_j)$, and so we have the inclusion:

$$|\mathbf{P}C_{P_k(X,L),0}| \subset \bigcup_j \kappa'(\hat{e}_0^{-1}(\lambda^{-1}(L)\cap W_j)).$$

For the other inclusion, recall that $\lambda^{-1}(L)\backslash\kappa^{-1}(0)$ is dense in $\lambda^{-1}(L)$ and so $\hat{e}_0^{-1}(\lambda^{-1}(L))$ is equal set theoretically to the closure in $E_0C(X)$ of $\hat{e}_0^{-1}(\lambda^{-1}(L)\backslash\kappa^{-1}(0))$. Then for any point $(0,[l],H)\in\hat{e}_0^{-1}(\lambda^{-1}(L)\cap\kappa^{-1}(0))$ there exists a sequence $\{(x_n,[x_n],H_n)\}$ in $\hat{e}_0^{-1}(\lambda^{-1}(L)\backslash\kappa^{-1}(0))$ converging to it. Now by commutativity of the diagram, we get that the sequence $\{(x_n,H_n)\}\subset\lambda^{-1}(L)$ and as such the sequence of points $\{x_n\}$ lies in the polar variety $P_k(X,L)$. This implies in particular, that the sequence $\{(x_n,[0x_n])\}$ is contained in $e_0^{-1}(P_k(X,L)\backslash\{0\})$ and the point $(0,[l])$ is in the projectivized tangent cone $|\mathbf{P}C_{P_k(X,L),0}|$.

The second and final step of the proof is to use that from *a)* and *b)* it follows that each $D_j\subset I\subset\mathbf{P}^{N-1}\times\check{\mathbf{P}}^{N-1}$ is the conormal space of V_j in $\mathbf{P}^{N-1}$, with the restriction of κ' to D_j being its conormal morphism.

Note that D_j is of dimension $N-2$, and since all the maps involved are just projections, we can take the cones over the V_j's and proceed as in Section 1.3.2. In this setting we get that since L is sufficiently general, by Proposition 1.3.14 and Definition 1.3.16:

- For the D_j's corresponding to cones $O(V_j)$ of dimension $d-k$ $(= \dim P_k(X,L))$, the intersection $D_j \cap (\mathbf{P}^{N-1} \times L)$ is not empty and as such its image is a polar variety $P_0(O(V_j),L) = O(V_j)$ which is independent of L.
- For the D_j's corresponding to cones $O(V_j)$ of dimension $d-k+\ell$ for some $1 \leqslant \ell \leqslant k$, the intersection $D_j \cap (\mathbf{P}^{N-1} \times L)$ is either empty or of dimension $d-k$ and as such its image is a polar variety of dimension $d-k$, which is $P_\ell(O(V_j),L)$ and varies with L if it is not empty.

You can find a detailed proof of these results in [35, §2], [55, Chap. IV]. $\square$

So for any reduced and purely d-dimensional complex analytic germ $(X,0)$, we have a method to "compute" or rather describe, the set of limiting positions of tangent hyperplanes. Between parentheses are the types of computations involved:

1) For all integers k, $0 \leqslant k \leqslant d-1$, compute the "general" polar varieties $P_k(X,L)$, leaving in the computation the coefficients of the equations of L as indeterminates. (Partial derivatives, Jacobian minors and residual ideals with respect to the Jacobian ideal);
2) Compute the tangent cones $C_{P_k(X,L),0}$ (computation of a standard basis with parameters);
3) Sort out those irreducible components of the tangent cone of each $P_k(X,L)$ which are independent of L (decomposition into irreducible components with parameters);
4) Take the projective duals of the corresponding projective varieties (Elimination).

We have noticed, that among the V_j's, there are those which are irreducible components of Proj $C_{X,0}$ and those that are of lower dimension.

Definition 1.5.3 The cones $O(V_j)$'s such that

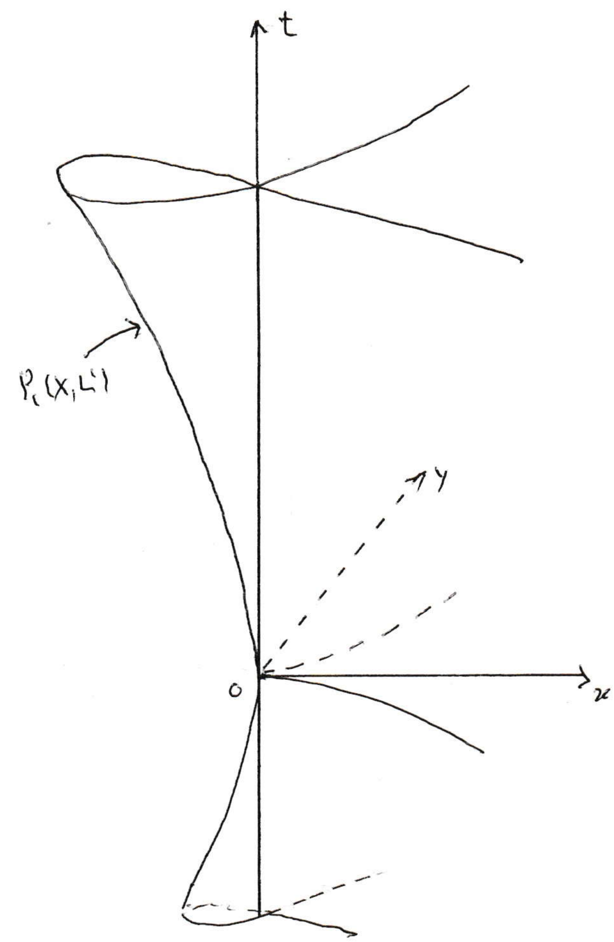
$$\dim. V_j < \dim. \operatorname{Proj} C_{X,0}$$

are called exceptional cones.

