Bernard TeissierA viewpoint on local resolution of singularities Teissier, Bernard

In the talk I gave a brief report on the progress made towards resolution of singularities in positive characteristic as it was presented by various groups during the RIMS workshop of December 2008. (see http://www.kurims.kyoto-u.ac.jp/ kenkyubu/proj08-mori/index.html for details and documents)

Apart from the work of Cossart-Piltant proving resolution of singularities in dimension 3 in the equicharacteristic case (see [2], [3]), all approaches follow the approach of Hironaka's fundamental paper, as modified by Villamayor to put the idealistic exponents, or basic objects, at the center of the process, as the only objects which need to be resolved (see [4], [8]).

An idealistic exponent on a non singular space W is a pair (J, b) of a coherent Ideal J on W and an integer $b \geq 1$. Its singular locus is the set of points x of W where $\frac{\nu_x(J)}{b} > 1$. The order $\nu_x(J)$ is the largest integer n such that $J_x \subseteq m_x^n$.

One then defines a permissible center for (J, b) as a non singular subvariety Y of W which is contained in the singular locus of (J, b). The transform of (J, b) by the blowing-up $W' \to W$ with center Y is then defined as $(J', b') = (I_Y \mathcal{O}_{W'})^{-b} J \mathcal{O}_{W'}, b)$.

The goal is then essentially to prove the existence of a finite sequence of permissible blowing-ups such that the final singular locus is empty. In fact all groups try to prove the existence of a canonical process, and one has to use a richer definition of idealistic exponents and their transforms, taking into account at each stage the exceptionnal divisors created by the previous blowing-ups. In order to produce a canonical process one associates to an idealistic exponent an "invariant" at each point of W, with values in an ordered set and such that the set of points of W where the invariant is the worst (largest) is non singular, or at least has simple normal crossings, and that blowing it up (or blowing up its components in some order) will make the worst invariant decrease strictly.

The main problem in positive characteristic is the non-existence of "hypersurfaces of maximal contact" with (J, b) in W. In characteristic zero, one can define on such non singular hypersurfaces a "trace" of the idealistic exponent which retains enough information about the order of the ideal and its behavior under permissible blowing-up to permit a proof by induction on the dimension. All the attempts to prove resolution in positive characteristic replace the idealistic exponent by (different) graded algebras which are stable under derivation, finitely generated and in several cases integrally closed. The generators are expected to play the role of maximal contact by allowing an inductive process.

The generators of the graded algebras just mentioned are monomials of the form $x_i^{p^{e_i}}$ where p is the characteristic, so that comparison with monomial ideals plays a role in all programs.

In the last years I have been led to try to prove local uniformization (a very local version of resolution) by a completely different method, in which the basic idea is to compare a given singular germ *by deformation* with a space whose resolution is easy and blind to the characteristic. The spaces in question are affine toric varieties, which are defined by prime binomial ideals. I refer to [7] for their toric

embedded resolution and to [1] for the proof of a canonical embedded resolution by composition of blowing-ups with equivariant non singular centers.
Given a base field k, which we assume to be algebraically closed, the algebra $k[t^\Gamma]$ of an affine toric variety over k is the semigroup algebra over k of a finitely generated semigroup Γ, for example a polynomial ring $k[u_1, \ldots, u_N]$. A binomial ideal in this algebra is an ideal generated by differences of terms, where a term is the product of a monomial with an element of k^*. It turns out that there is a deep relation between the most important valuations from the viewpoint of local uniformization and affine toric varieties. See [9].

Let R be a noetherian excellent equicharacteristic local domain with an algebraically closed residue field $k = R/m$. Let ν be a valuation on R, corresponding to an inclusion $R \subset R_\nu$ of R in a valuation ring of its field of fractions. We may assume that R_ν dominates R in the sense that $m_\nu \cap R = m$ and that the residual injection $R/m \hookrightarrow R_\nu/m_\nu$ is an isomorphism. This corresponds to the fact that the point picked by ν in all schemes birationally dominating $\mathrm{Spec}R$ by a proper map is a *closed* point.

In many important cases (see [9], [11]), one can check that there exists a formal embedding of $(\mathrm{Spec}R, m)$ in an affine space $(\mathbf{A}^N(k), 0)$ with the following properties: there is a system of coordinates such that the intersection of $(\mathrm{Spec}R, m)$ with the torus $T^N(k)$ consisting of the complement of the coordinate hyperplanes is dense, and a birational map of toric varieties $Z \to \mathbf{A}^N(k)$ with Z regular, which is equivariant with respect to $T^N(k)$ and such that the strict transform of $(\mathrm{Spec}R, m)$ in Z is non singular and transversal to the non dense orbits of Z *at the point of this strict transform picked by the valuation* ν.
Such a result is a constructive form of local uniformization, at least if one can effectively construct the embedding.

