Bernard Teissier Overweight deformations of weighted affine toric varieties Teissier, Bernard

I report on some recent progress in the program outlined in ([3], [4]) which tends to show that any excellent equicharacteristic germ of space X with an algebraically closed residue field k can be formally embedded in an affine space $\mathbf{A}^N(k)$ in such a way that it then has an embedded resolution by a toric modification of $\mathbf{A}^N(k)$.

The simplest example is the plane branch, say in characteristic zero, with equation $(U_2^2-U_1^3)^2-U_1^5U_2=0$. It cannot be resolved by any toric modification of the plane but if we embed it in $\mathbf{A}^3(k)$ as $U_2^2-U_1^3-U_3=0, U_3^2-U_1^5U_2=0$ it is resolved by any toric modification of $\mathbf{A}^3(k)$ which resolves the monomial curve with equations $U_2^2-U_1^3=0, U_3^2-U_1^5U_2=0$, to which it specializes flatly by putting a parameter in front of U_3 in the first equation and which is the toric curve corresponding to the semigroup of values $\langle 4,6,13\rangle$ taken on the local algebra of the curve by its unique k-valuation.

The approach I use is to begin by studying the analogous problem for local uniformization. This means that given a valuation of the local excellent equicharacteristic ring of our germ X, we wish to embed it so that a toric modification of $\mathbf{A}^N(k)$ makes the strict transform of X regular at the point picked by the valuation. After reductions which are outlined in [3]) one is led to study the following situation:

Let k be an algebraically closed field and Φ a totally ordered abelian group of finite rational rank r.

A *weight* on the rings $k[U_1,\dots,U_N]$ or $k[[U_1,\dots,U_N]]$ is an homomorphism of groups $\lambda\colon \mathbf{Z}^N\to\Phi$ which is induced by an homomorphism of semigroups $\mathbf{N}^N\to\Phi_+$. It induces a weight on monomials by $w(U^m)=\lambda(m)$ and a monomial order by $U^m<U^n\leftrightarrow\lambda(m)<\lambda(n)$.

This rather general situation can be reduced to a more familiar one thanks to the following result (see [3], Proposition 4.12):

The positive semigroup of a totally ordered semigroup if finite rational rank r is the union of a nested sequence of free subsemigroups of rank r:

$$\cdots\mathbf{N}^r_{(h)}\subset\mathbf{N}^r_{(h+1)}\subset\cdots\subset\Phi_+,$$

the inclusions being semigroup maps.

As explained in *loc.cit.* this result can be viewed as an avatar of the Jacobi-Perron algorithm for approximating vectors in $\mathbf{R}^N$ by integral vectors.

So we may assume in the sequel that $\Phi=\mathbf{Z}^r$ with a total order, but there are good reasons to begin as I did.

Then denoting by e_i the i-th basis vector of $\mathbf{Z}^N$ and setting $\gamma_i=\lambda(e_i)\in\mathbf{Z}^r$ we may consider the semigroup Γ generated by $\gamma_1,\dots,\gamma_N$ and the toric variety $X_0=\mathrm{Spec}k[t^\Gamma]$ where $k[t^\Gamma]$ is the semigroup algebra with coefficients in k. It is the closure of the orbit of the point $(1,1,\dots,1)\in\mathbf{A}^N(k)$ under the action of the torus k^{*r} determined by $(t,z_1,\dots,z_N)\mapsto(t^{\gamma_1}z_1,\dots,t^{\gamma_N}z_N)$ with $t=(t_1,\dots,t_r)\in k^{*r}$ and $t^{\gamma_j}=t_1^{\gamma_{j1}}\dots t_r^{\gamma_{jr}}$

Denoting by $\mathcal{L}$ the lattice which is the kernel of λ, we may choose a system of generators $(U^{m^\ell} - U^{n^\ell})_{\ell\in L}$ for $\mathcal{L}$, where all the exponents are non negative.
The ideal I_0 of $k[U_1,\dots,U_N]$ generated by the $(U^{m^\ell} - U^{n^\ell})_{\ell\in L}$ is a prime binomial ideal defining the embedding $X_0 \hookrightarrow \mathbf{A}^N(k)$. The Krull dimension of $k[t^\Gamma]$ is equal to r; it is the dimension of X_0. If we denote by H_ℓ the hyperplane of $\check{\mathbf{R}}^N$ which is dual to the vector $m^\ell - n^\ell \in \mathbf{R}^N$, it is shown in [2] that any regular fan Σ with support the first quadrant of $\check{\mathbf{R}}^N$ and compatible with the hyperplanes H_ℓ for $\ell \in L$ determines a toric modification $Z(\Sigma) \to \mathbf{A}^N(k)$ which gives an embedded resolution of the toric variety X_0.