Example 1.5.4
Let $X := y^2 - x^3 - t^2x^2 = 0 \subset \mathbf{C}^3$, so $\dim X = 2$, and thus $k = 0, 1$. An easy calculation shows that the singular locus of X is the t-axis, and $m_0(X) = 2$.

Note that for $k = 0$, D_3 is just the origin in $\mathbf{C}^3$, so the projection

$$\pi : X^0 \to \mathbf{C}^3$$



with kernel D_3 is the restriction to X^0 of the identity map, which is of rank 2 and we get that the whole X^0 is the critical set of such a map. Thus,

$$P_0(X, L^2) = X.$$

For $k = 1$, D_2 is of dimension 1. So let us take for instance $D_2 = y$-axis, so we get the projection

$$\pi : X^0 \to \mathbf{C}^2 \quad (x, y, t) \mapsto (x, t),$$

and we obtain that the set of critical points of the projection is given by

$$P_1(X, L^1) = \begin{cases} x = -t^2 \\ y = 0 \end{cases}$$

If we had taken for D_2 the line $t = 0,\ \alpha x + \beta y = 0$, we would have found that the polar curve is a nonsingular component of the intersection of our surface with the surface $2\alpha y = \beta x(3x + 2t^2)$. For $\alpha \neq 0$ all these polar curves are tangent to the t-axis. As we shall see in the next subsection, this means that the t-axis is an "exceptional cone" in the tangent cone $y^2 = 0$ of our surface at the origin, and therefore all the 2-planes containing it are limits at the origin of tangent planes at nonsingular points of our surface.

Remark 1.5.5 1) We repeat the remark on p. 567 of [35] to the effect that when $(X, 0)$ is analytically isomorphic to the germ at the vertex of a cone the polar varieties are themselves isomorphic to cones so that the families of tangent cones of polar varieties have no fixed components except when $k = 0$. Therefore in this case $(X, 0)$ has no exceptional cones.
2) The fact that the cone X over a nonsingular projective variety has no exceptional cones is thus related to the fact that the critical locus $P_1(X, 0)$ of the projection $\pi\colon X \to \mathbf{C}^d$, which is purely of codimension one in X if it is not empty, actually moves with the projection π; in the language of algebraic geometry, the ramification divisor of the projection is ample (see [61, Chap. I, cor. 2.14]) and even very ample (see [10]).
3) The dimension of $\kappa^{-1}(0)$ can be large for a singularity $(X, 0)$ which has no exceptional cones. This is the case for example if X is the cone over a projective variety of dimension $d - 1 < N - 2$ in $\mathbf{P}^{N-1}$ whose dual is a hypersurface.

Let $f : X \to S$ be a morphism of reduced analytic spaces, with purely d-dimensional fibers and such that there exists a closed nowhere dense analytic space such that the restriction to its complement X^0 in X :

$$f|_{X^0} : X^0 \longrightarrow S$$

has all its fibers smooth. They are manifolds of dimension $d = \dim. X - \dim. S$. Let us assume furthermore that the map f is induced, via a closed embedding $X \subset Z$ by a smooth map $F\colon Z \to S$. This means that locally on Z the map F is analytically isomorphic to the first projection $S \times \mathbf{C}^N \to S$. Locally on X, this is always the case because we can embed the graph of f, which lies in $X \times S$, into $\mathbf{C}^N \times S$.
Let us denote by $\pi_F\colon T^*(Z/S) \to Z$ the relative cotangent bundle of Z/S, which is a fiber bundle whose fiber over a point $z \in Z$ is the dual $T^*_{Z/S,x}$ of the tangent vector space at z to the fiber $F^{-1}(F(z))$. For $x \in X^0$, denote by $X^0(x)$ the submanifold $f^{-1}(f(x)) \cap X^0$ of X^0. Using this submanifold we will build the **conormal space of X relative to f**, denoted by $T^*_{X/S}(Z/S)$, by setting

$$N^*_{X^0(x),x} = \{\xi \in T^*Z/S, x | \xi(v) = 0,\ \forall v \in T_{X^0(x),x}\}$$

and
$$T^*_{X^0/S}(Z/S) = \{(x,\xi) \in T^*(Z/S)|\ x \in X^0,\ \xi \in N^*_{X^0(x),x}\},$$
and finally taking the closure of $T^*_{X^0/S}(Z/S)$ in $T^*(Z/S)$, which is a complex analytic space $T^*_{X/S}(Z/S)$ by an argument similar to the one we saw in Proposition 1.2.1. Since X^0 is dense in X, this closure maps onto X by the natural projection $\pi_F : T^*(Z/S) \to Z$.

Now we can projectivize with respect to the homotheties on ξ, as in the case where S is a point, which we have seen above. We obtain the (projectivized) relative conormal space $C_f(X) \subset \mathbf{P}T^*(Z/S)$ (also denoted by $C(X/S)$), naturally endowed with a map
$$\kappa_f : C_f(X) \longrightarrow X.$$
We can assume that locally the map f is the restriction of the first projection to $X \subset S \times U$, where U is open in $\mathbf{C}^n$. Then we have $T^*(S \times U/S) = S \times U \times \check{\mathbf{C}}^n$ and $\mathbf{P}T^*(S \times U/S) = S \times U \times \check{\mathbf{P}}^{N-1}$. This gives an inclusion $C_f(X) \subset X \times \check{\mathbf{P}}^{N-1}$ such that κ_f is the restriction of the first projection, and a point of $C_f(X)$ is a pair (x,H), where x is a point of X and H is a limit direction at x of hyperplanes of $\mathbf{C}^N$ tangent to the fibers of the map f at points of X^0. By taking for S a point we recover the classical case studied above.