In the case of plane branches (see [5]) and more generally of quasi-ordinary hypersurfaces (see [6]), the smallest embedding with this property can be explicitely constructed in characteristic zero from (generalized) Puiseux expansions.

Since it seems much easier to glue up the embeddings corresponding to various valuations (by compactness of the Zariski-Riemann manifold a finite number suffices) than to glue up *à la Zariski* various birational models, this led me to ask in [10] the following:
Question: *Given a noetherian excellent equicharacteristic local domain R with an algebraically closed residue field $k = R/m$, does there exist a formal embedding of* $(\mathrm{Spec}R, m)$ *with a toric birational map of toric varieties $Z \to \mathbf{A}^N(k)$ in appropriate coordinates on $\mathbf{A}^N(k)$ such that the strict transform of* $\mathrm{Spec}R$ *in Z is non singular and transversal to the non dense orbits at each point mapped to the closed point m of* $\mathrm{Spec}R$.
One may ask that in addition the singular locus of $\mathrm{Spec}R$ should be the union of intersections with $\mathrm{Spec}R$ of sets of coordinate hyperplanes in this new embedding. The map from the strict transform to $\mathrm{Spec}R$ is then an isomorphism outside of the singular locus. I think of this as a generalization of the condition of non-degeneracy with respect to a Newton polyhedron.

This would imply local resolution of singularities and the difficulty is moved from the study of the behaviour of the order of ideals under certain blowing ups to the search of functions in R having very special properties. The simplest non trivial example is the plane curve $(y^2 - x^3)^2 - x^5 y = 0$ which can be resolved by a single toric modification of the ambient space only after being embedded in $\mathbf{A}^3(k)$ by the functions $x, y, y^2 - x^3$. See [5], [9] and [11].

After hearing me mention this last January at the Workshop on Toric Geometry (see [11]), Jenia Tevelev kindly sent me a proof of the following:
Theorem (Tevelev) Let k be an algebraically closed field of characteristic zero and let $X \subset \mathbf{P}^n(k)$ be a projective algebraic variety. Then, for a sufficiently high order Veronese reembedding $X \subset \mathbf{P}^N(k)$ one can choose projective coordinates $z_0 : \ldots : z_N$ such that if $T^N(k)$ is the torus $(k^*)^N$ consisting of the complement of the coordinate hyperplanes in $\mathbf{P}^N(k)$,

- The intersection of X with $T^N(k)$ is dense in X,
- There exists a nonsingular toric variety Z and an equivariant map $Z \to \mathbf{P}^N(k)$ such that the strict transform of X is non singular and transversal to the non dense toric orbits in Z.

The proof uses resolution of singularities and answers the question in characteristic zero for algebraizable singularities while of course one would hope to prove local resolution in the manner I have described. Still, it is very encouraging.

References

[1] Bierstone, Edward; Milman, Pierre D. *Desingularization of toric and binomial varieties*, J. Algebraic Geom. 15 (2006), no. 3, 443–486.

[2] V. Cossart and O. Piltant, *Resolution of singularities of threefolds in positive characteristic, I ,Reduction to local uniformization on Artin-Schreier and purely inseparable coverings*, J. Algebra 320 (2008), no. 3, 1051–1082.

[3] V. Cossart and O. Piltant, *Resolution of singularities of threefolds in positive characteristic. II*, J. Algebra 321 (2009), no. 7, 1836–1976.

[4] S. D. Cutkosky, *Resolution of singularities*, Graduate Studies in Math. vol. 63, AMS 2004.

[5] R. Goldin and B. Teissier, *Resolving singularities of plane analytic branches with one toric morphism*, Resolution of singularities (Obergurgl, 1997), 315–340, Progr. Math., 181, Birkhuser, Basel, 2000.

[6] P. González Pérez, *Toric embedded resolutions of quasi-ordinary hypersurfaces*, Annales Inst. Fourier (Grenoble) 53, (2003), 1819-1881.

[7] P. González Pérez et B. Teissier *Embedded resolutions of non necessarily normal affine toric varieties*, Comptes-tendus Acad. Sci. Paris, Ser.1, 334, (2002), 379-382.

[8] J. Kollár, *Lectures on resolution of singularities*, Annals of Mathematics Studies, 166. Princeton University Press, Princeton, NJ, 2007.

[9] B. Teissier, *Valuations, deformations, and toric geometry*, Valuation Theory and its applications, Vol. II, Fields Inst. Commun. 33, AMS., Providence, RI., (2003), 361-459.

[10] B. Teissier, *Monomial ideals, binomial idals, polynomial ideals*, Trends in Commutative Algebra, MSRI publications, Cambridge University Press (2004), 211-246.

[11] B. Teissier, *Overweight deformations of weighted affine toric varieties*, Oberwolfach workshop on Toric Geometry, January 2009, Oberwolfach Report OWR 2009 01.

[12] J. Tevelev, *On a question of Bernard Teissier*. letter to the author, January 2009.