We are going to show that *some of these toric modifications also resolve the singularities of some special deformations of X_0, called* overweight*, at the point picked by a valuation which is determined by the weight w.*

An *overweight deformation* of our affine toric variety X_0 is described by a deformation of its equations of the following form:

$$F_\ell = U^{m^\ell} - U^{n^\ell} + \sum_p c_p^\ell U^p \quad \text{with } w(U^p) > w(U^m) = w(U^n),\ \ell \in L.$$

Here the F_ℓ are elements of $k[[U_1,\dots,U_N]]$. We denote by I the ideal which they generate.
The initial forms of the series F_ℓ with respect to the monomial order determined by the weight w are the binomials $(U^{m^\ell} - U^{n^\ell})_{\ell\in L}$. We assume as part of the definition that the dimension of the ring $R = k[[U_1,\dots,U_N]]/I$ is equal to r, or equivalently that the $(U^{m^\ell} - U^{n^\ell})_{\ell\in L}$ generate the initial ideal of I with respect to w.
Then one can check that the ring R is endowed with a valuation defined in the following manner:
Define the order $\nu(x)$ of an element of R as the maximum weight of one of its representatives in $k[[U_1,\dots,U_N]]$ (this maximum exists if $x \neq 0$). This order defines a filtration of R whose associated graded ring is the quotient of $k[U_1,\dots,U_N]$ by the initial ideal of I. By our assumption it is $k[U_1,\dots,U_N]/I_0$ and an integral domain, so that ν is a valuation.

It is shown in [3] that any complete equicharacteristic local ring endowed with a rational valuation whose associated graded ring is finitely generated over k is obtained in this manner.

Now we want to find a fan subdividing $\check{\mathbf{R}}^N_+$, compatible with the hyperplanes H_ℓ, and such that the strict transform of each F_ℓ is a deformation of the strict transform of its initial form at the point determined by the valuation. This will ensure the nonsingularity at that point of the strict transform.
The idea is very simple to explain in the case where the value group Φ (or the order on $\mathbf{Z}^r$) is of rank one. For simplicity let us look at one equation $F = U^m - U^n + \sum_p c_p U^p$.
Let

$$E' = \langle \{p - n / c_p \neq 0\}, m - n\rangle \subset \mathbf{R}^N,$$

where as above $\langle a, b, ...\rangle$ denotes the cone generated by $a, b, ...$ Since there are infinitely many exponents p, the strictly convex cone E' may not be rational, but the power series ring being noetherian E' is contained in a rational cone E, also strictly convex.

Given a regular cone $\sigma = \langle a^1, \ldots, a^N\rangle \subset \check{\mathbf{R}}^N$, set $Z(\sigma) = \mathrm{Spec} k[\check{\sigma} \cap M]$.
The map $Z(\sigma) \to \mathbf{A}^N(k)$ is monomial and we write it as:

$$U_i \mapsto Y_1^{a_i^1} \ldots Y_N^{a_i^N},\ 1 \leq i \leq N.$$

Assuming that Φ is of (real) rank one, choose an ordered embedding $\Phi \subset \mathbf{R}$ and using it define the *weight vector*

$$\mathbf{w} = (w(U_1), \ldots, w(U_N)) \in \check{\mathbf{R}}^N.$$

The center of the valuation ν is in $Z(\sigma)$ if and only if the weights $w(Y_i)$, which are uniquely determined by the monomial map since the a^j form a basis, are all ≥ 0, which is equivalent to the condition that $\mathbf{w} \in \sigma$. Note that $\mathbf{w}$ is in the hyperplane H of $\check{\mathbf{R}}^N$ corresponding to the vector $m - n$ and that our overweight hypothesis says precisely that $\mathbf{w}$ lies in the interior of the intersection with H of the convex dual $\check{E}$ of E. Note that $\check{E}$ is of dimension N.
So if Σ is a regular subdivision of $\check{\mathbf{R}}_+^N$ which is compatible with H and $\check{E}$, it will contain a regular cone σ of dimension N whose intersection with H is of dimension $N - 1$ and which contains $\mathbf{w}$. By compatibility with H the cone σ is entirely on one side of H so we may assume that the scalar products $\langle a^i, m - n\rangle$ which are not zero are all > 0. By compatibility with $\check{E}$ the cone σ is contained in $\check{E} \subseteq \check{E}'$, so that the $\langle a^i, p - n\rangle$ are all ≥ 0. In the corresponding chart $Z(\sigma)$ the transform of our equation F by the monomial map can then be written:

$$Y_1^{\langle a^1, n\rangle} \ldots Y_N^{\langle a^N, n\rangle}\big(Y_1^{\langle a^1, m-n\rangle} \ldots Y_N^{\langle a^N, m-n\rangle} - 1 + \sum_p c_p Y_1^{\langle a^1, p-n\rangle} \ldots Y_N^{\langle a^N, p-n\rangle}\big).$$

This shows that the strict transform of F, which is the quantity between parenthesis, is a deformation of the strict transform of its initial part, and this gives in this case the result we seek, since the initial part is obviously non singular.

The proof in the general case follows the same line but is somewhat less simple especially when the (real) rank of the value group is > 1.

One should note that unlike the case of plane branches or more generally of quasi-ordinary hypersurfaces (see [1], 5.3), one may have to choose a resolution of the toric variety adapted to the deformation and not just any resolution.

References

[1] P. González Pérez, *Toric embedded resolutions of quasi-ordinary hypersurfaces*, Annales Inst. Fourier (Grenoble) 53, (2003), 1819-1881.

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[3] B. Teissier, *Valuations, deformations, and toric geometry*, Valuation Theory and its applications, Vol. II, Fields Inst. Commun. 33, AMS., Providence, RI., (2003), 361-459.

[4] B. Teissier, *Monomial ideals, binomial idals, polynomial ideals*, Trends in Commutative Algebra, MSRI publications, Cambridge University Press (2004), 211-246.