Definition 1.5.6 Given a smooth morphism $F : Z \to S$ as above, the projection to S of $Z = S \times U$, with U open in $\mathbf{C}^n$, we shall say that a reduced complex subspace $W \subset T^*(Z/S)$ is **F-Lagrangian** (or **S-Lagrangian** if there is no ambiguity on F) if the fibers of the composed map $q := (\pi_F \circ F)|W : W \to S$ are purely of dimension $n = \dim. Z - \dim. S$ and the differential ω_F of the relative Liouville differential form α_F on $\mathbf{C}^N \times \check{\mathbf{C}}^N$ vanishes on all pairs of tangent vectors at smooth points of the fibers of the map q.

With this definition it is not difficult to verify that $T^*_{X/S}(Z/S)$ is F-Lagrangian, and by abuse of language we will say the same of $C_f(X)$. But we have more:

Proposition 1.5.7 (Lê-Teissier, see [35], proposition 1.2.6)*Let $F : Z \to S$ be a smooth complex analytic map with fibers of dimension n. Assume that S is reduced. Let $W \subset T^*(Z/S)$ be a reduced closed complex subspace and set as above $q = \pi_F \circ F|W : W \to S$. Assume that the dimension of the fibers of q over points of dense open analytic subsets U_i of the irreducible components S_i of S is n.*

1. *If the Liouville form on $T^*_{F^{-1}(s)} = (\pi_F \circ F)^{-1}(s)$ vanishes on the tangent vectors at smooth points of the fibers $q^{-1}(s)$ for $s \in U_i$ and all the fibers of q are of dimension n, then the Liouville form vanishes on tangent vectors at smooth points of all fibers of q.*
2. *The following conditions are equivalent:*

- *The subspace $W \subset T^*(Z/S)$ is F-Lagrangian;*

- *The fibers of q, once reduced, are all purely of dimension n and there exists a dense open subset U of S such that for $s \in U$ the fiber $q^{-1}(s)$ is reduced and is a Lagrangian subvariety of $(\pi_F \circ F)^{-1}(s)$;*
  If moreover W is homogeneous with respect to homotheties on $T^*(Z/S)$, these conditions are equivalent to:
- *All fibers of q, once reduced, are purely of dimension n and each irreducible component W_j of W is equal to $T^*_{X_j/S}(Z/S)$, where $X_j = \pi_F(W_j)$.*

The essential content of this is that an equidimensional specialization of Lagrangian varieties is a union of irreducible Lagrangian varieties. For more details see [35] or [13, Chap. I].

1.6 Whitney stratifications

1.6.1 Introduction

In this section we study Whitney stratifications!canonical minimal Whitney stratification of complex analytic spaces using the tools introduced in the preceding sections. For the history of the subject, including in real algebraic, real analytic, differentiable and definable geometry, we refer the reader to [58, §4.1] in Volume I of this Handbook. The complex analytic case has specific features which imply in particular that Whitney stratifications can be characterized by algebraic equimultiplicity conditions as well as topological equisingularity conditions, that they are also characterized by Lagrangian-type conditions for certain subspaces in auxiliary spaces, and finally that a complex analytic space has a canonical minimal Whitney stratification.

In his paper [60], Whitney gave a definition of a complex analytic stratification of a reduced complex analytic space X (see §18 of *loc.cit.*). The idea is to produce a locally finite decomposition $X = \bigsqcup_{\alpha \in A} S_\alpha$ of a reduced complex analytic space X into disjoint non-singular locally closed subspaces called *strata* such that the "local geometry" of X is the same at all points of the same stratum. To achieve this he proposed two types of conditions:

- Topological/Analytic conditions: each stratum $S_\alpha \subset X$ is a non-singular analytic space, its closure $\overline{S_\beta}$ is a closed analytic subspace of X and the frontier $\overline{S_\beta} \setminus S_\beta$ is a union of strata.
- Differential conditions: Consider a pair of strata (S_α, S_β) such that S_α is contained in the closure of S_β:

  $$S_\alpha \subset \overline{S}_\beta$$

  and consider a point $x \in S_\alpha$. We can assume that a neighborhood of x in X is a closed subset of a open subset U of an affine space $\mathbf{C}^N$. Now, consider a sequence x_n of points of $S_\beta \cap U$ which tends to x and a sequence y_n of points of $S_\alpha \cap U$ which also tends to x. By choosing good subsequences of (x_n) and (y_n), we may suppose that the limit of secant lines $\overline{x_n y_n}$ is ℓ and the limit of the tangents $T_{x_n}S_\beta$ is $\mathbf{T}$. Then one says the we have the Whitney condition for (S_α, S_β) at the point $x \in S_\alpha$, if for all sequences (x_n), (y_n), we have:

$$\ell \subset \mathbf{T}.$$

This is the same as [58, Def. 4.2.1]. Note that the first condition is equivalent to: $S_\alpha \cap \overline{S_\beta} \neq \varnothing$ implies $S_\alpha \subset \overline{S}_\beta$. This is known as the *frontier condition.*

Definition 1.6.1 One says that a locally finite partition $X = \bigsqcup_{\alpha \in A} S_\alpha$ is a Whitney stratification if the topological/analytic conditions are satisfied by the collection of strata and the differential condition is satisfied for all pairs of strata (S_α, S_β) such that $S_\alpha \subset \overline{S}_\beta$ and all points $x \in S_\alpha$.

Theorem 1.6.2 (Whitney) *Any reduced complex analytic space admits Whitney stratifications.*

Proof For the original proof see [60, Theorem 19.2]. For a different proof see [55, Chap. III, Proposition 2.2.2]. ▫

Remark 1.6.3 As we mentioned in Lemma 1.5.1, Whitney discovered (see [60, Theorem 22.1]) that an analytic space is *asymptotically conical* near any of its points. This means that given $x \in X$, a sequence of points $x_n \in X$ tending to x, and a (limit of) tangent space(s) T_n at each x_n (or a limit of limits at x_n of tangent spaces at points of X^0 if the x_n are singular points), up to taking a subsequence, the limit ℓ of secant lines $\overline{xx_n}$ is contained in the limit $\mathbf{T}$ of the T_n. Dealing with the case where the x_n are singular points necessitates the existence of Whitney stratifications of X; that is why the theorem appears at the very end of Whitney's paper.
A consequence of this is that if we take a sufficiently small sphere $\mathbf{S}_\epsilon$, boundary of a ball $\mathbf{B}_\epsilon$ around x in $\mathbf{C}^N$, since it is transversal to the secants $\overline{xx_n}$ it has to be transversal to X^0 and in fact to all the strata S_α containing x in their closure. From this one deduces that $X \cap \mathbf{B}_\epsilon$ is homeomorphic to the (real) cone with vertex x over $X \cap \mathbf{S}_\epsilon$. This is the *local conicity theorem.*

The differential part of the Whitney conditions extends this to the case where the point $x \in X$ is extended to be the stratun $S_\alpha \subset \overline{S_\beta}$, where, as we may, we assune S_α to be a linear subspace of an ambient $\mathbf{C}^N$, so that $\overline{S_\beta}$ is asymptotically like a cone with vertex S_α. That is, the product of the (linear) S_α by a cone. The intuition then is that if we take a sufficiently small tubular neighborhood $\mathbf{T}_\epsilon$ of S_α in $\mathbf{C}^N$, then $\overline{S_\beta}$ should be homeomorphic to the cone with vertex S_α over the intersection of $\overline{S_\beta}$ with the boundary of the tube. This ensures that at least topologically the local geometry of the $\overline{S_\beta}$ containing S_α is constant along S_α, and therefore also that of X.

This intuition turned out to be correct, and in fact more is true (see [43]), but the precise proofs, due to Thom and Mather, are far from easy; see [58].

Remark 1.6.4 In addition to the applications to the study of the topology of singular complex spaces, one must mention that complex Whitney stratifications play a key role in the theory of $\mathcal{D}$-modules (see [24, Chap. 6 and Appendix 2]) and constructible sheaves on complex spaces (see [38, Section 10.3.3]) and also in the theory of characteristic classes for singular complex varieties (see [5] and [6, Section 10]).

1.6.2 Whitney conditions and the normal/conormal diagram

In order to simplify notations we consider a pair of strata $Y \subset X \subset \mathbf{C}^N$ in the neighborhood of $0 \in \mathbf{C}^N$, with Y linear of dimension t. They represent $S_\alpha \subset \overline{S_\beta} \subset \mathbf{C}^N$ with $X^0 = S_\beta$. Since we have to consider limits of secants stating in Y, we consider the following generalization of the normal/conormal diagram:

$$\begin{tikzcd}
X \times \mathbf{P}^{N-t-1} \times \check{\mathbf{P}}^{N-1} \arrow[dd, "K"] & \supset & E_Y C(X) \arrow[r, "\hat{e}_Y"] \arrow[dd, "\kappa'"] \arrow[ddr, "\xi"] & C(X) \arrow[r, hook] \arrow[dd, "\kappa"] \arrow[dr, "\lambda"] & X \times \check{\mathbf{P}}^{N-1} \arrow[d, "pr_2"] \\
 & & & & \check{\mathbf{P}}^{N-1} \\
X \times \mathbf{P}^{N-t-1} & \supset & E_Y X \arrow[r, "e_Y"'] & X &
\end{tikzcd}$$

where now e_Y denotes the blowing-up of Y in X, which, as we remember from section 1.4, builds limits of directions of secant lines $\overline{x\rho(x)}$ for $x \in X \setminus Y$ and some local retraction $\rho\colon \mathbf{C}^N \to Y$. Remember that $E_Y C(X)$ is the blowing up of the subspace $\kappa^{-1}(Y)$ in $C(X)$, and κ' is obtained from the universal property of the blowing up, with respect to $E_Y X$ and the map ξ. Just as in the case where $Y = \{0\}$, it is worth mentioning that $E_Y C(X)$ lives inside the fiber product $C(X) \times_X E_Y X \subset X \times \mathbf{P}^{N-t-1} \times \check{\mathbf{P}}^{N-1}$ and can be described in the following way: take the inverse image of $E_Y X \setminus e_Y^{-1}(Y)$ in $C(X) \times_X E_Y X$ and close it, thus obtaining κ' as the restriction of the second projection to this space.

Looking at the definitions, it is not difficult to prove that, if we consider the divisor:

$$D = |\xi^{-1}(Y)| \subset E_Y C(X), \quad D \subset Y \times \mathbf{P}^{N-t-1} \times \check{\mathbf{P}}^{N-1},$$

and denote by $\check{\mathbf{P}}^{N-t-1} \subset \check{\mathbf{P}}^{N-1}$ the space of hyperplanes containing $T_0 Y$:

- The pair (X^0, Y) satisfies Whitney's condition a) along Y if and only if we have the set theoretical equality $|C(X) \cap C(Y)| = |\kappa^{-1}(Y)|$. It satisfies Whitney's condition a) at 0 if and only if $|\xi^{-1}(0)| \subset \mathbf{P}^{N-t-1} \times \check{\mathbf{P}}^{N-t-1}$.

Note that we have the inclusion $C(X) \cap C(Y) \subset \kappa^{-1}(Y)$, so it all reduces to having the inclusion $|\kappa^{-1}(Y)| \subset C(Y)$, and since we have already seen that every limit of tangent hyperplanes H contains a limit of tangent spaces T, we are just saying that every limit of tangent hyperplanes to X at a point $y \in Y$, must be a tangent hyperplane to Y at y. Following this line of thought, satisfying condition a) at 0 is then equivalent to the inclusion $|\kappa^{-1}(0)| \subset \{0\} \times \check{\mathbf{P}}^{N-t-1}$ which implies $|\xi^{-1}(0)| \subset \mathbf{P}^{N-t-1} \times \check{\mathbf{P}}^{N-t-1}$.

- The pair (X^0, Y) satisfies Whitney's condition b) at 0 if and only if $|\xi^{-1}(0)|$ is contained in the incidence variety $I \subset \mathbf{P}^{N-t-1} \times \check{\mathbf{P}}^{N-t-1}$.

This is immediate from the relation between limits of tangent hyperplanes and limits of tangent spaces and the interpretation of $E_Y C(X)$ as the closure of the inverse image of $E_Y X \backslash e_Y^{-1}(Y)$ in $C(X) \times_X E_Y X$ since we are basically taking limits as $x \to Y$ of couples (l, H) where l is the direction in $\mathbf{P}^{N-t-1}$ of a secant line $\overline{yx}$ with $x \in X^0 \backslash Y, y = \rho(x) \in Y$, where ρ is some local retraction of the ambient space to the nonsingular subspace Y, and H is a tangent hyperplane to X at x. So, in order to verify the Whitney conditions, it is important to control the geometry of the projection $D \to Y$ of the divisor $D \subset E_Y C(X)$.

Remark 1.6.5 Although it is beyond the scope of these notes, we point out to the interested reader that there is an algebraic definition of the Whitney conditions for X^0 along $Y \subset X$ solely in terms of the ideals defining $C(X) \cap C(Y)$ and $\kappa^{-1}(Y)$ in $C(X)$. Indeed, the inclusion $C(X) \cap C(Y) \subset \kappa^{-1}(Y)$ follows from the fact that the sheaf of ideals $\mathcal{J}_{C(X) \cap C(Y)}$ defining $C(X) \cap C(Y)$ in $C(X)$ contains the sheaf of ideals $\mathcal{J}_{\kappa^{-1}(Y)}$ defining $\kappa^{-1}(Y)$, which is generated by the pull-back by κ of the equations of Y in X. What was said above means that condition a) is equivalent to the second inclusion in:

$$\mathcal{J}_{\kappa^{-1}(Y)} \subseteq \mathcal{J}_{C(X) \cap C(Y)} \subseteq \sqrt{\mathcal{J}_{\kappa^{-1}(Y)}}.$$

It is proved in [35, Proposition 1.3.8] that having both Whitney conditions is equivalent to having the second inclusion in:

$$\mathcal{J}_{\kappa^{-1}(Y)} \subseteq \mathcal{J}_{C(X) \cap C(Y)} \subseteq \overline{\mathcal{J}_{\kappa^{-1}(Y)}}\ ,$$

where the bar denotes the integral closure of the sheaf of ideals, which is contained in the radical and is in general much closer to the ideal than the radical. The second inclusion is an algebraic expression of the fact that locally near every point of the common zero set the modules of local generators of the ideal $\mathcal{J}_{C(X) \cap C(Y)}$ are bounded, up to a multiplicative constant depending

only on the chosen neighborhood of the common zero set, by the supremum of the modules of generators of $\mathcal{J}_{\kappa^{-1}(Y)}$.
This result is used in [20] to produce an algorithm computing the Whitney stratification of a projective variety.

In the case where Y is a point x, the ideal defining $C(X) \cap C(\{x\})$ in $C(X)$ is just the pull-back by κ of the maximal ideal $m_{X,x}$, so it coincides with $\mathcal{J}_{\kappa^{-1}(x)}$ and Whitney's lemma for the smooth part X^0 follows.

Definition 1.6.6 Let $Y \subset X \subset \mathbf{C}^N$ as before. Then we say that the local polar variety $P_k(X; L^{d-k})$ is equimultiple along Y at a point $x \in Y$ if the map $y \mapsto m_y(P_k(X; L^{d-k}))$ is constant for $y \in Y$ in a neighborhood of x.
Note that this implies that if $(P_k(X; L^{d-k}), x) \neq \emptyset$, then $P_k(X; L^{d-k}) \supset Y$ in a neighborhood of x since the emptiness of a germ is equivalent to multiplicity zero.

We can now state the main theorem of this section, a complete proof of which can be found in [55, Chap. V, Thm. 1.2, p. 455].

Theorem 1.6.7 (Teissier; see also [18] for another proof) *Given $0 \in Y \subset X$ as before, the following conditions are equivalent, where ξ is the diagonal map in the normal/conormal diagram above:*

1) The pair (X^0, Y) satisfies Whitney's conditions at 0.
2) The local polar varieties $P_k(X, L)$, $0 \leqslant k \leqslant d-1$, are equimultiple along Y (at 0*), for general L.*
3) $\dim.\, \xi^{-1}(0) = N - t - 2$.

Note that since $\dim.\, D = N - 2$, condition 3) is open and the theorem implies that (X^0, Y) satisfies Whitney's conditions at 0 if and only if it satisfies Whitney's conditions in a neighborhood of 0.

Note also that by analytic semicontinuity of fiber dimension (see [11, Chap. 3, 3.6] or [30, §49]), condition 3) is satisfied outside of a closed analytic subspace of Y, which shows that Whitney's conditions are a stratifying condition in the sense of [55, Chap. III, Definition 1.4].

Moreover, since a blowing up does not lower dimension, the condition $\dim.\, \xi^{-1}(0) = N-t-2$ implies $\dim.\, \kappa^{-1}(0) \leqslant N-t-2$. So that, in particular $\kappa^{-1}(0) \not\supset \check{\mathbf{P}}^{N-t-1}$, where $\check{\mathbf{P}}^{N-t-1}$ denotes as before the space of hyperplanes containing T_0Y. This tells us that *a general hyperplane containing T_0Y is not a limit of tangent hyperplanes to X*. This fact is crucial in the proof that Whitney conditions are equivalent to the equimultiplicity of polar varieties since it allows the start of an inductive process. In the actual proof of [55], one reduces to the case where $\dim.\, Y = 1$ and shows by a geometric argument that the Whitney conditions imply that the polar curve has to be empty, which gives a bound on the dimension of $\kappa^{-1}(0)$. Conversely, the equimultiplicity condition on polar varieties gives bounds on the dimension of $\kappa^{-1}(0)$

by implying the emptiness of the polar curve and on the dimension of $e_Y^{-1}(0)$ by Hironaka's result, hence a bound on the dimension of $\xi^{-1}(0)$.

It should be noted that Hironaka had proved in [21, Corollary 6.2] that the Whitney conditions for X^0 along Y imply equimultiplicity of X along Y.

Finally, a consequence of the theorem is that *given a complex analytic space X, there is a unique minimal (coarsest) Whitney stratification*; any other Whitney stratification of X is obtained by adding strata inside the strata of the minimal one. A detailed explanation of how to construct this "canonical" Whitney stratification using Theorem 1.6.7, and the proof that this is in fact the coarsest one appears in [55, Chap. VI, §3]. The connected components of the strata of the minimal Whitney stratification give a minimal "Whitney stratification with connected strata"

1.6.3 The Whitney conditions are Lagrangian in nature

Consider the irreducible components $D_j \subset Y \times \mathbf{P}^{N-t-1} \times \check{\mathbf{P}}^{N-1}$ of the divisor $D = |\xi^{-1}(Y)|$, that is $D = \bigcup_j D_j$, and their images:

$$\begin{aligned} V_j &= \kappa'(D_j) \subset Y \times \mathbf{P}^{N-t-1}, \\ W_j &= \hat{e}_Y(D_j) \subset Y \times \check{\mathbf{P}}^{N-1}. \end{aligned}$$

We have $\kappa_X^{-1}(Y) = \bigcup_j W_j$ and $e_Y^{-1}(Y) = \bigcup_j V_j$:

Theorem 1.6.8 (Lê-Teissier, see [35, Thm. 2.1.1]) *The equivalent statements of Theorem 1.6.7 are also equivalent to the following one.*
For each j, the irreducible divisor D_j is the relative conormal space of its image $V_j \subset \mathrm{Proj}_Y C_{X,Y} \subset Y \times \mathbf{P}^{N-t-1}$ under the first projection $Y \times \mathbf{P}^{N-t-1} \to Y$ restricted to V_j, and all the fibers of the restriction $\xi|D_j : D_j \to Y$ have the same dimension near 0.

In particular, Whitney's conditions are equivalent to the equidimensionality of the fibers of the map $D_j \to Y$, plus the fact that each D_j is contained in $Y \times I \subset Y \times \mathbf{P}^{N-t-1} \times \check{\mathbf{P}}^{N-t-1}$, where $\check{\mathbf{P}}^{N-t-1}$ is the space of hyperplanes containing the tangent space $T_{Y,0}$ and I is the incidence subvariety. The new fact is that the contact form on $I \subset \mathbf{P}^{N-t-1} \times \check{\mathbf{P}}^{N-t-1}$ vanishes on the smooth points of $D_j(y)$ for $y \in Y$. This means that each D_j is Y-Lagrangian and is equivalent to a relative (or fiberwise) duality:

$$\begin{array}{ccl} & D_j & \longrightarrow W_j = Y\text{-dual of } V_j \subset Y \times \check{\mathbf{P}}^{N-t-1} \\ & \downarrow & \\ Y \times \mathbf{P}^{N-t-1} \supset & V_j & \end{array}$$

The proof uses that the Whitney conditions are stratifying in the sense of [55, Chap. III, Definition 1.4 and Proposition 2.2.2], and that Theorem 1.6.7 and the result of Remark 1.6.5 imply[5] that D_j is the conormal of its image over a dense open set of Y. The condition $\dim. \xi^{-1}(0) = N - t - 2$ then gives exactly what is needed, in view of Proposition 1.5.7, for D_j to be Y-Lagrangian.

Remark 1.6.9 As we have seen in subsection 1.6.2, the original definition of the Whitney conditions, translates as the fact that $|\xi^{-1}(Y)|$ is in $Y \times \mathbf{P}^{N-t-1} \times \check{\mathbf{P}}^{N-t-1}$ and not just $Y \times \mathbf{P}^{N-t-1} \times \check{\mathbf{P}}^{N-1}$ (condition a) and moreover lies in the product $Y \times I$ of Y with the incidence variety $I \subset \mathbf{P}^{N-t-1} \times \check{\mathbf{P}}^{N-t-1}$ (condition b)). Theorem 1.6.8 shows that they are in fact of a Lagrangian, or Legendrian, nature. This explains their stability by general sections (by non singular subspaces containing Y) as proved in [55, Chap. V] and linear projections, as proved in [35, Théorème 2.2.4].

The condition $\dim. \kappa^{-1}(y) \leqslant N - t - 2$ which follows from $\dim. \xi^{-1}(y) = N - t - 2$ corresponds to the fact that a general hyperplane of $\mathbf{C}^N$ containing $T_{Y,y}$ is not a limit of tangent hyperplanes to X^0, which is an important consequence of the Whitney conditions as we have already noted.

1.7 The multiplicities of local polar varieties and a Plücker type formula

In this section we relate the multiplicities of the local polar varieties of the closures of strata, which are algebraic invariants of singularities which can be computed by intersection theory in the normal/conormal diagram at a point, with vanishing Euler characteristics associated to the strata of a Whitney stratification.

As we shall see, when applied to the cone over a projective variety $Z \subset \mathbf{P}^{N-1}$ this formula yields a general Plücker type formula expressing the degree of the dual variety $\check{Z} \subset \check{\mathbf{P}}^{N-1}$ of Z in terms of the Euler characteristics of the strata of the minimal Whitney stratification $(Z_\alpha)_{\alpha \in A}$ of Z and their sections by general linear subspaces of all dimensions, and the vanishing Euler-Poincaré characteristics associated to pairs of strata $Z_\alpha \subset \overline{Z_\beta}$.

Proposition 1.7.1 (Lê-Teissier, see [36, §3])*Let $X = \bigsqcup_\alpha X_\alpha$ be a Whitney stratified complex analytic set of dimension d, with connected strata. Given $x \in X_\alpha$, choose a local embedding $(X, x) \subset (\mathbf{C}^N, 0)$. Set $d_\alpha = \dim. X_\alpha$. For each integer $i \in [d_\alpha + 1, d]$ there exists a Zariski open dense subset $W_{\alpha,i}$ in*

[5] The proof of this in [35] uses a lemma, p. 559, whose proof is incorrect, but easy to correct. There is an unfortunate mixup in notation. One needs to prove that $\sum_{t+1}^N \xi_k dz_k = 0$ and use the fact that the same vector remains tangent after the homothety $\xi_k \mapsto \lambda\xi_k,\ t+1 \leqslant k \leqslant N$. Since we want to prove that L_1 is Y-Lagrangian, we must take $dy_i = 0$.

the Grassmannian $G(N-i,N)$ and for each $L_i \in W_{\alpha,i}$ a semi-analytic subset E_{L_i} of the first quadrant of $\mathbf{R}^2$, of the form $\{(\epsilon,\eta)|0<\epsilon<\epsilon_0, 0<\eta<\phi(\epsilon)\}$ with $\phi(\epsilon)$ a certain Puiseux series in ϵ, such that the homotopy type of the intersection $X\cap(L_i+t)\cap\mathbf{B}(0,\epsilon)$ for $t\in\mathbf{C}^N$ is independent of $L_i\in W_{\alpha,i}$ and (ϵ,t) provided that $(\epsilon,|t|)\in E_{L_i}$. Moreover, this homotopy type depends only on the stratified set X and not on the choice of $x\in X_\alpha$ or the local embedding. In particular the Euler-Poincaré characteristics $\chi_i(X,X_\alpha)$ of these homotopy types are invariants of the stratified analytic set X.

Definition 1.7.2 The Euler-Poincaré characteristics $\chi_i(X,X_\alpha)$, for $i\in[d_\alpha+1,d]$ are called the local vanishing Euler-Poincaré characteristics of X along X_α.

The independence of the point $x\in X_\alpha$ is a consequence of the local topological triviality of the closures of the Whitney strata along the strata of their boundaries (The Thom-Mather Theorem). We shall not go into this here. See [58, Theorem 4.2.17]. The connection between the local vanishing Euler characteristics and the multiplicities of polar varieties is expressed as follows:

Theorem 1.7.3 (Lê-Teissier, see [34, théorème 6.1.9], [36, 4.11]) *With the conventions just stated, and for any Whitney stratified complex analytic set $X=\bigsqcup_\alpha X_\alpha\subset\mathbf{C}^N$, we have for $x\in X_\alpha$ the equality*

$$\chi_{d_\alpha+1}(X,X_\alpha)-\chi_{d_\alpha+2}(X,X_\alpha)=\\ \sum_{d_\beta>d_\alpha}(-1)^{d_\beta-d_\alpha-1}m_x(P_{d_\beta-d_\alpha-1}(\overline{X_\beta},x))(1-\chi_{d_\beta+1}(X,X_\beta)),$$

where it is understood that $m_x(P_{d_\beta-d_\alpha-1}(\overline{X_\beta},x))=0$ if $x\notin P_{d_\beta-d_\alpha-1}(\overline{X_\beta},x)$.

It follows that given a Whitney stratified complex analytic set $X=\bigsqcup_\alpha X_\alpha$ with connected strata, it is equivalent to give the collections of multiplicities of the local polar varieties of the closures $\overline{X_\beta}$ of strata at the points of the strata X_α in their boundary and to give the collections of vanishing Euler-Poincaré characteristics $\chi_i(\overline{X_\beta},X_\alpha)$. There is an invertible linear relation berween the two sets.

Let us now consider the special case where X is the cone over a projective variety Z, which we assume not to be contained in a hyperplane. The dual variety $\check{Z}$ of Z was defined in Subsection 1.3.2. Remember that every complex analytic space, and in particular Z, has a minimal Whitney stratification. We shall use the following facts, with the notation of Proposition 1.7.1 and those introduced after Proposition 1.3.8:

Proposition 1.7.4 (see [14, Section 8]) *Let $Z\subset\mathbf{P}^{N-1}$ be a projective variety of dimension d.*

1. *If $Z=\bigsqcup_\alpha Z_\alpha$ is a Whitney stratification of Z, denoting by $X_\alpha\subset\mathbf{C}^N$ the cone over Z_α, we have that $X=\{0\}\cup(\bigsqcup_\alpha X^*_\alpha)$, where $X^*_\alpha=X_\alpha\setminus\{0\}$, is a Whitney*

stratification of X. It may be that (Z_α) is the minimal Whitney stratification of V but $\{0\} \cup (\bigsqcup_\alpha X^*_\alpha)$ is not minimal, for example if Z is itself a cone.

2. *If $L_i + t$ is an i-codimensional affine space in $\mathbf{C}^N$ it can be written as $L_{i-1} \cap (L_1 + t)$ with vector subspaces L_i and for general directions of L_i we have, denoting by $\mathbf{B}(0,\epsilon)$ the closed ball with center 0 and radius ϵ, for small ϵ and $0 < |t| \ll \epsilon$:*

$$\chi_i(X,\{0\}) := \chi(X \cap (L_i + t) \cap \mathbf{B}(0,\epsilon)) = \chi(Z \cap H_{i-1}) - \chi(Z \cap H_{i-1} \cap H_1),$$

where $H_i = \mathbf{P}L_i \subset \mathbf{P}^{N-1}$.

3. *For every stratum X^*_α of X, we have the equalities $\chi_i(X, X^*_\alpha) = \chi_i(Z, Z_\alpha)$.*
4. *If the dual $\check{Z} \subset \check{\mathbf{P}}^{N-1}$ is a hypersurface, its degree is equal to $m_0(P_d(X,0))$, which is the number of non singular critical points of the restriction to Z of a general linear projection $\mathbf{P}^{N-1} \setminus L_2 \to \mathbf{P}^1$.*

Note that we will apply statements 2) and 3) not only to the cone X over Z but also to the cones $\overline{X_\beta}$ over the closed strata $\overline{Z_\beta}$.

If we now apply the theorem 1.7.3, we see that, using Proposition 1.7.4, we can rewrite in this case the formula of Theorem 1.7.3 as a generalized Plücker formula for any d-dimensional projective variety $Z \subset \mathbf{P}^{N-1}$ whose dual is a hypersurface:

Proposition 1.7.5 (Teissier, see [55, §5]) *Given the projective variety $Z \subset \mathbf{P}^{N-1}$ equipped with a Whitney stratification $Z = \bigsqcup_{\alpha\in A} Z_\alpha$, denote by d_α the dimension of Z_α. We have, if the projective dual $\check{Z}$ is a hypersurface in $\check{\mathbf{P}}^{N-1}$:*

$$\begin{aligned}(-1)^d \deg\check{Z} &= \chi(Z) - 2\chi(Z \cap H_1) + \chi(Z \cap H_2)\\ &- \sum_{d_\alpha < d} (-1)^{d_\alpha} \deg_{N-2} P_{d_\alpha}(\overline{Z_\alpha})(1 - \chi_{d_\alpha+1}(Z, Z_\alpha))\ ,\end{aligned}$$

where H_1, H_2 denote general linear subspaces of $\mathbf{P}^{N-1}$ of codimension 1 and 2 respectively, $\deg_{N-2} P_{d_\alpha}(\overline{Z_\alpha})$ is the number of nonsingular critical points of a general linear projection $\overline{Z_\alpha} \to \mathbf{P}^1$, which is the degree of $\check{\overline{Z_\alpha}}$ if it is a hypersurface and is set equal to zero otherwise. It is equal to 1 if $d_\alpha = 0$.

Here we remark that if $(Z_\alpha)_{\alpha\in A}$ is the minimal Whitney stratification of the projective variety $Z \subset \mathbf{P}^{N-1}$, and L is a general linear subspace in $\mathbf{P}^{N-1}$, the $Z_\alpha \cap L$ that are not empty constitute the minimal Whitney stratification of $Z \cap L$. See [55, Chap. III, Lemma 4.2.2] and use the fact that the minimal Whitney stratification is defined by equimultiplicity of polar varieties (see [55, Chap. VI, §3]) and that the multiplicity of polar varieties of dimension > 1 is preserved by general hyperplane sections as we saw before Theorem 1.7.3.

It is explained in [14, Section 8] that if the dual of Z is not a hypersurface, the dual of the intersection of Z with a general linear space of $\mathbf{P}^{N-1}$ of codimension $\delta(Z) = \mathrm{codim}_{\check{\mathbf{P}}^{N-1}} \check{Z} - 1$ is a hypersurface of the same degree as $\check{Z}$. Using this and an induction on the dimension by applying proposition 1.7.4, possibly after general linear sections, to compute the degrees of the $\check{\overline{Z_\alpha}}$, we see that *we have proved the existence of a general formula to compute the degree of* $\check{Z}$ *from the Euler-Poincaré characteristics of the closed strata* $\overline{Z_\alpha}$ *and their general linear sections, and the vanishing Euler-Poincaré characteristics* $\chi_i(\overline{Z_\beta}, Z_\alpha)$. We shall not write this formula explicitly, only remark that it is linear in the Euler-Poincaré characteristics of the strata and their general linear sections, and polynomial of degree bounded by the depth (the integer d in [58, Definition 4.1.1]) of the stratification in the local vanishing Euler-Poincaré characteristics. *The degree of the variety* $\check{Z}$ *of all limit tangent hyperplanes to a projective variety* Z *depends explicitly on basic topological characters of its minimal Whitney stratification.*

Remark 1.7.6 We note that as we compute the degree of the dual $\check{Z}$, we also compute the degrees of the duals of the closures of at least some of the strata of the canonical Whitney stratification $Z = \bigsqcup_\alpha Z_\alpha$. This suggests the definition of the *total dual* of the projective variety Z: it is the union of the duals of the closures of the strata of its canonical Whitney stratification. For example if Z is the dual of a general non singular projective plane curve its total dual is the union of that curve, its bitangents and its tangents of inflexion, corresponding respectively to the nodes and cusps of Z. The total dual gives a tangentially exploded view of the singularities of Z.

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